SPECTRAL PROPERTIES OF GENERAL ADVECTION OPERATORS AND WEIGHTED TRANSLATION SEMIGROUPS

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ABSTRACT. We investigate the spectral properties of a class of weighted shift semigroups $(\mathcal{U}(t))_{t\geqslant 0}$ associated to abstract transport equations with a Lipschitz–continuous vector field $\mathscr{F}$ and no–reentry boundary conditions. Generalizing the results of [25], we prove that the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ admits a canonical decomposition into three C_0-semigroups $(\mathcal{U}_1(t))_{t\geqslant 0}$, $(\mathcal{U}_2(t))_{t\geqslant 0}$ and $(\mathcal{U}_3(t))_{t\geqslant 0}$ with independent dynamics. A complete description of the spectra of the semigroups $(\mathcal{U}_i(t))_{t\geqslant 0}$ and their generators $\mathcal{T}_i$, $i=1,2$ is given. In particular, we prove that the spectrum of $\mathcal{T}_i$ is a left-half plane and that the Spectral Mapping Theorem holds: $\mathfrak{S}(\mathcal{U}_i(t)) = \exp\{t\mathfrak{S}(\mathcal{T}_i)\}$, $i=1,2$. Moreover, the semigroup $(\mathcal{U}_3(t))_{t\geqslant 0}$ extends to a C_0-group and its spectral properties are investigated by means of abstract results from positive semigroups theory. The properties of the flow associated to $\mathscr{F}$ are particularly relevant here and we investigate separately the cases of periodic and aperiodic flows. In particular, we show that, for periodic flow, the Spectral Mapping Theorem fails in general but $(\mathcal{U}_3(t))_{t\geqslant 0}$ and its generator $\mathcal{T}_3$ satisfy the so-called Annular Hull Theorem. We illustrate our results with various examples taken from collisionless kinetic theory.

1. INTRODUCTION AND PRELIMINARIES

We develop in the present paper a systematic approach to the spectral analysis in L^p-spaces $(1 \leqslant p < \infty)$ of a class of *weighted shift semigroups* arising in kinetic theory

$$\mathcal{U}(t)\ : f \longmapsto \mathcal{U}(t)f(\mathbf{x}) = \exp\left[-\int_0^t \nu(\Phi(\mathbf{x},-s))\mathrm{d}s\right] f(\Phi(\mathbf{x},-t))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}) \qquad (1.1)$$

where the flow $\Phi(\mathbf{x},t)$ is associated to a globally Lipschitz transport field $\mathscr{F}$ and $\nu(\mathbf{x})$ is given by

$$\nu(\mathbf{x}) = h(\mathbf{x}) + \mathrm{div}(\mathscr{F})(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}$$

where the functions h and $\mathscr{F}$ satisfy the following:

Assumption 1.1. *The field* $\mathscr{F}\ :\ \overline{\mathbf{\Omega}} \to \mathbb{R}^N$ *is Lipschitz-continuous with Lipschitz constant* $\kappa > 0$. *Moreover, its divergence* $\mathrm{div}(\mathscr{F})$ *is a bounded function on* $\mathbf{\Omega}$. *The absorption function* $h(\cdot)$ *is measurable and bounded below.*

Before explaining more in details the contents of this work, we have to explicit a bit the first properties of the different terms arising in Eq. (1.1). Since $\mathscr{F}$ is Lipschitz over $\mathbf{\Omega}$ (with constant $\kappa > 0$), it is known from Kirszbraun's extension theorem [17, p. 201], that $\mathscr{F}$ can be extended as a Lipschitz function (with the same Lipschitz constant $\kappa > 0$) over the whole space $\mathbb{R}^N$. We

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shall still denote this extension by $\mathscr{F}$. In Eq. (1.1), $\Phi(\mathbf{x},t)$ is the unique maximal solution of the characteristic equation

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t)=\mathscr{F}(\mathbf{X}(t)), \qquad (t\in\mathbb{R});\\ \mathbf{X}(0)=\mathbf{x},\end{cases} \tag{1.2}$$

which is well-defined since the (extended) field $\mathscr{F}$ is globally Lipschitz. In (1.1), $\tau_-(\mathbf{x})$ denotes the stay time in $\mathbf{\Omega}$ of the characteristic curves $t>0\mapsto\Phi(\mathbf{x},-t)$ starting from $\mathbf{x}\in\mathbf{\Omega}$:

$$\tau_{\pm}(\mathbf{x})=\inf\{s>0\,;\Phi(\mathbf{x},\pm s)\notin\mathbf{\Omega}\}, \tag{1.3}$$

with the convention that $\inf\varnothing=\infty$. In other words, given $\mathbf{x}\in\mathbf{\Omega}$, $I_{\mathbf{x}}=(-\tau_-(\mathbf{x}),\tau_+(\mathbf{x}))$ is the maximal time interval for which the solution $\mathbf{X}(t)$ lies in $\mathbf{\Omega}$ for any $t\in I_{\mathbf{x}}$. We shall denote by $\tau(\mathbf{x}):=\tau_+(\mathbf{x})+\tau_-(\mathbf{x})$ the length of the maximal interval $I_{\mathbf{x}}$.

The general strategy we adopt to describe the spectral properties of the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ consists in a canonical decomposition of the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ into three semigroups $(\mathcal{U}_i(t))_{t\geqslant 0}$, $(i=1,2,3)$ with *independent dynamics*, the third one $(\mathcal{U}_3(t))_{t\geqslant 0}$ extending to a C_0-group. Notice that it would be possible to investigate the spectral properties of $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$ within the general framework developed in [13, Chapter 6] (see also [22]). The approach of [13] uses sophisticated tools from dynamical systems theory while our approach is completely different and relies on general results concerning the spectral properties of C_0-groups in L^p-spaces given in the Appendix.

1.1. **Preliminaries and motivation.** If $\tau(\mathbf{x})$ is finite, then the function $\mathbf{X}\;:\;s\in I_{\mathbf{x}}\longmapsto\Phi(\mathbf{x},s)$ is bounded since $\mathscr{F}$ is Lipschitz continuous. Moreover, still by virtue of the Lipschitz continuity of $\mathscr{F}$, the only case when $\tau_{\pm}(\mathbf{x})$ is finite is when $\Phi(\mathbf{x},\pm s)$ reaches the boundary $\partial\mathbf{\Omega}$ so that $\Phi(\mathbf{x},\pm\tau_{\pm}(\mathbf{x}))\in\partial\mathbf{\Omega}$. We finally mention that it is not difficult to prove that the mappings $\tau_{\pm}:\mathbf{\Omega}\to\mathbb{R}^+$ are lower semi-continuous and therefore measurable [7, p. 301]. Note that, since the field $\mathscr{F}$ is not assumed to be divergence–free, then the transformation induced by the flow Φ is not measure–preserving. Precisely, one has the following [21, 6]:

Proposition 1.2. *Let ϱ_t denote the image of the N-dimensional Lebesgue measure $\mathfrak{m}$ through the transformation $T_t:\mathbf{x}\mapsto\Phi(\mathbf{x},-t)$, $(t\in I_{\mathbf{x}})$. Then, ϱ_t is absolutely continuous with respect to $\mathfrak{m}$, and its Radon-Nikodym derivative $\frac{\mathrm{d}\varrho_t}{\mathrm{d}\mathfrak{m}}$ with respect to $\mathfrak{m}$ is given by*

$$\frac{\mathrm{d}\varrho_t}{\mathrm{d}\mathfrak{m}}(\mathbf{x})=\exp\left[\int_0^t\mathrm{div}(\mathscr{F})(\Phi(\mathbf{x},s))\mathrm{d}s\right]\qquad \text{for } \mathfrak{m}-a.e.\;\; \mathbf{x}\in\mathbf{\Omega},\;\; t\in I_{\mathbf{x}}.$$

The semiflow Φ enjoys the following elementary properties [7]

Proposition 1.3. *Let $\mathbf{x}\in\mathbf{\Omega}$ and $t\in\mathbb{R}$ be fixed. Then,*

(i) $\Phi(\mathbf{x},0)=\mathbf{x}$.
(ii) $\Phi(\Phi(\mathbf{x},s_1),s_2)=\Phi(\mathbf{x},s_1+s_2),\quad\forall s_1\in I_{\mathbf{x}},s_2\in(-s_1-\tau_-(\mathbf{x}),\tau_+(\mathbf{x})-s_1)$.
(iii) $|\Phi(\mathbf{x}_1,t)-\Phi(\mathbf{x}_2,t)|\leqslant\exp(\kappa|t|)|\mathbf{x}_1-\mathbf{x}_2|$ *for any* $\mathbf{x}_1,\mathbf{x}_2\in\mathbf{\Omega}$, $t\in I_{\mathbf{x}_1}\cap I_{\mathbf{x}_2}$.

In all the paper, we *fix* $1\leqslant p<\infty$ and set

$$X=L^p(\mathbf{\Omega},\mathrm{d}\mathfrak{m}).$$

The following classical result (see, e.g., [7, 6, 8]) asserts that the family $(\mathcal{U}(t))_{t\geqslant 0}$ given by (1.1) is a strongly continuous semigroup of bounded operators in X.

Theorem 1.4. *Let*

$$\mathcal{U}(t)f(\mathbf{x}) = \exp\left[-\int_0^t \nu(\Phi(\mathbf{x},-s))\mathrm{d}s\right] f(\Phi(\mathbf{x},-t))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}), \qquad \mathbf{x}\in\mathbf{\Omega},\ f\in X,$$

where χ_A denotes the characteristic function of a set A. The family $(\mathcal{U}(t))_{t\geqslant 0}$ is a positive C_0-semigroup of bounded operators in X. We shall denote by $(\mathcal{T},\mathscr{D}(\mathcal{T}))$ its generator.

In the present paper, we do not need to explicit further the generator $(\mathcal{T},\mathscr{D}(\mathcal{T}))$ of $(\mathcal{U}(t))_t$ since our spectral analysis does not depend on its description. Note however that, if $\mathbf{\Omega}$ is a sufficiently smooth open subset of $\mathbb{R}^N$, then the generator $(\mathcal{T},\mathscr{D}(\mathcal{T}))$ is explicitly described in [6] for $h = 0$. In some sense, which we do not explicit here (see for instance [6, 5]), the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ governs the following advection equation:

$$\partial_t f(\mathbf{x},t) + \nabla_{\mathbf{x}}\cdot(\mathscr{F}(\mathbf{x})f(\mathbf{x},t)) + h(\mathbf{x})f(\mathbf{x},t) = 0 \qquad (\mathbf{x}\in\mathbf{\Omega},\ t>0), \tag{1.4a}$$

supplemented by the boundary condition

$$f_{|\Gamma_-}(\mathbf{y},t) = 0, \qquad (\mathbf{y}\in\Gamma_-, t>0), \tag{1.4b}$$

and the initial condition

$$f(\mathbf{x},0) = f_0(\mathbf{x}), \qquad (\mathbf{x}\in\mathbf{\Omega}), \tag{1.4c}$$

where Γ_- is the incoming part of the boundary of $\mathbf{\Omega}$ (we refer the reader to [8, 7] for details on the matter), i.e. $f(\mathbf{x},t) = [\mathcal{U}(t)f_0](\mathbf{x})$ at least for regular initial data f_0. A typical example of such an absorption equation is the so-called Vlasov equation for which:

i) The phase space $\mathbf{\Omega}$ is given by the cylindrical domain $\mathbf{\Omega} = \mathcal{D}\times\mathscr{V}\subset\mathbb{R}^6$ where $\mathcal{D}$ is a smooth open subset of $\mathbb{R}^3$, referred to as the *position space*, and $\mathscr{V}\subset\mathbb{R}^3$ is referred to as the *velocity space*.

ii) For any $\mathbf{x} = (x,v)\in\mathcal{D}\times\mathscr{V}$,

$$\mathscr{F}(\mathbf{x}) = (v,\mathbf{F}(x,v))\in\mathbb{R}^6 \tag{1.5}$$

where $\mathbf{F} = (\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3)$ is a time independent globally Lipschitz field (the force field) over $\mathcal{D}\times\mathscr{V}$.

With the above choice, Eq. (1.4a) reads:

$$\partial_t f(x,v,t) + v\cdot\nabla_x f(x,v,t) + \mathbf{F}(x,v)\cdot\nabla_v f(\mathbf{x},v,t) + \nu(x,v)f(x,v,t) = 0, \tag{1.6}$$

where $\nu(x,v) = h(x,v) + \mathrm{div}_v(\mathbf{F})(x,v)$, supplemented with suitable initial and boundary conditions (see (1.4b)). More general problems can be handled with. For instance, one can treat with our formalism the collisionless version of the linear Boltzmann arising in *semiconductor theory*:

$$\partial_t f(\mathbf{r},\mathbf{k},t) + \frac{1}{\hbar}\nabla_{\mathbf{k}}\epsilon(\mathbf{k})\cdot\nabla_{\mathbf{k}}f(\mathbf{r},\mathbf{k},t) + \frac{e}{\hbar}\mathcal{E}\cdot\nabla_{\mathbf{k}}f(\mathbf{r},\mathbf{k},t) + \nu(\mathbf{r},\mathbf{k})f(\mathbf{r},\mathbf{k},t) = 0$$

where the unknown $f(\mathbf{r},\mathbf{k},t)$ is the density of electrons having the position $\mathbf{r}\in\mathbb{R}^3$, the wave-vector $\mathbf{k}\in\mathbb{R}^3$ at time $t>0$. The parameters e and $2\pi\hbar$ are the positive electron charge and the Planck constant respectively while $\epsilon(\mathbf{k})$ represents the electron energy and $\mathcal{E} = \mathcal{E}(\mathbf{r},\mathbf{k})$ is an external electric field.

The abstract equation (1.4) allows also to consider collisionless kinetic equation for relativistic models for which

$$v = v(p) = \frac{p}{m\sqrt{1 + p^2/c^2m^2}},$$

where m stands for the mass of particles and c being the velocity of light. For all this kind of models, the solution f to the collisionless kinetic equation is given, under suitable conditions on the data, by $f(\mathbf{x}, t) = \mathcal{U}(t)f_0(\mathbf{x})$ where f_0 corresponds to the initial state of the system and $\mathcal{U}(t)$ is a weighted shift semigroup of the shape (1.1). We provide in the rest of the paper a large number of examples arising in kinetic theory for which our abstract results apply. Spectral properties of systems of scalar advective equations on the torus coupled by a pseudo-differential operator of order zero (motivated by fluid mechanics) are dealt with by Shvydkoy [28] (see also [27, 29, 30, 14]). We note certain similarities between some of those results and our results concerning the group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$.

