Diffusion versus absorption in semilinear elliptic equations

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Abstract

We study the limit behaviour of a sequence of singular solutions of a nonlinear elliptic equation with a strongly degenerate absorption term at the boundary of the domain. We give sharp conditions on the level of degeneracy in order the pointwise singularity not to propagate along the boundary.

1 Introduction

Let Ω be a bounded C^2 domain in $\mathbb{R}^N$. If $q > 1$ and $H \in C(\Omega)$ is a positive function, it is well-known that there exists a maximal solution U to

$$-\Delta u + H(x)u^q = 0 \quad \text{in } \Omega. \tag{1.1}$$

Furthermore, if $H(x) \leq \tilde{H}(\rho(x))$ where $\tilde{H}$ is nonincreasing, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and

$$\int_0^1 \sqrt{\tilde{H}(s)}ds < \infty, \tag{1.2}$$

then it is proved in [5] that U is a large solution in the sense that

$$\lim_{\rho(x)\to 0} U(x) = \infty. \tag{1.3}$$

If (1.2) holds, it is possible to construct a minimal large solution $\underline{U}$, and in many cases $U = \underline{U}$ (see [5], [9]). Let K be the Poisson kernel in Ω and $a \in \partial\Omega$. If

$$\int_\Omega H(x)K^q(x,a)\rho(x)dx < \infty \tag{1.4}$$

then for any $k > 0$ there exists a unique weak solution $u = u_{k,a}$ to

$$\left\{\begin{array}{rl} -\Delta u + H(x)u^q = 0 & \text{in } \Omega \\ u = k\delta_a & \text{on } \partial\Omega \end{array}\right. \tag{1.5}$$

in the sense that $u \in L^1(\Omega) \cap L^q_\rho(\Omega)$ and

$$\int_\Omega (-u\Delta\zeta + \zeta H(x)u^q)\,dx = -k\frac{\partial\zeta}{\partial n}(a) \tag{1.6}$$

for any $\zeta \in W^{2,\infty}(\Omega) \cap W^{1,\infty}_0(\Omega)$ (see [2]). Furthermore, the mapping $k \mapsto u_{k,a}$ is increasing. Since $u_{k,a} \le U$ it converges to some $u_{\infty,a}$ which is a positive solution of (1.1) in Ω. A natural question is to identify $u_{\infty,a}$. The following result is proved in [4]

Theorem 0. *Assume*

$$0 < H(x) \le \exp(-\tau/\rho(x)) \quad \forall x \in \Omega \tag{1.7}$$

for some $\tau > 0$*, then* $u_{\infty,a} = \underline{U}$.

This result means that the pointwise boundary blow-up at a has propagated along the whole $\partial\Omega$. In this article we give conditions which prevents this phenomenon and we prove the following.

Theorem 1. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *flat in the neighborhood of some boundary point* a*. Assume*

$$\liminf_{\rho(x)\to 0} \rho^\theta(x)\ln(H(x)) > -\infty \tag{1.8}$$

for some $0 < \theta < 1$*. Then* $\lim_{x\to x_0} u_{\infty,a}(x) = 0$ *for any* $x_0 \in \partial\Omega \setminus \{a\}$.

This means that the singularity remains localized at the point a. This theorem is a consequence of a much more general result in which the flatness condition of H near the boundary is expressed by mean of a Dini condition. This condition allows to replace (1.8) by

$$H(x) \ge h(\rho(x)) \quad \text{and} \quad \ln(1/h(\rho(x)) \in L^1(\Omega) \tag{1.9}$$

Contrary to the complete boundary blow-up phenomenon under assumption (1.7) which is obtained by constructing local subsolutions, the proof of Theorem1 is performed by local energy methods in the spirit of Saint-Venant principle. Similar results of propagations or confinement of singularities have been proved for parabolic equations of the type

$$\partial_t u - \Delta u + \exp(-\omega(t)/t)u^q = 0 \in \mathbb{R}^N_+ \times (0,\infty) \tag{1.10}$$

($q > 1$) in [3] and [7].

Aknowledgements The authors have been supported by INTAS grant Ref. No : 05-1000008-7921.

2 The general result

Let $\Omega \subset \mathbb{R}^N_+ = \{(x_1, x') \in \mathbb{R}^N : x_1 > 0\}$ be a bounded domain with C^2 boundary $\partial\Omega$, such that

$$\Gamma_\gamma : \{(0, x') : |x'| \le 2\gamma\} \subset \partial\Omega, \quad (0, 2\gamma) \times \Gamma_\gamma \subset \Omega. \tag{2.1}$$

for some $\gamma > 0$. Let $q > 1$ and $H \in C(\Omega)$ be a nonnegative function satisfying (1.4). We consider the following boundary value problem:

$$\begin{cases} -\Delta u + H(x)u^q = 0 & \text{in } \Omega \\ u = \bar{K}_j \delta & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\delta = \delta_0$ is the Dirac measure at 0, $\{\bar{K}_j\}$ is positive increasing sequence: $\bar{K}_j \to \infty$ as $j \to \infty$. Then for arbitrary $j \in \mathbb{N}$ problem (2.2) has a unique solution $u_j(x)$ ([2][8]) and the sequence $\{u_j\}$ is increasing. Furthermore, since there exists a maximal solution U of equation (2.2) which also satisfies $\lim_{\rho(x)\to 0} U(x) \to \infty$, u_j is smaller than U for any j. Our aim is to find sharp conditions on H, guaranteeing that the limit solution $u_\infty = \lim_{j\to\infty} u_j$ has a boundary singularity localized at $\{0\}$ and satisfies $\lim_{x\to y} u_\infty(x) = 0$ for all $y \in \partial\Omega \setminus \{0\}$. We shall assume that

$$H(x) \geq h(\rho(x)) \quad \forall x \in \Omega, \tag{2.3}$$

for some positive nondecreasing function h that we shall write under the form

$$h(s) = \exp\Big(-\frac{\omega(s)}{s}\Big) \quad \forall\, s \in (0, \gamma). \tag{2.4}$$

Our main result is the following.

