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DECAY AND NON-DECAY OF THE LOCAL ENERGY FOR THE WAVE EQUATION ON THE DE SITTER–SCHWARZSCHILD METRIC

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Abstract. We describe an expansion of the solution of the wave equation on the De Sitter–Schwarzschild metric in terms of resonances. The principal term in the expansion is due to a resonance at 0. The error term decays polynomially if we permit a logarithmic derivative loss in the angular directions and exponentially if we permit an ε derivative loss in the angular directions.

1. Introduction

There has been important progress in the question of local energy decay for the solution of the wave equation in black hole type space-times over the last years. The best results are now known in the Schwarzschild space-time. We refer to the papers of Blue–Soffer [4], Blue–Sterbenz [5], and Dafermos–Rodnianski [12] and references therein for an overview. See also the paper of Finster–Kamran–Smoller–Yau for the Kerr space-time [13]. Results on the decay of local energy are believed to be a prerequisite for a possible proof of the global nonlinear stability of these space-times. Today global nonlinear stability is only known for the Minkowski space-time (see [11]).

From our point of view one of the most efficient approaches to the question of local energy decay is the theory of resonances. Resonances correspond to the frequencies and rates of dumping of signals emitted by the black hole in the presence of perturbations (see [9, Chapter 4.35]). On the one hand these resonances are today an important hope of effectively detecting the presence of a black hole as we are theoretically able to measure the corresponding gravitational waves. On the other hand, the distance of the resonances to the real axis reflects the stability of the system under the perturbation: larger distances correspond to more stability. In particular the knowledge of the localization of resonances gives precise informations about the decay of the local energy and its rate. The aim of the present paper is to show how this method applies to the simplest model of a black hole: the De Sitter–Schwarzschild black hole.

In the euclidean space, such results are already known, especially for non trapping geometries. The first result is due to Lax and Phillips (see their book [15, Theorem III.5.4]). They have proved that the cut-off propagator associated to the wave equation outside an obstacle in odd dimension ≥ 3 (more precisely the Lax–Phillips semi-group $Z(t)$) has an expansion in terms of resonances if $Z(T)$ is compact for a given T. In particular, there is a uniform exponential decay of the local energy. From Melrose–Sjöstrand [18], this assumption is true

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for non trapping obstacles. Vaĭnberg [32] has obtained such results for general, non trapping, differential operators using different techniques. In the trapping case, we know, by the work of Ralston [21], that it is not possible to obtain uniform decay estimates without loss of derivatives. In the exterior domain of two strictly convex obstacles, the local energy decays exponentially with a loss of derivatives, by the work of Ikawa [14]. This situation is close to the one treated in this paper. We also mention the works Tang–Zworski [30] and Burq–Zworski [8] concerning the resonances close to the real line and the work of Christiansen–Zworski [10] for the wave equation on the modular surface and on the hyperbolic cylinder.

Thanks to the work of Sá Barreto and Zworski ([24]) we have a very good knowledge of the localization of resonances for the wave equation on the De Sitter–Schwarzschild metric. Using their results we can describe an expansion of the solution of the wave equation on the De Sitter–Schwarzschild metric in terms of resonances. The main term in the expansion is due to a resonance at 0. The error term decays polynomially if we permit a logarithmic derivative loss in the angular directions and exponentially if we permit an ε derivative loss in the angular directions. For initial data in the complement of a one-dimensional space the local energy is integrable if we permit a $(\ln\langle -\Delta_\omega\rangle)^\alpha$ derivative loss with $\alpha > 1$. This estimate is almost optimal in the sense that it becomes false for $\alpha < \frac{1}{2}$.

The method presented in this paper does not directly apply to the Schwarzschild case. This is not linked to the difficulty of the photon sphere which we treat in this paper, but to the possible accumulation of resonances at the origin in the Schwarzschild case.

The exterior of the De Sitter–Schwarzschild black hole is given by

$$(\mathcal{M}, g), \quad \mathcal{M} = \mathbb{R}_t \times X \text{ with } X =]r_-, r_+[_r \times \mathbb{S}^2_\omega \tag{1.1}$$

$$g = \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2 d\omega^2, \quad \alpha = \Big(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\Big)^{1/2}, \tag{1.2}$$

where $M > 0$ is the mass of the black holes and $\Lambda > 0$ with $9M^2\Lambda < 1$ is the cosmological constant. r_- and r_+ are the two positive roots of $\alpha = 0$. We also denoted by $d\omega^2$ the standard metric on $\mathbb{S}^2$.

The corresponding d'Alembertian is

$$\Box_g = \alpha^{-2}\big(D_t^2 - \alpha^2 r^{-2} D_r (r^2\alpha^2) D_r + \alpha^2 r^{-2}\Delta_\omega\big), \tag{1.3}$$

where $D_\bullet = \frac{1}{i}\partial_\bullet$ and $-\Delta_\omega$ is the positive Laplacian on $\mathbb{S}^2$. We also denote

$$\widehat{P} = \alpha^2 r^{-2} D_r (r^2\alpha^2) D_r - \alpha^2 r^{-2}\Delta_\omega,$$

the operator on X which governs the situation on $L^2(X, r^2\alpha^{-2} dr\, d\omega)$. We define

$$P = r\widehat{P}r^{-1},$$

on $L^2(X, \alpha^{-2} dr\, d\omega)$, and, in the coordinates (r, ω), we have

$$P = \alpha^2 D_r(\alpha^2 D_r) - \alpha^2 r^{-2}\Delta_\omega + r^{-1}\alpha^2(\partial_r \alpha^2).$$

We introduce the Regge–Wheeler coordinate given by

$$x'(r) = \alpha^{-2}. \tag{1.4}$$

In the coordinates (x, ω), the operator P is given by

$$P = D_x^2 - \alpha^2 r^{-2}\Delta_\omega + \alpha^2 r^{-1}(\partial_r \alpha^2), \tag{1.5}$$

on $L^2(\mathbb{R}\times\mathbb{S}^2, dx\, d\omega)$. Let $V = \alpha^2 r^{-2}$ and $W = \alpha^2 r^{-1}(\partial_r\alpha^2)$ be the potentials appearing in P. As stated in Proposition 2.1 of [24], the work of Mazzeo–Melrose [17] implies that for $\chi \in C_0^\infty(\mathbb{R})$

$$R_\chi(\lambda) = \chi(P-\lambda^2)^{-1}\chi,$$

has a meromorphic extension from the upper half plane to $\mathbb{C}$, whose poles λ are called resonances. The set of the resonances is denoted by $\operatorname{Res} P$. We recall the main result of [24]:

Theorem 1.1 (Sá Barreto–Zworski). *There exist $K > 0$ and $\theta > 0$ such that for any $C > 0$ there exists an injective map, $\tilde{b}$, from the set of pseudo-poles*

$$\frac{(1-9\Lambda M^2)^{\frac12}}{3^{\frac32}M}\left(\pm\mathbb{N}\pm\frac12 - i\frac12\left(\mathbb{N}_0+\frac12\right)\right),$$

into the set of poles of the meromorphic continuation of $(P-\lambda^2)^{-1} : L^2_{\rm comp} \to L^2_{\rm loc}$ such that all the poles in

$$\Omega_C = \{\lambda;\ \operatorname{Im}\lambda > -C,\ |\lambda| > K,\ \operatorname{Im}\lambda > -\theta|\operatorname{Re}\lambda|\},$$

are in the image of $\tilde{b}$ and for $\tilde{b}(\mu) \in \Omega_C$,

$$\tilde{b}(\mu) - \mu \to 0 \quad \textit{as} \quad |\mu| \to \infty.$$

If $\mu = \mu^\pm_{\ell,j} = 3^{-\frac32}M^{-1}(1-9\Lambda M^2)^{\frac12}\big((\pm\ell\pm\frac12)-i\frac12(j+\frac12)\big)$, $\ell\in\mathbb{N}$, $j\in\mathbb{N}_0$, then the corresponding pole, $\tilde{b}(\mu)$, has multiplicity $2\ell+1$.

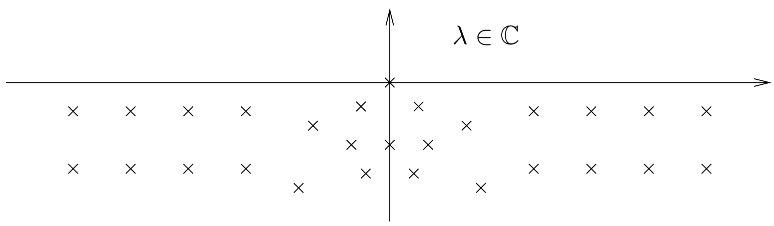

FIGURE 1. The resonances of P near the real axis.