1.2. **Main results and methodology.** To describe the spectral features of the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ and its generator $\mathcal{T}$, we generalize the approach initiated in a recent work of the second author [25]. The analysis of [25] is restricted to the neutron transport equation, corresponding to the choice of $\mathbf{F} = 0$ in the above equation (1.6). We generalize [25] to general vector field $\mathscr{F}$. Precisely, thanks to a suitable decomposition of the phase space $\mathbf{\Omega}$ according to the finiteness of $\tau_-(\cdot)$ and $\tau_+(\cdot)$, we show that a general weighted shift semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ admits a canonical decomposition into three semigroups $(\mathcal{U}_i(t))_{t\geqslant 0}$, $(i = 1, 2, 3)$ with *independent dynamics*. The third semigroup $(\mathcal{U}_3(t))_{t\geqslant 0}$ actually extends to a C_0-group, which corresponds to the global in time flow $\Phi(t, \cdot)$ already investigated in [22]. Concerning the spectral properties of the semigroups $(\mathcal{U}_1(t))_{t\geqslant 0}$ and $(\mathcal{U}_2(t))_{t\geqslant 0}$ (which correspond to *trajectories* $(\Phi(\mathbf{x}, t))_t$ *such that either* $\tau_+(\mathbf{x})$ *or* $\tau_-(\mathbf{x})$ *is finite*), they both enjoy the same spectral structure:

$$\mathfrak{S}(\mathcal{U}_i(t)) = \{\mu\,;\, |\mu| \leqslant \exp(-\gamma_i t)\}, \qquad \mathfrak{S}(\mathcal{T}_i) = \{\lambda\,;\, \mathrm{Re}\lambda \leqslant -\gamma_i\}, \quad i = 1, 2 \tag{1.7}$$

where γ_i $(i = 1, 2)$ are positive constants, depending on h and $\mathscr{F}$ (see Theorems 2.4 & 2.6). The above description relies on several abstract results on positive semigroups on Banach lattices and, in particular, on the following property, proved in [25]:

Proposition 1.5. *Let* $(T(t))_{t\geqslant 0}$ *be a* C_0*-semigroup of positive operators on a complex Banach lattice* $\mathfrak{X}$*. Let* $\mathfrak{Y}$ *denote the subspace of local quasinilpotence of* $(T(t))_{t\geqslant 0}$*:*

$$\mathfrak{Y} = \left\{ u \in \mathfrak{X}\,;\, \lim_{t\to\infty} \exp\left[\frac{1}{t}\log \|T(t)(|u|)\|_{\mathfrak{X}}\right] = 0 \right\}.$$

If $\mathfrak{Y}$ *is dense in* $\mathfrak{X}$ *then* $[0, \exp(\omega_0(T)t)] \subset \mathfrak{S}_{\mathrm{ap}}(T(t))$ *for any* $t \geqslant 0$ *while* $(-\infty, s(A)] \subset \mathfrak{S}_{\mathrm{ap}}(A)$*, where* $\omega_0(T) \in [-\infty, \infty)$ *is the type of* $(T(t))_{t\geqslant 0}$ *and* $s(A)$ *denotes the spectral bound of the generator* A *of* $(T(t))_{t\geqslant 0}$*.*

Such a result allows to describe very precisely the *real spectrum* of $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$. Then, to deal with the non-real spectrum, we prove the invariance by rotation of the spectrum of $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ and the invariance by translation along the imaginary axis of their generators. This is done in the spirit of [31], see Proposition 2.2.

The description (1.7) shows in particular that, whenever the flow Φ (and the geometry $\mathbf{\Omega}$) do not allow trajectories that are global in both positive and negative times (i.e. either $\tau_+(\mathbf{x})$ or $\tau_-(\mathbf{x})$ is finite), then the spectrum of the generator $\mathcal{T}$ is a left-half plane and the Spectral Mapping Theorem

$$\mathfrak{S}(\mathcal{U}(t)) \setminus \{0\} = \exp\left(\mathfrak{S}(\mathcal{T})t\right), \qquad t \geqslant 0, \tag{1.8}$$

holds. Such a result seems to be new. Actually, we show that only the existence of periodic orbits and/or stationary points could prevent the Spectral Mapping Theorem (1.8) to hold. Indeed, when dealing with the C_0-group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$, we show that, here again, this group admits a canonical decomposition into three groups with independent dynamics $(\mathcal{U}_{\mathrm{rest}}(t))_{t\in\mathbb{R}}$, $(\mathcal{U}_{\mathrm{per}}(t))_{t\in\mathbb{R}}$ and $(\mathcal{U}_{\infty}(t))_{t\in\mathbb{R}}$ corresponding respectively to stationary orbits, periodic orbits and infinite orbits which are neither stationary nor periodic. The spectral analysis of $(\mathcal{U}_{\mathrm{rest}}(t))_{t\in\mathbb{R}}$ is really easy to derive since $\mathcal{U}_{\mathrm{rest}}(t)$ acts as a *multiplication* group. On the other hand, the group $(\mathcal{U}_{\infty}(t))_{t\in\mathbb{R}}$ falls within the general theory of Mather groups associated to aperiodic flow investigated in [13] and [22]. Concerning the delicate case of periodic trajectories, we deal with a description of the spectrum of the generator $\mathcal{T}_{\mathrm{per}}$ and prove (thanks to a general result on positive C_0-*groups* on L^p-spaces) that $(\mathcal{U}_{\mathrm{per}}(t))_{t\in\mathbb{R}}$ fulfils the so-called Annular Hull Theorem

$$\mathbb{T}\cdot\mathfrak{S}(\mathcal{U}_{\mathrm{per}}(t)) = \mathbb{T}\cdot\exp\left(t\Big(\mathfrak{S}(\mathcal{T}_{\mathrm{per}})\cap\mathbb{R}\Big)\right), \qquad \forall t\in\mathbb{R}.$$

Notice that, in full generality, $\mathcal{U}_{\mathrm{per}}(t)$ does not satisfy the Spectral Mapping Theorem (1.8) (see Example 4.1).

The organization of the paper is as follows: in Section 2, we establish the aforementioned decomposition of $(\mathcal{U}(t))_{t\geqslant 0}$ into three semigroups $(\mathcal{U}_i(t))_{t\geqslant 0}$, $(i=1,2,3)$ with independent dynamics. Moreover, we provide a complete description of the spectrum of $(\mathcal{U}_1(t))_{t\geqslant 0}$, $(\mathcal{U}_2(t))_{t\geqslant 0}$ and their generators and illustrate our results by several examples from kinetic theory. The properties of the group $(\mathcal{U}_3(t))_{t\geqslant 0}$ are investigated in Section 3 where we deal only with stationary or aperiodic flow. In Section 4, we investigate the more delicate case of a periodic flow. Finally, in the Appendix, we state some known and new abstract results on the spectral properties of positive C_0-semigroups in L^p-spaces, $1\leqslant p<\infty$.

2. Spectral properties of the streaming semigroup $(\mathcal{U}(t))_{t\geqslant 0}$

Let us define the following subsets of $\mathbf{\Omega}$:

$$\mathbf{\Omega}_1 = \{\mathbf{x}\in\mathbf{\Omega}\,;\,\tau_+(\mathbf{x})<\infty\}, \qquad \mathbf{\Omega}_2 = \{\mathbf{x}\in\mathbf{\Omega}\,;\,\tau_+(\mathbf{x})=\infty \text{ and } \tau_-(\mathbf{x})<\infty\}$$

and

$$\mathbf{\Omega}_3 = \{\mathbf{x}\in\mathbf{\Omega}\,;\,\tau_+(\mathbf{x})=\tau_-(\mathbf{x})=\infty\}.$$

Moreover, define

$$X_i = \{f\in X\,;\,f(\mathbf{x})=0 \quad \mathfrak{m}-\text{a.e. } \mathbf{x}\in\mathbf{\Omega}\setminus\mathbf{\Omega}_i\}, \qquad i=1,2,3.$$

In the sequel, we shall *identify* X_i with $L^p(\mathbf{\Omega}_i,\mathrm{d}\mathfrak{m})$, $i=1,2,3$. Since $(\mathbf{\Omega}_i)_{i=1,\ldots,3}$ is a partition of $\mathbf{\Omega}$, it is clear that

$$X = X_1\oplus X_2\oplus X_3.$$

Of course, if $\mathfrak{m}(\mathbf{\Omega}_i)=0$, the space X_i reduces to $\{0\}$ and drops out in the direct sum $(i=1,2,3)$. Following [25], we can state the following:

Theorem 2.1. *For any $i=1,2,3$, X_i is invariant under $(\mathcal{U}(t))_{t\geqslant 0}$. Define $\mathcal{U}_i(t)=\mathcal{U}(t)_{|X_i}$ for any $t\geqslant 0$, $i=1,2,3$. Then, $(\mathcal{U}_i(t))_{t\geqslant 0}$ is a positive C_0-semigroup of X_i $(i=1,2,3)$ and*

$$\mathfrak{S}(\mathcal{U}(t))=\mathfrak{S}(\mathcal{U}_1(t))\cup\mathfrak{S}(\mathcal{U}_2(t))\cup\mathfrak{S}(\mathcal{U}_3(t)) \qquad (t\geqslant 0) \tag{2.1}$$

where $\mathfrak{S}(\mathcal{U}_i(t))$ stands for the spectrum of $\mathcal{U}_i(t)$ in the space X_i ($i=1,2,3$). Moreover, $(\mathcal{U}_3(t))_{t\geqslant 0}$ extends to a C_0-group in X_3.

Proof. Let $f\in X$ and recall that

$$\mathcal{U}(t)f(\mathbf{x})=\exp\left[-\int_0^t \nu(\Phi(\mathbf{x},-s))\mathrm{d}s\right]f(\Phi(\mathbf{x},-t))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}) \qquad (t\geqslant 0).$$

Consequently, if $\mathcal{U}(t)f(\mathbf{x})\neq 0$, then $t<\tau_-(\mathbf{x})$ and $f(\Phi(\mathbf{x},-t))\neq 0$. Moreover, one deduces easily from Proposition 1.3, that $\tau_+(\Phi(\mathbf{x},-t))=t+\tau_+(\mathbf{x})$ for any $t<\tau_-(\mathbf{x})$. Therefore, $\tau_+(\mathbf{x})<\infty$ if and only if $\tau_+(\Phi(\mathbf{x},-t))$ for any $t<\tau_-(\mathbf{x})$. As a direct consequence, one gets that $\mathcal{U}(t)f(\mathbf{x})=0$ $\mathfrak{m}$-a.e. on $\mathbf{\Omega}\setminus\mathbf{\Omega}_1$ for any $f\in X_1$. This shows that X_1 is invariant under $(\mathcal{U}(t))_{t\geqslant 0}$. In the same way, still using Proposition 1.3, one observes that

$$\tau_-(\Phi(\mathbf{x},-t))=\tau_-(\mathbf{x})-t \qquad \forall t<\tau_-(\mathbf{x})$$

so that $\tau_-(\mathbf{x})<\infty$ if and only if $\tau_-(\Phi(\mathbf{x},-t))$ for any $t<\tau_-(\mathbf{x})$. As before, this leads to the invariance of both X_2 and X_3 under $(\mathcal{U}(t))_{t\geqslant 0}$. Thus, the triplet (X_1,X_2,X_3) reduces $(\mathcal{U}(t))_{t\geqslant 0}$ and (2.1) follows. Finally, defining $\mathcal{U}_3(-t)f(\mathbf{x})$ for any $t\geqslant 0$, $\mathbf{x}\in\mathbf{\Omega}_3$ and $f\in X_3$ as

$$\mathcal{U}_3(-t)f(\mathbf{x})=\exp\left[\int_{-t}^0 \nu(\Phi(\mathbf{x},-s))\mathrm{d}s\right]f(\Phi(\mathbf{x},t))$$

one obtains easily the last assertion. □

From the above Theorem, to describe the spectra of $(T(t))_{t\geqslant 0}$ and $\mathcal{T}$, one can deal separately with the spectral properties of the various semigroups $(\mathcal{U}_i(t))_{t\geqslant 0}$ and their generator $\mathcal{T}_i$ on X_i, $i=1,2,3$. We will show in the rest of this paper that our analysis of $(\mathcal{U}_1(t))_{t\geqslant 0}$ and $(\mathcal{U}_2(t))_{t\geqslant 0}$ differs very much from that of the group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$. In particular, to show the rotational invariance of the spectra of the formers, we will make use of the following result [31]:

Proposition 2.2. *Let $\widetilde{\mathbf{\Omega}}$ be a measurable subset of $\mathbf{\Omega}$ such that*

$$\widetilde{X}=\left\{f\in X\,;\; f(\mathbf{x})=0 \quad \mathfrak{m}-a.e.\ \mathbf{x}\in\mathbf{\Omega}\setminus\widetilde{\mathbf{\Omega}}\right\}$$

is invariant under $(\mathcal{U}(t))_{t\geqslant 0}$ and let $(\widetilde{\mathcal{U}}(t))_{t\geqslant 0}$ and $\widetilde{\mathcal{T}}$ be the restrictions to $\widetilde{X}$ of $(\mathcal{U}(t))_{t\geqslant 0}$ and $\mathcal{T}$ respectively. Assume that there exists a ***measurable*** *mapping $\alpha(\cdot)\ :\ \mathbf{\Omega}\to\mathbb{R}$ such that*

(a) $|\alpha(\mathbf{x})|$ is finite for almost any $\mathbf{x}\in\widetilde{\mathbf{\Omega}}$;
(b) for almost any $\mathbf{x}\in\widetilde{\mathbf{\Omega}}$ such that $t<\tau_-(\mathbf{x})$,

$$\alpha(\Phi(\mathbf{x},-t))=\alpha(\mathbf{x})+t. \tag{2.2}$$

Then, for any $\eta \in \mathbb{R}$, the mapping

$$\mathcal{M}_\eta : \quad f \in \widetilde{X} \longmapsto [\mathcal{M}_\eta f](\mathbf{x}) = \exp(-i\eta\alpha(\mathbf{x}))f(\mathbf{x}) \in \widetilde{X}$$

is an isometric isomorphism of $\widetilde{X}$ such that

$$\mathcal{M}_\eta^{-1}\widetilde{\mathcal{T}}\mathcal{M}_\eta = \widetilde{\mathcal{T}} + i\eta\mathrm{Id} \qquad \text{and} \qquad \mathcal{M}_\eta^{-1}\widetilde{\mathcal{U}}(t)\mathcal{M}_\eta = e^{i\eta t}\widetilde{\mathcal{U}}(t) \qquad (t \geqslant 0). \tag{2.3}$$

Consequently, $\mathfrak{S}(\widetilde{\mathcal{T}}) = \mathfrak{S}(\widetilde{\mathcal{T}}) + i\mathbb{R}$ and $\mathfrak{S}(\widetilde{\mathcal{U}}(t)) = \mathfrak{S}(\widetilde{\mathcal{U}}(t)) \cdot \mathbb{T}$ for any $(t \geqslant 0)$ where $\mathbb{T}$ is the unit circle of the complex plane.

2.1. **Spectral properties of $(\mathcal{U}_1(t))_{t\geqslant 0}$.** To investigate the spectral properties of the restriction of the C_0-semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ to X_1 we employ a strategy inspired by [25] based upon Propositions 1.5 and 2.2. First, one gets the following invariance result as a straightforward application of Prop. 2.2:

Theorem 2.3. *The spectra of $(\mathcal{U}_1(t))_{t\geqslant 0}$ and $\mathcal{T}_1$ in X_1 enjoy the following invariance properties:*

$$\mathfrak{S}(\mathcal{T}_1) = \mathfrak{S}(\mathcal{T}_1) + i\mathbb{R} \qquad \text{and} \qquad \mathfrak{S}(\mathcal{U}_1(t)) = \mathfrak{S}(\mathcal{U}_1(t)) \cdot \mathbb{T} \qquad (t \geqslant 0)$$

where $\mathbb{T}$ is the unit circle of the complex plane.