Theorem 2. *Assume ω is a nondecreasing continuous function satisfying the technical condition*

$$s^{\gamma_1} \leq \omega(s) \leq \omega_0 = const < \infty \quad \forall\, s \in (0, \gamma),\ 0 < \gamma_1 < 1, \tag{2.5}$$

and the Dini condition,

$$\int_0^c \frac{\omega(s)}{s}\, ds < \infty, \tag{2.6}$$

and let h and H be subjects to (2.3) and (2.4). If u_j is the solution of problem (2.2), then $u_\infty = \lim_{j\to\infty} u_j$ is a solution of (1.1) with a boundary singularity at 0 and which satisfies

$$\lim_{x\to y} u(x) = 0 \quad \forall\, y \in \partial\Omega \setminus \{0\}. \tag{2.7}$$

Since the solution u_j on (2.2) is a decreasing function of the potential H, we shall assume in the sequel that $H(x) = h(\rho(x))$ for all $x \in \Omega$, thus the equation under consideration will be

$$-\Delta u + h(\rho(x))u^q = 0 \quad \text{in } \Omega, \tag{2.8}$$

and u_j denotes the solution subject to the boundary condition

$$u = \bar{K}_j \delta \quad \text{on } \partial\Omega. \tag{2.9}$$

2.1 Energy a priori estimates

The proof of Theorem 2 is based on some new variant of the local energy estimates method. For the study of the localized singular boundary regimes for the quasilinear second order parabolic equations energy method was first used in [6]. An adaptation of these methods to the study of the localization principle of initial singularities of singular solutions of parabolic equations with a strong absorption and a degenerate

t-dependent potential was elaborated in [7]. Here we propose the "elliptic" version of the above mentioned result.

$$\Omega_s := \{x \in \Omega : \rho(x) > s\}, \quad s \in \mathbb{R}^1_+,$$
$$\Omega^s := \{x \in \Omega : 0 < \rho(x) < s\}, \quad s \in \mathbb{R}^1_+,$$
$$\Omega^s(\tau) := \Omega^s \cap \{x = (x_1, x') : |x'| > \tau\}, \quad \tau > 0,\ 0 < s < \gamma.$$

Because $\partial\Omega$ is C^2, there exists $\tilde{s} > 0$ such that, for any $0 < s \le \tilde{s}$, $\partial\Omega^s \cap \Omega = \partial\Omega_s$ is C^2. Notice also that we can assume that $\rho(x) = x_1$ for any $x \in [0, 2\gamma] \times \Gamma_\gamma$. If u is a solution of (2.8) we set

$$I(s) := \int_{\Omega_s} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\, dx, \qquad s > 0. \tag{2.10}$$

Lemma 2.1. *The function I satisfies*

$$I(s) \le d_1 \left[\int_0^s h(r)^{\frac{2}{q+3}}\, dr \right]^{-\frac{q+3}{q-1}} \qquad \forall\, 0 < s \le \tilde{s}, \tag{2.11}$$

where constant d_1 does not depend u.

Proof. Multiplying equation (2.2) by u and integrating on Ω_s $(0 < s \le \tilde{s},)$, we get

$$I(s) = \int_{\partial\Omega_s} u \frac{\partial u}{\partial \overrightarrow{n}}\, d\sigma \le \left(\int_{\partial\Omega_s} |\nabla u|^2\, d\sigma \right)^{1/2} \left(\int_{\partial\Omega_s} |u|^2\, d\sigma \right)^{1/2}. \tag{2.12}$$

By Hölder's inequality,

$$\left(\int_{\partial\Omega_s} |u|^2\, dx \right)^{1/2} \le (\operatorname{mes} \partial\Omega_s)^{\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} \left(\int_{\partial\Omega_s} h(\rho(x))|u|^{q+1}\, d\sigma \right)^{\frac{1}{q+1}}. \tag{2.13}$$

Substituting estimate (2.13) in (2.12) and using Young inequality we obtain

$$I(s) \le c_1 h(s)^{-\frac{1}{q+1}} \left[\int_{\partial\Omega_s} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\, d\sigma \right]^{1 - \frac{q-1}{2(q+1)}}. \tag{2.14}$$

Because $\partial\Omega$ is C^2,

$$\frac{dI(s)}{ds} = -\int_{\partial\Omega_s} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\, d\sigma. \tag{2.15}$$

