Boundary singularities of positive solutions of some nonlinear elliptic equations

Singularités au bord de solutions positives d'équations elliptiques non-linéaires

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Abstract

We study the behavior near x_0 of any positive solution of (E) $-\Delta u = u^q$ in Ω which vanishes on $\partial\Omega \setminus \{x_0\}$, where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $q \geq (N+1)/(N-1)$ and $x_0 \in \partial\Omega$. Our results are based upon *a priori* estimates of solutions of (E) and existence, non-existence and uniqueness results for solutions of some nonlinear elliptic equations on the upper-half unit sphere. *To cite this article: M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).*

Résumé

Nous étudions le comportement quand x tend vers x_0 de toute solution positive de (E) $-\Delta u = u^q$ dans Ω qui s'annule sur $\partial\Omega \setminus \{x_0\}$, où $\Omega \subset \mathbb{R}^N$ est un domaine régulier, $q \geq (N+1)/(N-1)$ et $x_0 \in \partial\Omega$. Nos résultats sont fondés sur des estimations a priori des solutions de (E), et des résultats d'existence, de non existence et d'unicité de solutions de certaines équations elliptiques non linéaires sur la demi-sphère unité. *Pour citer cet article : M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).*

Version française abrégée Soit Ω un ouvert régulier de $\mathbb{R}^N$, $N \geq 2$, tel que $0 \in \partial\Omega$. Étant donné $q > 1$, nous considérons une fonction $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ qui vérifie (3). Nous nous intéressons à la description du comportement de u au voisinage de 0.

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Nous distinguerons les trois valeurs critiques de q données par (4). Si $1 < q < q_1$, le comportement en 0 des solutions est décrit dans [4] ; aussi supposerons-nous le plus souvent $q \geq q_1$. Si u est une solution de (3) dans $\mathbb{R}^N_+$ de la forme $u(x) = u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma)$, alors ω vérifie l'équation (6). Dans ce cas, nous avons le résultat suivant :

Théorème 0.1 *(i) Si $1 < q \leq q_1$, le problème (3) n'admet aucune solution.*
(ii) Si $q_1 < q < q_3$, (3) admet une unique solution, notée ω_0.
(iii) Si $q \geq q_3$, (3) n'admet aucune solution.

Le résultat d'unicité décrit en (ii) est en fait un cas particulier d'un résultat plus général :

Théorème 0.2 *Pour tous $q > 1$ et $\lambda \in \mathbb{R}$, il existe au plus une solution positive de (7).*

Ce résultat demeure si, dans (7), S^{N-1}_+ est remplacé par une boule dans $\mathbb{R}^N$, et Δ' par le laplacien ordinaire.

Par simplicité, nous pouvons supposer que $\partial\mathbb{R}^N_+$ est l'hyperplan tangent à Ω en 0. Le théorème ci-dessous donne une classification des singularités isolées du problème (3) :

Théorème 0.3 *Soit $q \geq q_1$, avec $q \neq q_2$. Supposons que la solution u du problème (3) vérifie*

$$0 \leq u(x) \leq C\,|x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \tag{1}$$

pour $C, a > 0$. Si $q_1 \leq q < q_3$, ou bien u est continue en 0, ou bien

$$u(r,\sigma) = \begin{cases} r^{-(N-1)}\left(\log(1/r)\right)^{\frac{1-N}{2}}\left(k_N\sigma_1 + o(1)\right) & si\ q = q_1, \\ r^{-2/(q-1)}\left(\omega_0(\sigma) + o(1)\right) & si\ q_1 < q < q_3, \end{cases} \tag{2}$$

lorsque $r \to 0$, uniformément par rapport à $\sigma \in S^{N-1}_+$; k_N est une constante qui dépend seulement de N. Si $q \geq q_3$, u est continue en 0.

L'estimation a priori (1) est obtenue pour $q_1 \leq q < q_2$:

Théorème 0.4 *Si $q_1 \leq q < q_2$, toute solution u de (3) vérifie (1) pour $C = C(N, q, \Omega) > 0$.*

Les démonstrations détaillées sont présentées dans [2].

