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RELAXATION APPROXIMATION OF THE KERR MODEL FOR THE IMPEDANCE INITIAL-BOUNDARY VALUE PROBLEM

Gilles Carbou, Bernard Hanouzet

Mathématiques Appliquées de Bordeaux
UMR 5466, CNRS
Université Bordeaux 1
351, cours de la Libération
33405 Talence cedex, France

Abstract. The Kerr-Debye model is a relaxation of the nonlinear Kerr model in which the relaxation coefficient is a finite response time of the nonlinear material. We establish the convergence of the Kerr-Debye model to the Kerr model when this relaxation coefficient tends to zero.

1. **Physical context.** In this paper we study different models for the electromagnetic waves propagation in an isotropic nonlinear material (a cristal for example). The wave propagation is described by Maxwell's equations

$$\begin{aligned} &\partial_t D - \text{curl } H = 0,\\ &\partial_t B + \text{curl } E = 0,\\ &\text{div } D = \text{div } B = 0. \end{aligned}$$

where E is the electric field, H is the magnetic field, D is the electric displacement and B is the magnetic induction. Once nondimensionalized, the constitutive relations for a nonlinear Kerr medium are given by

$$B = H \text{ and } D = E + P,$$

where P is the nonlinear polarization.

For the Kerr model, the medium exhibits an instantaneous response :

$$P = |E|^2 E.$$

For the Kerr-Debye model, the medium exhibits a finite response time ε :

$$P = \chi E,$$

where

$$\partial_t \chi + \frac{1}{\varepsilon}\chi = \frac{1}{\varepsilon}|E|^2$$

(see [10, 13]).

Formally, when ε tends to zero, χ tends to $|E|^2$, that is the Kerr-Debye model is a relaxation approximation of the Kerr model.

Concerning the Cauchy problem with initial data (D_0, H_0, χ_0) satisfying div $D_0 =$ div $H_0 = 0$ and $\chi_0 \geq 0$, the convergence for the smooth solutions is proved by [6] using the results of [11]. Generally a boundary layer in time appears because of the non compatibility of the initial data with the equilibrium condition $\chi = |E|^2$.

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In order to describe realistic physical situations it is more convenient to study the initial boundary value problem. We denote by $\Omega = I\!R^+ \times I\!R^2$ the domain in which the nonlinear material is confined, and by $\Gamma = \{0\} \times I\!R^2$ its boundary. We consider the Kerr and the Kerr-Debye models in the domain $I\!R_t^+ \times \Omega$ with the impedance boundary condition on $I\!R_t^+ \times \Gamma$ and with null initial data.

In this case the Kerr model, denoted by (K), becomes, for $(t,x) \in R^+ \times \Omega$

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \end{cases} \tag{1}$$

with the constitutive relation :

$$D = (1 + |E|^2)E. \tag{2}$$

We suppose that the initial data vanishes :

$$D(0,x) = H(0,x) = 0 \text{ for } x \in \Omega \tag{3}$$

so that we obtain the conservative relations

$$\text{div } D = \text{div } H = 0. \tag{4}$$

We denote by $n = {}^t(-1,0,0)$ the outer unit normal on Γ. We consider the impedance boundary condition

$$H \wedge n + a((E \wedge n) \wedge n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma, \tag{5}$$

where a is a positive endomorphism acting on Γ.

The initial boundary value problem for the Kerr-Debye model (KD) writes, for $(t,x) \in R^+ \times \Omega$

$$\begin{cases} \partial_t D_\varepsilon - \text{curl } H_\varepsilon = 0, \\ \partial_t H_\varepsilon + \text{curl } E_\varepsilon = 0, \\ \partial_t \chi_\varepsilon = \frac{1}{\varepsilon}(|E_\varepsilon|^2 - \chi_\varepsilon), \end{cases} \tag{6}$$

with the constitutive relation :

$$D_\varepsilon = (1 + \chi_\varepsilon)E_\varepsilon. \tag{7}$$

We suppose that the initial data vanishes :

$$D_\varepsilon(0,x) = H_\varepsilon(0,x) = 0,\ \chi_\varepsilon(0,x) = 0 \text{ for } x \in \Omega, \tag{8}$$

and we have also

$$\text{div } D_\varepsilon = \text{div } H_\varepsilon = 0. \tag{9}$$

In addition we suppose that we have the same impedance boundary condition

$$H_\varepsilon \wedge n + a((E_\varepsilon \wedge n) \wedge n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma. \tag{10}$$

Two dimensional models

Following [12] we can also introduce the two-dimensional transverse magnetic (TM) and transverse electric (TE) models.

