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GLOBAL EXTERNALITIES, ENDOGENOUS GROWTH AND SUNSPOT FLUCTUATIONS

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Abstract: *We consider a two-sector economy with Cobb-Douglas technologies, labor-augmenting global external effects and increasing social returns. We prove the existence of a normalized balanced growth path and we give conditions for the occurrence of sunspot fluctuations that are compatible with both types of capital intensity configuration at the private level provided the elasticity of intertemporal substitution in consumption admits intermediary values. We finally show that the existence of period-two cycles requires the consumption good to be physical capital intensive at the private level.*

Keywords: *Global externalities, increasing returns, endogenous growth, intertemporal substitution in consumption, indeterminacy, sunspot fluctuations, period-two cycles.*

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1 Introduction

Over the last decade, a large literature has focussed on the existence of local indeterminacy, i.e. a continuum of equilibrium paths converging toward a steady state, and sunspot fluctuations in endogenous growth models. These studies provide a possible explanation of the facts that economic growth rates are volatile over time and dispersed across countries. Within infinite-horizon two-sector models, the occurrence of multiple equilibrium paths is related to the existence of market imperfections such that productive externalities. Two different strategies have been followed to modelize these learning-by-doing effects. A first set of contributions builds upon Benhabib and Farmer [1] and consider sector-specific externalities with constant returns at the social level. Most of the papers are based on continuous-time models and local indeterminacy is shown to arise if the final good sector is human capital (labor) intensive at the private level but physical capital intensive at the social level.[1] It is worth noticing however that Mino *et al.* [14] have recently considered a discrete-time two-sector analogue model and show that the existence of local indeterminacy follows from a more complex set of conditions based on capital intensities differences at the private and social level, the rate of depreciation of capital and the discount factor.

A second set of contributions build upon the initial endogenous growth framework developed by Lucas [12] and Romer [17]. Local indeterminacy is derived from the consideration of global externalities and increasing returns at the social level. Most of the papers are however based on specific functional forms with a Leontief technology in the investment good sector, and/or assume a degenerate allocation of human capital (labor) across sectors.[2] In such a framework, there is no clear condition in terms of capital intensities differences to ensure the occurrence of multiple equilibria. Considering a non-trivial allocation mechanism of labor between the two sectors, Drugeon *et al.* [10] have shown within a general continuous-time

[1] Benhabib *et al.* [2], Mino [13]. See also Bond *et al.* [7] in which similar results are obtained from the consideration of distortionary factor taxes.

[2] See Benhabib and Perli [4], Boldrin and Rustichini [6], Boldrin *et al.* [5], Xie [19].

model with labor-augmenting global external effects borrowed from Boldrin and Rustichini [6] that the occurrence of local indeterminacy necessarily requires a physical capital intensive investment good sector at the private level. This condition appears to be the complete opposite to the one derived with sector-specific externalities. The aim of this paper is then to explore the robustness of this result by considering a discrete-time formulation.

Considering a discrete-time analogue of the Drugeon *et al.* [10] model, Goenka and Poulsen [11] suggest that local indeterminacy may arise under both configurations for the capital intensity difference at the private level. However, their conditions are quite complex as they involve the endogenous growth rate, and they do not show that a non-empty set of economies may satisfy these conditions. We then consider a discrete-time model with Cobb-Douglas technologies and labor-augmenting global externalities. We prove the existence of a balanced growth path and we give conditions on the Cobb-Douglas coefficients for the occurrence of sunspot fluctuations that are compatible with both types of capital intensity configuration at the private level provided the elasticity of intertemporal substitution in consumption admits intermediary values. However, the occurrence of period-two cycles requires the consumption good to be physical capital intensive at the private level. All these results are finally illustrated through numerical examples.

The rest of the paper if organized as follows: Section 2 sets up the basic model. In Section 3 we study the existence of a balanced growth path. Section 4 provides the main results on local indeterminacy. Section 5 contains concluding comments. The proofs are gathered in a final Appendix.