Proof. The proof consists in applying the above Proposition 2.2 to the subset $\mathbf{\Omega}_1$ and the subspace X_1 using the function

$$\alpha(\mathbf{x}) = -\tau_+(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_1.$$

One checks immediately that $\alpha(\cdot)$ fulfills (2.2). Note that $\alpha(\cdot)$ is finite over $\mathbf{\Omega}_1$. □

According to the above Theorem, to describe the spectral features of $(\mathcal{U}_1(t))_{t\geqslant 0}$, it is sufficient to describe *its real spectrum*. We shall denote

$$\Sigma_p(\mathbf{x}) = h(\mathbf{x}) + \frac{1}{p^\star}\mathrm{div}(\mathscr{F})(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega},$$

where $p^\star$ is the conjugate of the exponent $1 \leqslant p < \infty$ we fixed at the beginning, $\frac{1}{p^\star} + \frac{1}{p} = 1$.

Theorem 2.4. *Assume $\mathfrak{m}(\mathbf{\Omega}_1) \neq 0$. One has*

$$\mathfrak{S}(\mathcal{U}_1(t)) = \mathfrak{S}_{\mathrm{ap}}(\mathcal{U}_1(t)) = \{\xi \in \mathbb{C}\,;\, |\xi| \leqslant \exp(-\gamma_1 t)\} \qquad (t \geqslant 0)$$

and

$$\mathfrak{S}(\mathcal{T}_1) = \mathfrak{S}_{\mathrm{ap}}(\mathcal{T}_1) = \{\lambda\,;\, \mathrm{Re}\lambda \leqslant -\gamma_1\}$$

where

$$\gamma_1 = \lim_{t\to\infty} \inf \left\{ t^{-1} \int_0^t \Sigma_p(\Phi(\mathbf{x},-s))\mathrm{d}s\,;\, \mathbf{x} \in \mathbf{\Omega}_1,\ t < \tau_-(\mathbf{x}) \right\}.$$

Proof. Let us denote by $\omega_0(\mathcal{U}_1)$ the type of $(\mathcal{U}_1(t))_{t\geqslant 0}$ and let us fix $t \geqslant 0$ and $f \in X_1$. One has

$$\|\mathcal{U}_1(t)f\|_{X_1}^p = \int_{\{\tau_+(\mathbf{x})<\infty\}} \exp\left[-p\int_0^t \nu(\Phi(\mathbf{x},-s))\mathrm{d}s\right] \left|f(\Phi(\mathbf{x},-t))\right|^p \chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x})\mathrm{d}\mathfrak{m}(\mathbf{x}).$$

Recalling that the finiteness of $\tau_+(\mathbf{x})$ is equivalent to that of $\tau_+(\Phi(\mathbf{x},-t))$ for any $t<\tau_-(\mathbf{x})$, the change of variable $\mathbf{x}\mapsto\mathbf{y}=\Phi(\mathbf{x},-t)$ maps $\mathbf{\Omega}_1$ onto itself. Moreover, according to Proposition 1.2, the Jacobian of the transformation is given by

$$G(t,\mathbf{y})=\exp\left[\int_0^t \mathrm{div}(\mathscr{F})(\Phi(\mathbf{y},s))\mathrm{d}s\right].$$

Now, using that $\Phi(\mathbf{y},s)=\Theta(\mathbf{x},t,s)=\Phi(\mathbf{x},s-t)\in\mathbf{\Omega}$ as long as $0<t-s<\tau_-(\mathbf{x})$, one sees that

$$\chi_{\{\tau_+(\mathbf{x})<\infty\}\cap\{t<\tau_-(\mathbf{x})\}}(\mathbf{x})=\chi_{\{\tau_+(\mathbf{y})<\infty\}\cap\{\tau_+(\mathbf{y})>t\}}(\mathbf{y}).$$

Therefore,

$$\begin{aligned}\|\mathcal{U}_1(t)f\|_{X_1}^p&=\int_{\{\tau_+(\mathbf{y})<\infty\}}\exp\left[-p\int_0^t\nu(\Phi(\mathbf{y},r))\mathrm{d}r\right]|f(\mathbf{y})|^p\chi_{\{t<\tau_+(\mathbf{y})\}}(\mathbf{y})G(t,\mathbf{y})\mathrm{d}\mathfrak{m}(\mathbf{y})\\&=\int_{\{\tau_+(\mathbf{y})<\infty\}}\exp\left[-p\int_0^t\Sigma_p(\Phi(\mathbf{y},r))\mathrm{d}r\right]|f(\mathbf{y})|^p\chi_{\{t<\tau_+(\mathbf{y})\}}(\mathbf{y})\mathrm{d}\mathfrak{m}(\mathbf{y}).\end{aligned}\tag{2.4}$$

Consequently,

$$\|\mathcal{U}_1(t)\|_{\mathscr{B}(X_1)}=\sup\left\{\exp\left[-\int_0^t\Sigma_p(\Phi(\mathbf{y},r))\mathrm{d}r\right]\,;\,\tau_+(\mathbf{y})<\infty\,,\,t<\tau_+(\mathbf{y})\right\}$$

or, equivalently,

$$\log\|\mathcal{U}_1(t)\|_{\mathscr{B}(X_1)}=-\inf\left\{\int_0^t\Sigma_p(\Phi(\mathbf{y},r))\mathrm{d}r\,;\,\tau_+(\mathbf{y})<\infty\,,\,t<\tau_+(\mathbf{y})\right\}.$$

Performing now the converse change of variable $\mathbf{y}\mapsto\mathbf{x}=\Phi(\mathbf{y},t)$ one sees as previously that

$$\log\|\mathcal{U}_1(t)\|_{\mathscr{B}(X_1)}=-\inf\left\{\int_0^t\Sigma_p(\Phi(\mathbf{x},-s))\mathrm{d}s\,;\,\tau_+(\mathbf{x})<\infty\,,\,t<\tau_-(\mathbf{x})\right\}.$$

Finally, recalling that $\omega_0(\mathcal{U}_1)=\lim_{t\to\infty}t^{-1}\log\|\mathcal{U}_1(t)\|_{\mathscr{B}(X_1)}$, one deduces that $\omega_0(\mathcal{U}_1)=-\gamma_1$. Moreover, from the positiveness of $(\mathcal{U}_1(t))_{t\geqslant 0}$ and [33], one gets $s(\mathcal{T}_1)=-\gamma_1$ where $s(\mathcal{T}_1)$ is the spectral bound of $\mathcal{T}_1$. Let us now show that the set $\mathfrak{Y}_1$ of local quasi-nilpotence of $(\mathcal{U}_1(t))_{t\geqslant 0}$ is dense in X_1. Indeed, set $\mathbf{\Omega}_1^m=\{\mathbf{x}\in\mathbf{\Omega}\,;\,\tau_+(\mathbf{x})\leqslant m\}$, $m\in\mathbb{N}$. One has $\mathbf{\Omega}_1=\cup_{m\geqslant 1}\mathbf{\Omega}_1^m$. Set $Y_1=\cup_{m\geqslant 1}X_1^m$ where $X_1^m=\{f\in X_1\,;\,f(\mathbf{x})=0\ \text{a.e. }\mathbf{x}\in\mathbf{\Omega}_1\setminus\mathbf{\Omega}_1^m\}$. One checks easily that Y_1 is a dense sublattice of X_1. Now, let $f\in Y_1$. There exists some integer $m\geqslant 1$ such that $f(\mathbf{x})=0$ for almost every $\mathbf{x}\in\mathbf{\Omega}_1$ with $\tau_+(\mathbf{x})>m$. From (2.4), one gets

$$\|\mathcal{U}_1(t)(|f|)\|_{X_1}=0\qquad\forall t>m$$

which clearly shows that Y_1 is a subset of $\mathfrak{Y}_1$ and that $\mathfrak{Y}_1$ is dense in X_1. From Proposition 1.5, one gets then that

$$(-\infty,-\gamma_1]\subset\mathfrak{S}_{\mathrm{ap}}(\mathcal{T}_1)\qquad\text{and}\qquad[0,\exp(-\gamma_1 t)]\subset\mathfrak{S}_{\mathrm{ap}}(\mathcal{U}_1(t))\qquad(t\geqslant 0).$$

Now, the result follows from the invariance of $\mathfrak{S}(\mathcal{T}_1)$ and $\mathfrak{S}(\mathcal{U}_1(t))$ under vertical translations along the imaginary axis and under rotations respectively (Theorem 2.3). □

2.2. **Spectral properties of** $(\mathcal{U}_2(t))_{t\geqslant 0}$**.** Let us investigate now the spectral properties of the restriction of $(\mathcal{U}(t))_{t\geqslant 0}$ to the subspace X_2. As in the previous case, the main ingredient of our proof is the following special version of Proposition 2.2.

Theorem 2.5. *The spectra of* $(\mathcal{U}_2(t))_{t\geqslant 0}$ *and* $\mathcal{T}_2$ *in* X_2 *enjoy the following invariance properties:*

$$\mathfrak{S}(\mathcal{T}_2) = \mathfrak{S}(\mathcal{T}_2) + i\mathbb{R} \qquad \textit{and} \qquad \mathfrak{S}(\mathcal{U}_2(t)) = \mathfrak{S}(\mathcal{U}_2(t)) \cdot \mathbb{T} \qquad (t \geqslant 0)$$

where $\mathbb{T}$ *is the unit circle of the complex plane.*

Proof. The proof consists in applying the above Proposition 2.2 to $(\mathcal{U}_2(t))_{t\geqslant 0}$ by choosing

$$\alpha(\mathbf{x}) = \tau_-(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_2.$$

One checks immediately that $\alpha(\cdot)$ fulfills (2.2) and that $\alpha(\cdot)$ is finite $\mathbf{\Omega}_2$. □

This leads to the following result whose proof is a technical generalization of [25] that we repeat here for the self-consistency of the paper.

Theorem 2.6. *Assume* $\mathfrak{m}(\mathbf{\Omega}_2) \neq 0$. *One has*

$$\mathfrak{S}(\mathcal{U}_2(t)) = \{\xi \in \mathbb{C}\,;\, |\xi| \leqslant \exp(-\gamma_2 t)\} \qquad \textit{and} \qquad \mathfrak{S}(\mathcal{T}_2) = \{\lambda\,;\, \mathrm{Re}\lambda \leqslant -\gamma_2\}$$

where

$$\gamma_2 = \lim_{t\to\infty} \inf \left\{ t^{-1}\int_0^t \Sigma_p(\Phi(\mathbf{x},s))\mathrm{d}s\,;\, \mathbf{x} \in \mathbf{\Omega}_2,\ t < \tau_-(\mathbf{x}) \right\}.$$

Proof. The proof relies on duality arguments. We only prove it in the L^1 case, the case $1 < p < \infty$ being simpler (see [25]). Since the dual semigroup $(\mathcal{U}_2^\star(t))_{t\geqslant 0}$ of $(\mathcal{U}_2(t))_{t\geqslant 0}$ is not strongly continuous on $L^\infty(\mathbf{\Omega}_2)$, we have to introduce the space of strong continuity

$$X_2^{\odot} = \left\{ g \in L^\infty(\mathbf{\Omega}_2)\,;\, \lim_{t\to 0^+} \sup_{\mathbf{x}\in\mathbf{\Omega}_2} |\mathcal{U}_2^\star(t)g(\mathbf{x}) - g(\mathbf{x})| = 0 \right\}.$$

The general theory of C_0-semigroups [16, Chapter 2, Section 2.6] ensures that $X_2^{\odot}$ is a closed subset of $L^\infty(\mathbf{\Omega}_2)$ invariant under $(\mathcal{U}_2^\star(t))_{\tau\geqslant 0}$. For any $t \geqslant 0$, define $\mathcal{U}_2^{\odot}(t)$ as the restriction of $\mathcal{U}_2^\star(t)$ to $X_2^{\odot}$, i.e. $(\mathcal{U}_2^{\odot}(t))_{t\geqslant 0}$ is the sun–dual semigroup of $(\mathcal{U}_2(t))_{t\geqslant 0}$.. Then, the semigroup $(\mathcal{U}_2^{\odot}(t))_{t\geqslant 0}$ is a C_0-semigroup of $X_2^{\odot}$ whose generator is denoted by $\mathcal{T}_2^{\odot}$ with [16, Chapter 2, Section 2.6],

$$\mathscr{D}(\mathcal{T}_2^{\odot}) = \{g \in \mathscr{D}(\mathcal{T}_2^\star)\,;\, \mathcal{T}_2^\star g \in X_2^{\odot}\} \qquad \text{and} \qquad \mathcal{T}_2^{\odot} g = \mathcal{T}_2^\star g, \qquad \forall g \in \mathscr{D}(\mathcal{T}_2^{\odot}),$$

where $\mathcal{T}_2^\star$ denotes the dual operator of $\mathcal{T}_2$ whose domain $\mathscr{D}(\mathcal{T}_2^\star)$ is dense in $X_2^{\odot}$. Moreover [16, Chapter 4, Proposition 2.18],

$$\mathfrak{S}(\mathcal{T}_2) = \mathfrak{S}(\mathcal{T}_2^\star) = \mathfrak{S}(\mathcal{T}_2^{\odot}) \qquad \text{and} \qquad \mathfrak{S}(\mathcal{U}_2(t)) = \mathfrak{S}(\mathcal{U}_2^\star(t)) = \mathfrak{S}(\mathcal{U}_2^{\odot}(t))$$

where the spectra of $\mathcal{T}_2^\star$and $\mathcal{U}_2^\star(t)$ are intended in the space $L^\infty(\mathbf{\Omega}_2)$ while that of $\mathcal{T}_2^{\odot}$ and $\mathcal{U}_2^{\odot}(t)$ stand for spectra in $X_2^{\odot}$. Define

$$\widehat{X_2^{\odot}} = \left\{ g \in X_2^{\odot}\,;\, \lim_{r\to\infty} \sup_{\tau_-(\mathbf{x})\geqslant r} |g(\mathbf{x})| = 0 \right\}$$

and let us show that $\widehat{X_2^\odot}$ is invariant under $(\mathcal{U}_2^\odot(t))_{t\geqslant 0}$. Indeed, let $g\in\widehat{X_2^\odot}$ and $f=(\lambda-T_2^\odot)^{-1}g$ where $\lambda>0$ is large enough. Using again Proposition 1.2, one can check without difficulty that the dual of $\mathcal{U}_2(t)$ is given by

$$\mathcal{U}_2^\star(t)g(\mathbf{x})=\exp\left[-\int_0^t h(\Phi(\mathbf{x},s))\mathrm{d}s\right]g(\Phi(\mathbf{x},t))$$

for any $t\geqslant 0$, $\mathbf{x}\in\mathbf{\Omega}_2$ and $g\in L^\infty(\mathbf{\Omega}_2)$ where we recall that $\tau_+(\mathbf{x})=\infty$ for any $\mathbf{x}\in\mathbf{\Omega}_2$. Consequently,

$$f(\mathbf{x})=(\lambda-\mathcal{T}_2^\odot)^{-1}g(\mathbf{x})=\int_0^\infty\exp\left[-\lambda t-\int_0^t h(\Phi(\mathbf{x},s))\mathrm{d}s\right]g(\Phi(\mathbf{x},t))\mathrm{d}t.$$