For the transverse magnetic case we assume that

$$\begin{aligned} H(x_1,x_2,x_3) &= {}^t(0, H_2(x_1,x_3), 0), \\ E(x_1,x_2,x_3) &= {}^t(E_1(x_1,x_3), 0, E_3(x_1,x_3)), \end{aligned} \tag{11}$$

in the domain $(x_1,x_3) \in \{x_1 > 0\} \times I\!R$. The Maxwell system becomes

$$\begin{cases} \partial_t D_1 + \partial_3 H_2 = 0, \\ \partial_t D_3 - \partial_1 H_2 = 0, \\ \partial_t H_2 + \partial_3 E_1 - \partial_1 E_3 = 0, \end{cases} \tag{12}$$

with the divergence conservation condition

$$\partial_1 D_1 + \partial_3 D_3 = 0 \tag{13}$$

(in this case the divergence condition for H is irrelevant). The impedance boundary condition writes

$$H_2 - aE_3 = \varphi \text{ with } a \geq 0. \tag{14}$$

This system is coupled with (2) for the Kerr model and with (7) and the third equation in (6) for the Kerr-Debye model.

In the transverse electric case, we assume that

$$\begin{array}{l} E(x_1, x_2, x_3) = {}^t(0, E_2(x_1, x_3), 0), \\ H(x_1, x_2, x_3) = {}^t(H_1(x_1, x_3), 0, H_3(x_1, x_3)). \end{array} \tag{15}$$

We obtain

$$\begin{cases} \partial_t D_2 - \partial_3 H_1 + \partial_1 H_3 = 0, \\ \partial_t H_1 - \partial_3 E_2 = 0, \\ \partial_t H_3 + \partial_1 E_2 = 0, \end{cases} \tag{16}$$

with the divergence conservation condition

$$\partial_1 H_1 + \partial_3 H_3 = 0, \tag{17}$$

and the impedance boundary condition becomes

$$H_3 + aE_2 = \varphi \text{ with } a \geq 0. \tag{18}$$

In the case of a fixed finite response time, numerical simulations are obtained for these two-dimensional models by finite-difference methods in [12] and by finite-element methods in [7].

One dimensional model

In [1] the one dimensionnal model is introduced :

$$\begin{array}{l} E(x_1, x_2, x_3) = {}^t(0, e(x_1), 0), \\ H(x_1, x_2, x_3) = {}^t(0, 0, h(x_1)). \end{array} \tag{19}$$

In this case the Maxwell system becomes :

$$\begin{cases} \partial_t d + \partial_1 h = 0, \\ \partial_t h + \partial_1 e = 0, \end{cases} \tag{20}$$

with the impedance boundary condition

$$h(t, 0) + ae(t, 0) = \varphi(t), \; a \geq 0. \tag{21}$$

We can also remark that the divergence conditions on h and d are irrelevant.

2. Mathematical Properties.

Properties of the Kerr model:

We recall the initial-boundary value problem for the general Kerr model.

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \\ D = (1 + |E|^2)E, \end{cases} \tag{22}$$

for $(t, x) \in I\!R^+ \times \Omega$,

$$D(t = 0) = E(t = 0) = 0 \text{ for } x \in \Omega, \tag{23}$$

$$H \wedge n + a((E \wedge n) \wedge n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma. \tag{24}$$

The energy density given by

$$\mathcal{E}_K(E,H) = \frac{1}{2}(|E|^2 + |H|^2 + \frac{3}{2}|E|^4) \tag{25}$$

is a strictly convex entropy, so (22) is a quasilinear hyperbolic symmetrizable system.

In the three dimensional case, the eigenvalues are, for $\xi \neq 0$,

$$\lambda_1(E,\xi) \leq \lambda_2(E,\xi) < \lambda_3 = \lambda_4 = 0 < \lambda_5 = -\lambda_2 \leq \lambda_6 = -\lambda_1,$$

so the boundary $I\!R^+ \times \Gamma$ is characteristic of constant multiplicity equal to two.