2 The model

We consider a discrete-time two-sector economy having an infinitely-lived representative agent with a single period utility function given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where c is consumption and $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution in consumption. The labor supply is inelastic. There are two goods: the consumption good, c, and the capital good, k. Each good is produced with a Cobb-Douglas technology. We assume that the production functions contain positive global externalities given by the average capital stock in the economy and which can be interpreted as a labor augmenting technical progress. We denote by y and c the outputs of sectors k and c:

$$c = K_c^{\alpha_1}(eL_c)^{\alpha_2}, \quad y = AK_y^{\beta_1}(eL_y)^{\beta_2}$$

with $e = \bar{k}$ and $A > 0$ a normalization constant. Labor is normalized to one, $L_c + L_y = 1$ and the total stock of capital is $K_c + K_y = k$. We assume that the economy-wide average $\bar{k}$ is taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{k} = k$.

We assume constant returns to scale at the private level, i.e. $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$. At the social level the returns to scale are therefore increasing. It can be easily shown that if $\beta_1/\beta_2 > (<)\alpha_1/\alpha_2$ the investment (consumption) good sector is capital intensive from the private perspective.

We assume complete depreciation of capital in one period so that the capital accumulation equation is $y_t = k_{t+1}$. Optimal factor allocations across sectors are obtained by solving the following program:

$$\begin{array}{cl} \max\limits_{K_{ct}, L_{ct}, K_{yt}, L_{yt}} & K_{ct}^{\alpha_1}(e_t L_{ct})^{\alpha_2} \\ s.t. & k_{t+1} = AK_{yt}^{\beta_1}(e_t L_{yt})^{\beta_2} \\ & 1 = L_{ct} + L_{yt} \\ & k_t = K_{ct} + K_{yt} \\ & e_t \quad given \end{array} \tag{1}$$

Denote by p_t, w_t and r_t respectively the price of the capital good, the wage rate of labor and the rental rate of capital, all in terms of the price of the consumption good. For any (k_t, k_{t+1}, e_t), solving the associated first order conditions gives inputs as $\tilde{K}_c(k_t, k_{t+1}, e_t)$, $\tilde{L}_c(k_t, k_{t+1}, e_t)$, $\tilde{K}_y(k_t, k_{t+1}, e_t)$ and $\tilde{L}_y(k_t, k_{t+1}, e_t)$. We then define the efficient production frontier as

$$T(k_t, k_{t+1}, e_t) = \tilde{K}_c(k_t, k_{t+1}, e_t)^{\alpha_1}[e_t\tilde{L}_c(k_t, k_{t+1}, e_t)]^{\alpha_2}$$

which describes the standard trade-off between consumption and investment: for a given e_t, $T(k_t, k_{t+1}, e_t)$ is increasing with respect to k_t and decreasing with respect to k_{t+1}. Using the envelope theorem we then derive the equilibrium prices

$$p_t = -T_2(k_t, k_{t+1}, e_t), \quad r_t = T_1(k_t, k_{t+1}, e_t) \tag{2}$$

where $T_1 = \frac{\partial T}{\partial k}$ and $T_2 = \frac{\partial T}{\partial y}$. The representative consumer's optimization program is finally given by

$$\begin{array}{cc} \max\limits_{\{k_t\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{\infty} \delta^t \frac{[T(k_t, k_{t+1}, e_t)]^{1-\sigma}}{1-\sigma} \\ s.t. & k_0 = \hat{k}_0, \{e_t\}_{t=0}^{+\infty} \ \ given \end{array} \tag{3}$$

with $\delta \in (0,1)$ the discount factor. The corresponding Euler equation is

$$-c_t^{-\sigma} p_t + \delta c_{t+1}^{-\sigma} r_{t+1} = 0 \tag{4}$$

Let $\{k_t\}_{t=0}^{+\infty}$ denote a solution of (4) which obviously depends on $\{e_t\}_{t=0}^{\infty}$, i.e. $k_t = k(t, \{e_t\}_{t=0}^{\infty})$ for all $t \geq 0$. As we have assumed that $e_t = \bar{k}_t$ with $\bar{k}_t$ the economy-wide average capital stock, expectations are realized if there exists a solution of an infinite-dimensional fixed-point problem such that $e_t = k(t, \{e_t\}_{t=0}^{\infty})$ for any $t = 0, 1, 2, ...$ Assuming that such a solution exists,[3] prices may now be written as