Substituting this equality in (2.14) we derive that I satisfies the differential inequality

$$I(s) \le c_1 h(s)^{-\frac{1}{q+1}} (-I'(s))^{1 - \frac{q-1}{2(q+2)}}.$$

Solving this inequality we obtain estimate (2.11). $\square$

Let $\tilde{u}_j$, $j = 1, 2, \ldots$, be the solution of equation (2.8) subject to the regularized boundary condition:

$$\tilde{u}_j = \bar{K}_j \delta_j \quad \text{on } \partial\Omega, \tag{2.16}$$

where the δ_j are C^1-smooth functions such that:

$$\begin{cases} \operatorname{supp}\delta_j \subset \{x' \subset \mathbb{R}^{N-1} : |x'| < j^{-1}\}, \quad 0 \le \delta_j(x') \le 2j^{N-1}, \\ \|\delta_j\|^{q+1}_{L_{q+1}(\mathbb{R}^{N-1})} \le 2j^{q(N-1)}, \qquad \|\nabla_{x'}\delta_j\|^2_{L_2(\mathbb{R}^{N-1})} \le 2j^{N+1}, \\ \qquad \|\delta_j\|_{L_1(\mathbb{R}^{N-1})} = 1 \quad \text{and} \quad \delta_j(x') \rightharpoonup \delta(x) \quad \text{as } j \to \infty. \end{cases} \tag{2.17}$$

The next lemma provides a global energy estimate on $\tilde{u}_j$.

Lemma 2.2. *The solution $\tilde{u}_j$ of problem* (2.8), (2.16) *satisfies*

$$\int_\Omega (|\nabla \tilde{u}_j|^2 + h(\rho(x))|\tilde{u}_j|^{q+1})\, dx \le K_j, \tag{2.18}$$

with $K_j \le c(\bar{K}_j^{q+1}\gamma j^{q(N-1)} + \bar{K}_j^2\gamma j^{N+1} + \bar{K}_j^2\gamma^{-1}j^{N-1})$, where the constant $c > 0$ does not depend on j.

Proof. Let us introduce a C^2 cut-off function ζ such that $\zeta(r) = 1$ if $r \le 0$, $\zeta(r) = 0$ if $r \ge \gamma$ (γ is from condition (2.1)). Let us denote for simplicity $\tilde{u}_j = u$. If we multiply (2.2) by

$$v_j(x) = u(x) - \bar{K}_j\delta_j(x')\zeta(x_1)$$

and integrate on Ω, we obtain for all $j > j_0 = \gamma^{-1}$, since $v_j(x) = 0$ on $\partial\Omega$,

$$\begin{aligned} \int_\Omega (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\, dx &= \int_\Omega \bar{K}_j(\nabla u, \nabla(\delta_j(x')\zeta(x_1)))\, dx \\ &\quad + \int_\Omega h(\rho(x))u^q\bar{K}_j\delta_j(x')\zeta(x_1)\, dx := A_1 + A_2. \end{aligned} \tag{2.19}$$

By Young's inequality and properties (2.17), we derive

$$\begin{aligned} |A_1| &\le \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + c_1\bar{K}_j^2(\gamma j^{N+1} + \gamma^{-1}j^{N-1}), \\ |A_2| &\le \frac{1}{2}\int_\Omega h(\rho(x))u^{q+1}\, dx + c_1\bar{K}_j^{q+1}\gamma j^{q(N-1)}. \end{aligned} \tag{2.20}$$

Estimate (2.18) follows from (2.19), (2.20)with

$$K_j = g(\bar{K}_j) := 2c_1(\bar{K}_j^{q+1}\gamma j^{q(N-1)} + \bar{K}_j^2\gamma j^{N+1} + \bar{K}_j^2\gamma^{-1}j^{N-1}). \tag{2.21}$$

□

We introduce a family of cut-off functions ζ_s with

$$\begin{cases} \zeta_s(r) = 1 \quad \text{if } r \le s, \qquad \zeta_s(r) = 0 \quad \text{if } r \ge 2s \\ \qquad \left|\dfrac{d}{dr}\zeta_s(r)\right| \le c_2 s^{-1} \quad \forall\, s > 0, \end{cases} \tag{2.22}$$

and define the additional family of energy functions, for any solution of (2.8),

$$J(s,\tau) := \int_{\Omega^{2s}(\tau)} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\zeta_s(\rho(x))\, dx, \qquad J(s) := J(s,0). \tag{2.23}$$

We shall denote by $I_j(s)$ and $J_j(s,\tau)$ the energy functions $I(s)$ and $J(s,\tau)$ associated with the solution $\tilde{u}_j(x)$.

Lemma 2.3. *The following differential inequality holds:*

$$J_j(s,\tau) \le d_2 s\Big(-\frac{d}{d\tau}J_j(s,\tau)\Big) + d_3 F(I_j(s),h(s),s) \quad \forall\,\tau\in(j^{-1},2\gamma),\ \forall\, s\in(0,\gamma), \tag{2.24}$$

where the constants d_2, d_3 do not depend on j and $F(I,h,s)$ is defined by

$$F(I,h,s) := \frac{I^{1-\frac{q-1}{2(q+1)}}}{s^{\frac{q+3}{2(q+1)}}h^{\frac{1}{q+1}}} + \frac{I^{1-\frac{q-1}{q+1}}}{s^{\frac{2}{q+1}}h^{\frac{2}{q+1}}}. \tag{2.25}$$

Proof. We consider (2.2) satisfied by $u=\tilde{u}_j$, multiply the equation by $\tilde{u}_j\zeta_s(\rho(x))$ and integrate on the domain $\Omega^{2s}(\tau),\ 2\gamma>\tau>j^{-1}$. As result we have the following

$$\begin{aligned} J_j(s,\tau) &= \int_{\Omega^{2s}(\tau)} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\zeta_s(\rho(x))\,dx \\ &= \int_{\Gamma^{2s}(\tau)} u\frac{\partial u}{\partial n}\zeta_s(\rho(x))\,d\sigma - \int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} (\nabla u, \nabla\zeta_s(\rho(x)))u\,dx \\ &:= R_1 + R_2, \end{aligned} \tag{2.26}$$

where $\Gamma^{2s}(\tau)=\{\rho(x)<2s,\ |x'|=\tau\}$. Let us estimate the terms R_1, R_2 from above.