In the two dimensional cases, TM and TE, the eigenvalues are of the form:

$$\lambda_1(E,\xi) < \lambda_2 = 0 < \lambda_3 = -\lambda_1,$$

so the boundary $I\!R^+ \times \Gamma$ is characteristic of constant multiplicity equal to one.

In the one dimensional case, the system is strictly hyperbolic and the boundary is non characteristic. We have

$$\lambda_1(E) < 0 < \lambda_2 = -\lambda_1.$$

The impedance boundary condition (24) is maximal dissipative thus we can apply the existence results in [9] and also the more general results in [4]. We precise these results in the 3-d case. We assume that the source term φ is compactly supported in $I\!R_t^+ \times \Gamma$. We denote by H^s the classical Sobolev spaces and we suppose that φ belongs to $H^s(I\!R_t \times \Gamma)$ for s great enough. So the boundary condition (24) and the initial data (23) match one each other and we obtain smooth local solutions.

Proposition 1. *Under the previous assumptions there exists a maximal smooth solution (E,H) for the IBVP (22)-(23)-(24) which lifespan is denoted by T^* and such that*

$$\partial_t^i(E,H) \in \mathcal{C}^0([0,T^*[; H^{3-i}(\Omega)) \text{ for } i = 0,1,2,3. \tag{26}$$

We have analogous results in the 2-d and the 1-d cases.

Properties of the Kerr-Debye models.

These models write, for $(t,x) \in I\!R^+ \times \Omega$

$$\begin{cases} \partial_t D_\varepsilon - \text{curl } H_\varepsilon = 0, \\ \partial_t H_\varepsilon + \text{curl } E_\varepsilon = 0, \\ \partial_t \chi_\varepsilon = \frac{1}{\varepsilon}(|E_\varepsilon|^2 - \chi_\varepsilon), \\ D_\varepsilon = (1+\chi_\varepsilon)E_\varepsilon. \end{cases} \tag{27}$$

$$D_\varepsilon(0,x) = H_\varepsilon(0,x) = 0,\ \chi_\varepsilon(0,x) = 0 \text{ for } x \in \Omega, \tag{28}$$

$$H_\varepsilon \wedge n + a((E_\varepsilon \wedge n) \wedge n) = \varphi \text{ for } (t,x) \in I\!R^+ \times .\Gamma, \tag{29}$$

By the third equation in (27) we observe that we have

$$\chi_\varepsilon \geq 0. \tag{30}$$

The energy density given by

$$\mathcal{E}_{KD}(E,H,\chi) = \frac{1}{2}(1+\chi)^{-1}|D|^2 + \frac{1}{2}|H|^2 + \frac{1}{4}\chi^2 \tag{31}$$

is a strictly convex entropy in the domain $\{\chi \geq 0\}$. So (27) is a quasilinear symmetrizable hyperbolic system.

In the three dimensional case the eigenvalues are, for $\xi \neq 0$,

$$\mu_1(E,\chi,\xi) = \mu_2 < \mu_3 = \mu_4 = \mu_5 = 0 < \mu_6 = \mu_7 = -\mu_1,$$

so the boundary is characteristic of constant multiplicity equal to three.

In the two dimensional cases we obtain

$$\mu_1(E,\chi,\xi) < \mu_2 = \mu_3 = 0 < \mu_4 = -\mu_1,$$

so the boundary is characteristic of constant multiplicity equal to two.

In the one dimensional case, the system is strictly hyperbolic and the boundary is characteristic of constant multiplicity one. The eigenvalues are

$$\mu_1(E,\chi) < \mu_2 = 0 < \mu_3 = -\mu_1.$$

The impedance boundary condition (29) is maximal dissipative and we apply the existence results of [4].