$$r(k_t, k_{t+1}) = T_1(k_t, k_{t+1}, k_t), \quad p(k_t, k_{t+1}) = -T_2(k_t, k_{t+1}, k_t) \tag{5}$$

and consumption at time t is given by a linear homogeneous function.[4]

$$c(k_t, k_{t+1}) = T(k_t, k_{t+1}, k_t) \tag{6}$$

Equation (4) finally becomes:

$$p(k_t, k_{t+1},)c(k_t, k_{t+1})^{-\sigma} = \delta r(k_{t+1}, k_{t+2})c(k_{t+1}, k_{t+2})^{-\sigma} \tag{7}$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ of (7) which also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t c(k_t, k_{t+1})^{-\sigma} p(k_t, k_{t+1}) k_{t+1} = 0 \tag{8}$$

and the summability condition

$$\sum_{t=0}^{\infty} \delta^t \frac{[c(k_t, k_{t+1})]^{1-\sigma}}{1-\sigma} < +\infty \tag{9}$$

is called an equilibrium path.[5]

[3] A detailed treatment of the existence of such a solution within a discrete-time version of the Lucas [12] model is provided in Mitra [15].

[4] See Proposition 1 in Drugeon and Venditti [9] or Lemma 1 in Drugeon *et al.* [10].

[5] See Lemma 1 and Corollary 1 in Boldrin *et al.* [5].

3 Balanced growth path

We call a path $\{k_t\}_{t=0}^{+\infty}$ satisfying $k_0 = \hat{k}_0$ and $k_t \leq k_{t+1} \leq Ak_t$, a feasible path (from $k_0 = \hat{k}_0$). We call $\{k_t\}_{t=0}^{+\infty}$ a balanced growth path if it is in equilibrium and $k_{t+1}/k_t = \theta$ for $t = 0, 1, ...$.

Lemma 1. *Along an equilibrium path, prices satisfy*

$$r(k_t, k_{t+1}) = A\alpha_1 \left[\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\left(\frac{k_{t+1}}{A\bar{g}(k_t,k_{t+1})}\right)^{1/\beta_2}\right]^{\alpha_2}, \quad p(k_t, k_{t+1}) = \frac{r_t\bar{g}(k_t,k_{t+1})}{\beta_1 k_{t+1}} \tag{10}$$

with

$$\bar{g}(k_t, k_{t+1}) = \left\{K_y \in (0, Ak)/\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \frac{[k_t - K_y][(k_{t+1}/A)^{1/\beta_2}k_t^{-1}K_y^{-\beta_1/\beta_2}]}{[1-(k_{t+1}/A)^{1/\beta_2}k_t^{-1}K_y^{-\beta_1/\beta_2}]K_y}\right\}$$

a linear homogeneous function.

Define $\theta_t = k_{t+1}/k_t (= c_{t+1}/c_t)$ the growth factor of capital (and thus consumption) at time t. Notice that $ln\theta_t$ is the growth rate of capital. By the feasibility condition $k_t \leq k_{t+1} \leq Ak_t$, it holds that $\theta_t \in (0, A)$. Denoting $g(\theta_t^{-1}) = \bar{g}(\theta_t^{-1}, 1)$ and using (10), the Euler equation (7) can be transformed into an implicit recursive equation as follows

$$\delta\beta_1 g(\theta_{t+1}^{-1})^{-\alpha_2/\beta_2} = g(\theta_t^{-1})^{1-(\alpha_2/\beta_2)}\theta_t^{\sigma} \tag{11}$$

A balanced growth factor is defined by $\theta_t = \theta_{t+1} = \theta$ and satisfies equation (11) together with conditions (8)-(9). Using the normalization constant A we will show that there exists such a normalized balanced growth factor.

Proposition 1. *Let* $\tilde{\sigma} = ln\beta_1/ln(\delta\beta_1) > 0$. *There exists a normalized balanced growth factor (NBGF)* $\theta^* = (\delta\beta_1)^{-1}$ *solution of the Euler equation (11) if* $\sigma > \tilde{\sigma}$ *and the normalization constant* A *is set at the following value*

$$A^* = \left[\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\frac{1-(\delta\beta_1)^{\sigma}\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}{(\delta\beta_1)^{(1+\sigma)/\beta_2}}\right]^{\beta_2} \tag{12}$$

Proposition 1 establishes the existence of a normalized balanced growth rate (NBGR) $ln\theta^*$ from which we define a normalized balanced growth path (NBGP), namely $k_t = \hat{k}_0\theta^{*t} = \hat{k}_0(\delta\beta_1)^{-t}$.