The natural energy space $\mathcal{E}$ for the wave equation is given by the completion of $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\times C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ in the norm

$$\|(u_0,u_1)\|^2_{\mathcal{E}} = \|u_1\|^2 + \langle Pu_0, u_0\rangle. \tag{1.6}$$

It turns out that this is not a space of distributions. The problem is very similar to the problem for the wave equation in dimension 1. We therefore introduce another energy space $\mathcal{E}^{\rm mod}_{a,b}$ $(-\infty < a < b < \infty)$ defined as the completion of $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\times C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ in the norm

$$\|(u_0,u_1)\|^2_{\mathcal{E}^{\rm mod}_{a,b}} = \|u_1\|^2 + \langle Pu_0,u_0\rangle + \int_a^b\int_{\mathbb{S}^2}|u_0(s,\omega)|^2 ds\, d\omega.$$

Note that for any $-\infty < a < b < \infty$ the norms $\mathcal{E}_{a,b}^{\mathrm{mod}}$ and $\mathcal{E}_{0,1}^{\mathrm{mod}}$ are equivalent. We will therefore only work with the space $\mathcal{E}_{0,1}^{\mathrm{mod}}$ in the future and note it from now on $\mathcal{E}^{\mathrm{mod}}$. Let us write the wave equation $\Box_g u = 0$ as a first order system in the following way:

$$\begin{cases} i\partial_t v = Lv \\ v(0) = v_0 \end{cases} \quad \text{with} \quad L = \begin{pmatrix} 0 & i \\ -iP & 0 \end{pmatrix}.$$

Let $\mathcal{H}^k$ be the scale of Sobolev spaces associated to P. We denote $\mathcal{H}_c^2$ the completion of $\mathcal{H}^2$ in the norm $\|u\|_2^2 := \langle Pu, u\rangle + \|Pu\|^2$. Then $(L, D(L) = \mathcal{H}_c^2 \oplus \mathcal{H}^1)$ is selfadjoint on $\mathcal{E}$. We denote $\mathcal{E}^k$ the scale of Sobolev spaces associated to L. Note that because of

$$(L-\lambda)^{-1} = (P-\lambda^2)^{-1} \begin{pmatrix} \lambda & i \\ -iP & \lambda \end{pmatrix}, \tag{1.7}$$

the meromorphic extension of the cut-off resolvent of P entails a meromorphic extension of the cut-off resolvent of L and the resonances of L coincide with the resonances of P.

Recall that $(-\Delta_\omega, H^2(\mathbb{S}^2))$ is a selfadjoint operator with compact resolvent. Its eigenvalues are $\ell(\ell+1)$, $\ell \geq 0$ with multiplicity $2\ell+1$. We denote

$$P_\ell = r^{-1} D_x r^2 D_x r^{-1} + \alpha^2 r^{-2} \ell(\ell+1) \tag{1.8}$$

the operator restricted to $\mathcal{H}_\ell = L^2(\mathbb{R}) \times Y_\ell$ where Y_ℓ is the eigenspace of the eigenvalue $\ell(\ell+1)$. In the following, P_ℓ will be identified with the operator on $L^2(\mathbb{R})$ given by (1.8). The operators L_ℓ and the spaces $\mathcal{E}_\ell$, $\mathcal{E}_\ell^{\mathrm{mod}}$, $\mathcal{E}_\ell^k$ are defined similarly to the operator L and the spaces $\mathcal{E}$, $\mathcal{E}^{\mathrm{mod}}$, $\mathcal{E}^k$. Let Π_ℓ be the orthogonal projector on $\mathcal{E}_\ell^{\mathrm{mod}}$. For $\ell \geq 1$, the space $\mathcal{E}_\ell^{\mathrm{mod}}$ and $\mathcal{E}_\ell$ are the same and the norms are equivalent uniformly with respect to ℓ.

Using Proposition II.2 of Bachelot and Motet-Bachelot [3], the group e^{-itL} preserves the space $\mathcal{E}^{\mathrm{mod}}$ and there exist $C, k > 0$ such that

$$\|e^{-itL} u\|_{\mathcal{E}^{\mathrm{mod}}} \leq C e^{k|t|} \|u\|_{\mathcal{E}^{\mathrm{mod}}}.$$

From the previous discussion, the same estimate holds for L_ℓ with $k = 0$ uniformly in $\ell \geq 1$. In particular, $(L-z)^{-1}$ is bounded on $\mathcal{E}^{\mathrm{mod}}$ for $\operatorname{Im} z > k$, and we denote $\mathcal{E}^{\mathrm{mod},-\mathrm{j}} = (L-z)^j \mathcal{E}^{\mathrm{mod}} \subset \mathcal{D}'(\mathbb{R} \times \mathbb{S}^2) \times \mathcal{D}'(\mathbb{R} \times \mathbb{S}^2)$ for $j \in \mathbb{N}_0$.

We first need a result on P:

Proposition 1.2. *For $\ell \geq 1$, the operator P_ℓ has no resonance and no eigenvalue on the real axis.*

For $\ell = 0$, P_0 has no eigenvalue in $\mathbb{R}$ and no resonance in $\mathbb{R} \setminus \{0\}$. But, 0 is a simple resonance of P_0, and, for z close to 0, we have

$$(P_0 - z^2)^{-1} = \frac{i\gamma}{z} r \langle r | \cdot \rangle + H(z), \tag{1.9}$$

where $\gamma \in]0, +\infty[$ and $H(z)$ is a holomorphic (bounded) operator near 0. Equation (1.9) is an equality between operators from L^2_{comp} to L^2_{loc}.

The proof of Proposition 1.2 is given in Section 2.1. For $\chi \in C_0^\infty(\mathbb{R})$ we denote henceforth:

$$\widehat{R}_\chi(\lambda) = \chi(L-\lambda)^{-1}\chi.$$

For a resonance λ_j we define $m(\lambda_j)$ by the Laurent expansion of the cut-off resolvent near λ_j:

$$\widehat{R}_\chi(\lambda) = \sum_{k=-(m(\lambda_j)+1)}^{\infty} A_k(\lambda-\lambda_j)^k.$$

We also define $\pi_{j,k}^{\chi}$ by

$$\pi_{j,k}^{\chi} = \frac{-1}{2\pi i}\oint \frac{(-i)^k}{k!}\widehat{R}_\chi(\lambda)(\lambda-\lambda_j)^k d\lambda. \tag{1.10}$$

The main result of this paper is the following:

Theorem 1.3. *Let $\chi \in C_0^\infty(\mathbb{R})$.*

(i) Let $0 < \mu \notin \frac{(1-9\Lambda M^2)^{1/2}}{3^{1/2}M}\frac{1}{2}\left(\mathbb{N}_0+\frac{1}{2}\right)$ such that there is no resonance with $\operatorname{Im} z = -\mu$. Then there exists $M > 0$ with the following property. Let $u \in \mathcal{E}^{\mathrm{mod}}$ such that $\langle -\Delta_\omega\rangle^M u \in \mathcal{E}^{\mathrm{mod}}$. Then we have:

$$\chi e^{-itL}\chi u = \sum_{\substack{\lambda_j\in\operatorname{Res}P \\ \operatorname{Im}\lambda_j>-\mu}} \sum_{k=0}^{m(\lambda_j)} e^{-i\lambda_j t} t^k \pi_{j,k}^{\chi} u + E_1(t)u, \tag{1.11}$$

with

$$\|E_1(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\mu t}\|\langle -\Delta_\omega\rangle^M u\|_{\mathcal{E}^{\mathrm{mod}}}, \tag{1.12}$$

and the sum is absolutely convergent in the sense that

$$\sum_{\substack{\lambda_j\in\operatorname{Res}P \\ \operatorname{Im}\lambda_j>-\mu}} \sum_{k=1}^{m(\lambda_j)} \|\pi_{j,k}^{\chi}\langle -\Delta_\omega\rangle^{-M}\|_{\mathcal{L}(\mathcal{E}^{\mathrm{mod}})} \lesssim 1. \tag{1.13}$$

(ii) There exists $\varepsilon > 0$ with the following property. Let $g \in C([0,+\infty[)$, $\lim_{|x|\to\infty} g(x) = 0$, positive, strictly decreasing with $x^{-1} \le g(x)$ for x large. Let $u = (u_1, u_2) \in \mathcal{E}^{\mathrm{mod}}$ be such that $\big(g(-\Delta_\omega)\big)^{-1}u \in \mathcal{E}^{\mathrm{mod}}$. Then we have

$$\chi e^{-itL}\chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + E_2(t)u, \tag{1.14}$$

with

$$\|E_2(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{1.15}$$

Remark 1.4. a) By the results of Sá Barreto and Zworski we know that there exists $\mu > 0$ such that 0 is the only resonance in $\operatorname{Im} z > -\mu$. Choosing this μ in (i) the sum on the right hand side contains a single element which is

$$\gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix}.$$

b) Again by the paper of Sá Barreto and Zworski we know that $\lambda_j = \widetilde{b}(\mu^{\varepsilon}_{\ell,\widetilde{\jmath}})$ for all the λ_j's outside a compact set (see Theorem 1.1). For such λ_j, we have $m_j(\lambda_j) = 0$ and $\pi_{j,k}^{\chi} = \Pi_\ell \pi_{j,k}^{\chi} = \pi_{j,k}^{\chi}\Pi_\ell$ is an operator of rank $2\ell+1$.

c) Let $\mathcal{E}^{\mathrm{mod},\perp} = \{u \in \mathcal{E}^{\mathrm{mod}};\ \langle r, \chi u_2 \rangle = 0\}$. By part (ii) of the theorem, for $u \in \mathcal{E}^{\mathrm{mod},\perp}$, the local energy is integrable if $(\ln\langle -\Delta_\omega\rangle)^\alpha u \in \mathcal{E}^{\mathrm{mod}}$, for some $\alpha > 1$, and decays exponentially if $\langle -\Delta_\omega\rangle^\varepsilon u \in \mathcal{E}^{\mathrm{mod}}$ for some $\varepsilon > 0$.

d) In fact, we can replace $\langle -\Delta_\omega\rangle^M$ by $\langle P\rangle^{2M}$ in the first part of the theorem. And, by an interpolation argument, we can obtain the following estimate: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\chi e^{-itL}\chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + E_3(t)u, \tag{1.16}$$

with

$$\|E_3(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\delta t}\big\|\langle P\rangle^\varepsilon u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{1.17}$$

Remark 1.5. In the Schwarzschild case the potential $V(x)$ is only polynomially decreasing at infinity and we cannot apply the result of Mazzeo–Melrose. Therefore we cannot exclude a possible accumulation of resonances at 0. This difficulty has nothing to do with the presence of the photon sphere which is treated by the method presented in this paper.