$$|R_1| \le \left(\int_{\Gamma^{2s}(\tau)} |\nabla u|^2\zeta_s\,d\sigma\right)^{1/2}\left(\int_{\Gamma^{2s}(\tau)} u^2\zeta_s\,d\sigma\right)^{1/2} := \left(R_1^{(1)}\right)^{1/2}\left(R_1^{(2)}\right)^{1/2}. \tag{2.27}$$

We decompose $R_1^{(2)}$ as follows

$$R_1^{(2)} = \int_{\Gamma^{2s}(\tau)\setminus\Gamma^s(\tau)} u^2\zeta_s\,d\sigma + \int_{\Gamma^s(\tau)} u^2\zeta_s\,d\sigma := R_1^{(2,1)} + R_1^{(2,2)}.$$

In order to estimate $R_1^{(2,1)}$, we use a standard trace interpolation inequality (see e.g. [1]), and get

$$\begin{aligned} &\int_{|x'|=\tau} u(x_1,x')^2\,d\sigma' \\ &\qquad \le c_1\left(\int_{\tau<|x'|<2\gamma} |\nabla_{x'}u(x_1,x')|^2dx'\right)^{1/2}\left(\int_{\tau<|x'|<2\gamma} u(x_1,x')^2dx'\right)^{1/2} \\ &\qquad\qquad + c_2\int_{\tau<|x'|<2\gamma} u(x_1,x')^2dx' \qquad \forall\,\tau<\gamma,\ \forall\,x_1\in(s,2s). \end{aligned}$$

Integrating the last inequality in x_1 over $(s,2s)$, we obtain

$$\begin{aligned} R_1^{(2,1)} &\le c_1\left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} |\nabla u|^2\,dx\right)^{1/2}\left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} u^2\,dx\right)^{1/2} \\ &\qquad + c_2 s^{-1}\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} u^2\,dx \\ &:= c_1\left(R_1^{(2,1,1)}\right)^{1/2}\left(R_1^{(2,1,2)}\right)^{1/2} + c_2 R_1^{(2,1,2)}. \end{aligned} \tag{2.28}$$

By Hölder's inequality,

$$\begin{aligned} R_1^{(2,1,2)} &\le d_4 \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} u^{q+1}\, dx \right)^{\frac{2}{q+1}} \left(\mathrm{mes}(\Omega^{2s}(\tau) \setminus \Omega^s(\tau)) \right)^{\frac{q-1}{q+1}} \\ &\le d_5 s^{\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} h(\rho(x))|u|^{q+1}\, dx \right)^{\frac{2}{q+1}}. \end{aligned} \tag{2.29}$$

Therefore it follows from (2.28) and (2.29),

$$\begin{aligned} R_1^{(2,1)} &\le d_6 s^{\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} h(\rho(x))|u|^{q+1}\, dx \right)^{\frac{2}{q+1}} \\ &+ d_7 s^{\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} |\nabla u|^2\, dx \right)^{1/2} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} h(\rho(x))|u|^{q+1}\, dx \right)^{\frac{1}{q+1}} \\ &\le d_8 s^{\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} \tilde{R}^{1-\frac{q-1}{2(q+1)}} + d_8 s^{-\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} \tilde{R}^{1-\frac{q-1}{q+1}}, \end{aligned}$$

where

$$\tilde{R} = \int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\, dx.$$

Using the definition of $I_j(s)$ we derive

$$\begin{aligned} R_1^{(2,1)} &\le d_8 s^{\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{2(q+1)}} \\ &\quad + d_8 s^{-\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{q+1}}. \end{aligned} \tag{2.30}$$

Since $u(0,x') = u_j(0,x') = 0\ \forall x' : j^{-1} < |x'| < \gamma$, we derive by Poincaré's inequality,

$$R_1^{(2,2)} = \int_{\Gamma^s(\tau)} u^2\, d\sigma \le d_9 s^2 \int_{\Gamma^s(\tau)} \left| \frac{\partial u}{\partial x_1} \right|^2 d\sigma \le d_9 s^2 \int_{\Gamma^s(\tau)} |\nabla u|^2\, d\sigma. \tag{2.31}$$

Plugging (2.30) and (2.31) into (2.27) and using Young's inequality leads to

$$\begin{aligned} |R_1| &\le d_{10} \left(\int_{\Gamma^{2s}(\tau)} |\nabla u|^2 \zeta_s\, d\sigma \right)^{1/2} \Bigg[s^{\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} \left(I_j(s) - I_j(2s) \right)^{1-\frac{q-1}{2(q+1)}} \\ &\quad + s^{\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} \left(I_j(s) - I_j(2s) \right)^{1-\frac{q-1}{q+1}} + s^2 \int_{\Gamma^s(\tau)} |\nabla u|^2\, d\sigma \Bigg]^{1/2} \\ &\le d_{11} \Bigg[s \int_{\Gamma^{2s}(\tau)} |\nabla u|^2 \zeta_s\, d\sigma + s^{-1+\frac{q-1}{2(q+1)}} h(s)^{-\frac{1}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{2(q+1)}} \\ &\quad + s^{-1+\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{q+1}} \Bigg]. \end{aligned} \tag{2.32}$$