4 Local indeterminacy

The local stability properties of the NBGP are obtained from the linearization of the Euler equation (11) around θ^*.

Assumption 1. $\alpha_1\beta_2 \neq \alpha_2\beta_1$

Assumption 1, which is equivalent to a non-zero capital intensity difference at the private level, implies that the technologies are not identical.

Lemma 2. *Under Assumption 1, if $\sigma > \tilde{\sigma}$ and $A = A^*$ as defined in Proposition 1, the linearization of the Euler equation (11) around θ^* gives*

$$\frac{d\theta_{t+1}}{d\theta_t} = \sigma\beta_2 \frac{1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1} + \frac{1-(\delta\beta_1)^\sigma\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}{\beta_2(\delta\beta_1)^\sigma}}{\alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)} - \beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right) \tag{13}$$

Notice that the Euler equation (11) is forward looking since the initial growth rate $\theta_0 = k_1/\hat{k}_0$ is not pre-determined. As a result, if $d\theta_{t+1}/d\theta_t \in (-1, 1)$, then every initial point in the neighborhood of the NBGR has indeterminate equilibrium paths satisfying (8) and (9), and we say that the NBGP is locally indeterminate. If on the contrary $|d\theta_{t+1}/d\theta_t| > 1$, the NBGR is locally unstable and starting from $\hat{k}_0$ one possible equilibrium consists in jumping at $t = 1$ on the NBGP. In such a case the NBGP is locally determinate.

Proposition 2. *Let $\tilde{\sigma} = ln\beta_1/ln(\delta\beta_1)$. Under Assumption 1, if one of the following set of conditions is satisfied:*

i) the investment good is capital intensive at the private level with

$$\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] - \alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\left[1+\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] < 0 \tag{14}$$

ii) the consumption good is capital intensive at the private level with

$$\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] + \alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\left[1-\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] < 0 \tag{15}$$

then there exist $\underline{\sigma} \geq \tilde{\sigma}$ and $\bar{\sigma} > \underline{\sigma}$ such that the NBGP $k_t = \hat{k}_0(\delta\beta_1)^{-t}$ is locally indeterminate when $\sigma \in (\underline{\sigma}, \bar{\sigma})$.

We easily derive from Proposition 2 that there exists some case in which the growth rate of capital exhibits period-two cycles.

Corollary 1. *Let $\tilde{\sigma} = ln\beta_1/ln(\delta\beta_1)$. Under Assumption 1, if the consumption good is capital intensive at the private level and condition (15) holds, then there exist $\underline{\sigma} \geq \tilde{\sigma}$ and $\bar{\sigma} > \underline{\sigma}$ such that the NBGP is locally indeterminate when $\sigma \in (\underline{\sigma}, \bar{\sigma})$. Moreover, when σ crosses $\bar{\sigma}$ from below the NBGP becomes locally determinate and there exist locally indeterminate (determinate) period-two growth cycles in a right (left) neighborhood of $\bar{\sigma}$.*

Corollary 1 shows that the existence of endogenous fluctuations requires a capital intensive consumption good. The intuition for this result, initially provided by Benhabib and Nishimura [3], may be summarized as follows. Consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. The Rybczinsky theorem thus implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}, and thus implies a decrease of the balanced growth factor $\theta_t = k_{t+1}/k_t$.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. Indeed, the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to lower the investment and the capital stock in period $t+2$, k_{t+2}, and thus implies an increase of the balanced growth factor $\theta_t = k_{t+2}/k_{t+1}$.

Boldrin *et al.* [5] consider a similar two-sector model but assume a Leontief technology in the investment good sector with a degenerate allocation of labor across sectors as the investment good is only produced from physical capital. They prove the existence of global indeterminacy of equilibria and construct robust examples of both topological and ergodic chaos for the dynamics of balanced growth paths. We may expect similar results from Corollary 1. However, showing the existence of chaotic dynamics with a Cobb-Douglas technology in both sector is beyond the goal of the current paper and is left for future research.