Remark 1.6. Let $u \in \mathcal{E}^{\mathrm{mod},\perp}$ be such that $(\ln\langle -\Delta_\omega\rangle)^\alpha u \in \mathcal{E}^{\mathrm{mod}}$ for some $\alpha > 1$. Then we have from part (ii) of the theorem, for $\lambda \in \mathbb{R}$,

$$\Big\|\int_0^\infty \chi e^{-it(L-\lambda)}\chi u\, dt\Big\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim \|(\ln\langle -\Delta_\omega\rangle)^\alpha u\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{1.18}$$

This estimate is almost optimal since it becomes false for $\alpha < \frac{1}{2}$. Indeed we have ($\lambda \in \mathbb{R}$):

$$\widehat{R}_\chi(\lambda)u = i\int_0^\infty \chi e^{-it(L-\lambda)}\chi u\, dt.$$

Thus from (1.18) we obtain the resolvent estimate

$$\|\widehat{R}_\chi(\lambda)(\ln\langle -\Delta_\omega\rangle)^{-\alpha}\|_{\mathcal{L}(\mathcal{E}^{\mathrm{mod},\perp},\mathcal{E}^{\mathrm{mod}})} \lesssim 1.$$

It is easy to see that this entails the resolvent estimate

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi(\ln\langle \ell(\ell+1)\rangle)^{-\alpha}\| \lesssim \frac{1}{|\lambda|},$$

for $\ell \geq 1$. We introduce the semi-classical parameter $h^2 = (\ell(\ell+1))^{-1}$ and $\widetilde{P} = h^2 D_x^2 + V(x) + h^2 W(x)$ as in Section 2.3. Then, for $R > 0$, the above estimate gives the semi-classical estimate:

$$\|\chi(\widetilde{P} - z)^{-1}\chi\| \lesssim \frac{|\ln h|^\alpha}{h},$$

for $1/R \leq z \leq R$ (see (2.25) and (2.26)). Such an estimate is known to be false for $\alpha < \frac{1}{2}$ and $z = z_0$, the maximum value of the potential $V(x)$ (see [1, Proposition 2.2]).

Remark 1.7. Let $\mathcal{P}_1$ be the projection on the first variable, $\mathcal{P}_1(u_1, u_2) = u_1$. If $u \in \mathcal{E}^{\mathrm{mod}}$ is such that $\big(g(-\Delta_\omega)\big)^{-1}(L+i)u \in \mathcal{E}^{\mathrm{mod}}$, then $\mathcal{P}_1\chi e^{-itL}\chi u \in C^0(\mathbb{R}\times\mathbb{S}^2)$ and the remainder term in (1.14) satisfies

$$\|\mathcal{P}_1 E_2(t)u\|_{L^\infty(\mathbb{R}\times\mathbb{S}^2)} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}(L+i)u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{1.19}$$

Moreover, if $u \in \mathcal{E}^{\mathrm{mod}}$ is such that $\big(g(-\Delta_\omega)\big)^{-1}(L+i)^2 u \in \mathcal{E}^{\mathrm{mod}}$, then $\chi e^{-itL}\chi u \in C^0((\mathbb{R}\times \mathbb{S}^2)\times(\mathbb{R}\times\mathbb{S}^2))$ and the remainder term in (1.14) satisfies

$$\|E_2(t)u\|_{L^\infty((\mathbb{R}\times\mathbb{S}^2)\times(\mathbb{R}\times\mathbb{S}^2))} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}(L+i)^2 u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{1.20}$$

The proof of the theorem is based on resolvent estimates. Using (1.7) we see that it is sufficient to prove resolvent estimates for $\chi(P_\ell-\lambda^2)^{-1}\chi$. This is the purpose of the next section.

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2. Estimate for the cut-off resolvent.

In this section, we obtain estimates for the cut-off resolvent of P_ℓ, the operator P restricted to the spherical harmonic ℓ. We will use the description of the resonances given in Sá Barreto–Zworski [24]. Recall that

$$R_\chi(\lambda) = \chi(P-\lambda^2)^{-1}\chi, \tag{2.1}$$

has a meromorphic extension from the upper half plane to $\mathbb{C}$. The resonances of P are defined as the poles of this extension. We treat only the case $\operatorname{Re}\lambda > -1$ since we can obtain the same type of estimates for $\operatorname{Re}\lambda < 1$ using $\big(R_\chi(-\overline{\lambda})\big)^* = R_\chi(\lambda)$.

Theorem 2.1. *Let $C_0 > 0$ be fixed. The operators $\chi(P_\ell-\lambda^2)^{-1}\chi$ satisfy the following estimates uniformly in ℓ.*

i) For all $R>0$, the number of resonances of P is bounded in $B(0,R)$. Moreover, there exists $C>0$ such that

$$\|\chi(P_\ell-\lambda^2)^{-1}\chi\| \leq \|\chi(P-\lambda^2)^{-1}\chi\| \leq C \prod_{\substack{\lambda_j\in \operatorname{Res} P\\ |\lambda_j|<2R}} \frac{1}{|\lambda-\lambda_j|} \tag{2.2}$$

for all $\lambda\in B(0,R)$. As usual, the resonances are counted with their multiplicity.

ii) For R large enough, P_ℓ has no resonance in $[R,\ell/R]+i[-C_0,0]$. Moreover, there exists $C>0$ such that

$$\|\chi(P_\ell-\lambda^2)^{-1}\chi\| \leq \frac{C}{\langle\lambda\rangle^2}, \tag{2.3}$$

for $\lambda\in[R,\ell/R]+i[-C_0,C_0]$.

iii) Let R be fixed. For ℓ large enough, the resonances of P_ℓ in $[\ell/R,R\ell]+i[-C_0,0]$ are the $\widetilde{b}(\mu^+_{\ell,j})$ given in Theorem 1.1 (in particular their number is bounded uniformly in ℓ). Moreover, there exists $C>0$ such that

$$\|\chi(P_\ell-\lambda^2)^{-1}\chi\| \leq C\langle\lambda\rangle^C \prod_{\substack{\lambda_j\in \operatorname{Res} P_\ell\\ |\lambda-\lambda_j|<1}} \frac{1}{|\lambda-\lambda_j|}, \tag{2.4}$$

for $\lambda \in [\ell/R, R\ell] + i[-C_0, C_0]$.

Furthermore, P_ℓ *has no resonance in* $[\ell/R, R\ell] + i[-\varepsilon, 0]$, *for some* $\varepsilon > 0$, *and we have*

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq C\frac{\ln\langle\lambda\rangle}{\langle\lambda\rangle}e^{C|\operatorname{Im}\lambda|\ln\langle\lambda\rangle}, \tag{2.5}$$

for $\lambda \in [\ell/R, R\ell] + i[-\varepsilon, 0]$.

iv) Let $C_1 > 0$ *be fixed. For* R *large enough,* P_ℓ *has no resonance in* $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > R\ell,\ \text{and } 0 \geq \operatorname{Im}\lambda \geq -C_0 - C_1\ln\langle\lambda\rangle\}$. *Moreover, there exists* $C > 0$ *such that*

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq \frac{C}{\langle\lambda\rangle}e^{C|\operatorname{Im}\lambda|}, \tag{2.6}$$

for $\operatorname{Re}\lambda > R\ell$ *and* $C_0 \geq \operatorname{Im}\lambda \geq -C_0 - C_1\ln\langle\lambda\rangle$.

The results concerning the localization of the resonances in this theorem are proved in [3] and [24], the following figure summarizes the different estimates of the resolvent.

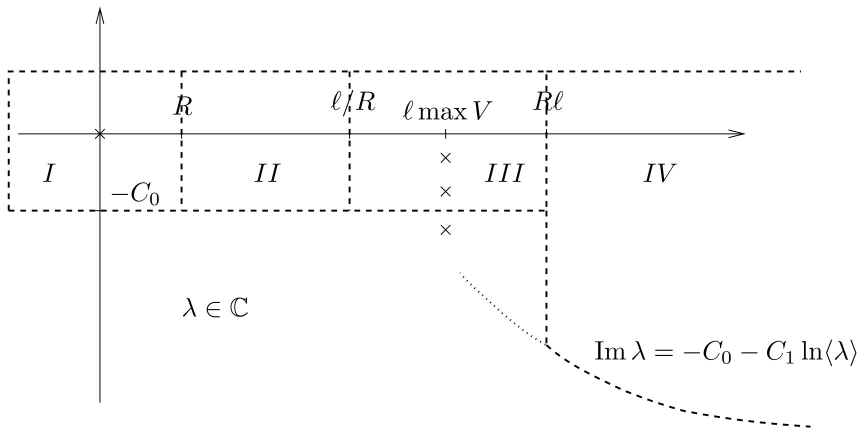

FIGURE 2. The different zones in Theorem 2.1.

In zone I which is compact, the result of Mazzeo–Melrose [17] gives a bound uniform with respect to ℓ (away from the possible resonances). In particular, part *i)* of Theorem 2.1 is a direct consequence of this work.

In zone II, the result of Zworski [33] gives us a good (uniform with respect to ℓ) estimate of the resolvent. Here, we use the exponential decay of the potential at $+\infty$ and $-\infty$. By comparison, the corresponding potential for the Schwarzschild metric does not decay exponentially, and our present work cannot be extended to this setting. Note that this problem concerns only zones I and II, but zones III and IV can be treated in the same way.

In zone III, we have to deal with the so called "photon sphere". The estimate (2.4) follows from a general polynomial bound of the resolvent in dimension 1 (see [6]).

In zone IV, the potentials $\ell(\ell+1)V$ and W are very small in comparison to λ^2. So they do not play any role, and we obtain the same estimate as in the free case of $-\Delta$ (or as for non trapping geometries).

2.1. **Estimate close to** 0.