Recalling that $\tau_-(\Phi(\mathbf{x},t))=\tau_-(\mathbf{x})+t$ and setting $\underline{h}=\operatorname{ess\,inf}_{\mathbf{y}\in\mathbf{\Omega}}h(\mathbf{y})$, one has

$$\sup_{\tau_-(x)\geqslant r}|f(\mathbf{x})|\leqslant\int_0^\infty\exp(-(\lambda+\underline{h})t)\mathrm{d}t\sup_{\tau_-(\mathbf{y})\geqslant r}|g(\mathbf{y})|,\qquad\forall r>0$$

which proves that $f\in\widehat{X_2^\odot}$ and implies that $\widehat{X_2^\odot}$ is invariant under the action of $(\mathcal{U}_2^\odot(t))_{t\geqslant 0}$. Denote $(\widehat{\mathcal{U}}_2^\odot(t))_{t\geqslant 0}$ the restriction of $(\mathcal{U}_2^\odot(t))_{t\geqslant 0}$ to $\widehat{X_2^\odot}$. Define now $\widehat{Y}_2$ as the set of $g\in\widehat{X_2^\odot}$ for which there is some $r\geqslant 0$ such that $g(\mathbf{x})=0$ if $\tau_-(\mathbf{x})\geqslant r$. One claims that $\widehat{Y}_2$ is dense in $\widehat{X_2^\odot}$. Indeed, for any integer $m\geqslant 1$, let $\gamma_m(\cdot)$ be a continuous function from $[0,\infty)$ to $[0,1]$ such that $\gamma_m(s)=1$ for any $0\leqslant s\leqslant m$ and $\gamma_m(s)=0$ for any $s\geqslant 2m$. For any $g\in\widehat{X_2^\odot}$, one can define $g_m(\mathbf{x})=\gamma_m(\tau_-(\mathbf{x}))\,g(\mathbf{x})$ and prove without major difficulties that $(g_m)_m\subset\widehat{Y}_2$ and $\|g_m-g\|_{L^\infty(\mathbf{\Omega}_2)}\to 0$ as $m\to\infty$. This proves our claim. Now, the set $\widehat{\mathfrak{Y}}_2$ of local quasi–nilpotence of $(\widehat{\mathcal{U}}_2^\odot(t))_{t\geqslant 0}$ obviously contains $\widehat{Y}_2$ and is therefore dense in $\widehat{X_2^\odot}$. According to Proposition 1.5, one gets then

$$[0,\exp(\omega_0(\widehat{\mathcal{U}}_2^\odot)\,t)]\subset\mathfrak{S}_{\mathrm{ap}}(\widehat{\mathcal{U}}_2^\odot(t))\cap\mathbb{R}\qquad\text{and}\qquad(-\infty,\omega_0(\widehat{\mathcal{U}}_2^\odot)]\subset\mathfrak{S}_{\mathrm{ap}}(\widehat{\mathcal{T}}_2^\odot)\cap\mathbb{R}$$

where $\widehat{\mathcal{T}}_2^\odot$ is the restriction of $\mathcal{T}_2^\odot$ to $\widehat{X_2^\odot}$ and where we used the identity between the type $\omega_0(\widehat{\mathcal{U}}_2^\odot)$ of $(\widehat{\mathcal{U}}_2^\odot(t))_{t\geqslant 0}$ and the spectral bound of $\widehat{\mathcal{T}}_2^\odot$. Since $\widehat{X_2^\odot}\subset X_2^\odot$, one obtains the following inclusion

$$[0,\exp(\omega_0(\widehat{\mathcal{U}}_2^\odot)\,t)]\subset\mathfrak{S}_{\mathrm{ap}}(\mathcal{U}_2^\odot(t))\cap\mathbb{R}\qquad\text{and}\qquad(-\infty,\omega_0(\mathcal{U}_2^\odot)]\subset\mathfrak{S}_{\mathrm{ap}}(\mathcal{T}_2^\odot)\cap\mathbb{R}$$

and Theorem 2.5 implies that

$$\mathfrak{S}(\mathcal{U}_2(t))=\mathfrak{S}(\mathcal{U}_2^\odot(t))=\mathfrak{S}_{\mathrm{ap}}(\mathcal{U}_2^\odot(t))=\{\xi\in\mathbb{C}\,;\,|\xi|\leqslant\exp(\omega_0(\widehat{\mathcal{U}}_2^\odot)\,t)\}$$

and

$$\mathfrak{S}(\mathcal{T}_2)=\mathfrak{S}(\mathcal{T}_2^\odot)=\mathfrak{S}_{\mathrm{ap}}(\mathcal{T}_2^\odot)=\{\lambda\,;\,\mathrm{Re}\lambda\leqslant\omega_0(\widehat{\mathcal{U}}_2^\odot)\}.$$

To conclude the proof, it remains only to show that $\omega_0(\widehat{\mathcal{U}}_2^\odot)=-\gamma_2$. To do so, one notes easily that

$$\|\widehat{\mathcal{U}}_2^\odot(t)\|_{\mathscr{B}(\widehat{X_2^\odot})}\leqslant\omega(t):=\sup\left\{\exp\left[-\int_0^t h(\Phi(\mathbf{x},s))\mathrm{d}s\right]\,;\,\mathbf{x}\in\mathbf{\Omega}_2,\ t<\tau_-(\mathbf{x})\right\}.$$

Moreover, with the above notations, let $\psi_m(\mathbf{x}) = \gamma_m(\tau_-(\mathbf{x}))$, $m \geqslant 1$. One has $\psi_m \in \widehat{X_2^{\odot}}$ and $\psi_m = 1$ on the set $\{\mathbf{x} \in \mathbf{\Omega}_2\,;\, \tau_-(\mathbf{x}) \leqslant m\}$. Therefore,

$$\|\widehat{\mathcal{U}}_2^{\odot}(t)\|_{\mathscr{B}(\widehat{X_2^{\odot}})} \geqslant \|\widehat{\mathcal{U}}_2^{\odot}(t)\psi_m\|_{\widehat{X_2^{\odot}}} \geqslant \sup\left\{\exp\left[-\int_0^t h(\Phi(\mathbf{x},s))\mathrm{d}s\right]\,;\, \mathbf{x} \in \mathbf{\Omega}_2,\ t < \tau_-(\mathbf{x}) \leqslant m\right\}.$$

Letting $m \to \infty$, one gets $\|\widehat{\mathcal{U}}_2^{\odot}(t)\|_{\mathscr{B}(\widehat{X_2^{\odot}})} = \omega(t)$. One obtains immediately that $\omega_0(\widehat{\mathcal{U}}_2^{\odot}) = -\gamma_2$. □

The above results give a complete picture of the spectra of $(\mathcal{U}(t))_{t\geqslant 0}$ and $\mathcal{T}$ when $\mathfrak{m}(\mathbf{\Omega}_3) = 0$:

Theorem 2.7. *Assume that $\mathfrak{m}(\mathbf{\Omega}_3) = 0$. Then,*

$$\mathfrak{S}(\mathcal{U}(t)) = \{\xi \in \mathbb{C}\,;\, |\xi| \leqslant \exp(-\gamma t)\} \qquad \textit{and} \qquad \mathfrak{S}(\mathcal{T}) = \{\lambda\,;\, \mathrm{Re}\lambda \leqslant -\gamma\}$$

*where $\gamma = \min\{\gamma_1, \gamma_2\}$, γ_1, γ_2 being given by Theorems 2.4 and 2.6. In particular, $(\mathcal{U}(t))_{t\geqslant 0}$ satisfies a **Spectral Mapping Theorem** $\mathfrak{S}(\mathcal{U}(t)) \setminus \{0\} = \exp(t\mathfrak{S}(\mathcal{T}))$ for any $t \geqslant 0$.*

We shall see in the following section that the picture is drastically different when $\mathfrak{m}(\mathbf{\Omega}_3) \neq 0$. First, we shall illustrate the above results by several examples.

2.3. **Examples.** The above result is of particular relevance for applications when $\mathbf{\Omega}$ is bounded in some directions.

Example 2.8. Let us consider the case of the classical Vlasov equation described in (1.6) with a Lorentz force. Namely, let $\mathbf{\Omega} = \mathcal{D} \times \mathbb{R}^3$ where $\mathcal{D}$ is a smooth open subset of $\mathbb{R}^3$. For any $\mathbf{x} = (x, v) \in \mathcal{D} \times \mathbb{R}^3$, let $\mathscr{F}(\mathbf{x}) = (v, \mathbf{F}(v)) \in \mathbb{R}^6$ where the force field $\mathbf{F}$ is given by the *Lorentz force*:

$$\mathbf{F} = \mathbf{F}(v) = q(\mathcal{E} + v \times \mathcal{B})$$

where $\mathcal{E} \in \mathbb{R}^3$ stands for some given electric field, $\mathcal{B} \in \mathbb{R}^3$ denotes a given magnetic field and q is the electric charge of the particle. One assumes in this example that $\mathcal{E}$ and $\mathcal{B}$ are two ***constant*** fields such that $\langle \mathcal{E}, \mathcal{B}\rangle \neq 0$ and that $\mathcal{D}$ is bounded in the $\mathcal{B}$-direction, i.e.

$$\sup\{|\langle x, \mathcal{B}\rangle|, x \in \mathcal{D}\} < \infty,$$

then $\mathfrak{m}(\mathbf{\Omega}_3) = 0$. Indeed, one sees easily that the solution $(x(t), v(t))$ to the characteristic system

$$\begin{cases} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= q(\mathcal{E} + v(t) \times \mathcal{B}), \qquad t \in \mathbb{R} \end{cases} \tag{2.5}$$

with initial condition $x(0) = x$, $v(0) = v$, is such that

$$\langle x(t), \mathcal{B}\rangle = \frac{q}{2}\langle \mathcal{E}, \mathcal{B}\rangle t^2 + \langle v, \mathcal{B}\rangle t + \langle x, \mathcal{B}\rangle.$$

Since the right-hand side is unbounded as $t \to \pm\infty$ for any $(x, v) \in \mathbf{\Omega}$, necessarily $x(t)$ escapes $\mathcal{D}$ in finite time and $\mathbf{\Omega}_3 = \varnothing$.

Remark 2.9. *Note that, in general, the solution $(x(t), v(t))$ to the characteristic system (2.5) describes an helix with axis directed along $\mathcal{B}$ and radius proportional to $1/|q||\mathcal{B}|$ (Larmor radius).*

Example 2.10. Consider now the following one dimensional relativistic transport equation

$$\partial_t f(x,p,t) + v(p)\partial_x f(x,v,t) - \left(pv(p) + (1+p^2)^{-1/2})\right)\phi'(x)\partial_p f(x,p,t) \\ + \nu(x,p)f(x,p,t) = 0, \qquad (x,v) \in (0,1)\times\mathbb{R} \quad (2.6)$$

where the potential ϕ is a given smooth function, say $\phi \in W^{2,\infty}(0,1)$, and the relativistic velocity $v(p)$ corresponding to the impulsion $p \in \mathbb{R}$ is given by $v(p) = \frac{p}{\sqrt{1+p^2}}$.

The study of the above equation (2.6) is related to the relativistic Nordström-Vlasov systems for plasma (see [12, 11] and the references therein). We can then define a smooth vector field over $\mathbf{\Omega} = (0,1)\times\mathbb{R}$ by

$$\mathscr{F}(x,p) = \left(v(p), -\left(pv(p) + (1+p^2)^{-1/2})\right)\phi'(x)\right) = \left(\frac{p}{\sqrt{1+p^2}}, -\sqrt{1+p^2}\phi'(x)\right)$$

It can be proved [12, Corollary 2] that, whenever ϕ is *convex*, the exit time associated to the above field is finite for all characteristic curves with non zero impulsion, i.e., with the notations of the above section,

$$\tau_{\pm}(x,p) < \infty, \qquad \forall (x,p) \in (0,1)\times(\mathbb{R}\setminus\{0\}).$$

Therefore, Theorem 2.7 applies to such a problem.

3. Spectral properties of $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$

We are dealing in this section with the spectrum of the C_0-group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$ and its generator $\mathcal{T}_3$. We assume therefore for this section that $\mathfrak{m}(\mathbf{\Omega}_3) \neq 0$. For any $\mathbf{x} \in \mathbf{\Omega}_3$, the mapping $s \mapsto \Phi(\mathbf{x}, s)$ is defined over $\mathbb{R}$. For simplicity, for any $t \in \mathbb{R}$, we will denote by φ_t the mapping

$$\varphi_t \,:\, \mathbf{x} \in \mathbf{\Omega}_3 \mapsto \varphi_t(\mathbf{x}) = \Phi(\mathbf{x}, -t).$$

With this notation, the group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$ is given by

$$\mathcal{U}_3(t)f(\mathbf{x}) = \exp\left[-\int_0^t \nu(\varphi_s(\mathbf{x}))\mathrm{d}s\right] f(\varphi_t(\mathbf{x})), \qquad \forall \mathbf{x} \in \mathbf{\Omega}_3,\;\; t \in \mathbb{R}.$$

Notice that, arguing as above, it is possible to compute explicitly the type of both the semigroups $(\mathcal{U}_3(t))_{t\geqslant 0}$ and $(\mathcal{U}_3(-t))_{t\geqslant 0}$ (see [34, Theorem 1]). As we shall see further on, the arguments of the previous section do not apply in general to the group $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$. However, for very peculiar cases, it is possible to proceed in the same way. More precisely, we recall the following definition of section, taken from [9].