Proposition 2. *Under the previous assumptions there exists a maximal smooth solution $(D_\varepsilon, H_\varepsilon, \chi_\varepsilon)$ for the IBVP (27)-(28)-(29) which lifespan is denoted by T^*_ε and such that*

$$\partial_t^i(D_\varepsilon, H_\varepsilon, \chi_\varepsilon) \in \mathcal{C}^0([0,T^*_\varepsilon[; H^{3-i}(\Omega)) \text{ for } i = 0,1,2,3. \tag{32}$$

If for a fixed ε we have $T^*_\varepsilon < +\infty$ the behaviour of the solution is described in [4] extending the results of Majda [8] on the general quasilinear hyperbolic systems. In the one dimensional case, the Kerr-Debye model behaves like a semilinear system. We prove in [2] that it does not exhibit schock waves: if the gradient of the solution blows up, the solution itself blows up:

$$\sup_{[0,T^*_\varepsilon[} \left(\|d_\varepsilon\|_{L^\infty(I\!R^+)} + \|h_\varepsilon\|_{L^\infty(I\!R^+)} + \|\chi_\varepsilon\|_{L^\infty(I\!R^+)} \right) = +\infty. \tag{33}$$

3. **Convergence Results.** In order to prove the convergence results it is more convenient to use the entropic variables as it is proposed in [5]. These variables are obtained taking the gradient of the convex entropy (31).

$$\begin{cases} \partial_D \mathcal{E}_{KD} = (1+\chi)^{-1} D = E, \\ \partial_H \mathcal{E}_{KD} = H, \\ \partial_\chi \mathcal{E}_{KD} = \dfrac{1}{2}(\chi - |E|^2) := v, \end{cases}$$

The IBVP (27) (28) (29) becomes

$$A_0(W_\varepsilon)\partial_t W_\varepsilon + \sum_{j=1}^{3} A_j \partial_j W_\varepsilon = \frac{1}{\varepsilon} Q(W_\varepsilon) \tag{34}$$

for $(t,x) \in I\!R^+ \times \Omega$, where

- $W_\varepsilon = \begin{pmatrix} E_\varepsilon \\ H_\varepsilon \\ v_\varepsilon \end{pmatrix}$
- $A_0(W_\varepsilon) = \begin{pmatrix} (|E_\varepsilon|^2 + 2v_\varepsilon + 1)I_3 + 2E_\varepsilon\, {}^tE_\varepsilon & 0 & 2E_\varepsilon \\ 0 & I_3 & 0 \\ 2\, {}^tE_\varepsilon & 0 & 2 \end{pmatrix}$

- $\displaystyle\sum_{j=1}^{3} A_j\partial_j = \begin{pmatrix} 0 & -\text{curl} & 0 \\ \text{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q(W_\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ -2v_\varepsilon \end{pmatrix}$

with the initial data

$$E_\varepsilon(0,x) = H_\varepsilon(0,x) = 0,\ v_\varepsilon(0,x) = 0 \text{ for } x \in \Omega, \tag{35}$$

and with the boundary condition

$$H_\varepsilon \wedge n + a((E_\varepsilon \wedge n) \wedge n) = \varphi \text{ for } (t,x) \in I\!R^+ \times \Gamma. \tag{36}$$

We observe that the boundary condition is linear for the variables (E, H). In addition the equilibrium manifold $\Big\{(D,H,\chi),\ \chi = |E|^2\Big\}$ is linearized as $\Big\{(E,H,v),\ v = 0\Big\}$ and the relaxation term is linear.

For the three dimensional case the main convergence results are the two following theorems (see [3]):

Theorem 1. *There exist $\tilde{T} > 0$ and a constant $K > 0$ such that for all $\varepsilon > 0$, $T^*_\varepsilon \geq \tilde{T}$ and the solution W_ε of the (KD) boundary value problem (34)- (35)- (36) satisfies*

$$\|\partial_t^i W_\varepsilon\|_{\mathcal{C}^0([0,\tilde{T}];H^{3-i}(I\!R^+))} \leq K \text{ for } i = 0,1,2,3, \tag{37}$$

$$\|\partial_t^i v_\varepsilon\|_{\mathcal{C}^0([0,\tilde{T}];H^{2-i}(I\!R^+))} \leq K\varepsilon \text{ for } i = 0,1,2. \tag{38}$$

Theorem 1 shows the strong convergence of $v_\varepsilon = \dfrac{1}{2}(\chi_\varepsilon - |E_\varepsilon|^2)$ to zero. The convergence of $(E_\varepsilon, H_\varepsilon)$ to the solution (E, H) of the IBVP (1)-(2)-(3)-(5) is described in the following statement:

Theorem 2. *For $T \leq \tilde{T}$ and $T < T^*$, there exists a constant $K > 0$ such that for all $\varepsilon > 0$,*

$$\|(E_\varepsilon, H_\varepsilon) - (E,H)\|_{\mathcal{C}^0([0,T];H^1(I\!R^+))} \leq K\varepsilon. \tag{39}$$

In our study we remark that no boundary layer appears in the time variable because the null initial data belongs to the equilibrium manifold defined by

$$\mathcal{V} = \Big\{(D,H,\chi) \text{ such that } \chi - (1+\chi)^{-2}|D|^2 = 2v = 0\Big\}.$$

For the space variable, we have the same boundary condition for the system (K) and for the system (KD), so no space boundary layer appears again.