This part is devoted to the proof of Proposition 1.2 and of part *i)* of Theorem 2.1. Since $\chi(P-\lambda^2)^{-1}\chi$ has a meromorphic extension to $\mathbb{C}$, the number of resonances in $B(0,R)$ is always bounded and point *i)* of Theorem 2.1 is clear. It is a classic result (see Theorem XIII.58 in [22]) that P_ℓ has no eigenvalue in $\mathbb{R}\setminus\{0\}$. On the other hand, from Proposition II.1 of the work of Bachelot and Motet-Bachelot [3], 0 is not an eigenvalue of the operators P_ℓ. Moreover, by the limiting absorption principle [19],

$$\|\langle x\rangle^{-\alpha}(P_\ell-(z+i0))^{-1}\langle x\rangle^{-\alpha}\|<\infty, \tag{2.7}$$

for $z\in\mathbb{R}\setminus\{0\}$ and any $\alpha>1$, we know that P_ℓ has no resonance in $\mathbb{R}\setminus\{0\}$.

We now study the resonance 0 using a technique specific to the one dimensional case. We start by recalling some facts about outgoing Jost solutions. Let

$$Q=-\Delta+\widetilde{V}(x), \tag{2.8}$$

be a Schrödinger operator with $\widetilde{V}\in C^\infty(\mathbb{R})$ decaying exponentially at infinity. For $\operatorname{Im}\lambda>0$, there exists a unique pair of functions $e_\pm(x,\lambda)$ such that

$$\begin{cases}(Q-\lambda^2)e_\pm(x,\lambda)=0\\ \lim\limits_{x\to\pm\infty}\big(e_\pm(x,\lambda)-e^{\pm i\lambda x}\big)=0\end{cases}$$

The function $e_\pm$ is called the outgoing Jost solution at $\pm\infty$. Since $\widetilde{V}\in C^\infty(\mathbb{R})$ decays exponentially at infinity, the functions $e_\pm$ can be extended, as $C^\infty(\mathbb{R})$ functions of x, analytically in a strip $\{\lambda\in\mathbb{C};\ \operatorname{Im}\lambda>-\varepsilon\}$, for some $\varepsilon>0$. Moreover, in such a strip, they satisfy

$$|e_\pm(x,\lambda)-e^{\pm i\lambda x}|=\mathcal{O}(e^{-|x|(\operatorname{Im}\lambda+\delta)})\ \text{for}\ \pm x>0 \tag{2.9}$$

$$|\partial_x e_\pm(x,\lambda)\mp i\lambda e^{\pm i\lambda x}|=\mathcal{O}(e^{-|x|(\operatorname{Im}\lambda+\delta)})\ \text{for}\ \pm x>0, \tag{2.10}$$

for some $\delta>0$. All these properties can be found in Theorem XI.57 of [23].

Using these Jost solutions, the kernel of $(Q-\lambda^2)^{-1}$, for $\operatorname{Im}\lambda>0$ takes the form

$$R(x,y,\lambda)=\frac{1}{w(\lambda)}\big(e_+(x,\lambda)e_-(y,\lambda)H(x-y)+e_-(x,\lambda)e_+(y,\lambda)H(y-x)\big), \tag{2.11}$$

where $H(x)$ is the Heaviside function

$$H(x)=\begin{cases}1\ \text{for}\ x>0\\ 0\ \text{for}\ x\le 0\end{cases},$$

and

$$w(\lambda)=(\partial_x e_-)e_+-(\partial_x e_+)e_-, \tag{2.12}$$

is the wronskian between e_- and e_+ (the right hand side of (2.12) does not depend on x). In particular, $w(\lambda)$ is an analytic function on $\{\lambda\in\mathbb{C};\ \operatorname{Im}\lambda>-\varepsilon\}$. Since the $e_\pm$ are always non-zero thanks to (2.9), the resonances are the zeros of $w(\lambda)$. Such a discussion can be found in the preprint of Tang–Zworski [31].

Remark that P_ℓ is of the form (2.8). If 0 is a resonance of one of the P_ℓ's with $\ell\ge 1$, the Jost solutions $e_\pm(x,0)$ are collinear. In particular, from (2.9) and (2.10), the C^∞ function

$e_+(x,0)$ converge to two non zero limits at $\pm\infty$ and $\partial_x e_+(x,0)$ goes to 0 as $x \to \pm\infty$. Since

$$P_\ell = r^{-1} D_x r^2 D_x r^{-1} + \alpha^2 r^{-2} \ell(\ell+1), \tag{2.13}$$

we get, by an integration by parts,

$$\begin{aligned} 0 =& \int_{-R}^{R} (P_\ell e_+) \overline{e_+} \, dx \\ =& \ell(\ell+1) \int_{-R}^{R} |\alpha r^{-1} e_+|^2 \, dx + \int_{-R}^{R} |r D_x (r^{-1} e_+)|^2 \, dx - \Big[i r^{-1} \overline{e_+} D_x (r^{-1} e_+) \Big]_{-R}^{R}. \end{aligned} \tag{2.14}$$

Since $\partial_x (r^{-1} e_+) = r^{-1} \partial_x e_+ - r^{-2} \alpha^2 e_+$, the last term in (2.14) goes to 0 as R goes to $+\infty$. Thus, if $\ell \geq 1$, (2.14) gives $e_+ = 0$ and 0 is not a resonance of P_ℓ.

We now study the case $\ell = 0$. If $u \in C^2(\mathbb{R})$ satisfies $P_0 u = 0$, we get from (2.13)

$$r^2 D_x r^{-1} u = -i\beta,$$

where $\beta \in \mathbb{C}$ is a constant. Then

$$u(x) = \alpha r(x) + \beta r(x) \int_0^x \frac{1}{r^2(t)} dt,$$

where $\alpha, \beta \in \mathbb{C}$ are constants. Note that

$$\widetilde{r}(x) := r(x) \int_0^x \frac{1}{r^2(t)} dt = \frac{x}{r_\pm} + \mathcal{O}(1),$$

as $x \to \pm\infty$. Since $e_\pm(x,0)$ are C^∞ functions bounded at $\pm\infty$ from (2.9) which satisfy $P_0 u = 0$, the two functions $e_\pm(x,0)$ are collinear to r and then $w(0) = 0$ which means that 0 is a resonance of P_0. The resolvent of P_0 thus has the form

$$(P_0 - \lambda^2)^{-1} = \frac{\Pi_J}{\lambda^J} + \cdots + \frac{\Pi_1}{\lambda} + H(\lambda),$$

where $H(\lambda)$ is an analytic family of bounded operators near 0 and $\Pi_J \neq 0$.

For all $\lambda = i\varepsilon$ with $\varepsilon > 0$, we have

$$\| \lambda^2 (P_0 - \lambda^2)^{-1} \|_{L^2 \to L^2} = \| \varepsilon^2 (P_0 + \varepsilon^2)^{-1} \|_{L^2 \to L^2} \leq 1,$$

from the functional calculus. This inequality implies that $J \leq 2$ and

$$\| \Pi_2 \|_{L^2 \to L^2} \leq 1.$$

If $f(x) \in L^2_{\mathrm{loc}}$ is in the range of Π_2, we have $f \in L^2$ and $P_0 f = 0$. Then, $f \in H^s$ for all s and f is an eigenvector of P_0 for the eigenvalue 0. This is impossible because P_0 has no eigenvalue. Thus $\Pi_2 = 0$ and $J = 1$.

So $w(\lambda)$ has a zero of order 1 at $\lambda = 0$. Since $e_\pm(x,0) = r(x)/r_\pm$, (2.11) implies that the kernel of Π_1 is given by

$$\Pi_1(x,y) = \frac{1}{w'(0) r_+ r_-} r(x) r(y) = i\gamma r(x) r(y). \tag{2.15}$$

Finally, since $i\varepsilon (P_0 + \varepsilon^2)^{-1} \to \Pi_1$ as $\varepsilon \to 0$ and since $P_0 + \varepsilon^2$ is a strictly positive operator, we get $\langle -i\Pi_1 u, u \rangle \geq 0$ for all $u \in L^2_{\mathrm{comp}}$. In particular, $-ii\gamma > 0$ and then $\gamma \in]0, +\infty[$.

2.2. Estimate for λ small in comparison to ℓ.

In this section, we give an upper bound for the cut-off resolvent for $\lambda \in [R, \ell/R]+i[-C_0, C_0]$. We assume that $\lambda \in [N, 2N] + i[-C_0, C_0]$ with $N \in [R, \ell/R]$, and define a new semi-classical parameter $h = N^{-1}$, a new spectral parameter $z = h^2\lambda^2 \in [1/4, 4] + i[-4C_0h, 4C_0h]$ and

$$\widetilde{P} = -h^2\Delta + h^2\ell(\ell+1)V(x) + h^2W(x). \tag{2.16}$$

With these notations, we have

$$(P_\ell - \lambda^2)^{-1} = h^2(\widetilde{P} - z)^{-1}. \tag{2.17}$$

We remark that $\beta^2 := h^2\ell(\ell+1) \gg 1$ in our window of parameters. The potentials V and W have a holomorphic extension in a sector

$$\Sigma = \{x \in \mathbb{C};\ |\operatorname{Im} x| \leq \theta_0|\operatorname{Re} x| \text{ and } |\operatorname{Re} x| \geq C\}, \tag{2.18}$$

for some $C, \theta_0 > 0$. From the form of α^2 (see (1.2)), there exist $\kappa_\pm > 0$ and functions $f_\pm \in C^\infty(\mathbb{R}^\pm; [1/C, C])$, $C > 0$, analytic in Σ such that

$$V(x) = e^{\mp\kappa_\pm x} f_\pm(x), \tag{2.19}$$

for $x \in \Sigma$ and $\pm \operatorname{Re} x > 0$. Moreover, $f_\pm$ have a (non zero) limit for $x \to \pm\infty$, $x \in \Sigma$.