Remark: Considering a discrete-time two-sector optimal endogenous growth model with two capital goods, one being consumable while the other is not, Drugeon [8] shows that the occurrence of period-two cycles requires a non-unitary rate of capital depreciation in one sector at least. In a similar formulation but extended to include sector-specific externalities, Mino *et al.* [14] show that the existence of local indeterminacy is based on the same necessary condition. We prove that the consideration of global externalities allows to get sunspot fluctuations even under full depreciation of capital.

Numerical illustration:
In order to check whether all the conditions of Proposition 2 can be satisfied simultaneously, we perform some numerical simulations.

i) Let $\delta = 0.98$, $\beta_1 = 0.85$ and $\alpha_1 = 0.55$. The investment good is thus capital intensive at the private level and the NBGF is equal to $\theta^* \approx 1.2$. The NBGP is locally indeterminate for any $\sigma \in (\tilde{\sigma}, \bar{\sigma})$ with $\tilde{\sigma} \approx 0.8894345$ and $\bar{\sigma} \approx 1.07066$.

ii) Let $\delta = 0.95$, $\beta_1 = 0.35$ and $\alpha_1 = 0.88$. The consumption good is thus capital intensive at the private level and $\theta^* \approx 3.0075$. The NBGP is locally indeterminate for any $\sigma \in (\tilde{\sigma}, \bar{\sigma})$ with $\tilde{\sigma} \approx 0.953417$ and $\bar{\sigma} \approx 1.07$. Moreover, $\bar{\sigma}$ is a flip bifurcation value so that there exist locally indeterminate (resp. determinate) period-two cycles in a right (resp. left) neighborhood of $\bar{\sigma}$.

5 Concluding comments

We have considered a discrete-time two-sector endogenous growth model with Cobb-Douglas technologies augmented to include labor-augmenting global externalities. We have proved the existence of a normalized balanced growth path and we have shown that sunspot fluctuations arise under both types of capital intensity configuration at the private level provided the elasticity of intertemporal substitution in consumption admits intermediary values. Moreover, the dynamics of growth rates exhibits period-two cycles if the consumption good is capital intensive at the private level.

6 Appendix

6.1 Proof of Lemma 1

The Lagrangian associated with program (1) is:

$$\begin{aligned}\mathcal{L} &= K_{ct}^{\alpha_1}(e_tL_{ct})^{\alpha_2} + p_t[AK_{yt}^{\beta_1}(e_tL_{yt})^{\beta_2} - k_{t+1}] + \omega_t[1 - L_{ct} - L_{yt}] \\ &+ r_t[k_t - K_{ct} - K_{yt}]\end{aligned}$$

The first order conditions are:

$$\alpha_1 c_t/K_{cc} = p_t\beta_1 y_t/K_{yt} = r_t \quad (16)$$

$$\alpha_2 c_t/L_{ct} = p_t\beta_2 y_t/L_{yt} = w_t \quad (17)$$

Solving $y_t = AK_{yt}^{\beta_1}(k_tL_{yt})^{\beta_2}$ with respect to L_{yt} gives

$$L_{yt} = (y_t/A)^{1/\beta_2}k_t^{-1}K_{yt}^{-\beta_1/\beta_2} \quad (18)$$

Using $K_{ct} = k_0 - K_{yt}$, $L_{yt} = 1 - L_{ct}$, and merging (16)-(18) we get:

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \frac{K_{ct}}{L_{ct}}\frac{L_{yt}}{K_{yt}} = \frac{k_t - K_{yt}}{1-(y_t/A)^{1/\beta_2}k_t^{-1}K_{yt}^{-\beta_1/\beta_2}}\frac{(y_t/A)^{1/\beta_2}k_t^{-1}K_{yt}^{-\beta_1/\beta_2}}{K_{yt}} \quad (19)$$

The solution K_{yt}^* is obtained as an implicit linear homogeneous function $\bar{g}(k_t, y_t)$ as shown in Drugeon and Venditti [9].[6] We then derive from (16)

$$r_t = A\alpha_1\left(\frac{k_tL_{ct}}{K_{ct}}\right)^{\alpha_2}, \quad p_t = \frac{r_tK_{yt}}{\beta_1 y_t}$$