1. Introduction and main result

Let Ω be a smooth open subset of $\mathbb{R}^N$, $N \geq 2$, such that $0 \in \partial\Omega$ and let $q > 1$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{3}$$

Our goal in this paper is to describe the behavior of u in a neighborhood of 0.

This problem has similar features with the case where $x_0 \in \Omega$, which has been studied by Gidas-Spruck [7]. In our case, we encounter three critical values of q in describing the local behavior of u:

$$q_1 := \frac{N+1}{N-1}, \qquad q_2 := \frac{N+2}{N-2} \ \text{ if } N \geq 3 \qquad \text{and} \qquad q_3 := \frac{N+1}{N-3} \ \text{ if } N \geq 4. \tag{4}$$

When $1 < q < q_1$, it is proved in [4] that for every solution u of (3) there exists $\alpha \geq 0$ (depending on N and u) such that

$$u(x) = \alpha\,|x|^{-N}\,\rho(x)\,\big(1 + o(1)\big) \quad \text{as } x \to 0, \tag{5}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$, $\forall x \in \Omega$. For this reason, we shall mainly restrict ourselves to $q \geq q_1$.

Let us first consider the case where $\Omega = \mathbb{R}^N_+$ and we look for solutions of (3) of the form $u(x) = u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma)$, where $r = |x|$ and $\sigma \in S^{N-1}_+$. An easy computation shows that ω must satisfy

$$\begin{cases} -\Delta'\omega = \ell_{N,q}\omega + \omega^q & \text{in } S^{N-1}_+, \\ \quad \omega \geq 0 & \text{in } S^{N-1}_+, \\ \quad \omega = 0 & \text{on } \partial S^{N-1}_+, \end{cases} \tag{6}$$

where Δ' denotes the Laplacian in S^{N-1} and $\ell_{N,q} = \frac{2(N-q(N-2))}{(q-1)^2}$. Concerning equation (6), we prove

Theorem 1.1 *(i) If $1 < q \leq q_1$, then (6) admits no positive solution.*
(ii) If $q_1 < q < q_3$, then (6) admits a unique positive solution.
(iii) If $q \geq q_3$, then (6) admits no positive solution.

One of the main ingredients in the proof of Theorem 1.1 *(ii)* is the following

Theorem 1.2 *If $q > 1$ and $\lambda \in \mathbb{R}$, then there exists at most one positive solution of*

$$\begin{cases} -\Delta' v = \lambda v + v^q & \text{in } S^{N-1}_+, \\ \quad v = 0 & \text{on } \partial S^{N-1}_+. \end{cases} \tag{7}$$

Remark 1 We emphasize that in Theorem 1.2 we *do not* assume that q is subcritical. The conclusion above remains valid if, in (7), S^{N-1}_+ is replaced by $B_1 \subset \mathbb{R}^N$ and Δ' by the usual Laplacian in $\mathbb{R}^N$. Theorem 1.2 extends a previous result of Kwong-Li [8].

We now return to the case where $\Omega \subset \mathbb{R}^N$ is an arbitrary smooth set such that $0 \in \partial\Omega$. For simplicity, we may assume that $\partial\mathbb{R}^N_+$ is the tangent hyperplane of Ω at 0. Using Theorem 1.2, we provide a classification of isolated singularities of solutions of (3):

Theorem 1.3 *Let $q \geq q_1$, $q \neq q_2$, and let u be a solution of (3). Assume that u satisfies*

$$0 \leq u(x) \leq C\,|x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \tag{8}$$

for some $C, a > 0$. If $q_1 \leq q < q_3$, then either u is continuous at 0 or

$$u(r,\sigma) = \begin{cases} r^{-(N-1)}\big(\log(1/r)\big)^{\frac{1-N}{2}}\big(k_N\sigma_1 + o(1)\big) & \text{if } q = q_1, \\ r^{-2/(q-1)}\big(\omega_0(\sigma) + o(1)\big) & \text{if } q_1 < q < q_3, \end{cases} \tag{9}$$

as $r \to 0$, uniformly with respect to $\sigma \in S^{N-1}_+$; k_N denotes a constant depending only on N and ω_0 is the unique positive solution of (6).
If $q \geq q_3$, then u is continuous at 0.