The last terms to estimate is R_2. By Hölder's inequality and (2.22), we have,

$$\begin{aligned} |R_2| &\le c s^{-1} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} |\nabla u|^2\, dx \right)^{1/2} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} u^2\, dx \right)^{1/2} \\ &:= c s^{-1} (R_2^{(1)})^{1/2} \left(R_2^{(2)} \right)^{1/2}. \end{aligned} \tag{2.33}$$

From (2.28), the term $R_2^{(2)}$ coincides with $R_1^{(2,1,2)}$; thus $R_2^{(2)}$ satisfies

$$R_2^{(2)} \leq d_5 s^{\frac{q-1}{q+1}} h(s)^{-\frac{2}{q+1}} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} h(\rho(x))|u|^{q+1}\,dx \right)^{\frac{2}{q+1}}. \tag{2.34}$$

using (2.34) and Young's inequality, we derive from (2.33),

$$|R_2| \leq c_1 s^{-(1-\frac{q-1}{2(q+1)})} h(s)^{-\frac{1}{q+1}} \left(\int_{\Omega^{2s}(\tau)\setminus\Omega^s(\tau)} (|\nabla u|^2 + h(\rho(x))|u|^{q+1})\,dx \right)^{1-\frac{q-1}{2(q+1)}}. \tag{2.35}$$

Thus, due to estimates (2.32) and (2.35), it follows from (2.26),

$$\begin{aligned} J_j(s,\tau) \leq cs \int_{\Gamma^{2s}(\tau)} |\nabla u|^2 \zeta_s\,d\sigma + c_1 s^{-\frac{q+3}{2(q+1)}} h(s)^{-\frac{1}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{2(q+1)}} \\ + c_2 s^{-\frac{2}{q+1}} h(s)^{-\frac{2}{q+1}} (I_j(s) - I_j(2s))^{1-\frac{q-1}{q+1}}. \end{aligned} \tag{2.36}$$

It is easy to see that

$$\int_{\Gamma^{2s}(\tau)} (|\nabla u_j|^2 + h(\rho(x))|u_j|^{q+1}) \zeta_s\,d\sigma \leq -c\frac{d}{d\tau} J_j(s,\tau), \tag{2.37}$$

where c does not depend on τ, s, j. Substituting (2.37) into (2.36) we obtain (2.24). □

In order to estimate from above the function $F(I_j(s), h(s), s)$ in the right-hand side of (2.24), we first prove the following technical result.

Lemma 2.4. *Let $a > 0$ and $\omega(s)$ be a nonnegative nondecreasing function satisfying the following condition:*

$$\mu(s) := \frac{s}{\omega(s)} \to 0 \quad as \quad s \to 0.$$

Then the following inequality holds:

$$\int_0^s \exp\Big(-\frac{a\omega(t)}{t}\Big)\,dt \geq \frac{s^2}{a\omega(s)(1+\frac{2}{a}\mu(s))} \exp\Big(-\frac{a\omega(s)}{s}\Big). \tag{2.38}$$

Proof. Since $\mu(0) = 0$, an integration by parts yields to

$$\int_0^s t\exp\Big(-\frac{a\omega(t)}{t}\Big)dt = \frac{s^2}{2}\exp\Big(\frac{-a\omega(s)}{a}\Big) + \frac{a}{2}\int_0^s \exp\Big(-\frac{a\omega(t)}{t}\Big)(t\omega'(t)-\omega(t))\,dt.$$

Due to the monotonicity of $\omega(t)$, inequality (2.38) follows from the last relation. □

Using Lemma 2.4 and identity (2.4), we obtain

$$\int_0^s h(r)^{\frac{2}{q+3}}\,dr \geq c_0 \frac{s^2}{\omega(s)} \exp\Big(-\frac{2}{q+3}\frac{\omega(s)}{s}\Big), \tag{2.39}$$

where $c_0 > 0$ does not depend on j, s, and this transforms (2.11) into

$$I_j(s) \leq \frac{d_1}{c_0^{\frac{q+3}{q+1}}} \frac{\omega(s)^{\frac{q+3}{q-1}}}{s^{\frac{2(q+3)}{q-1}}} \exp\Big(\frac{2}{(q-1)}\frac{\omega(s)}{s}\Big) := C\frac{\omega(s)^{\frac{q+3}{q-1}}}{s^{\frac{2(q+3)}{q-1}}} h(s)^{-\frac{2}{q-1}}. \tag{2.40}$$

Substituting this estimate into (2.25) we derive

$$F(I_j(s),h(s),s) \le C_1 h_0(s)^{-\frac{2}{q-1}} \left(\frac{\omega(s)^{\frac{(q+3)(q+3)}{2(q-1)(q+1)}}}{s^{\frac{(q+3)(3q+5)}{2(q+1)(q-1)}}} + \frac{\omega(s)^{\frac{2(q+3)}{(q+1)(q-1)}}}{s^{\frac{2(3q+5)}{(q+1)(q-1)}}} \right) \quad \forall\, s>0,\ \forall\, j\in\mathbb{N}. \tag{2.41}$$

In turn, (2.4), assumption (2.5) jointly with (2.41) yields to

$$F(I_j(s),h(s),s) \le C_2(\delta) h(s)^{-\frac{2}{q-1}-\delta} \qquad \forall\, s>0,\ \forall\, \delta>0, \tag{2.42}$$

where $C_2(\delta) \to \infty$ as $\delta \to 0$. Plugging this inequality into (2.24), we finally obtain

$$J_j(s,\tau) \le d_2 s\Big(-\frac{d}{d\tau}J_j(s,\tau)\Big) + d_3 C_2(\delta) h(s)^{-\frac{2}{q-1}-\delta} \quad \forall\, \delta>0,\ \forall\, s>0,\ \forall\, \tau\in(j^{-1},2\gamma). \tag{2.43}$$