Using (18) and (19) with $y_t = k_{t+1}$ gives the final results. □

6.2 Proof of Proposition 1

Consider the Euler equation (11):

$$\delta\beta_1 g(\theta_{t+1}^{-1})^{-\alpha_2/\beta_2} = g(\theta_t^{-1})^{1-(\alpha_2/\beta_2)}\theta_t^{\sigma}$$

Along a balanced growth path with $\theta_t = \theta_{t+1} = \theta$, we get $\delta\beta_1 = g(\theta^{-1})\theta^{\sigma}$. Now consider equation (19) with $K_y = \bar{g}(k_t, k_{t+1}) = k_{t+1}\bar{g}(k_t/k_{t+1}, 1) \equiv k_{t+1}g(\theta_t^{-1})$. We derive

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \frac{\theta_t^{-1} - g(\theta_t^{-1})}{A^{1\beta_2}\theta_t^{-1}g(\theta_t^{-1})^{1/\beta_2} - g(\theta_t^{-1})} \quad (20)$$

If $\theta_t = \theta$ and thus $g(\theta^{-1}) = \delta\beta_1\theta^{-\sigma}$ we get after simplifications

[6] See also Drugeon *et al.* [10].

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \frac{1-\delta\beta_1\theta^{1-\sigma}}{A^{1\beta_2}(\delta\beta_1\theta_t^{-\sigma})^{1/\beta_2}-\delta\beta_1\theta^{1-\sigma}}$$

It follows that $\theta = (\delta\beta_1)^{-1} \equiv \theta^*$ is a solution of this equation if and only if the normalization constant A satisfies $A = A^*$ as defined by (12). Along the stationary balanced growth path $k_t = (\delta\beta_1)^{-t}\hat{k}_0$, using the fact that $c(k, y)$ is homogeneous of degree one and $p(k, y)$ is homogeneous of degree zero, the transversality condition (8) becomes

$$\delta^{-1}\hat{k}_0^{1-\sigma}c(\delta\beta_1, 1)^{-\sigma}p(\delta\beta_1, 1)\lim_{t\to+\infty}(\delta^\sigma\beta_1^{\sigma-1})^{t+1} = 0$$

It will be satisfied if $\delta^\sigma < \beta_1^{1-\sigma}$ or equivalently $\sigma > \tilde{\sigma} = ln\beta_1/ln(\delta\beta_1) > 0$. Similarly the summability condition (9) becomes

$$\delta^{-1}\hat{k}_0^{1-\sigma}\frac{[c(\delta\beta_1,1)]^{1-\sigma}}{1-\sigma}\sum_{t=0}^{\infty}(\delta^\sigma\beta_1^{\sigma-1})^{t+1} < +\infty$$

and is satisfied if $\delta^\sigma < \beta_1^{1-\sigma}$ or equivalently $\sigma > \tilde{\sigma} = ln\beta_1/ln(\delta\beta_1) > 0$. Finally, the feasibility condition requires $(\delta\beta_1)^{-1} \leq A^*$ or equivalently

$$\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\frac{1-(\delta\beta_1)^\sigma\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}{(\delta\beta_1)^{\sigma/\beta_2}} \geq 1 \quad\Leftrightarrow\quad 1-(\delta\beta_1)^\sigma + (\delta\beta_1)^\sigma\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\left[1-(\delta\beta_1)^{(\sigma\beta_1)/\beta_2}\right] \geq 0$$

This inequality is always satisfied. □

6.3 Proof of Lemma 2

Notice first that $g(\theta^{*-1}) = g(\delta\beta_1) = (\delta\beta_1)^{1+\sigma}$. Total differentiation of the Euler equation (11) around $\theta = \theta^*$ with $A = A^*$ gives

$$\frac{d\theta_{t+1}}{d\theta_t} = \frac{\sigma\beta_2(\delta\beta_1)^\sigma}{\alpha_2 g'(\theta^{*-1})} + \frac{\beta_2-\alpha_2}{\alpha_2} \qquad (21)$$

Consider now equation (20) and let us denote $X = \theta^{-1}$. We get:

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \frac{X-g(X)}{A^{1\beta_2}Xg(X)^{1/\beta_2}-g(X)}$$

Total differenciation with respect to X gives

$$\frac{dg}{dX} = g'(X) = \frac{(\delta\beta_1)^\sigma\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}{1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}+\frac{1-(\delta\beta_1)^\sigma\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}{\beta_2(\delta\beta_1)^\sigma}}$$

Substituting this expression in (21) and using the fact that $\beta_2 - \alpha_2 = -\alpha_2\beta_1(1 - \alpha_1\beta_2/\alpha_2\beta_1)$ give the final result. □

6.4 Proof of Proposition 2

Consider the expression (13) which may be expressed as follows

$$\frac{d\theta_{t+1}}{d\theta_t} = \frac{\sigma}{\alpha_2}\left[\frac{\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{-1}}{(\delta\beta_1)^\sigma} - \beta_1\right] - \beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right) \tag{22}$$

Local indeterminacy of the NBGP will be obtained if and only if $d\theta_{t+1}/d\theta_t \in (-1,1)$. We first obtain that

$$\begin{aligned} \lim_{\sigma\to+\infty} \frac{d\theta_{t+1}}{d\theta_t} &= +\infty \Leftrightarrow 1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1} > 0 \\ &= -\infty \Leftrightarrow 1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1} < 0 \end{aligned} \tag{23}$$

Let $\sigma = \tilde{\sigma}$, or equivalently $(\delta\beta_1)^{\tilde{\sigma}} = \beta_1$, so that (13) becomes

$$\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\sigma=\tilde{\sigma}} = \frac{1}{\alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)}\left\{\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] - \alpha_2\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^2\right\}$$

i) Consider first the case in which the investment good is capital intensive at the private level, i.e. $1-\alpha_1\beta_2/\alpha_2\beta_1 > 0$. We get

$$\begin{aligned} &\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\sigma=\tilde{\sigma}} > -1 \\ \Leftrightarrow\ &\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] + \alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\left[1-\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] > 0 \end{aligned}$$

This inequality is always satisfied since $1-\alpha_1\beta_2/\alpha_2\beta_1 < 1$. We also have

$$\begin{aligned} &\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\sigma=\tilde{\sigma}} < 1 \\ \Leftrightarrow\ &\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] - \alpha_2\left(1+\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\left[1-\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] < 0 \end{aligned}$$

This last inequality corresponds to condition (14). If it holds we conclude from (23) that there exists $\bar{\sigma} > \tilde{\sigma}$ such that the NBGP is locally indeterminate when $\sigma \in (\tilde{\sigma}, \bar{\sigma})$.

ii) Consider now the case in which the consumption good is capital intensive at the private level, i.e. $1-\alpha_1\beta_2/\alpha_2\beta_1 < 0$. We get

$$\begin{aligned} &\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\sigma=\tilde{\sigma}} > -1 \\ \Leftrightarrow\ &\frac{\tilde{\sigma}}{\beta_1}\left[1-\beta_1^2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] + \alpha_2\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\left[1-\beta_1\left(1-\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)\right] < 0 \end{aligned}$$

This last inequality corresponds to condition (15). If it holds we conclude from (23) that there exist $\underline{\sigma} \geq \tilde{\sigma}$ and $\bar{\sigma} > \underline{\sigma}$ such that the NBGP is locally indeterminate when $\sigma \in (\underline{\sigma}, \bar{\sigma})$. We may indeed have $\underline{\sigma} > \tilde{\sigma}$ since it may be the case that $d\theta_{t+1}/d\theta_t|_{\sigma=\tilde{\sigma}} > 1$. □

6.5 Proof of Corollary 1

Let the consumption good be capital intensive at the private level and condition (15) holds. Then $d\theta_{t+1}/d\theta_t|_{\sigma=\tilde{\sigma}} > -1$. Considering (23) we derive from Proposition 2 that $d\theta_{t+1}/d\theta_t|_{\sigma=\bar{\sigma}} = -1$ and $d\theta_{t+1}/d\theta_t < -1$ when $\sigma > \bar{\sigma}$. Therefore $\bar{\sigma}$ is a flip bifurcation value (see Ruelle [18]) and the result follows. □

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