Under these hypotheses, and following Proposition 4.4 of [24], we can use the specific estimate developed by Zworski in [33] for operators like (2.16) with V satisfying (2.19). In the beginning of Section 4 of [33], Zworski defines a subtle contour Γ_θ briefly described in the following figure.

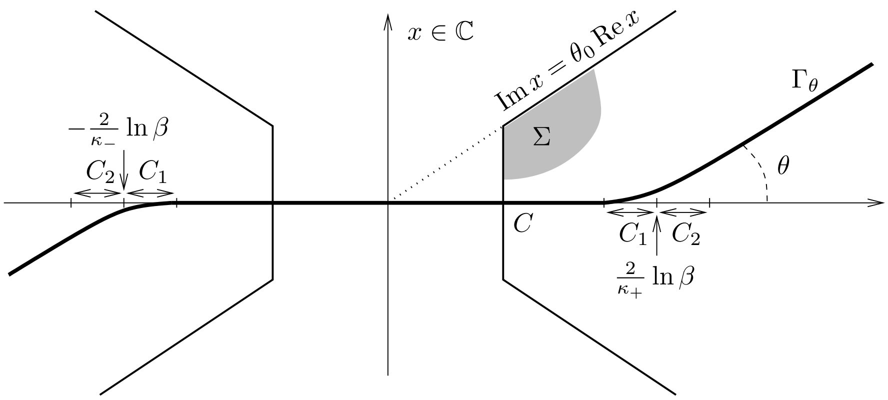

FIGURE 3. The set Σ and the contour Γ_θ.

Recall that the distorted operator $\widetilde{P}_\theta = \widetilde{P}_{|\Gamma_\theta}$ is defined by

$$\widetilde{P}_\theta u = (\widetilde{P}u)_{|\Gamma_\theta} \tag{2.20}$$

for all u analytic in Σ and then extended as a differential operator on $L^2(\Gamma_\theta)$ by means of almost analytic functions. The resonances of $\widetilde{P}$ in the sector $S_\theta = \{e^{-2is}r;\ 0 < s < \theta \text{ and } r \in]0, +\infty[\} = e^{2i]-\theta,0]}]0, +\infty[$ are then the eigenvalues of $\widetilde{P}_\theta$ in that set. For the general theory of resonances, see the paper of Sjöstrand [26] or his book [27].

For θ large enough, Proposition 4.1 of [33] proves that $\widetilde{P}$ has no resonance in $[1/4, 4] + i[-4C_0h, 4C_0h]$. Moreover, for z in that set, this proposition gives the uniform estimate

$$\|(\widetilde{P}_\theta - z)^{-1}\| \leq C. \tag{2.21}$$

Since Γ_θ coincides with $\mathbb{R}$ for $x \in \operatorname{supp} \chi$, we have

$$\chi(\widetilde{P} - z)^{-1}\chi = \chi(\widetilde{P}_\theta - z)^{-1}\chi, \tag{2.22}$$

from Lemma 3.5 of [28]. Using (2.16), we immediately obtain

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq \frac{C}{\langle\lambda\rangle^2}, \tag{2.23}$$

which is exactly (2.3).

2.3. Estimate for λ of order ℓ.

In this part, we study the cut-off resolvent for the energy $\lambda \in [\ell/R, R\ell] + i[-C_0, C_0]$. In this zone, we have to deal with the photon sphere. We define the new semi-classical parameter $h = (\ell(\ell+1))^{-1/2}$ and

$$\widetilde{P} = -h^2\Delta + V(x) + h^2W(x). \tag{2.24}$$

As previously, we have

$$(P_\ell - \lambda^2)^{-1} = h^2(\widetilde{P} - z)^{-1}, \tag{2.25}$$

where

$$z = h^2\lambda^2 \in [1/2R^2, R^2] + i[-3RC_0h, 0] \subset [a, b] + i[-ch, ch], \tag{2.26}$$

with $0 < a < b$ and $0 < c$. Note that V is of the form:

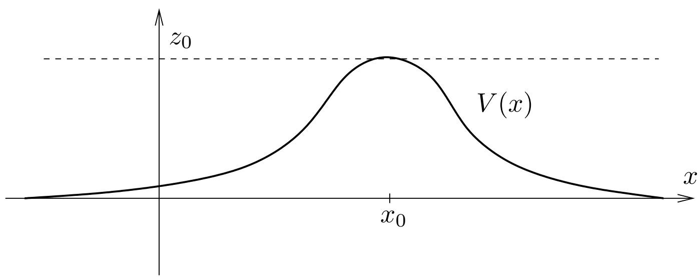

FIGURE 4. The potential $V(x)$.

In particular, V admits at x_0 a non-degenerate maximum at energy $z_0 > 0$. On the other hand, for $z \neq z_0$, $z > 0$, the energy level z is non trapping for $\widetilde{p}_0(x, \xi) = \xi^2 + V(x)$, the principal semi-classical symbol of $\widetilde{P}$. We define $\widetilde{P}_\theta$ by standard distortion (see Sjöstrand [26]) and can apply the following general upper bound on the cut-off resolvent in dimension one.

Lemma 2.2 (**Lemma 6.5 of** [6]). *We assume that $n = 1$ and that the critical points of $\widetilde{p}_0(x, \xi)$ on the energy level E_0 are non-degenerate (i.e. the points $(x, \xi) \in \widetilde{p}_0^{-1}(\{E_0\})$ such* *that* $\nabla\widetilde{p}_0(x,\xi)=0$ *satisfy* $\operatorname{Hess}\widetilde{p}_0(x,\xi)$ *is invertible). Then, there exists* $\varepsilon>0$ *such that, for* $E\in[E_0-\varepsilon,E_0+\varepsilon]$ *and* $\theta=Nh$ *with* $N>0$ *large enough,*

$$\|(\widetilde{P}_\theta-z)^{-1}\|=\mathcal{O}(h^{-M})\prod_{\substack{z\in\operatorname{Res}\widetilde{P}\\ |z-z_j|<\varepsilon\theta}}\frac{h}{|z-z_j|}\tag{2.27}$$

for $|z-E|<\varepsilon\theta/2$ *and some* $M>0$ *which depends on* N*.*

Note that there is a slight error in the statement of the lemma in [6]. Indeed, M depends on N, and in the proof of this lemma, the right hand side of (6.18), $\mathcal{O}(\ln(1/\theta))$, must be replaced by $\mathcal{O}(\theta h^{-1}\ln(1/\theta))$.

Recall that, from Proposition 4.3 [24], which is close to the work of Sjöstrand [25] on the the resonances associated to a critical point, there exists an injective map $b(h)$ from

$$\Gamma_0(h)=\big\{\mu_j=z_0-ih\sqrt{|V''(x_0)|/2}(j+1/2);\ j\in\mathbb{N}_0\big\},\tag{2.28}$$

into the set of resonances of $\widetilde{P}$ such that

$$b(h)(\mu)-\mu=o(h),\ \mu\in\Gamma_0(h),\tag{2.29}$$

and such that all the resonances in $[a/2,2b]+i[-ch,ch]$ are in the range of $b(h)$. In particular, the number of resonances of $\widetilde{P}$ is bounded in $[a/2,2b]+i[-ch,ch]$. Furthermore, the operator $\widetilde{P}$ has no resonance in

$$\Omega(h)=[a/2,2b]+i[-\varepsilon h,ch],$$

for any $\varepsilon>0$ and h small enough.

Using a compactness argument, we get (2.27) for all $z\in[a,b]+i[-ch,ch]$. Thus, from (2.25), (2.26), $\chi(\widetilde{P}-z)^{-1}\chi=\chi(\widetilde{P}_\theta-z)^{-1}\chi$, the estimate $\langle\lambda\rangle\lesssim h^{-1}=\sqrt{\ell(\ell+1)}\lesssim\langle\lambda\rangle$ for $\lambda\in[\ell/R,R\ell]+i[-C_0,0]$, Lemma 2.2 and the previous discussion, we get

$$\|\chi(P_\ell-\lambda^2)^{-1}\chi\|\le C\langle\lambda\rangle^C\prod_{\substack{z_j\in\operatorname{Res}P\\ |\lambda-\lambda_j|<1}}\frac{1}{|\lambda-\lambda_j|},\tag{2.30}$$

for $\lambda\in[\ell/R,R\ell]+i[-C_0,C_0]$ and (2.4) follows.

On the other hand, $\widetilde{P}$ has no resonance in $\Omega(h)$ and in this set

$$\|\chi(\widetilde{P}-z)^{-1}\chi\|\lesssim\begin{cases}h^{-M} & \text{on }\Omega(h),\\ \dfrac{1}{|\operatorname{Im}z|} & \text{on }\Omega(h)\cap\{\operatorname{Im}z>0\}.\end{cases}\tag{2.31}$$

We can now apply the following version, due to Burq [7], of the so-called "semi-classical maximum principle" introduced by Tang–Zworski [29].

Lemma 2.3 (Burq). *Suppose that* $f(z,h)$ *is a family of holomorphic functions defined for* $0<h<1$ *in a neighborhood of*

$$\Omega(h)=[a/2,2b]+i[-ch,ch],$$

with $0 < a < b$ *and* $0 < c$*, such that*

$$|f(z,h)| \lesssim \begin{cases} h^{-M} & \text{on } \Omega(h), \\ \dfrac{1}{|\operatorname{Im} z|} & \text{on } \Omega(h) \cap \{\operatorname{Im} z > 0\}. \end{cases}$$

Then, there exists $h_0, C > 0$ *such that, for any* $0 < h < h_0$,

$$|f(z,h)| \leq C \frac{|\ln h|}{h} e^{C|\operatorname{Im} z||\ln h|/h}, \tag{2.32}$$

for $z \in [a,b] + i[-ch, 0]$.