2.2 Proof of Theorem 2

Our proof will be based on the careful analysis of the vanishing properties of the energy functions $J_j(s,\tau)$, satisfying inequality (2.43). Notice that $J_j(s,\tau)$ satisfies the following initial condition, which follows from (2.18), (2.21)

$$J_j(s,j^{-1}) \le K_j = g(\bar{K}_j, j) \qquad \forall\, j\in\mathbb{N}, \tag{2.44}$$

Let us fix j large enough. If $0<\delta_0<1$, we shall define s_j by the identity

$$F_0(s_j) := d_3 C_2(\delta_0) h(s_j)^{-\frac{2}{q-1}-\delta_0} = K_j^{\varepsilon}, \tag{2.45}$$

where $0<\varepsilon<1$ will be made explicit later on. Then it follows from (2.43), (2.44) that $J_j(s_j,\tau)$ satisfies the following differential inequalities

$$\begin{cases} J_j(s_j,\tau) \le d_2 s_j\Big(-\frac{d}{d\tau}J_j(s_j,\tau)\Big) + K_j^{\varepsilon} \quad \forall\, \tau > j^{-1}, \\ J_j(s_j,j^{-1}) \le K_j. \end{cases} \tag{2.46}$$

Let us define now the value τ_j by the identity

$$J_j(s_j, j^{-1}+\tau_j) = 2K_j^{\varepsilon}, \tag{2.47}$$

where ε has been introduced in (2.45). In order to find an upper estimate for τ_j, we observe that

$$J_j(s_j,\tau) > 2K_j^{\varepsilon} \quad \forall\, \tau\in(j^{-1}, j^{-1}+\tau_j).$$

Therefore, (2.46) reads as

$$J_j(s_j,\tau) \le 2d_2 s_j\Big(-\frac{dJ_j(s_j,\tau)}{d\tau}\Big) \qquad \forall\, \tau\in(j^{-1}, j^{-1}+\tau_j). \tag{2.48}$$

Solving this differential inequality and taking into account the initial condition into (2.46), we obtain

$$J_j(s_j,\tau) \le K_j \exp\Big(-\frac{\tau - j^{-1}}{2d_2 s_j}\Big) \qquad \forall\, \tau\in(j^{-1}, j^{-1}+\tau_j). \tag{2.49}$$

By (2.47) and (2.49),

$$2K_j^{\varepsilon} \le K_j \exp\Big(-\frac{\tau_j}{2d_2 s_j}\Big).$$

Consequently, τ_j satisfies the following upper bound:

$$\tau_j \le 2d_2 s_j(-\ln 2 + (1-\varepsilon)\ln K_j). \tag{2.50}$$

Next, we notice that

$$\int_{\Omega(j^{-1}+\tau_j)} (|\nabla u_j|^2 + h(\rho(s))|u_j|^{q+1})\,dx \le I_j(s_j) + J_j(s_j, j^{-1}+\tau_j). \tag{2.51}$$

with $\Omega(\tau) := \{x : |x'| > \tau\}$. From estimate (2.40), it follows

$$I_j(s_j) \le C_3(\delta_0) h(s_j)^{-\frac{2}{q-1}-\delta_0}, \tag{2.52}$$

where δ_0 has been introduced in (2.45) and $C_3(\delta_0)$ depends on various parameters of the problem, but not on j. Using now the definition (2.45) of s_j and (2.47) of τ_j, we deduce, from (2.51) and (2.52),

$$\int_{\Omega(j^{-1}+\tau_j)} (|\nabla u_j|^2 + h(\rho(x))|u_j|^{q+1})\,dx \le (2 + \frac{C_3(\delta_0)}{d_3 C_2(\delta_0)})K_j^{\varepsilon}. \tag{2.53}$$

Because of (2.21), we can fix the sequence $\{\bar{K}_i\}$ such that

$$K_i = e^{e^i}, \qquad i = 1, 2, \ldots, j, \ldots. \tag{2.54}$$

Actually, $\bar{K}_i \approx e^{e^i/(q-1)}$. We fix ε (see definition (2.45) in order the next inequality be satisfied for j large enough,

$$(2 + C_4)K_j^{\varepsilon} \le K_{j-1}, \qquad C_4 := \frac{C_3(\delta_0)}{d_3 C_2(\delta_0)}. \tag{2.55}$$

Because of (2.54), (2.55) is equivalent to

$$\ln(2 + C_4) + \varepsilon \exp j \le e^{-1}\exp j, \tag{2.56}$$

and it is sufficient to take

$$\varepsilon = (2e)^{-1},$$

in order condition (2.56) be satisfied for all $j \ge j_0 = 1 + \ln 2 + \ln\ln(2 + C_4)$. With such a choice of ϵ and K_j, s_j is uniquely defined by identity (2.45). Therefore, from (2.53) and (2.55), it follows

$$\int_{\Omega(j^{-1}+\tau_j)} (|\nabla u_j|^2 + h(\rho(x))u_j^{q+1})\,dx \le K_{j-1}, \tag{2.57}$$

which will be the starting point for the second round of computations. From the first round, we can obtain sharper upper estimates of τ_j, s_j defined by (2.45), (2.47). First, (2.45) gives,

$$d_3 C_2 \exp\Big(\Big(\frac{2}{q-1} + \delta_0\Big)\frac{\omega(s_j)}{s_j}\Big) = K_j^{\varepsilon} \Longrightarrow \frac{\varepsilon}{2}\ln K_j \le \Big(\frac{2}{q-1} + \delta_0\Big)\frac{\omega(s_j)}{s_j} \le \varepsilon \ln K_j \qquad \forall\, j > j' = j'(C_2). \tag{2.58}$$