Definition 3.1. *A set $\mathcal{S} \subset \mathbf{\Omega}_3$ is said to be a section of $\mathbf{\Omega}_3$ (associated to the flow φ) if, for any $\mathbf{x} \in \mathbf{\Omega}_3$, there exists a unique real number $\theta(\mathbf{x})$ such that $\varphi_{\theta(\mathbf{x})}(\mathbf{x}) \in \mathcal{S}$.*

One can state the following:

Theorem 3.2. *If there exists a* ***measurable*** *section $\mathcal{S}$ associated to the flow φ, then*

$$\mathfrak{S}(\mathcal{T}_3) = \mathfrak{S}(\mathcal{T}_3) + i\mathbb{R} \qquad \textit{and} \qquad \mathfrak{S}(\mathcal{U}_3(t)) = \mathfrak{S}(\mathcal{U}_3(t)) \cdot \mathbb{T}, \qquad (t \in \mathbb{R}).$$

As a consequence,

$$\mathfrak{S}(\mathcal{U}_3(t)) = \exp\left(t\mathfrak{S}(\mathcal{T}_3)\right), \qquad (t \in \mathbb{R}).$$

Proof. As in the previous section, the proof consists in exhibiting a function $\alpha(\cdot)$ satisfying (2.2). Precisely, set

$$\alpha(\mathbf{x}) = -\theta(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_3$$

where $\theta(\cdot)$ is provided by Definition 3.1. Since the section $\mathcal{S}$ is measurable, it is easy to see that $\alpha(\cdot)$ is measurable and is finite almost everywhere. Moreover, from the definition of $\theta(\mathbf{x})$ as the unique 'time' at which a trajectory starting from $\mathbf{x}$ meets $\mathcal{S}$, one has

$$\alpha(\varphi_t(\mathbf{x})) = t + \alpha(\mathbf{x})$$

and we conclude the proof thanks to Proposition 2.2. □

Remark 3.3. *Notice that there exists a section $\mathcal{S}$ associated to the flow $(\varphi_t)_{t\in\mathbb{R}}$ if and only if $(\varphi_t)_{t\in\mathbb{R}}$ does not admit periodic orbits nor rest points* [9, p. 48]. *However, it is a difficult task to provide sufficient conditions on the flow $(\varphi_t)_{t\in\mathbb{R}}$ ensuring the section $\mathcal{S}$ to be measurable. Let us however mention that, whenever the flow $(\varphi_t)_{t\in\mathbb{R}}$ is dispersive over $\mathbf{\Omega}_3$ in the sense of* [9, Chapter IV] *then $\mathcal{S}$ is measurable (and the function $\theta(\cdot)$ provided by Definition 3.1 is continuous).*

Example 3.4. Adopting notations from neutron transport theory, assume $\mathbf{\Omega} = \mathbb{R}^N \times V$ where V is a closed subset of $\mathbb{R}^N$. Then, for any $\mathbf{x} = (x, v) \in \mathbf{\Omega}$, with $x \in \mathbb{R}^N$ and $v \in V$, define $\mathscr{F}(\mathbf{x}) = (v, 0)$. Then, the mapping $\alpha(\mathbf{x}) = (x \cdot v)/|v|^2$ for any $v \neq 0$, fulfills (2.2) and the spectrum of $(\mathcal{U}(t))_{t\in\mathbb{R}}$ and $\mathcal{T}$ are invariant by rotations and by vertical translations along the imaginary axis respectively (see [25, 31] for more details).

Apart from the very special result above, one has to provide an alternative approach to deal with the spectral properties of $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$. The aim of this section is to get a more precise picture of the spectrum of both $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$ and $\mathcal{T}_3$. Recall now that, for a point $\mathbf{x} \in \mathbf{\Omega}_3$, since $\mathscr{F}$ is globally Lipschitz, three situations may occur:

(1) $\mathbf{x}$ is a rest point of the flow $(\varphi_t)_t$, i.e. $\mathscr{F}(\mathbf{x}) = 0$;
(2) $\mathbf{x}$ belongs to a periodic orbit of $(\varphi_t)_t$, i.e. there is some $t_0 > 0$ such that $\varphi_{t_0}(\mathbf{x}) = \mathbf{x}$.
(3) $\mathbf{x}$ belongs to an infinite but non closed orbit.

This leads to the following splitting of $\mathbf{\Omega}_3$:

$$\mathbf{\Omega}_3 = \mathbf{\Omega}_{\text{rest}} \cup \mathbf{\Omega}_{\text{per}} \cup \mathbf{\Omega}_{\infty}$$

where

$$\mathbf{\Omega}_{\text{rest}} = \{\mathbf{x} \in \mathbf{\Omega}_3\,;\, \varphi_t(\mathbf{x}) = \mathbf{x} \;\; \forall t \in \mathbb{R}\}, \qquad \mathbf{\Omega}_{\text{per}} = \{\mathbf{x} \in \mathbf{\Omega}_3 \;:\; \exists t > 0, \varphi_t(\mathbf{x}) = \mathbf{x}\}$$

and

$$\mathbf{\Omega}_{\infty} = \mathbf{\Omega}_3 \setminus (\mathbf{\Omega}_{\text{rest}} \cup \mathbf{\Omega}_{\text{per}}).$$

Notice that $\mathbf{\Omega}_{\text{rest}}$ is clearly a closed subset of $\mathbf{\Omega}$ while $\mathbf{\Omega}_{\text{per}}$ is measurable (this follows from the fact that the sets $\mathbf{\Omega}_{\text{per},n}$ defined by (4.2) are closed for any $n \in \mathbb{N}$ according to [3, p. 314]). It is not difficult to see that these sets are all invariant under the flow $(\varphi_t)_{t\in\mathbb{R}}$ and consequently, under the action of $\mathcal{U}_3(t)$ for any $t\in\mathbb{R}$. Therefore, defining X_{rest}, X_{per} and X_∞ as the set of functions in X_3 which are null almost everywhere outside of the sets $\mathbf{\Omega}_{\text{rest}}$, $\mathbf{\Omega}_{\text{per}}$ and $\mathbf{\Omega}_\infty$ respectively, one can define the following restrictions of $\mathcal{U}_3(t)$:

$$\mathcal{U}_{\text{rest}}(t)=\mathcal{U}_3(t)_{|X_{\text{rest}}},\quad \mathcal{U}_{\text{per}}(t)=\mathcal{U}_3(t)_{|X_{\text{per}}}\quad\text{and}\quad \mathcal{U}_\infty(t)=\mathcal{U}_3(t)_{|X_\infty}.$$

Clearly, $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$, $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$ and $(\mathcal{U}_\infty(t))_{t\in\mathbb{R}}$ are positive C_0-groups of X_{rest}, X_{rest} and X_∞ respectively, whose generators will be denoted respectively by $\mathcal{T}_{\text{rest}}$, $\mathcal{T}_{\text{per}}$ and $\mathcal{T}_\infty$. Arguing as in Theorem 2.1, the spectra of $\mathcal{U}_3(t)$ and $\mathcal{T}_3$ are given respectively by

$$\begin{aligned}\mathfrak{S}(\mathcal{U}_3(t))&=\mathfrak{S}(\mathcal{U}_{\text{rest}}(t))\cup\mathfrak{S}(\mathcal{U}_{\text{per}}(t))\cup\mathfrak{S}(\mathcal{U}_\infty(t)),\\ \mathfrak{S}(\mathcal{T}_3)&=\mathfrak{S}(\mathcal{T}_{\text{rest}})\cup\mathfrak{S}(\mathcal{T}_{\text{per}})\cup\mathfrak{S}(\mathcal{T}_\infty).\end{aligned}\tag{3.1}$$

Because of the possible existence of periodic orbits, it is not clear *a priori* that a result analogous to Theorems 2.3 and 2.5 can be proved in this case (even if $\nu(\cdot)=0$). Indeed, if $\mathfrak{m}(\mathbf{\Omega}_{\text{per}})\neq 0$, then there is no mapping $\alpha(\cdot)$ satisfying (2.2) on $\mathbf{\Omega}_3$. Indeed, if such a mapping were exist, in particular, for any $\mathbf{x}\in\mathbf{\Omega}_{\text{per}}$,

$$\alpha(\varphi_t(\mathbf{x}))=\alpha(\mathbf{x})+t,\qquad\forall t\geqslant 0.$$

However, there exists a period $t_0\neq 0$ such that $\varphi_{t_0}(\mathbf{x})=\mathbf{x}$ so that $t_0=\alpha(\varphi_{t_0}(\mathbf{x}))-\alpha(\mathbf{x})=0$ which is a contradiction. Of course, the impossibility to construct a function $\alpha(\cdot)$ with the above properties does not imply that the spectrum of $\mathcal{T}_3$ is not invariant by vertical translations along the imaginary axis.

Example 3.5. Let us consider the planar field $\mathscr{F}(\mathbf{x})=(-y,x)$ for any $\mathbf{x}=(x,y)\in\mathbf{\Omega}=\mathbb{R}^2$. In such a case, the characteristic curves are *circular*, namely

$$\Phi(\mathbf{x},s)=(x\cos s-y\sin s,x\sin s+y\cos s),\qquad\mathbf{x}=(x,y),\qquad s\in\mathbb{R}.$$

In particular, for any $\mathbf{x}=(x,y)\in\mathbf{\Omega}$ one has $\tau_\pm(\mathbf{x})=\infty$ and the mapping:

$$t\in\mathbb{R}\longmapsto\varphi_t(\mathbf{x})=(x\cos t+y\sin t,y\cos t-x\sin t)$$

is 2π–periodic. This means that $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are both empty sets. Consider for a while $\nu(\cdot)=0$. The semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ extends to a group and one has

$$\mathcal{U}(t)f(\mathbf{x})=f(\varphi_t(\mathbf{x})),\qquad\forall t\in\mathbb{R},\ f\in X.$$

Clearly, $(\mathcal{U}(t))_{t\in\mathbb{R}}$ is a positive and periodic C_0–group of X with period 2π. As a consequence [26], $\mathfrak{S}(\mathcal{T})=i\mathbb{Z}$. In particular, $\mathfrak{S}(\mathcal{T})\neq\mathfrak{S}(\mathcal{T})+i\mathbb{R}$.

The previous example shows that one cannot hope to deduce the spectral properties of $\mathcal{T}_3$ or $(\mathcal{U}_3(t))_{t\in\mathbb{R}}$ from their respective real spectrum as it was the case for the restrictions of $(\mathcal{U}(t))_{t\geqslant 0}$ to X_1 and X_2.

3.1. **Stationary flow.** We first deal with the spectral structure of the part $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$ of the C_0-group $(\mathcal{U}(t))_{t\in\mathbb{R}}$ acting on the set of rest points $\mathbf{\Omega}_{\text{rest}}$. Since $\varphi_t(\mathbf{x}) = \mathbf{x}$ for any $\mathbf{x} \in \mathbf{\Omega}_{\text{rest}}$ and any $t \in \mathbb{R}$, it is clear that $\mathcal{U}_{\text{rest}}(t)$ is given by:

$$\mathcal{U}_{\text{rest}}(t)f(\mathbf{x}) = \exp(-t\nu(\mathbf{x}))f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_{\text{rest}},\ f \in X_{\text{rest}}.$$

Hence, $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$ is a positive C_0-group of multiplication. This leads naturally to the following description of its generator:

$$\mathcal{T}_{\text{rest}}f = -\nu(\cdot)f, \qquad\qquad f \in \mathscr{D}(\mathcal{T}_{\text{rest}})$$

which can easily be deduced from the definition of $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$. Since the spectrum of $\mathcal{T}_{\text{rest}}$ is real, one can deduce from the spectral mapping theorem for the real spectrum (see Theorem A. 3) the following:

Theorem 3.6. *Assume that $\mathfrak{m}(\mathbf{\Omega}_{\text{rest}}) \neq 0$. The spectra of $(\mathcal{U}_{\text{rest}}(t))_{t\in\mathbb{R}}$ and its generator $\mathcal{T}_{\text{rest}}$ satisfy the following spectral mapping theorem:*

$$\mathfrak{S}(\mathcal{U}_{\text{rest}}(t)) = \exp(t\mathfrak{S}(\mathcal{T}_{\text{rest}})), \qquad\qquad \forall t \in \mathbb{R}$$

where the spectrum of the generator is given by

$$\mathfrak{S}(\mathcal{T}_{\text{rest}}) = \mathcal{R}_{\text{ess}}(-\nu(\cdot)) \subset \mathbb{R}$$

where $\mathcal{R}_{\text{ess}}(-\nu(\cdot))$ denotes the essential range of $-\nu(\cdot)$.

Remark 3.7. *Note that the spectrum of $\mathcal{T}_{\text{rest}}$ is not necessarily connected.*

3.2. **Aperiodic flow.** The study of the spectral properties of the C_0-group associated to an aperiodic flow has been investigated in a framework of continuous functions in [3] and in the more general framework of Mather semigroups in [13] under the supplementary assumption:

Assumption 3.8. *The mapping $\mathbf{x} \in \mathbf{\Omega}_\infty \longmapsto \int_0^t h(\varphi_s(\mathbf{x}))\mathrm{d}s$ is **continuous** for any $t \in \mathbb{R}$.*

Namely, one can prove the following which is a consequence of [13, Theorem 6.37, p.188]:

Theorem 3.9. *Assume that $\mathfrak{m}(\mathbf{\Omega}_\infty) \neq 0$ and Assumption 3.8 holds true. Then,*

$$\mathfrak{S}(\mathcal{T}_\infty) = \mathfrak{S}(\mathcal{T}_\infty) + i\,\mathbb{R} \qquad \textit{and} \qquad \mathfrak{S}(\mathcal{U}_\infty(t)) = \mathfrak{S}(\mathcal{U}_\infty(t)) \cdot \mathbb{T}, \qquad (t \in \mathbb{R}).$$

As a consequence,

$$\mathfrak{S}(\mathcal{U}_\infty(t)) = \exp\left(t\mathfrak{S}(\mathcal{T}_\infty)\right), \qquad (t \in \mathbb{R}). \tag{3.2}$$

Proof. Using the terminology of [13], the group $(\mathcal{U}_\infty(t))_{t\in\mathbb{R}}$ is an evolution Mather group induced by a cocycle $(\mathbf{\Psi}_t)_t$ over a flow $(\psi_t)_t$. Indeed,

$$\mathcal{U}_\infty(t)f(\mathbf{x}) = J_t(\mathbf{x})\mathbf{\Psi}_t(\psi_{-t}(x))\, f(\psi_{-t}(x)), \qquad t \in \mathbb{R},\ \mathbf{x} \in \mathbf{\Omega}_\infty,\ f \in X_\infty$$

where the flow $(\psi_t)_t$ is given by $\psi_{-t} = \varphi_t$ while

$$\begin{cases} \mathbf{\Psi} \quad : \quad & \mathbf{\Omega}_\infty \times \mathbb{R} \to \mathbb{R} \\ & (\mathbf{x}, t) \mapsto \mathbf{\Psi}_t(\mathbf{x}) = \exp\left(-\displaystyle\int_0^t h(\varphi_s(\mathbf{x}))\mathrm{d}s\right) \end{cases}$$

and $J_t(\mathbf{x}) = \exp\left[-\int_0^t \mathrm{div}(\mathscr{F})(\varphi_{-s}(\mathbf{x}))\mathrm{d}s\right]$ is the Radon-Nikodym derivative of the $\mathrm{d}\mathfrak{m} \circ \psi_t$ with respect to $\mathfrak{m}$ (see Proposition 1.2). The fact that $(\mathbf{\Psi}_t)_t$ is a cocycle of $\mathbf{\Omega}_\infty$ over the flow $(\psi_t)_t$ is a consequence of the following straightforward identities

$$\mathbf{\Psi}_{t+s}(\mathbf{x}) = \mathbf{\Psi}_t(\psi_{-s}(\mathbf{x}))\mathbf{\Psi}_s(\mathbf{x}) \qquad \text{and} \qquad \mathbf{\Psi}_0(\mathbf{x}) = 1 \qquad \forall \mathbf{x} \in \mathbf{\Omega}_\infty,\ t, s \in \mathbb{R}.$$

Therefore, one sees that [13, Theorem 6.37] applies since Assumption 3.8 implies that the mapping $(\mathbf{x}, t) \in \mathbf{\Omega}_\infty \times \mathbb{R} \mapsto \mathbf{\Psi}_t(\mathbf{x}) \in \mathbb{R}$ is continuous. □

Remark 3.10. *Notice that a more precise picture of the spectrum of $\mathfrak{S}(\mathcal{T}_\infty)$ is still missing. We also point out that that the rotational invariance of the spectrum $\mathfrak{S}(\mathcal{U}_\infty(t))$ for any $t \in \mathbb{R}$ has been proved in* [20] *in a direct way.*

Remark 3.11. *Assumption 3.8 is needed here in order to apply directly the results of* [13, Chapter 6]. *One may wonder if it is possible to get rid of such an assumption.*

Remark 3.12. *From the results of the previous sections, one can deduces that, when the flow $\varphi(\cdot)$ is aperiodic then the semigroup $(\mathcal{U}(t))_{t \geqslant 0}$ satisfies the spectral mapping theorem. Precisely, if $\mathfrak{m}(\mathbf{\Omega}_{\mathrm{per}}) = 0$ and Assumption 3.8 holds, then*

$$\mathfrak{S}(\mathcal{U}(t)) \setminus \{0\} = \exp(t\mathfrak{S}(\mathcal{T})), \qquad \forall t \geqslant 0. \tag{3.3}$$

If moreover, $\mathfrak{m}(\mathbf{\Omega}_{\mathrm{rest}}) = 0$, then $\mathfrak{S}(\mathcal{T}) = \mathfrak{S}(\mathcal{T}) + i\mathbb{R}$.