For the one dimensionnal case the analogous results are proved in [1]. This case is simpler because we can use the semi-linear behaviour of the K-D model (see (33)) and because the boundary is characteristic of multiplicity one. In the 3-d case, the boundary is characteristic of multiplicity three, so we have to use the conservation conditions (9). The second one div $H_\varepsilon = 0$ is linear, but the first one div $D_\varepsilon = 0$ is nonlinear in the entropic variables, which entails additional technical difficulties. We guess that in the 2-d case, the TE case is similar to the 1-d case, and the TM case is analogous to the 3-d case.

4. **Sketch of the proof for the 3-d case.** Replacing the magnetic field H_ε by $H_\varepsilon + \varphi(t,x_2,x_3)\eta(x_1)$, where η is a cut off function we are led to study the following homogeneous IBVP:

$$A_0(W_\varepsilon)\partial_t W_\varepsilon + \sum_{j=1}^{3} A_j\partial_j W_\varepsilon = \frac{1}{\varepsilon}Q(W_\varepsilon) + G \text{ for } (t,x) \in I\!R^+ \times \Omega, \tag{40}$$

$$W_\varepsilon(0,x) = 0 \text{ for } x \in \Omega, \tag{41}$$

$$H_\varepsilon \wedge n + a((E_\varepsilon \wedge n)\wedge n) = 0 \text{ for } (t,x) \in I\!R^+ \times \Gamma, \tag{42}$$

So the conservation properties are preserved:

$$\text{div } ((|E_\varepsilon|^2 + 2v_\varepsilon + 1)E_\varepsilon) = \text{div } H_\varepsilon = 0. \tag{43}$$

The tangential derivatives $\partial_t = \partial_0,\ \partial_2,\ \partial_3$ and the normal derivative ∂_1 are estimated by different ways. The tangential derivatives are measured by ψ_ε:

$$\psi_\varepsilon(t) = \left(\|W_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i\neq 1} \|\partial_i W_\varepsilon\|^2_{L^2(\Omega)} + ... + \sum_{i,j,k\neq 1} \|\partial_{ijk} W_\varepsilon(t)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{44}$$

The normal derivatives are measured by λ_ε:

$$\lambda_\varepsilon(t) = \left(\|\partial_1 W_\varepsilon\|^2_{L^2(\Omega)} + \sum_i \|\partial_{1i} W_\varepsilon(t)\|^2_{L^2(\Omega)} + \sum_{i,j} \|\partial_{1ij} W_\varepsilon(t)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{45}$$

The source term G is measured by γ:

$$\gamma(t) = \left(\|G\|^2_{L^2(\Omega)} + \ldots + \sum_{i,j} \|\partial_{ij} G\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{46}$$

We also introduce :

$$\Psi_\varepsilon(t) = \sup_{[0,t]} \psi_\varepsilon, \quad \Lambda_\varepsilon(t) = \sup_{[0,t]} \lambda_\varepsilon, \quad \Gamma(t) = \sup_{[0,t]} \gamma. \tag{47}$$

Let us introduce T^1_ε defined by

$$T^1_\varepsilon = \max\left\{t \le T^*_\varepsilon, \Gamma(t) \le 1,\ \Psi_\varepsilon(t) \le 1,\ \Lambda_\varepsilon(t) \le 1\right\}. \tag{48}$$

Variational estimates for the tangential derivatives

Taking the inner product of (40) with W_ε we first obtain the L^2 estimate on $[0,T^1_\varepsilon]$:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega A_0(W_\varepsilon)W_\varepsilon \cdot W_\varepsilon + \frac{1}{\varepsilon}\int_\Omega |v_\varepsilon|^2 \le K. \tag{49}$$

Let us remark that we have used the dissipative property of the boundary condition (42):

$$\int_\Omega \sum_{j=1}^3 A_j \partial_j W_\varepsilon \cdot W_\varepsilon = \int_\Gamma a E^T_\varepsilon \cdot E^T_\varepsilon \ge 0,$$

where $E^T_\varepsilon = {}^t(E^2_\varepsilon, E^3_\varepsilon)$.