Remark 2 We do not know whether Theorem 1.3 is true when $q = q_2$. In this case, the equation is conformally invariant and thus other techniques are required. If $\Omega = \mathbb{R}^N_+$, then it can be proved that any solution of (3) depends only on the variables $r = |x|$ and $\theta = \cos^{-1}(x_1/|x|)$.

The next result establishes the existence of an *a priori* estimate for the solutions of (3). According to Theorem 1.4 below, assumption (8) is always fulfilled when $q_1 \leq q < q_2$:

Theorem 1.4 *Let $q_1 \leq q < q_2$ and let u be a solution of (3). Then,*

$$0 \leq u(x) \leq C\rho(x)\,|x|^{-2/(q-1)-1} \quad \forall x \in \Omega \cap B_1(0), \tag{10}$$

where C depends on N, q and Ω.

Remark 3 According to the Doob Theorem [6], any positive superharmonic function v in Ω satisfies $\int_\Omega |\Delta v|\,\rho < \infty$ and admits a boundary trace, which is a Radon measure on $\partial\Omega$. If u is a solution of (3), then its trace must be of the form $k\delta_{x_0}$, for some $k \geq 0$. We may have $k > 0$ if $1 < q < q_1$ (see [1]), but k is necessarily equal to 0 if $q \geq q_1$. Indeed, by the maximum principle, u satisfies $u \geq kP_\Omega(x, 0)$, where P_Ω denotes the Poisson potential of Ω. Since $u^q \in L^1_\rho(\Omega)$ (by the Doob Theorem), we must have $k = 0$ if $q \geq q_1$.

Detailed proofs will appear in [2].

2. Sketch of the proofs

Proof of Theorem 1.1. Assertion (i) is proved by multiplying (6) by $\phi(\sigma) = \sigma_1$. Note that ϕ is the first eigenfunction of $-\Delta'$ on S_+^{N-1}, with eigenvalue $\lambda_1 = N - 1$. Integrating the resulting expression over S_+^{N-1}, and using the fact that $1 < q \leq q_1 \Longrightarrow \ell_{N,q} \geq \lambda_1$, we obtain (i).
The existence part in (ii) is obtained by using the Mountain Pass Theorem; the uniqueness is a consequence of Theorem 1.2.
Assertion (iii) can be deduced from the following Pohožaev-type identity:

Proposition 2.1 *Assume $N \geq 4$ and $q > 1$. Then, any solution of* (7) *satisfies*

$$\frac{N-3}{q+1}(q-q_3)\int_{S_+^{N-1}} |\nabla' v|^2\,\phi\,d\sigma - \frac{(N-1)(q-1)}{q+1}\left(\lambda + \frac{N-1}{q-1}\right)\int_{S_+^{N-1}} v^2\phi\,d\sigma =$$
$$= -\int_{\partial S_+^{N-1}} |\nabla' v|^2\,d\tau.$$

This identity is obtained by computing the divergence of the vector field $P = \langle\nabla'\phi, \nabla' v\rangle\nabla' v$, where ∇' is the gradient on S^{N-1}, and then using the fact that the first eigenfunction satisfies $D^2\phi + \phi g_0 = 0$, where g_0 is the tensor of the standard metric on S^{N-1}. In order to establish (iii), it suffices to observe that $\ell_{N,q} \leq -\frac{N-1}{q-1} \Longleftrightarrow q \geq q_3$.