Note that practical criteria ensuring $\mathbf{\Omega}_{\mathrm{per}}$ to be *empty* are well-known. For a $\mathscr{C}^1$-planar field $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2)$ (i.e. when $N = 2$), one can mention the so-called *Dulac's criterion* [1, Proposition 24.14] which states that, if $\mathbf{\Omega}$ is simply connected, and if there exists $\varrho \in \mathscr{C}^1(\mathbf{\Omega}, \mathbb{R})$ such that $\mathrm{div}(\mathscr{F}\varrho)$ is not identically zero and does not change sign in $\mathbf{\Omega}$, then there are no periodic orbits lying entirely in $\mathbf{\Omega}$. Generalizations to higher dimension ($N \geqslant 3$) can also be provided (see e.g., [23, 10]). For planar C^1 field, one has also the following useful criterion:

Corollary 3.13. *Let I be an open interval of $\mathbb{R}$ and let $F \in \mathscr{C}^1(I \times \mathbb{R}\,;\,\mathbb{R})$. Consider the planar field $\mathscr{F}(\mathbf{x}) = (v, F(x, v))$ for any $\mathbf{x} = (x, v) \in \mathbf{\Omega} := I \times \mathbb{R}$. If*

$$F(x, 0) \neq 0, \qquad \forall x \in I,$$

then $\mathbf{\Omega}_{\mathrm{rest}} = \mathbf{\Omega}_{\mathrm{per}} = \varnothing$ and the spectral mapping theorem (3.3) *holds for any h such that Assumption 3.8 is met.*

Proof. Under the above assumption, $\mathscr{F}$ does not possess any equilibrium point in $\mathbf{\Omega}$, i.e. $\mathbf{\Omega}_{\mathrm{rest}} = \varnothing$. According to [1, Corollary 24.22, p. 346] there is no periodic orbit whose interior lies completely in $\mathbf{\Omega} = I \times \mathbb{R}$. Since I is an interval of $\mathbb{R}$, $\mathbf{\Omega}_{\mathrm{per}} = \varnothing$ and we get the conclusion from Remark 3.12. □

Finally, it is also known that planar *gradient flows* do not exhibit any periodic orbit [1, p. 241] leading to the following useful corollary

Corollary 3.14. *Assume that there exists $V : \mathbb{R}^N \to \mathbb{R}$ such that $\mathscr{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^N$. Then $\mathfrak{S}(U(t)) \setminus \{0\} = \exp(t\mathfrak{S}(T))$, for any $t \geqslant 0$ as soon as Assumption 3.8 is met.*

We finally illustrate the above results by examples taken from various kinetic equations.

3.3. **Examples.** Let us begin with the classical Vlasov equation with a quadratic potential

Example 3.15. One considers, as in (1.5), a cylindrical domain $\mathbf{\Omega} = D \times \mathscr{V} \subset \mathbb{R}^6$ where D is a smooth open subset of $\mathbb{R}^3$. For any $\mathbf{x} = (x, v) \in D \times \mathscr{V}$, let

$$\mathscr{F}(\mathbf{x}) = (v, \mathbf{F}(x, v)) \in \mathbb{R}^6 \tag{3.4}$$

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is a time independent globally Lipschitz field (the force field) over $D \times \mathscr{V}$ given by

$$\mathbf{F}(x, v) = -x - \nabla_v \mathbf{U}(v), \qquad \forall (x, v) \in \mathbf{\Omega}$$

where $\mathbf{U}(v)$ is a Lipschitz space homogeneous potential $\mathbf{U} : \mathscr{V} \to \mathbb{R}$. Then, $\mathscr{F}$ is a gradient flow associated to the potential $V(x, v) = \langle v, x \rangle + \mathbf{U}(v)$. Therefore, the associated transport operator and semigroups are satisfying Corollary 3.14.

Example 3.16. Of course, the classical neutron transport equation for which $\mathbf{\Omega} = D \times \mathscr{V} \subset \mathbb{R}^6$ and

$$\mathscr{F}(x, v) = (v, 0), \qquad \forall \mathbf{x} = (x, v) \in \mathbf{\Omega}$$

is such that $\mathfrak{m}(\mathbf{\Omega}_{\text{per}}) = 0$.

More surprisingly, the above results also apply to kinetic equations of second order via a suitable use of Fourier transform [15] :

Example 3.17. Consider as in [15] the Vlasov-Fokker-Planck equation with quadratic confining potential:

$$\partial_t f + v \cdot \nabla_x f - x \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + v f) \tag{3.5}$$

where $f = f(x, v, t) \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ for any $t \geqslant 0$. The above equation is unitarily equivalent to the following first order equation in ξ and η:

$$\partial_t \hat{f} + \eta \cdot \nabla_\xi \hat{f} + (\eta - \xi) \cdot \nabla_\eta \hat{f} + |\eta|^2 \hat{f} = 0 \tag{3.6}$$

where

$$\hat{f}(\xi, \eta, t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \exp(-i(x \cdot \xi + v \cdot \eta)) f(x, v, t) \mathrm{d}v \mathrm{d}x$$

denotes the L^2 Fourier transform (in x and v) of f. The above equation (3.6) falls within the theory we developed in the previous sections. Precisely, let

$$\mathscr{F}(\mathbf{x}) = \mathscr{F}(\xi, \eta) = (\eta, \eta - \xi), \qquad \forall \mathbf{x} = (\xi, \eta) \in \mathbf{\Omega} = \mathbb{R}^N \times \mathbb{R}^N.$$

Notice that $\mathrm{div}\mathscr{F}(\mathbf{x}) = N$ for any $\mathbf{x} \in \mathbf{\Omega}$ and, according to (3.6), $h(\mathbf{x}) = h(\xi, \eta) = |\eta|^2 - N$. The characteristic system

$$\dot{\xi}(t) = \eta(t), \qquad \dot{\eta}(t) = (\eta(t) - \xi(t)), \qquad t \in \mathbb{R}$$

with initial condition $\xi(0) = \xi_0$, $\eta(0) = \eta_0$ is explicitly solvable [15] with

$$\xi(t) = \frac{2}{\sqrt{3}} \exp(t/2) \left\{ \left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \xi_0 + \sin\left(\frac{\sqrt{3}}{2} t\right) \eta_0 \right\}$$

$$\eta(t) = \frac{2}{\sqrt{3}} \exp(t/2) \left\{ \left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \eta_0 - \sin\left(\frac{\sqrt{3}}{2} t\right) \xi_0 \right\}.$$

The flow $\varphi_t : (\xi_0, \eta_0) \mapsto (\xi(t), \eta(t))$ does not possess any periodic orbit while $(0,0)$ is the unique rest point. Therefore, according to the above Theorem 3.12, the C_0-group governing (3.6) satisfies the Spectral Mapping Theorem 3.3. Turning back to the original variables, the Vlasov-Fokker-Planck $(\mathcal{U}_{VFP}(t))_{t\in\mathbb{R}}$ governing Eq. (3.5) also satisfies the Spectral Mapping Theorem 3.3:

$$\mathfrak{S}(\mathcal{U}_{VFP}(t)) = \exp(t\mathfrak{S}(L_{VFP})), \qquad \forall t \in \mathbb{R},$$

where L_{VFP} is the Vlasov-Fokker-Planck operator:

$$L_{VFP}f(x,v) = -v\cdot\nabla_x f + x\cdot\nabla_v f + \nabla_v\cdot(\nabla_v f + vf),$$

with its maximal domain in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$.

4. Periodic flow

We now deal with the study of the periodic part of the group $\mathcal{U}(t)$, by studying the spectral properties of $(\mathcal{U}_{\text{per}}(t))_{t\in\mathbb{R}}$ on the space X_{per}. In contrast to what happens for aperiodic flow, for periodic flow the spectral mapping theorem

$$\mathfrak{S}(\mathcal{U}_{\text{per}}(t)) = \exp\left(t\mathfrak{S}(\mathcal{T}_{\text{per}})\right), \qquad \forall t \in \mathbb{R} \tag{4.1}$$

does not hold. Indeed, let us consider the following example

Example 4.1. We turn back to the rotation group in $\mathbb{R}^2$ introduced in Example 3.5. Recall that, for such an example $\mathbf{\Omega} = \mathbb{R}^2$, $\mathscr{F}(\mathbf{x}) = (-y, x)$ for any $\mathbf{x} = (x,y)$ and the associated C_0-group in $L^p(\mathbb{R}^2, \mathrm{d}\mathfrak{m})$ is given by $\mathcal{U}(t)f(\mathbf{x}) = f(\varphi_t(\mathbf{x}))$, $t \in \mathbb{R}$ where the flow φ_t is given by the rotation of angle $t \in \mathbb{R}$:

$$\varphi_t(\mathbf{x}) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{x} = (x,y) \in \mathbb{R}^2.$$

According to [26], the spectrum of the generator $\mathcal{T}$ of $(\mathcal{U}(t))_{t\in\mathbb{R}}$ is given by $\mathfrak{S}(\mathcal{T}) = i\mathbb{Z}$. Moreover, for any $t \in \mathbb{R}$, $\mu_n(t) = \exp(int)$ is an eigenvalue of $\mathcal{U}(t)$ for any $n \in \mathbb{Z}$. If $t/2\pi$ is irrational, the eigenvalues $\{\mu_n(t),\ n \in \mathbb{Z}\}$ describe a dense subset of $\mathbb{T}$ and the closedness of the spectrum of $\mathcal{U}(t)$ implies that $\mathfrak{S}(\mathcal{U}(t)) = \mathbb{T}$ while $\exp(t\mathfrak{S}(\mathcal{T})) = \exp(it\mathbb{Z}) \neq \mathbb{T}$ for such a t. This shows that, in general, the Spectral Mapping Theorem 4.1 fails for periodic flows.

For any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$, one can define the *prime period* of $\mathbf{x}$ as

$$\mathfrak{p}(\mathbf{x}) = \inf\{t > 0,\ \varphi_t(\mathbf{x}) = \mathbf{x}\}.$$

The main properties of the prime period are listed in the following Proposition. We refer the reader to [1, 3] for the first assertions while the last one is referred to as Yorke's Theorem [32].

Proposition 4.2. *The mapping $\mathfrak{p}(\cdot) : \mathbf{x} \in \mathbf{\Omega}_{\text{per}} \mapsto \mathfrak{p}(\mathbf{x}) \in (0,\infty)$ enjoys the following properties:*

(i) For any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$, $\varphi_t(\mathbf{x}) = \mathbf{x}$ if and only if $t = n\mathfrak{p}(\mathbf{x})$ for some $n \in \mathbb{Z}$.
(ii) $\mathfrak{p}(\cdot)$ is lower semicontinuous, and thus measurable.
(iii) $\mathfrak{p}(\cdot)$ is invariant under the flow φ_t, i.e. $\mathfrak{p}(\varphi_t(\mathbf{x})) = \mathfrak{p}(\mathbf{x})$ for any $t \in \mathbb{R}$.

(iv) $\mathfrak{p}(\cdot)$ *is bounded away from zero. Namely,*

$$\mathfrak{p}(\mathbf{x}) \geqslant 2\pi/\kappa \qquad \textit{for any} \quad \mathbf{x} \in \mathbf{\Omega}_{\text{per}},$$

where $\kappa > 0$ *is the Lispchitz constant of the field* $\mathscr{F}$.

Notice that, *a priori*, the prime period $\mathfrak{p}$ is an unbounded function. However, we shall prove that the description of $\mathfrak{S}(\mathcal{T}_{\text{per}})$ relies actually on the behavior of $\mathcal{T}_{\text{per}}$ on functions supported on sets where $\mathfrak{p}$ is bounded. Precisely, for any $n > 0$, define $\mathbf{\Omega}_{\text{per},n}$ as :

$$\mathbf{\Omega}_{\text{per},n} := \{\mathbf{x} \in \mathbf{\Omega}_{\text{per}}\,;\, \mathfrak{p}(\mathbf{x}) \leqslant n\}\,. \tag{4.2}$$

Proposition 4.2 asserts that $\mathbf{\Omega}_{\text{per},n} = \varnothing$ for any $0 < n < 2\pi/\kappa$ and $\mathbf{\Omega}_{\text{per},n}$ is invariant under the action of the flow $(\varphi_t)_t$ for any $n > 0$. As above, one can define $X_{\text{per},n}$ as

$$X_{\text{per},n} = \{f \in X_{\text{per}}\,;\, f(\mathbf{x}) = 0 \quad \mathfrak{m}-\text{a.e. } \mathbf{x} \in \mathbf{\Omega}_{\text{per}} \setminus \mathbf{\Omega}_{\text{per},n}\}\,, \qquad \forall n \geqslant 2\pi/\kappa,$$

and let $(\mathcal{U}_{\text{per},n}(t))_{t\in\mathbb{R}}$ be the restriction of $(\mathcal{U}_{\text{per}}(t))_{t\in\mathbb{R}}$ on $X_{\text{per},n}$. We denote by $\mathcal{T}_{\text{per},n}$ its generator. One has the following abstract result:

Theorem 4.3. *Let* $\lambda \in \mathbb{C}$ *be given. Then,* $\lambda \in \varrho(\mathcal{T}_{\text{per}})$ *if and only if*

$$\lambda \in \bigcap_{n\in\mathbb{N}} \varrho(\mathcal{T}_{\text{per},n}), \qquad \textit{and} \qquad \sup_{n\in\mathbb{N}} \left\|(\lambda - \mathcal{T}_{\text{per},n})^{-1}\right\|_{\mathscr{B}(X_{\text{per},n})} < \infty. \tag{4.3}$$