This lemma is strictly analogous to Lemma 4.7 of [7]. Combining (2.25), (2.26), $\langle \lambda \rangle \lesssim h^{-1} \lesssim \langle \lambda \rangle$ with this lemma, we obtain

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq C \frac{\ln\langle \lambda \rangle}{\langle \lambda \rangle} e^{C|\operatorname{Im} \lambda| \ln\langle \lambda \rangle}, \tag{2.33}$$

for $\lambda \in [\ell/R, R\ell] + i[-\varepsilon, 0]$, for some $\varepsilon > 0$.

2.4. **Estimate for the very large values of λ.**

Here, we study the resolvent for $|\lambda| \gg \ell$. More precisely, we assume that

$$\lambda \in [N, 2N] + i[-C \ln N, C_0],$$

for some $C > 0$ fixed and $N \gg \ell$. We define the new semi-classical parameter $h = N^{-1}$ and

$$z = h^2\lambda^2 \in h^2[N^2/2, 4N^2] + ih^2[-4CN \ln N, 4C_0 N^{-1}] \subset [a,b] + i[-ch|\ln h|, ch],$$

for some $0 < a < b$ and $0 < c$. Then, P_ℓ can be written

$$P_\ell - \lambda^2 = h^{-2}(\widetilde{P} - z),$$

where

$$\widetilde{P} = -h^2\Delta + \mu V(x) + \nu W(x),$$

with $\mu = \ell(\ell+1)h^2$, $\nu = h^2$. For $N \gg \ell$, the coefficients μ, ν are small, and the operator $\widetilde{P}$ is uniformly non trapping for $z \in [a,b]$. We can expect a uniform bound of the cut-off resolvent in $[a,b] + i[-ch|\ln h|, ch]$. Such a result is proved in the following lemma.

Lemma 2.4. *For all* $\chi \in C_0^\infty(\mathbb{R})$*, there exist* $\mu_0, \nu_0, h_0, C > 0$ *such that, for all* $\mu < \mu_0$, $\nu < \nu_0$ *and* $h < h_0$, $\widetilde{P}$ *has no resonance in* $[a,b] + i[-ch|\ln h|, ch]$*. Moreover*

$$\|\chi(\widetilde{P} - z)^{-1}\chi\| \leq \frac{C}{h} e^{C|\operatorname{Im} z|/h}, \tag{2.34}$$

for all $z \in [a,b] + i[-ch|\ln h|, ch]$.

Assume first Lemma 2.4. For $\lambda \in [N, 2N] + i[-C \ln N, C_0]$, we have

$$\begin{aligned} \|\chi(P_\ell - \lambda^2)^{-1}\chi\| =& \|h^2\chi(\widetilde{P} - z)^{-1}\chi\| \\ \leq& Che^{C|\operatorname{Im} z|/h} \\ \leq& \frac{C}{|\lambda|} e^{4C|\operatorname{Im} \lambda|}, \end{aligned}$$

and the estimate (2.6) follows.

Proof of Lemma 2.4. For μ and ν small and fixed, the estimate (2.34) is already known. The proof can be found in the book of Vaĭnberg [32] in the classical case and in the paper of Nakamura–Stefanov–Zworski [20] in our semi-classical setting. To obtain Lemma 2.4, we only have to check the uniformity (with respect to μ and ν) in the proof of [20, Proposition 3.1].

• *Limiting absorption principle.*
The point is to note that

$$A = xhD_x + hD_x x, \tag{2.35}$$

is a conjugate operator for all $\mu, \nu \ll 1$. Let $g \in C_0^\infty([a/3, 3b]; [0,1])$ be equal to 1 near $[a/2, 2b]$. The operator $g(\widetilde{P})Ag(\widetilde{P})$ is well defined on $D(A)$, and its closure, $\mathcal{A}$, is self-adjoint. The operator $\widetilde{P}$ is of class $C^2(\mathcal{A})$. Recall that $\widetilde{P}$ is of class $C^r(\mathcal{A})$ if there exists $z \in \mathbb{C} \setminus \sigma(\widetilde{P})$ such that

$$\mathbb{R} \ni t \to e^{it\mathcal{A}}(\widetilde{P} - z)^{-1} e^{-it\mathcal{A}},$$

is C^r for the strong topology of L^2 (see [2, Section 6.2] for more details).

We have

$$ih^{-1}[\widetilde{P}, A] = 4\widetilde{P} - 4\mu V - 4\nu W - 2\mu x V' - 2\nu x W'. \tag{2.36}$$

In particular, for μ and ν small enough, we easily obtain

$$1_{[a/2,2b]}(\widetilde{P}) i[\widetilde{P}, \mathcal{A}] 1_{[a/2,2b]}(\widetilde{P}) \geq ah 1_{[a/2,2b]}(\widetilde{P}). \tag{2.37}$$

Note that this Mourre estimate is uniform with respect to μ, ν.

It is also easy to check that

$$\begin{aligned} \|\langle x\rangle^{-1}\mathcal{A}\| &\leq C \\ \|(\widetilde{P} + i)^{-1}[\widetilde{P}, A]\| &\leq Ch \\ \|(\widetilde{P} + i)^{-1}[[\widetilde{P}, A], A]\| &\leq Ch^2 \\ \|(\widetilde{P} + i)^{-1}[\widetilde{P}, [\widetilde{P}, A]]\| &\leq Ch^2 \\ \|(\widetilde{P} + i)^{-1}A[\widetilde{P}, [\widetilde{P}, A]]\| &\leq Ch^2, \end{aligned} \tag{2.38}$$

uniformly in μ, ν.

The regularity $\widetilde{P} \in C^2(\mathcal{A})$, the Mourre estimate (2.37) and the upper bound (2.38) are the key assumptions for the limiting absorption principle. In particular, from, for instance, the proof of Proposition 3.2 in [1] which is an adaptation of the theorem of Mourre [19], we obtain the following estimate: For $\alpha > 1/2$, there exist $\mu_0, \nu_0, h_0, C > 0$, such that

$$\|\langle x\rangle^{-\alpha}(\widetilde{P} - z)^{-1}\langle x\rangle^{-\alpha}\| \leq Ch^{-1}, \tag{2.39}$$

for all $\mu < \mu_0$, $\nu < \nu_0$, $h < h_0$ and $z \in [a/2, 2b] + i]0, ch]$. In particular,

$$\|\chi(\widetilde{P} - z)^{-1}\chi\| \leq Ch^{-1}. \tag{2.40}$$

for $z \in [a/2, 2b] + i[0, ch]$.

• *Polynomial estimate in the complex.*
The second point of the proof is to obtain a polynomial bound of the distorted resolvent. To obtain such bounds, we use the paper of Martinez [16]. In this article, the author studies the resonances of $Q = -h\Delta + \widetilde{V}(x)$ where $\widetilde{V}$ is a $C^\infty(\mathbb{R}^n)$ function which can be extended

analytically in a domain like Σ (see (2.18)) and decays in this domain. If the energy level z_0 is non trapped for the symbol $q(x,\xi) = \xi^2 + \widetilde{V}(x)$, the operator Q has no resonance in $[z_0 - \delta, z_0 + \delta] + i[Ah \ln h, 0]$ for a δ small enough and any $A > 0$. Moreover,

$$\|(Q_\theta - z)^{-1}\| \leq Ch^{-C} \tag{2.41}$$

for $z \in [z_0 - \delta, z_0 + \delta] + i[Ah \ln h, 0]$. Here Q_θ denotes the distorted operator outside of a large ball of angle $\theta = Bh|\ln h|$, with $B \gg A$.

Of course, $\widetilde{P}$ satisfies the previous assumption on Q, for μ and ν fixed small enough. But, following line by line the proof of (2.41) in [16, Section 4], one can prove that (2.41) is uniformly true for $\mu, \nu \ll 1$. This means that there exist $\mu_0, \nu_0, h_0, C > 0$ such that

$$\|\chi(\widetilde{P} - z)^{-1}\chi\| = \|\chi(\widetilde{P}_\theta - z)^{-1}\chi\| \leq Ch^{-C}, \tag{2.42}$$

for all $\mu < \mu_0$, $\nu < \nu_0$, $h < h_0$ and $z \in [a/2, 2b] + i[ch \ln h, 0]$.

• *Semi-classical maximum principle.*
To finish the proof, we use a version of the semi-classical maximum principle. This argument can be found in [20, Proposition 3.1], but we give it here for the convenience of the reader.

We can construct a holomorphic function $f(z,h)$ with the following properties:

$$\begin{aligned}
&|f| \leq C \quad \text{for } z \in [a/2, 2b] + i[ch \ln h, 0],\\
&|f| \geq 1 \quad \text{for } z \in [a, b] + i[ch \ln h, 0],\\
&|f| \leq h^M \quad \text{for } z \in [a/2, 2b] \setminus [2a/3, 3b/2] + i[ch \ln h, 0],
\end{aligned}$$

where M is the constant C given in (2.42). We can then apply the maximum principle in $[a/2, 2b] + i[ch \ln h, 0]$ to the subharmonic function

$$\ln \|\chi(\widetilde{P} - z)^{-1}\chi\| + \ln|f(z,h)| + \frac{M}{c}\frac{\operatorname{Im} z}{h},$$

proving the lemma with (2.40) and (2.42). $\square$

3. Proof of the main theorem

3.1. Resolvent estimates for L_ℓ.

The cut-off resolvent estimates for P_ℓ give immediately cut-off resolvent estimates for L_ℓ.