We can derivate the IBVP with respect to the tangential derivatives $\partial_i,\ i \neq 1$, We obtain the same initial data and same boundary condition for $\partial_i W$ and the same form for the relaxation term . Using also the uniform elliptic relation

$$C_1|\xi|^2 \le A_0(W)\xi\cdot\xi \le C_2|\xi|^2,$$

we obtain the estimation on ψ_ε: there exists K_1 such that for all ε and for $t \in [0, T_\varepsilon^1]$ we have

$$\frac{d}{dt}\psi_\varepsilon^2 + \frac{2}{\varepsilon}\int_\Omega (|v_\varepsilon|^2 + \ldots + |\partial_{ijk} v_\varepsilon|^2) \leq K_1. \tag{50}$$

Estimates for the normal derivatives

In order to estimate v_ε we solve the last equation in (40) using Duhamel formula:

$$v_\varepsilon(t,x) = -\frac{1}{2}|E_\varepsilon|^2(t,x) + \frac{1}{2\varepsilon}\int_0^t \exp\frac{s-t}{\varepsilon}|E_\varepsilon|^2(s,x)ds,$$

and from (50) we obtain that

$$\sum_{i=0}^{3} \|\partial_t^i v_\varepsilon\|_{H^{3-i}(\Omega)} \leq K(\Psi_\varepsilon + \Lambda_\varepsilon)^2. \tag{51}$$

We estimate the normal derivatives using either the equation (40) or the divergence equations (43). For example, from curl $E_\varepsilon = G_2 - \partial_t H_\varepsilon$ we deduce that

$$\|\partial_1 E_\varepsilon^2\|_{L^2(\Omega)} \leq K\psi + \gamma.$$

From div $E_\varepsilon = -\text{div}\left((|E_\varepsilon|^2 + 2v_\varepsilon)E_\varepsilon\right)$ we obtain

$$\|\partial_1 E_\varepsilon^1\|_{L^2(\Omega)} \leq K\psi_\varepsilon + K(\psi_\varepsilon + \lambda_\varepsilon)^2.$$

To sum up we obtain the following estimates for the normal derivatives : there exists a constant K_2 such that on $[0, T_\varepsilon^1]$,

$$\Lambda_\varepsilon(t) \leq K_2(\Psi_\varepsilon(t) + \Gamma(t)) + K_2\Lambda_\varepsilon^2(t). \tag{52}$$

End of the proof.

Integrating (49) we obtain that on $[0, T_\varepsilon^1]$

$$\Psi_\varepsilon^2(t) \leq K_1 t. \tag{53}$$

Let us define the polynomial map $P_\delta(\xi) = K_2\xi^2 - \xi + K_2\delta$. If $\delta \leq \dfrac{1}{2K_2}$, the smallest rooth of P_δ is less than 1. So while $\Gamma(t) \leq \frac{1}{4K_2}$ and $\Psi_\varepsilon(t) \leq \frac{1}{4K_2}$, then $\Lambda_\varepsilon(t) \leq 1$. We fix $\widetilde{T_1} > 0$ such that

$$\forall\, t \leq \widetilde{T_1},\ \Gamma(t) \leq \frac{1}{4K_2},$$

and we introduce $\widetilde{T_2} = \dfrac{1}{16K_1K_2^2}$. By (53) we have that

$$\forall\, t \leq \widetilde{T_2},\ \Psi_\varepsilon(t) \leq \frac{1}{4K_2}.$$

Then we obtain that $T_\varepsilon^1 \geq \widetilde{T} = \min\{\widetilde{T_1}, \widetilde{T_2}\}$, and we deduce (37).

The estimate (38) is obtained by

$$v_\varepsilon(t) = \int_0^t \exp\frac{s-t}{\varepsilon} E_\varepsilon \partial_t E_\varepsilon(s,x)ds.$$

Theorem 2 is a straightforward corollary of Theorem 1.

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E-mail address: carbou@math.u-bordeaux1.fr, hanouzet@math.u-bordeaux1.fr