Proof of Theorem 1.2. We first notice that any positive solution of (7) depends only on the variable $\theta = \cos^{-1}(x_1/|x|) \in [0, \pi/2]$; this follows from a straightforward adaptation of the Gidas-Ni-Nirenberg moving plane method to S_+^{N-1} (see [9]). Thus, v satisfies

$$\begin{cases} v'' + (N-2)\cot\theta\, v' + \lambda v + v^q = 0 & \text{in } (0, \pi/2), \\ v'(0) = 0, \quad v(\pi/2) = 0. \end{cases} \tag{11}$$

Let $w(\theta) := \sin^\alpha\theta\, v(\theta)$, where $\alpha > 0$. By choosing $\alpha = 2(N-2)/(q+3)$, then w satisfies

$$\big(w'(\pi/2)\big)^2 = \int_0^{\pi/2} G'(\theta) w^2(\theta)\,d\theta, \tag{12}$$

where G is a function of the form $G(\theta) = \sin^{\beta'}\theta\,\big(\alpha_1\sin^2\theta + \alpha_2\big)$; the parameters $\alpha_1, \alpha_2, \beta' \in \mathbb{R}$ can be explicitly computed in terms of λ, N and q.
Assume, by contradiction, that v_1 and v_2 are two distinct solutions of (11). Then,

$$\int_0^{\pi/2} v_1 v_2\big(v_2^{q-1} - v_1^{q-1}\big)\,d\theta = 0. \tag{13}$$

Therefore, their graphs must intersect at some $\theta_0 \in (0, \pi/2)$. We claim that v_1 and v_2 intersect at least twice in $(0, \pi/2)$. If there is only one intersection point, then it can be shown that there exists $\gamma \geq 0$ such that the function $\theta \mapsto G'(\theta)\big(w_2^2(\theta) - \gamma w_1^2(\theta)\big)$ never vanishes in $(0, \pi/2)$. We then let $L(t) := (t^2 - \gamma)^{-1}$, $\forall t \in \mathbb{R} \setminus \{\gamma\}$. By (12) and the Mean Value Theorem, there exists $\theta_1 \in (0, \pi/2)$ such that

$$L\left(\frac{w_2'(\pi/2)}{w_1'(\pi/2)}\right) = \frac{\int_0^{\pi/2} G'(\theta) w_1^2(\theta)\, d\theta}{\int_0^{\pi/2} G'(\theta)\left[w_2^2(\theta) - \gamma w_1^2(\theta)\right] d\theta} = L\left(\frac{w_2(\theta_1)}{w_1(\theta_1)}\right).$$

Since L is injective in $\mathbb{R}_+$, this implies

$$\frac{w_2'(\pi/2)}{w_1'(\pi/2)} = \frac{w_2(\theta_1)}{w_1(\theta_1)}. \tag{14}$$

On the other hand, by the Sturm-Liouville Theory, the function $\theta \mapsto w_2(\theta)/w_1(\theta)$ is (strictly) monotone. L'Hôpital's Rule yields a contradiction as we let $\theta \to \pi/2$. Therefore, v_1 and v_2 must intersect at least twice. This fact leads to another contradiction by using the Shooting Method (see [8]). Thus, $v_1 = v_2$ in $(0, \pi/2)$.

Remark 4 The method above follows the lines of the proof of Kwong-Li [8]. The main difference is that we use an alternative argument based on the Mean Value Theorem in order to deduce (14). In [8], they have to assume that the exponent q is subcritical.

Proof of Theorem 1.3. It follows from methods developed in [7] and [3]. For simplicity, we shall assume that $a = 1$ and $\partial\Omega \cap B_1 = \partial\mathbb{R}_+^N \cap B_1$. We set

$$w(t, \sigma) = r^{2/(q-1)} u(r, \sigma), \quad t = \log(1/r) \in (0, \infty) \times S_+^{N-1} := Q.$$

Then, w satisfies

$$w_{tt} - \left(N - 2\frac{q+1}{q-1}\right) w_t + \Delta' w + \ell_{N,q} w + w^q = 0 \quad \text{in } Q \tag{15}$$

and w vanishes on $(0, \infty) \times \partial S_+^{N-1}$. Since w is uniformly bounded on Q, standard *a priori* estimates for elliptic problems yield

$$\left|\partial_t^k \nabla'^j w\right| \leq M_{k,j} \quad \text{in } (1, \infty) \times S_+^{N-1}$$