From (2.58), (2.5) and (2.54) we obtain,

$$s_j \le 2\Big(\delta_0 + \frac{2}{q-1}\Big)\varepsilon^{-1}(\ln K_j)^{-1}\omega(s_j) \le 2\Big(\delta_0 + \frac{2}{q-1}\Big)\omega_0 \exp(-j), \tag{2.59}$$

and, by the monotonicity of $\omega(s)$,

$$\omega(s_j) \le \omega(C_5 \exp(-j)), \qquad C_5 = 2\Big(\delta_0 + \frac{2}{q-1}\Big)\omega_0. \tag{2.60}$$

As for τ_j, we deduce from (2.50) and (2.58):

$$\tau_j \le 2d_2(1-\varepsilon)s_j \ln K_j \le C_6\omega(s_j), \qquad C_6 := \frac{4d_2(1-\varepsilon)(\delta_0 + \frac{2}{q-1})}{\varepsilon}. \tag{2.61}$$

Substituting (2.60) into (2.61) we get:

$$\tau_j \le C_6\omega(C_5\exp(-j)). \tag{2.62}$$

Thus we can initiate the second circle of computations. We define s_{j-1} similarly to (2.45) by the identity

$$F_0(s_{j-1}) = d_3C_2(\delta_0)h(s_{j-1})^{-\frac{2}{q-1}-\delta_0} = K^{\varepsilon}_{j-1}, \tag{2.63}$$

with $\varepsilon = 1/2e$). Then $J_j(s_{j-1}, \tau)$ satisfies, instead of (2.46), the following differential inequality,

$$\begin{cases} J_j(s_{j-1},\tau) \le d_2 s_{j-1}\Big(-\dfrac{d}{d\tau}J_j(s_{j-1},\tau)\Big) + K^{\varepsilon}_{j-1} & \forall\,\tau > \tau_j, \\ J_j(s_{j-1}, j^{-1}+\tau_j) \le K_{j-1}. \end{cases} \tag{2.64}$$

Observe that the initial value condition follows from estimate (2.57) resulting first round of computations. Next we define τ_{j-1} by the following analog of (2.47)

$$J_j(s_{j-1}, j^{-1}+\tau_j+\tau_{j-1}) = 2K^{\varepsilon}_{j-1}. \tag{2.65}$$

Thus, we obtain the following analog of (2.48):

$$J_j(s_{j-1},\tau) \le 2d_2 s_{j-1}\Big(-\frac{d}{d\tau}J_j(s_{j-1},\tau)\Big) \qquad \forall\,\tau \in (j^{-1}+\tau_j, j^{-1}+\tau_j+\tau_{j-1}). \tag{2.66}$$

Solving this inequality with the initial condition of (2.64), we obtain, in the same way as for (2.49),

$$J_j(s_{j-1},\tau) \le K_{j-1}\exp\Big(-\frac{\tau-\tau_j-j^{-1}}{2d_2 s_{j-1}}\Big) \qquad \forall\,\tau \in (j^{-1}+\tau_j, j^{-1}+\tau_j+\tau_{j-1}). \tag{2.67}$$

Definition (2.65) of τ_{j-1} and estimate (2.67) lead to the following estimate of τ_{j-1}

$$\tau_{j-1} \le 2d_2 s_{j-1}(-\ln 2 + (1-\varepsilon)\ln K_{j-1}), \tag{2.68}$$

and finally, to the estimates on s_{j-1} and τ_{j-1},

$$\begin{aligned} &(i) \quad s_{j-1} \le C_5\exp(-(j-1)) \\ &(ii) \quad \tau_{j-1} \le C_6\omega(C_5\exp(-j+1)). \end{aligned} \tag{2.69}$$

The final energy estimate, similar to (2.57) with index $j-1$ follows,

$$\int_{\Omega(j^{-1}+\tau_j+\tau_{j-1})} (|\nabla u_j|^2 + h(\rho(x))|u_j|^{q+1})\,dx \le K_{j-2}. \tag{2.70}$$

The described circles of computations can be repeated i times with a unique restriction on i already observed, namely $j-i \ge j_0 = 1+\ln 2+\ln\ln(2+C_4)$. Thus, performing $(j-j_0)$ times our computation, we obtain at end

$$\int_{\Omega(j^{-1}+\sum_{i=j_0}^{j}\tau_i)} (|\nabla u_j|^2 h(\rho(x))|u_j|^{q+1})\,dx \le K_{j_0}. \tag{2.71}$$

The key point in our construction is to prove that $j^{-1}+\sum_{i=j_0}^{j}\tau_i$ remains uniformly bounded. It is clear from (2.69)-(ii) that, because of the monotonicity of ω,

$$\begin{aligned}\sum_{i=j_0}^{j}\tau_i &\le C_6\sum_{i=j_0}^{j}\omega(C_5\exp(-i))\\ &\le C_6\int_{j_0-1}^{j}\omega(C_5\exp(-s))\,ds\\ &\le C_6C_5^{-1}\int_{C_5\exp(-j)}^{C_5\exp(-j_0+1)} r^{-1}\omega(r)\,dr \le C_7 \quad \forall\, j\in\mathbb{N}.\end{aligned} \tag{2.72}$$