Proposition 3.1. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the operator $\chi(L_\ell - \lambda)^{-1}\chi$ sends $\mathcal{E}_\ell^{\mathrm{mod}}$ into itself and we have uniformly in ℓ:*

$$\|\chi(L_\ell - z)^{-1}\chi\|_{\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}})} \lesssim \langle z\rangle \|\chi(P_\ell - z^2)^{-1}\chi\| \tag{3.1}$$

Proof. Using Theorem 2.1, (1.7), the equivalence of the norms $\mathcal{E}_{a,b}^{\mathrm{mod}}$ as well as the fact that we can always replace u by $\widetilde{\chi}u$, $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$, $\widetilde{\chi}\chi = \chi$ we see that it is sufficient to show:

$$\|\chi(P_\ell - z^2)^{-1}\chi u\|_{H^1} \lesssim \|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{H^1}, \tag{3.2}$$

$$\|\chi(P_\ell - z^2)^{-1}\chi u\|_{H^1} \lesssim \langle z\rangle\|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{L^2}, \tag{3.3}$$

$$\|\chi(P_\ell - z^2)^{-1}P_\ell\chi u\|_{L^2} \lesssim \langle z\rangle\|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{H^1}. \tag{3.4}$$

Using complex interpolation we see that it is sufficient to show:

$$\|\chi(P_\ell - z^2)^{-1}\chi u\|_{H^2} \lesssim \|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{H^2}, \tag{3.5}$$

$$\|\chi(P_\ell - z^2)^{-1}\chi u\|_{H^2} \lesssim \langle z\rangle^2\|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{L^2}, \tag{3.6}$$

$$\|\chi(P_\ell - z^2)^{-1}P_\ell\chi u\|_{L^2} \lesssim \|\widetilde{\chi}(P_\ell - z^2)^{-1}\widetilde{\chi}\|\|u\|_{H^2}. \tag{3.7}$$

We start with (3.7) which follows from

$$\chi(P_\ell - z^2)^{-1}P_\ell\chi = \chi(P_\ell - z^2)^{-1}\chi P_\ell + \chi(P_\ell - z^2)^{-1}[P_\ell, \chi]u.$$

Let us now observe that

$$\begin{aligned} P_\ell\chi(P_\ell - z^2)^{-1}\chi u =& [P_\ell, \chi](P_\ell - z^2)^{-1}\chi + \chi(P_\ell - z^2)^{-1}P_\ell\chi u \\ =& \widetilde{\chi}(P_\ell + i)^{-1}[P_\ell, [P_\ell, \chi]](P_\ell - z^2)^{-1}\chi u \\ &+ \widetilde{\chi}(P_\ell + i)^{-1}[P_\ell, \chi](P_\ell - z^2)^{-1}(P_\ell + i)\chi u + \chi(P_\ell - z^2)^{-1}P_\ell\chi u. \end{aligned}$$

From this identity we obtain (3.5) and (3.6) using (3.7) (for (3.5)) and the uniform boundedness of $(P_\ell + i)^{-1}[P_\ell, [P_\ell, \chi]]$. □

3.2. **Resonance expansion for the wave equation.**

For the proof of the main theorem we follow closely the ideas of Vaĭnberg [32, Chapter X.3]. If $\mathcal{N}$ is a Hilbert space we will denote by $L^2_\nu(\mathbb{R}; \mathcal{N})$ the space of all functions $v(t)$ with values in $\mathcal{N}$ such that $e^{-\nu t}v(t) \in L^2(\mathbb{R}; \mathcal{N})$. Let $u \in \mathcal{E}^{\mathrm{mod}}_\ell$ and

$$v(t) = \begin{cases} e^{-itL_\ell}u & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Then $v \in L^2_\nu(\mathbb{R}; \mathcal{E}_\ell)$ for all $\nu > 0$. We can define

$$\tilde{v}(k) = \int_0^\infty v(t)e^{ikt}dt$$

as an element of $\mathcal{E}$ for all k with $\operatorname{Im} k > 0$. The function $\tilde{v}$ depends analytically on k when $\operatorname{Im} k > 0$. Also, on the line $\operatorname{Im} k = \nu$ the function belongs to $L^2(\mathbb{R}; \mathcal{E}_\ell)$. We have the inversion formula:

$$v(t) = \frac{1}{2\pi}\int_{-\infty+i\nu}^{\infty+i\nu} e^{-ikt}\tilde{v}(k)\, dk$$

and the integral converges in $L^2_\nu(\mathbb{R}; \mathcal{E}_\ell)$ for all $\nu > 0$. From the functional calculus we know that

$$\tilde{v}(k) = -i(L_\ell - k)^{-1}u$$

for all k with $\operatorname{Im} k > 0$. We therefore obtain for all $t \geq 0$:

$$e^{-itL_\ell}u = \frac{1}{2\pi i}\int_{-\infty+i\nu}^{\infty+i\nu} (L_\ell - k)^{-1}e^{-ikt}u\, dk, \tag{3.8}$$

where the integral is convergent in $L^2_\nu(\mathbb{R}; \mathcal{E}_\ell)$. In the following, we denote by $\widehat{R}^\ell_\chi(k)$ the meromorphic extension of $\chi(L_\ell - k)^{-1}\chi$.

Lemma 3.2. *Let $\chi \in C_0^\infty(\mathbb{R})$, $N \geq 0$. Then, there exist bounded operators $B_j \in \mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod},-q}; \mathcal{E}_\ell^{\mathrm{mod},-j-q})$, $j = 0, \dots, N$, $q \in \mathbb{N}_0$ and $B \in \mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod},-q}; \mathcal{E}_\ell^{\mathrm{mod},-N-1-q})$, $q \in \mathbb{N}_0$ such that*

$$\widehat{R}_\chi^\ell(k) = \sum_{j=0}^{N} \frac{1}{(k - i(\nu+1))^{j+1}} B_j + \frac{1}{(k - i(\nu+1))^{N+1}} B \widehat{R}_{\widetilde{\chi}}^\ell(k)\chi, \tag{3.9}$$

for some $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\chi\widetilde{\chi} = \chi$.

Proof. We proceed by induction over N. For $N = 0$, we write

$$(L_\ell - k)^{-1} + \frac{1}{k - i(\nu+1)} = \frac{1}{k - i(\nu+1)}(L_\ell - i(\nu+1))(L_\ell - k)^{-1}.$$

and choose $B_0 = -\chi^2$. Then

$$\widehat{R}_\chi^\ell(k) - \frac{1}{k - i(\nu+1)} B_0 = \frac{1}{k - i(\nu+1)} \widetilde{B}_{\chi,\widetilde{\chi}} \widehat{R}_{\widetilde{\chi}}^\ell(k)\chi, \tag{3.10}$$

where $\widetilde{B}_{\chi,\widetilde{\chi}} = \chi(L_\ell - i(\nu+1))\widetilde{\chi}$, with $\chi = \chi\widetilde{\chi}$, is in the space $\mathcal{L}(\mathcal{E}_\ell^{m,-q}; \mathcal{E}_\ell^{m,-1-q})$.

Let us suppose that the lemma is proved for $N \geq 0$. We put

$$B_{N+1} = \frac{1}{(k - i(\nu+1))^{N+1}} B \widetilde{\chi}^2 \chi \tag{3.11}$$

Using (3.10), we get

$$\begin{aligned} \widehat{R}_\chi^\ell(k) &= \sum_{j=0}^{N} \frac{1}{(k - i(\nu+1))^{j+1}} B_j + \frac{1}{(k - i(\nu+1))^{N+1}} B \widehat{R}_{\widetilde{\chi}}^\ell \chi \\ &= \sum_{j=0}^{N+1} \frac{1}{(k - i(\nu+1))^{j+1}} B_j + \frac{1}{(k - i(\nu+1))^{N+2}} B \widetilde{B}_{\widetilde{\chi},\widetilde{\widetilde{\chi}}} \widehat{R}_{\widetilde{\widetilde{\chi}}}^\ell \chi, \end{aligned} \tag{3.12}$$

with $\widetilde{\widetilde{\chi}} \in C_0^\infty(\mathbb{R})$ with $\widetilde{\widetilde{\chi}}\widetilde{\chi} = \widetilde{\chi}$. This proves the lemma. □

Let us define

$$\widetilde{R}_\chi^\ell(k) = \widehat{R}_\chi^\ell(k) - \sum_{j=0}^{1} \frac{1}{(k - i(\nu+1))^{j+1}} B_j.$$

Then, Lemma 3.2 implies, for $\operatorname{Im} k \leq \nu$,

$$\| \widetilde{R}_\chi^\ell(k) \|_{\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}}; \mathcal{E}_\ell^{\mathrm{mod},-2})} \lesssim \frac{1}{\langle k \rangle^2} \| \widehat{R}_\chi^\ell(k) \|_{\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}}; \mathcal{E}_\ell^{\mathrm{mod}})}. \tag{3.13}$$

Now observe that

$$\int_{-\infty+i\nu}^{\infty+i\nu} \frac{B_j}{(k - i(\nu+1))^{-j-1}} e^{-ikt} dk = 0. \tag{3.14}$$

Therefore (3.8) becomes:

$$\chi e^{-itL_\ell} \chi u = \frac{1}{2\pi i} \int_{-\infty+i\nu}^{\infty+i\nu} \widetilde{R}_\chi^\ell(k) e^{-ikt} u \, dk,$$

where the previous integral is absolutely convergent in $\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}}; \mathcal{E}_\ell^{\mathrm{mod},-2})$.