Proof. Assume first that $\lambda \in \varrho(\mathcal{T}_{\text{per}})$ and let $n \in \mathbb{N}$. Given $g \in X_{\text{per},n} \subset X_{\text{per}}$, there is a unique $f \in X_{\text{per}}$ such that $(\lambda - \mathcal{T}_{\text{per}})f = g$. It is clear that, $f \in X_{\text{per},n}$ which proves that $\lambda \in \varrho(\mathcal{T}_{\text{per},n})$. Moreover, $\|f\|_{X_{\text{per},n}} \leqslant \|(\lambda - \mathcal{T}_{\text{per}})^{-1}\|_{\mathscr{B}(X_{\text{per}})}\|g\|_{X_{\text{per},n}}$, which proves that

$$\sup_{n\in\mathbb{N}} \left\|(\lambda - \mathcal{T}_{\text{per},n})^{-1}\right\|_{\mathscr{B}(X_{\text{per},n})} \leqslant \left\|(\lambda - \mathcal{T}_{\text{per}})^{-1}\right\|_{\mathscr{B}(X_{\text{per}})}.$$

Conversely, assume Eq. (4.3) to holds. Let $g \in X_{\text{per}}$. For any $n \in \mathbb{N}$, let $g_n = g\chi_{\mathbf{\Omega}_{\text{per},n}}$. It is clear that $g_n \in X_{\text{per},n}$. Since $\lambda \in \varrho(\mathcal{T}_{\text{per},n})$, there is a unique $f_n \in X_{\text{per},n}$ such that $(\lambda - \mathcal{T}_{\text{per},n})f_n = g_n$. Now, one notes that, for any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$, $\lim_{n\to\infty} \chi_{\mathbf{\Omega}_{\text{per},n}}(\mathbf{x}) = 1$. Consequently, g_n converges to g in X_{per}. Let us prove that $(f_n)_n$ also converge in X_{per}. Given $n_1 \leqslant n_2$, one has $f_{n_1} - f_{n_2} \in X_{\text{per},n_2}$ and

$$\begin{aligned}\|f_{n_1} - f_{n_2}\|_{X_{\text{per}}} = \|f_{n_1} - f_{n_2}\|_{X_{\text{per},n_2}} &\leqslant \sup_{n\in\mathbb{N}} \left\|(\lambda - \mathcal{T}_{\text{per},n})^{-1}\right\|_{\mathscr{B}(X_{\text{per},n})} \|g_{n_1} - g_{n_2}\|_{X_{\text{per},n_2}}\\ &\leqslant \sup_{n\in\mathbb{N}} \left\|(\lambda - \mathcal{T}_{\text{per},n})^{-1}\right\|_{\mathscr{B}(X_{\text{per},n})} \|g_{n_1} - g_{n_2}\|_{X_{\text{per}}}, \qquad n_1 \leqslant n_2.\end{aligned}$$

Therefore, $(f_n)_n$ is a Cauchy sequence in X_{per}. Let f denote its limit. Since $f_n \longrightarrow f$ and $g_n \longrightarrow g$, one has $\mathcal{T}_{\text{per},n}f_n$ converges to $\lambda f - g$. Since $\mathcal{T}_{\text{per}}$ is a closed operator, one gets $f \in \mathscr{D}(\mathcal{T}_{\text{per}})$ with $(\lambda - \mathcal{T}_{\text{per}})f = g$. Now, let $h \in \mathscr{D}(\mathcal{T}_{\text{per}})$ be another solution to the spectral problem $(\lambda - \mathcal{T}_{\text{per}})h = g$. Then, since $\lambda \in \varrho(\mathcal{T}_{\text{per},n})$ for any n, one sees that $h\chi_{\mathbf{\Omega}_{\text{per},n}} = f_n$ for any $n \in \mathbb{N}$. Using again the fact that $\chi_{\mathbf{\Omega}_{\text{per},n}}(\mathbf{x}) \to 1$ for any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$, one sees that $h(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$. This proves that $\lambda \in \varrho(\mathcal{T}_{\text{per}})$. □

We describe now more precisely the spectrum $\mathfrak{S}(\mathcal{T}_{\mathrm{per},n})$. For any $\lambda \in \mathbb{C}$, define

$$\vartheta(\mathbf{x}) = -\frac{1}{\mathfrak{p}(\mathbf{x})}\int_0^{\mathfrak{p}(\mathbf{x})} \nu(\varphi_s(\mathbf{x}))\mathrm{d}s \quad \text{and} \quad M_\lambda(\mathbf{x}) = \exp\left\{-\mathfrak{p}(\mathbf{x})(\lambda - \vartheta(\mathbf{x}))\right\}, \qquad \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}.$$

Note that, for any $t \in \mathbb{R}$ and any $\mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}$, $\mathfrak{p}(\varphi_t(\mathbf{x})) = \mathfrak{p}(\mathbf{x})$ so that

$$\vartheta(\varphi_t(\mathbf{x})) = \vartheta(\mathbf{x}), \quad \text{and} \quad M_\lambda(\varphi_t(\mathbf{x})) = M_\lambda(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}.$$

One has,

Theorem 4.4. *For any $n \in \mathbb{N}$, $\mathfrak{S}(\mathcal{T}_{\mathrm{per},n}) \subset \{\lambda \in \mathbb{C}\,;\, 1 \in \mathcal{R}_{\mathrm{ess}}(M_\lambda)\}$ where $\mathcal{R}_{\mathrm{ess}}(M_\lambda)$ denotes the essential range of $M_\lambda\,:\,\mathbf{\Omega}_{\mathrm{per},n} \to \mathbb{C}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. First, for any $\lambda \in \mathbb{C}$, let

$$\mathcal{J}_\lambda\,:\, f \in X_{\mathrm{per},n} \mapsto \mathcal{J}_\lambda f(\mathbf{x}) = \int_0^{\mathfrak{p}(\mathbf{x})} \exp(-\lambda t)\mathcal{U}_{\mathrm{per},n}(t) f(\mathbf{x})\mathrm{d}t, \qquad \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n}.$$

The proof of the Proposition is based on the fact that $\mathcal{J}_\lambda f \in \mathscr{D}(\mathcal{T}_{\mathrm{per},n})$ for any $f \in X_{\mathrm{per},n}$ with

$$(\lambda - \mathcal{T}_{\mathrm{per},n})\mathcal{J}_\lambda f(\mathbf{x}) = \left(1 - M_\lambda(\mathbf{x})\right) f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n}. \tag{4.4}$$

Indeed, given $f \in X$, since $\mathfrak{p}(\mathbf{x}) \leqslant n$ for any $\mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n}$, one sees that $\mathcal{J}_\lambda f \in X_{\mathrm{per},n}$ with

$$\|\mathcal{J}_\lambda f\|_{X_{\mathrm{per},n}} \leqslant \int_0^n \exp(-\mathrm{Re}\lambda t)\|\mathcal{U}_{\mathrm{per},n}(t) f\|_{X_{\mathrm{per},n}} dt.$$

Now, given $t \in \mathbb{R}$ and $f \in X_{\mathrm{per},n}$, since $\mathfrak{p}(\cdot)$ is invariant under the action of the flow $(\varphi_t)_t$, one sees easily that

$$\begin{aligned}\mathcal{U}_{\mathrm{per},n}(t)\mathcal{J}_\lambda f(\mathbf{x}) &= \int_0^{\mathfrak{p}(\mathbf{x})} \exp(-\lambda s)\mathcal{U}_{\mathrm{per},n}(t+s) f(\mathbf{x})\mathrm{d}s \\ &= \exp(\lambda t)\int_t^{t+\mathfrak{p}(\mathbf{x})} \exp(-\lambda s)\mathcal{U}_{\mathrm{per},n}(s) f(\mathbf{x})\mathrm{d}s, \qquad \forall \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n},\end{aligned}$$

and,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{U}_{\mathrm{per},n}(t)\mathcal{J}_\lambda f(\mathbf{x})\right]\Big|_{t=0} = \lambda\mathcal{J}_\lambda f(\mathbf{x}) + \exp\left(-\lambda\mathfrak{p}(\mathbf{x})\right)\mathcal{U}_{\mathrm{per},n}(\mathfrak{p}(\mathbf{x})) f(\mathbf{x}) - f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n}.$$

Now, since

$$[\mathcal{U}_{\mathrm{per},n}(\mathfrak{p}(\mathbf{x}))f](\mathbf{x}) = \exp\left(-\int_0^{\mathfrak{p}(\mathbf{x})} \nu(\varphi_s(\mathbf{x}))\mathrm{d}s\right) f(\mathbf{x})$$

one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{U}_{\mathrm{per},n}(t)\mathcal{J}_\lambda f(\mathbf{x})\right]\Big|_{t=0} = \lambda\mathcal{J}_\lambda f(\mathbf{x}) + \left(M_\lambda(\mathbf{x}) - 1\right) f(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per},n}.$$

This proves that $\mathcal{J}_\lambda f \in \mathscr{D}(\mathcal{T}_{\mathrm{per},n})$ and Eq. (4.4) holds. Now, let us prove that $\lambda \in \mathfrak{S}(\mathcal{T}_{\mathrm{per},n})$ implies $1 \in \mathcal{R}_{\mathrm{ess}}(M_\lambda)$. To do so, assume $1 \notin \mathcal{R}_{\mathrm{ess}}(M_\lambda)$, and define the operator

$$\mathscr{R}_\lambda f(\mathbf{x}) = (1 - M_\lambda(\mathbf{x}))^{-1}\mathcal{J}_\lambda f(\mathbf{x}).$$

Since $1 \notin \mathcal{R}_{\text{ess}}(M_\lambda)$, one sees that $\mathscr{R}_\lambda$ is a bounded operator in $X_{\text{per},n}$. Moreover, using the fact that $\mathfrak{p}(\cdot)$ and $M_\lambda(\cdot)$ are invariant under the flow $(\varphi_t)_t$, one easily sees that

$$\mathscr{R}_\lambda \mathcal{U}_{\text{per},n}(t) = \mathcal{U}_{\text{per},n}(t)\mathscr{R}_\lambda, \qquad t \in \mathbb{R}.$$

Classically, this implies that $\mathscr{R}_\lambda \mathscr{D}(\mathcal{T}_{\text{per},n}) \subset \mathscr{D}(\mathcal{T}_{\text{per},n})$ and

$$\mathscr{R}_\lambda \mathcal{T}_{\text{per},n} f = \mathcal{T}_{\text{per},n}\mathscr{R}_\lambda f, \qquad \forall f \in \mathscr{D}(\mathcal{T}_{\text{per},n}).$$

Using again the fact that $M_\lambda(\cdot)$ is invariant under the flow $(\varphi_t)_t$, one sees that $\mathcal{T}_{\text{per},n}\mathscr{R}_\lambda f = (1 - M_\lambda(\cdot))^{-1}\mathcal{T}_{\text{per},n}\mathcal{J}_\lambda f$ and Eq. (4.4) asserts that

$$\mathscr{R}_\lambda(\lambda - \mathcal{T}_{\text{per},n})f = (\lambda - \mathcal{T}_{\text{per},n})\mathscr{R}_\lambda f, \qquad \forall f \in \mathscr{D}(\mathcal{T}_{\text{per},n}).$$

This proves that $\lambda \in \varrho(\mathcal{T}_{\text{per},n})$ with $(\lambda - \mathcal{T}_{\text{per},n})^{-1} = \mathscr{R}_\lambda$. □

Remark 4.5. *We conjecture that the inclusion in the above Theorem is an equality. More precisely, if there is some $T \geqslant 0$ such that $\mathfrak{p}(\mathbf{x}) \leqslant T$ for any $\mathbf{x} \in \mathbf{\Omega}$, then, we conjecture that*

$$\lambda \in \mathfrak{S}(\mathcal{T}_{\text{per}}) \iff 1 \in \mathcal{R}_{\text{ess}}(M_\lambda). \tag{4.5}$$

The following reasoning comforts us in our belief. If $1 \in \mathcal{R}_{\text{ess}}(M_\lambda)$, then, for any fixed $\epsilon > 0$, the set $\mathbf{\Omega}_\epsilon = \{\mathbf{x} \in \mathbf{\Omega}_{\text{per},n}\,;\, |1 - M_\lambda(\mathbf{x})| < \epsilon\}$ is of non zero measure. Let $f_\epsilon \in X_{\text{per}}$ be such that $\|f_\epsilon\| = 1$ and Supp $f_\epsilon \subset \mathbf{\Omega}_\epsilon$. Let $g_\epsilon = \mathcal{J}_\lambda f_\epsilon$. Then, Eq. (4.4) implies that

$$(\lambda - \mathcal{T}_{\text{per}})g_\epsilon(\mathbf{x}) = (1 - M_\lambda(\mathbf{x}))f_\epsilon(\mathbf{x}),$$

so that

$$\|(\lambda - \mathcal{T}_{\text{per}})g_\epsilon\| \leqslant \epsilon.$$

To prove that $\lambda \in \mathfrak{S}(\mathcal{T}_{\text{per}})$, it would suffice to prove that $\|g_\epsilon\| \geqslant C > 0$ for some constant $C > 0$ that does not depend on ϵ. We did not succeed in proving this point. Notice however that the identity (4.5) holds true in space of continuous functions [3].