The last estimate follows from condition (2.6). Moreover, from (2.6) follows that $C_7 = C_7(j_0) \to 0$ as $j_0 \to \infty$. Therefore for arbitrary small $\nu > 0$ we can find $j_0 = j_0(\nu)$ such that

$$\int_{\Omega(\nu)} (|\nabla u_j|^2 + h(\rho(x))|u_j|^{q+1})\,dx \le K_{j_0(\nu)} \qquad \forall\, j > j_0. \tag{2.73}$$

Validity of the statement of Theorem 2 follows from (2.73) by standard way. First of all (2.73) yields

$$\|u_j\|_{H^1(\Omega(\nu),\partial\Omega(\nu)\cap\Omega)} \le c = c(\nu) \qquad \forall\, j\in\mathbb{N}, \tag{2.74}$$

where for arbitrary set $S\subset\partial\Omega$ by $H^1(\Omega,S)$ we denote, as usually, the closure in the norm $H^1(\Omega)$ of the set $C^1(\Omega,S) := \{f\in C^1(\Omega) : f|_S = 0\}$. Therefore for arbitrary $\nu > 0$ limiting solution $u(x)$ is weak limit of some subsequence $\{u_i(x)\}$ in the space $H^1(\Omega(\nu),\partial\Omega(\nu)\cap\Omega)$. As result:

$$u \in H^1(\omega(\nu),\partial\Omega(\nu)\cap\Omega) \qquad \forall\,\nu > 0, \tag{2.75}$$

thus, u satisfies boundary condition (2.7) in the weak sense. Next, since $h(\rho(x)) \ge 0$, each function $u_j(x)$ is subsolution of Laplace equation:

$$\Delta u_j \ge 0 \qquad \forall\, x\in\Omega,\ \forall\, j\in\mathbb{N}. \tag{2.76}$$

Therefore due to well known inner a priory estimate (see, for example [1]):

$$(\sup_{\Omega(2\nu)} u_j)^2 \le c_1(\nu)\int_{\Omega(\nu)} |u_j(x)|^2 dx \qquad \forall\,\nu > 0,\ \forall\, j\in\mathbb{N}, \tag{2.77}$$

where $c_1 = c_1(\nu)$ does not depend on $j \in \mathbb{N}$. From (2.73) and (2.77) follows:

$$\sup_{\Omega(\nu)} u_j \le c_2 = c_2(\nu) \qquad \forall\, j \in \mathbb{N},\ \forall\, \nu > 0. \tag{2.78}$$

Next, function $u_j(x)$ is the solution of the boundary problem:

$$\Delta u_j = f_j(x) := h(\rho(x)) u_j(x)^q \quad \text{in } \Omega(\nu) \tag{2.79}$$

$$u_j|_{\partial\Omega(\nu)\cap\Omega} = 0,\ \forall\, j > j_0(\nu), \tag{2.80}$$

where, due to (2.78),

$$\|f_j\|_{L_p(\Omega(\nu))} \le c_3(\nu) \qquad \forall\, j \in \mathbb{N},\ \forall\, p > 1. \tag{2.81}$$

Therefore due to classical local L_p a priory estimate (see, for example, [1]),

$$\|u_j\|_{W^{2,p}(\Omega(2\nu))} \le c_4(\nu) \qquad \forall\, j \in \mathbb{N},\ \forall\, p > 1, \tag{2.82}$$

as consequence,

$$u \in C^{1,\lambda}(\overline{\Omega}(\nu)) \qquad \forall\, \nu > 0. \tag{2.83}$$

Finally, it follows from to (2.75) and (2.83), that u satisfies the boundary condition (2.7) in a strong sense.

□

2.3 Further extensions

Problem 1. Although the construction should be much more technical, it looks clear that local flatness condition on $\partial\Omega$ near a must be of a technical aspect.

Problem 2. A related problem is the following. Let $k > 0$, $r > 0$ and $u = u_k$ be the solution of

$$\begin{cases} -\Delta u + H(x)u^q = 0 & \text{in } \Omega \\ \qquad\qquad u = k\chi_{\Gamma_r(a)} & \text{on } \partial\Omega \end{cases} \tag{2.84}$$

where $a \in \partial\Omega$ and $\Gamma_r(a) = B_r(a) \cap \partial\Omega$. Are conditions (2.5)(2.6) sufficient in order to garantee that $u_\infty := \lim_{k\to\infty} u_k$ satisfies $\lim_{x\to y} u_\infty(y) = 0$, for all $y \in \Omega \setminus \Gamma_r(a)$.

Problem 3. Assume Ω and Ω' are two bounded C^2 domains such that $\partial\Omega$ and $\partial\Omega'$ are tangent at some point a. Assume also that $\overline{\Omega} \subset \Omega' \cup \{a\}$ and $H \in C(\overline{\Omega}')$ is positive in Ω, vanishes on $\Omega' \setminus \overline{\Omega}$. Under what condition on H and the tangency order of $\partial\Omega$ and $\partial\Omega'$, is the solution $u = u_{k,a}$ of

$$\begin{cases} -\Delta u + H(x)u^q = 0 & \text{in } \Omega' \\ \qquad\qquad u = k\delta_a & \text{on } \partial\Omega' \end{cases} \tag{2.85}$$

satisfy $u_{\infty,a} := \lim_{k\to\infty} u_{k,a}$ a solution in Ω' ? has $u_{\infty,a}$ zero limit on $\partial\Omega' \setminus \{a\}$?

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