for any integers $k, j \geq 0$, where ∇'^j stands for the covariant derivative on S^{N-1}. Thus, the trajectory $\mathcal{T}_w = \{w(t, \cdot) : t \geq 1\}$ is relatively compact in $C^2\big(\overline{S_+^{N-1}}\big)$. Multiplying (15) by w_t and integrating over S_+^{N-1}, we obtain

$$\frac{d}{dt} H(t) = \left(N - 2\frac{q+1}{q-1}\right) \int_{S_+^{N-1}} w_t^2 \, d\sigma, \tag{16}$$

where

$$H(t) := \frac{1}{2} \int_{S_+^{N-1}} \left(w_t^2 - |\nabla' v|^2 - \ell_{N,q} w^2 + \frac{2}{q+1} w^{q+1}\right) d\sigma.$$

Since $q \neq q_2$, we know that $N - 2(q+1)/(q-1) \neq 0$. Thus, iterated energy estimates imply that $w_t(t, \cdot)$, $w_{tt}(t, \cdot) \to 0$ in $L^2(S_+^{N-1})$ as $t \to \infty$. Therefore, the limit set Γ_w of $\mathcal{T}$ is a connected subset of the set of solutions of (6). By Theorem 1.1, we deduce that

$$\Gamma_w = \begin{cases} \{0\} & \text{if } q = q_1 \text{ or } q \geq q_3, \\ \{0\} \text{ or } \{\omega_0\} & \text{if } q_1 < q < q_3. \end{cases}$$

Then, a linearization argument as in [3] leads to the conclusion if $q > q_1$.
We now consider the case $q = q_1$; we borrow some ideas from [1] and [11]. We first prove, by ODE techniques, that

$$X(t) := \int_{S_+^{N-1}} w(t, \cdot) \phi \, d\sigma \leq C t^{-(N-1)/2}. \tag{17}$$

Using (8) and the boundary Harnack inequality (see [5]), we derive

$$0 \leq w(t,\sigma) \leq Ct^{-(N-1)/2} \quad \text{in } (1,\infty) \times S_+^{N-1}. \tag{18}$$

Set $\eta(t,\sigma) := t^{(N-1)/2} w(t,\sigma)$. We verify as above that the limit set Γ_η in $C^2\big(\overline{S_+^{N-1}}\big)$ of the trajectory $\mathcal{T}_\eta$ of η is an interval of the form $\big\{\kappa\phi : 0 \leq \kappa_0 \leq \kappa \leq \kappa_1\big\}$. In order to show that $\mathcal{T}_\eta$ is reduced to a single point, we prove that $\|r(t,\cdot)\|_{L^2} \leq Ct^{-1}$, where

$$r(t,\cdot) := \eta(t,\cdot) - z(t)\phi \quad \text{and} \quad z(t) = \int_{S_+^{N-1}} \eta(t,.)\phi \, d\sigma.$$

Writing the equation satisfied by z as a non-homogeneous second order linear ODE, we prove that either $z(t) \to 0$, which implies that u is continuous at 0, or $z(t) \to \tilde{k}_N$ as $t \to \infty$, for some constant depending only on N.

Proof of Theorem 1.4. It is an application of the Doubling Lemma Method introduced in [10], from which we derive the following local estimate:

Lemma 2.1 *Let $1 < q < q_2$ and let u be a solution of* (3)*. Then, for every $x_0 \in \partial\Omega\backslash\{0\}$ and $0 < R < |x_0|$, we have*

$$0 \leq u(x) \leq C\big(R - |x - x_0|\big)^{-2/(q-1)} \quad \forall x \in B_R(x_0) \cap \Omega, \tag{19}$$

for some constant $C > 0$ depending only on Ω.

Apply this lemma with $x_0 \in \partial\Omega \setminus \{0\}$ and $R = |x_0|/2$. Using elliptic regularity theory, we obtain

$$0 \leq u(x) \leq C\rho(x)\,|x|^{-2/(q-1)-1} \quad \forall x \in \Omega \text{ such that } 0 < \rho(x) < |x|/2.$$

If $\rho(x) \geq |x|/2$, then we use Gidas-Spruck's internal estimates (see [7]). We thus obtain (10).

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