The following Proposition provides a complete picture of the set of λ for which $1 \in \mathcal{R}_{\text{ess}}(M_\lambda)$. Its proof is inspired by similar calculations already used in the study of $2D$ neutron transport equations [24]:

Proposition 4.6. *Assume that $\mathfrak{m}(\mathbf{\Omega}_{\text{per}}) \neq 0$ and there exists some $T > 0$ such that $\mathfrak{p}(\mathbf{x}) \leqslant T$ for any $\mathbf{x} \in \mathbf{\Omega}_{\text{per}}$. Then,*

$$1 \in \mathcal{R}_{\text{ess}}(M_\lambda) \quad \textit{if and only if} \quad \lambda \in \overline{\bigcup_{k\in\mathbb{Z}} \mathcal{R}_{\text{ess}}(F_k)}$$

where, for any $k \in \mathbb{Z}$, $F_k \,:\, \mathbf{\Omega}_{\text{per}} \to \mathbb{C}$ is a measurable mapping given by

$$F_k(\mathbf{x}) = \vartheta(\mathbf{x}) + i\frac{2k\pi}{\mathfrak{p}(\mathbf{x})}, \qquad \mathbf{x} \in \mathbf{\Omega}_{\text{per}}.$$

Proof. Let us pick $\lambda \notin \bigcup_{k\in\mathbb{Z}} \mathcal{R}_{\text{ess}}(F_k)$. Then, for any $k \in \mathbb{Z}$, there exists $\beta_k > 0$ such that

$$|\lambda - F_k(\mathbf{x})| \geqslant \beta_k \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\text{per}},$$

i.e.

$$|-\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - 2ik\pi - \lambda\,\mathfrak{p}(\mathbf{x})| \geqslant \mathfrak{p}(\mathbf{x})\,\beta_k \geqslant 2\pi\beta_k/\kappa \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\text{per}}$$

where we made use of Yorke's Theorem for the last estimate. This means that, for any integer $n \geqslant 0$, there exists $c_n > 0$ such that

$$\left|\frac{-\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\,\mathfrak{p}(\mathbf{x})}{2\pi\, n} \pm i\right| \geqslant c_n \quad \text{a. e.} \quad \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}},\ n \geqslant 1, \tag{4.6}$$

and

$$|\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\,\mathfrak{p}(\mathbf{x})| \geqslant c_0 \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}. \tag{4.7}$$

Now, since

$$e^u - 1 = u e^{\frac{u}{2}} \prod_{n=1}^{\infty}\left(1 + \frac{u^2}{4\pi^2\, n^2}\right), \qquad u \in \mathbb{C},$$

and

$$\prod_{n=N}^{\infty}\left(1 + \frac{u^2}{4\pi^2\, n^2}\right) \to 1 \quad (N \to \infty)$$

uniformly on any compact subset of $\mathbb{C}$, for any $M > 0$, there exists $N > 0$ such that

$$|e^u - 1| \geqslant \frac{1}{2}\,|u|\left|e^{\frac{u}{2}}\right| \prod_{n=1}^{N}\left|i - \frac{u}{2\pi\, n}\right|\left|i + \frac{u}{2\pi\, n}\right|, \qquad |u| < M.$$

Now, since ν is bounded and $\mathfrak{p}(\mathbf{x}) \leqslant T$, there exists $M > 0$ (large enough), such that

$$|\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\,\mathfrak{p}(\mathbf{x})| \leqslant M \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}},$$

one gets form (4.6) and (4.7) that

$$|\exp\{\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\,\mathfrak{p}(\mathbf{x})\} - 1| \geqslant \frac{C}{2}\prod_{n=1}^{N} c_n^2 \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}},$$

where $C = \operatorname{ess\,inf}_{\mathbf{x}\in\mathbf{\Omega}_{\mathrm{per}}}\left|\exp\left\{\frac{\mathfrak{p}(\mathbf{x})}{2}(\vartheta(\mathbf{x}) - \lambda)\right\}\right|$. Hence, $\operatorname{ess\,inf}_{\mathbf{x}\in\mathbf{\Omega}_{\mathrm{per}}} |M_\lambda(\mathbf{x}) - 1| > 0$ which proves the first inclusion. To prove the second inclusion, it is enough to notice, from the continuity of the exponential function that, if there is some constant $C > 0$ such that

$$|\exp(\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\mathfrak{p}(\mathbf{x})) - 1| \geqslant C > 0, \qquad \text{a. e. } \mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}},$$

then, for any $k \in \mathbb{Z}$, there exists $c_k > 0$ such that $|\mathfrak{p}(\mathbf{x})\vartheta(\mathbf{x}) - \lambda\mathfrak{p}(\mathbf{x}) - 2ik\pi| \geqslant c_k$ for a. e. $\mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}$. Now, since $\mathfrak{p}(\mathbf{x}) \leqslant T$ for any $\mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}$, one sees that $|\lambda - F_k(\mathbf{x})| \geqslant c_k/T$ for a. e. $\mathbf{x} \in \mathbf{\Omega}_{\mathrm{per}}$. □

It remain to investigate the spectral properties of the group $(\mathcal{U}_{\mathrm{per}}(t))_{t\in\mathbb{R}}$. As we already saw it, the Spectral Mapping Theorem fails to be true in general. However, one can deduce from Proposition A. 2 the following version of the Annular Hull Theorem:

Theorem 4.7. *Assume* $\mathfrak{m}(\mathbf{\Omega}_{\mathrm{per}}) \neq 0$. *Then,*

$$\mathbb{T} \cdot \mathfrak{S}\big(\mathcal{U}_{\mathrm{per}}(t)\big) = \mathbb{T} \cdot \exp\Big(t\mathfrak{S}\big(\mathcal{T}_{\mathrm{per}}\big) \cap \mathbb{R}\Big), \qquad \forall t \in \mathbb{R}. \tag{4.8}$$

In particular, the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ fulfils the Annular Hull Theorem:

$$\mathbb{T}\cdot\left(\mathfrak{S}(\mathcal{U}(t))\setminus\{0\}\right)=\mathbb{T}\cdot\exp\left(t\mathfrak{S}(\mathcal{T})\right),\qquad \forall t\geqslant 0.$$

Remark 4.8. *The above Annular Hull Theorem is known to be true for general weighted shift semigroups* [22, 13]. *Notice that the proof of* [22, 13] *rely on completely different arguments and involve very sophisticated tools. On the contrary, our result is a very easy consequence of a more general result on positive groups on L^p spaces. Of course, the simplification relies on the fact that we are dealing here with C_0-groups rather than semigroups.*

In the L^1 case, one can strengthen this result thanks to the Weak Spectral Mapping Theorem for positive C_0–group by W. Arendt and G. Greiner [3, Corollary 1.4]. Precisely,

Theorem 4.9. *Assume $p=1$, i.e. $X_{\mathrm{per}}=L^1(\mathbf{\Omega}_{\mathrm{per}},\mathrm{d}\mathfrak{m})$. For any $t\in\mathbb{R}$, the following weak spectral mapping theorem holds: $\mathfrak{S}(\mathcal{U}_{\mathrm{per}}(t))=\overline{\exp\left(t\mathfrak{S}(\mathcal{T}_{\mathrm{per}})\right)}$ and, under the assumption 3.8,*

$$\mathfrak{S}(\mathcal{U}(t))=\overline{\exp\left(t\mathfrak{S}(\mathcal{T})\right)},\qquad \forall t\geqslant 0.$$

Appendix: Spectral properties of general positive semigroups

Let $\mathfrak{X}$ be a complex Banach lattice with positive cone $\mathfrak{X}_+$ and let $(T(t))_{t\in\mathbb{R}}$ be a positive C_0-group in $\mathscr{B}(\mathfrak{X})$ with generator A. Recall that the positivity of the group $(T(t))_{t\in\mathbb{R}}$ means that $\mathfrak{X}_+$ is invariant under $T(t)$ for any $t\in\mathbb{R}$. We establish in this Appendix several abstract results on (positive) C_0-groups we used in the paper. The key point is the following spectral decomposition result for strongly continuous groups of positive operators due to Arendt [2, Theorem 4.2] and Greiner [18].

Theorem A. 1. Let $(T(t))_{t\in\mathbb{R}}$ be a C_0–group of positive operators with generator A on some Banach lattice $\mathfrak{X}$. Let $\mu\in\varrho(A)\cap\mathbb{R}$. Then, $\mathfrak{X}$ is the direct sum of the orthogonal projection bands:

$$I_\mu=\{x\in\mathfrak{X}\,;\,\mathscr{R}(\mu,A)|x|\geqslant 0\}\qquad\text{and}\qquad J_\mu=\{x\in\mathfrak{X}\,;\,\mathscr{R}(\mu,A)|x|\leqslant 0\}.$$

Moreover, I_μ and J_μ are invariant under $T(t)$ $(t\in\mathbb{R})$ and $\mathfrak{S}(A_{|I_\mu})=\{\lambda\in\mathfrak{S}(A)\,;\,\mathrm{Re}\lambda<\mu\}$, $\mathfrak{S}(A_{|J_\mu})=\{\lambda\in\mathfrak{S}(A)\,;\,\mathrm{Re}\lambda>\mu\}$ where $A_{|I_\mu}$ (respectively $A_{|J_\mu}$) denotes the generator of $(T(t)_{|I_\mu})_{t\in\mathbb{R}}$ (resp. of $(T(t)_{|J_\mu})_{t\in\mathbb{R}}$).

Using this result, G. Greiner [18] has been able to prove a *Spectral Mapping Theorem* for the *real spectrum* (Theorem A. 3 hereafter). Borrowing the ideas of Greiner, it is possible to prove the following more general result. Actually, we did not find this result in the literature and give here a simple proof of it.

Proposition A. 2. Let (Σ,ϖ) be a $\mathfrak{S}$-measured space and let $(T(t))_{t\in\mathbb{R}}$ be a *positive* C_0-group in $L^p(\Sigma,\mathrm{d}\varpi)$ $(1\leqslant p<\infty)$, with generator A. Then,

$$\{|\mu|\,;\,\mu\in\mathfrak{S}(T(t))\}=\exp\{t\left(\mathfrak{S}(A)\cap\mathbb{R}\right)\}\qquad\text{for any } t\geqslant 0.$$

In particular, the following Annular Hull Theorem holds true:

$$\mathbb{T}\cdot\mathfrak{S}(T(t))=\mathbb{T}\cdot\exp\left(t\big(\mathfrak{S}(A)\cap\mathbb{R}\big)\right)=\mathbb{T}\cdot\exp\left(t\mathfrak{S}(A)\right),\qquad\forall t\in\mathbb{R}.\tag{4.9}$$

Proof. It is clear that $\exp\{t\,(\mathfrak{S}(A)\cap\mathbb{R})\}\subset\{|\mu|\,;\,\mu\in\mathfrak{S}(T(t))\}$. Let us show the converse inclusion. Let $z\in\mathbb{R}^+\setminus\exp\{t\,(\mathfrak{S}(A)\cap\mathbb{R})\}$. Then, $z=\exp(\alpha t)$, with $\alpha\in\varrho(A)\cap\mathbb{R}$. Then, according to Theorem A. 1, there exist two projection bands I_α and J_α such that $L^p(\Omega,\mathrm{d}\varpi)=I_\alpha\oplus J_\alpha$ and

$$\mathfrak{S}(A_{|I_\alpha})=\{z\in\mathfrak{S}(A)\,;\,\mathrm{Re}z<\alpha\},\qquad\text{while}\qquad\mathfrak{S}(A_{|J_\alpha})=\{z\in\mathfrak{S}(A)\,;\,\mathrm{Re}z>\alpha\}.$$

Consequently, $s(A_{|I_\alpha})<\alpha$ and $s(-A_{|J_\alpha})<-\alpha$ where $s(\cdot)$ denotes the spectral bound. Since, in L^p–spaces $(1\leqslant p<\infty)$, the type of any positive C_0-semigroup coincide with the spectral bound of its generator [33], one gets

$$r(T(t)_{|I_\alpha})<\exp(\alpha t),\qquad r(T(-t)_{|J_\alpha})<\exp(-\alpha t).$$

Hence, using that $\mathfrak{S}(T(t)_{|J_\alpha})=\{z\in\mathbb{C}\,;\,1/z\in\mathfrak{S}(T(-t)_{|J_\alpha})\}$,

$$\mathfrak{S}(T(t)_{|I_\alpha})\subset\{z\in\mathbb{C}\,;\,|z|<\exp(\alpha t)\}\quad\text{whereas}\quad\mathfrak{S}(T(t)_{|J_\alpha})\subset\{z\in\mathbb{C}\,;\,|z|>\exp(\alpha t)\}.$$

Now, since $\mathfrak{S}(T(t))=\mathfrak{S}(T(t)_{|I_\alpha})\cup\mathfrak{S}(T(t)_{|J_\alpha})$, any $\mu\in\mathbb{C}$ with $|\mu|=\exp(\alpha t)$ is such that $\mu\notin\mathfrak{S}(T(t))$ which achieves the proof. The proof of the Annular Hull Theorem is then obvious since any $\mu\in\mathfrak{S}(T(t))$ writes $\mu=|\mu|\exp(i\theta)$ for some $\theta\in\mathbb{R}$ while $|\mu|=\exp(\alpha t)$ for some $\alpha\in\mathfrak{S}(A)\cap\mathbb{R}$. □

The *Spectral Mapping Theorem* for the *real spectrum* of Greiner [18] (see also [2] and [26, Corollary 4.10]) is now a direct consequence of the above Proposition.

Theorem A. 3. Let (Σ,ϖ) be a $\mathfrak{S}$-measured space and let $(T(t))_{t\in\mathbb{R}}$ be a *positive* C_0-group in $L^p(\Sigma,\mathrm{d}\varpi)$ $(1\leqslant p<\infty)$, with generator A. Then,

$$\mathfrak{S}(T(t))\cap\mathbb{R}_+=\exp\{t\,(\mathfrak{S}(A)\cap\mathbb{R})\}\qquad\qquad\text{for any } t\geqslant 0.$$

Proof. It is clear that $\mathfrak{S}(T(t))\cap\mathbb{R}_+\supseteq\exp\{t\,(\mathfrak{S}(A)\cap\mathbb{R})\}$ for any $t\in\mathbb{R}$. Now, since $\mathfrak{S}(T(t))\cap\mathbb{R}_+\subset\{|\mu|\,;\,\mu\in\mathfrak{S}(T(t))\}$, the converse inclusion follows immediately from Proposition A.2. □

Another consequence of Proposition A. 2 is the following spectral mapping theorem which applies to generator whose approximate spectrum is invariant by vertical translations:

Theorem A. 4. Let (Σ,ϖ) be a $\mathfrak{S}$-measured space and let $(T(t))_{t\in\mathbb{R}}$ be a *positive* C_0-group in $L^p(\Sigma,\mathrm{d}\varpi)$ $(1\leqslant p<\infty)$, with generator A. If $\mathfrak{S}_{\mathrm{ap}}(A)=\mathfrak{S}_{\mathrm{ap}}(A)+i\mathbb{R}$ then

$$\mathfrak{S}(T(t))=\exp(t\mathfrak{S}(A))\qquad\qquad\text{for any } t\in\mathbb{R}.$$

Proof. It clearly suffices to prove the "$\subset$" inclusion. We first note that $\mathfrak{S}(A)=\mathfrak{S}(A)+i\mathbb{R}$. Indeed, let $\lambda\in\mathfrak{S}(A)$, assume that $\lambda+i\mathbb{R}\not\subseteq\mathfrak{S}(A)$. Then, there is $\alpha\in\mathbb{R}$ such that $\lambda+i\alpha$ lies in the boundary of $\mathfrak{S}(A)$. In particular, $\lambda+i\alpha\in\mathfrak{S}_{\mathrm{ap}}(A)$, and by assumption, $\lambda+i\mathbb{R}\subset\mathfrak{S}(A)$ which is a contradiction. Therefore, $\mathfrak{S}(A)=\mathfrak{S}(A)+i\mathbb{R}$. Now, let $z\notin\exp(t\mathfrak{S}(A))$. Then, there is $\lambda\in\varrho(A)$ such that $z=\exp(\lambda t)$, $\lambda=\alpha+i\beta$. Since the spectrum of A is invariant by vertical translations, $\alpha\in\varrho(A)$. Hence $\exp(\alpha t)\notin\exp\{t\,(\mathfrak{S}(A)\cap\mathbb{R})\}$. According to Proposition A.2, and since $|z|=\exp(\alpha t)$, this means that $z\in\varrho(T(t))$. □

Remark A. 5. As a consequence of the above result, one sees that, if $(T(t))_{t\in\mathbb{R}}$ is a positive groups of operator with generator A in some L^p-space $(1 \leqslant p < \infty)$ and if $\mathfrak{S}(A) = \mathfrak{S}(A) + i\mathbb{R}$, then $\mathfrak{S}(T(t)) = \mathfrak{S}(T(t)) \cdot \mathbb{T}$ $(t \in \mathbb{R})$ where $\mathbb{T}$ is the unit circle of $\mathbb{C}$.

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