We first show part (i) of the theorem. Integrating along the path indicated in Figure 5 we obtain by the Cauchy theorem:

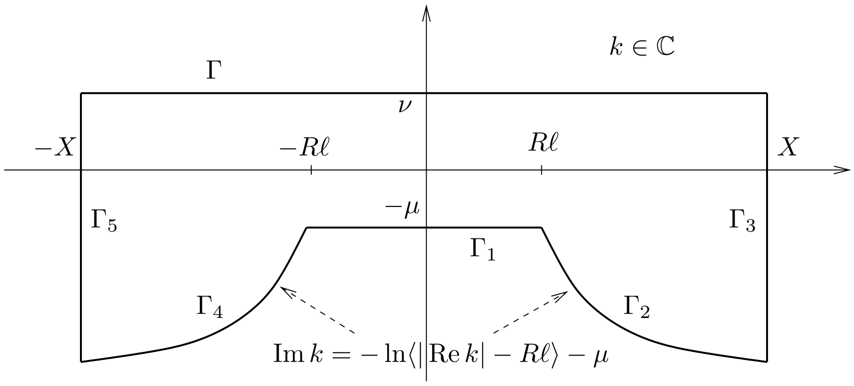

FIGURE 5. The paths Γ_j.

$$\frac{1}{2\pi i}\int_{-X+i\nu}^{X+i\nu} e^{-ikt}\widetilde{R}_\chi^\ell(k)u\,dk = \sum_{\substack{\lambda_j\in\operatorname{Res}P_\ell\\ \operatorname{Im}\lambda_j>-\mu}}\sum_{k=0}^{m(\lambda_j)} e^{-i\lambda_j t}t^k\pi_{j,k}^\chi u+\sum_{j=1}^{5}\frac{1}{2\pi i}\int_{\Gamma_j} e^{-it\lambda}\widetilde{R}_\chi^\ell(\lambda)u\,d\lambda. \tag{3.15}$$

Let $I_j = \frac{1}{2\pi i}\int_{\Gamma_j} e^{-it\lambda}\widetilde{R}_\chi^\ell(\lambda)u\,d\lambda$. Using (2.6), (3.1) and (3.13), we have, for t large enough,

$$\begin{aligned}\|I_3\|_{\mathcal{E}_\ell^{\mathrm{mod},-2}} &\lesssim \int_{X-i\ln\langle X\rangle}^{X+i\nu} \|e^{-ist}\widetilde{R}_\chi^\ell(s)u\|_{\mathcal{E}_\ell^{\mathrm{mod},-2}}\,ds\\ &\lesssim \int_{-\ln\langle X\rangle}^{\nu}\frac{1}{\langle X\rangle^2}e^{ts+C|s|}ds\,\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}} \lesssim \frac{e^{t\nu}}{t}X^{-2}\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}}.\end{aligned} \tag{3.16}$$

We now take the limit X goes to $+\infty$ in the $\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}};\mathcal{E}_\ell^{\mathrm{mod},-2})$ sense in (3.15). The integrals I_3 and I_5 go to 0 thanks to (3.16) and, in the integrals I_2 and I_4, the paths $\Gamma_\bullet$ are replaced by paths which extend $\Gamma_\bullet$ in a natural way and which go to ∞. We denote them again by $\Gamma_\bullet$. We remark that

$$\int_{\Gamma_4\cup\Gamma_1\cup\Gamma_2}\frac{B_j}{(k-i(\nu+1))^{-j-1}}e^{-ikt}dk = 0, \tag{3.17}$$

where the integral is absolutely convergent in $\mathcal{L}(\mathcal{E}_\ell^{\mathrm{mod}};\mathcal{E}_\ell^{\mathrm{mod},-2})$. On the other hand, we have the estimate, for t large enough,

$$\begin{aligned}\|I_1\|_{\mathcal{E}_\ell^{\mathrm{mod}}} &\lesssim \int_{-R\ell}^{R\ell}\|e^{-\mu t}\widehat{R}_\chi^\ell(s-i\mu)u\|_{\mathcal{E}_\ell^{\mathrm{mod}}}\,ds\\ &\lesssim e^{-\mu t}\int_{-R\ell}^{R\ell}\langle s\rangle^C ds\,\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}} \lesssim e^{-\mu t}\ell^{C+1}\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\|I_2\|_{\mathcal{E}_\ell^{\mathrm{mod}}} &\lesssim \int_0^{+\infty}\left\|e^{-i(R\ell+s-i(\mu+\ln\langle s\rangle))t}\widehat{R}_\chi^\ell(R\ell+s-i(\mu+\ln\langle s\rangle))u\right\|_{\mathcal{E}_\ell^{\mathrm{mod}}}ds\\ &\lesssim \int_0^\infty e^{-\mu t}e^{-\ln\langle s\rangle t}e^{C(\ln\langle s\rangle+\mu)}\,ds\,\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}} \lesssim e^{-\mu t}\|u\|_{\mathcal{E}_\ell^{\mathrm{mod}}},\end{aligned} \tag{3.19}$$

and a similar estimate holds for I_4. Since all these estimates hold in $\mathcal{L}(\mathcal{E}_\ell^{\rm mod})$, (3.18) and (3.19) give the estimate of the rest (1.12) with $M = (C+1)/2$. The estimate (1.13) follows from (1.10), Theorem 2.1 *iii)* and Proposition 3.1.

Let us now show part (*ii*) of the theorem. We choose $0 > -\mu > \sup\{\operatorname{Im}\lambda;\ \lambda \in (\operatorname{Res} P)\backslash\{0\}\}$ and the integration path as in part (*i*) of the theorem. We first suppose $e^{\varepsilon' t} > \ell$ for some $\varepsilon' > 0$ to be chosen later. Then the estimate for I_1 can be replaced by

$$\|I_1\|_{\mathcal{E}_\ell^{\rm mod}} \lesssim e^{((C+1)\varepsilon' - \mu)t}\|u\|_{\mathcal{E}_\ell^{\rm mod}}.$$

Let us now suppose $\ell \geq e^{\varepsilon' t}$. On the one hand we have the inequality:

$$\|\chi e^{-itL_\ell}\chi\|_{\mathcal{L}(\mathcal{E}_\ell^{\rm mod})} \lesssim 1,$$

since the norms on $\mathcal{E}_\ell^{\rm mod}$ and on $\mathcal{E}_\ell$ are uniformly equivalent for $\ell \geq 1$. On the other hand by the hypotheses on g we have

$$1 \leq \frac{g(e^{2\varepsilon' t})}{g(\ell(\ell+1))}.$$

It follows:

$$\|\chi e^{-itL}\chi\|_{\mathcal{L}(\mathcal{E}_\ell^{\rm mod})} \lesssim \frac{g(e^{2\varepsilon' t})}{g(\ell(\ell+1))}.$$

This concludes the proof of the theorem if we choose ε' sufficiently small and put $\varepsilon := \min\{2\varepsilon', \mu - (C+1)\varepsilon'\}$.

Proof of Remark 1.4 d). We note that for $u_\ell \in D(P_\ell)$, we have

$$\begin{aligned}\langle P_\ell u_\ell, u_\ell\rangle =& \big\langle \big(r^{-1}D_x r^2 D_x r^{-1} + V\ell(\ell+1)\big)u_\ell, u_\ell\big\rangle \\ \geq& \langle V\ell(\ell+1)\big)u_\ell, u_\ell\rangle, \end{aligned} \tag{3.20}$$

and then

$$\|\ell\sqrt{V}u_\ell\|^2 \leq \|(P+1)u_\ell\|^2. \tag{3.21}$$

Estimate (1.12) can be written

$$\|E_1(t)\|_{\mathcal{E}^{\rm mod}} \lesssim e^{-\mu t}\|\langle -\Delta_\omega\rangle^M \chi_0 u\|_{\mathcal{E}^{\rm mod}},$$

with $\chi_0 \in C_0^\infty(\mathbb{R})$ and $\chi_0\chi = \chi$. Let $\chi_j \in C_0^\infty(\mathbb{R})$, $j = 1, \ldots, 2M$ with $\chi_{j+1}\chi_j = \chi_j$ for $j = 0, \ldots, 2M-1$. Remark that there exists $C > 0$ such that $\sqrt{V} > 1/C$ on the support of χ_{2M}. Using the radial decomposition $u = \sum_\ell u_\ell$, we get

$$\begin{aligned}\|\langle -\Delta_\omega\rangle^M \chi_0 u\|_{\mathcal{E}^{\rm mod}} \lesssim& \sup_\ell \|\ell^{2M}\chi_0 u_\ell\|_{\mathcal{E}^{\rm mod}} \\ \lesssim& \sup_\ell \|\ell^{2M-1}(P+1)\chi_0 u_\ell\|_{\mathcal{E}^{\rm mod}} = \sup_\ell \|\ell^{2M-1}\chi_1(P+1)\chi_0 u_\ell\|_{\mathcal{E}^{\rm mod}} \\ \lesssim& \sup_\ell \|\chi_{2M}(P+1)\chi_{2M-1}(P+1)\cdots\chi_1(P+1)\chi_0 u_\ell\|_{\mathcal{E}^{\rm mod}} \\ \lesssim& \|(P+1)^{2M}u\|_{\mathcal{E}^{\rm mod}}. \end{aligned} \tag{3.22}$$

Finally, for the interpolation argument, we use the fact that

$$\|e^{-itL_\ell}\|_{\mathcal{L}(\mathcal{E}_\ell^{\rm mod};\mathcal{E}_\ell^{\rm mod})} \lesssim \|e^{-itL_\ell}\|_{\mathcal{L}(\mathcal{E}_\ell;\mathcal{E}_\ell)} = 1, \tag{3.23}$$

for $\ell \geq 1$. $\square$

Proof of Remark 1.7. We only prove the first part since the proof of the second part is analogous. As we are localized in space, we can use the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow C^0(\mathbb{R}^3)$. Using $\|Pu_1\| \leq \big\|L\big(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\big)\big\|_{\mathcal{E}^{\mathrm{mod}}}$, it is sufficient to consider $(L+i)\chi e^{-itL}\chi$. For this purpose, we write

$$\begin{aligned} L\chi e^{-itL}\chi =&(L+i)^{-1}[L,[L,\chi]]e^{-itL}\chi + (L+i)^{-1}[L,\chi]e^{-itL}[L,\chi] \\ &+ (L+i)^{-1}[L,\chi]e^{-itL}\chi(L+i) + \chi e^{-itL}[L,\chi] + \chi e^{-itL}\chi L. \end{aligned} \tag{3.24}$$

It is easy to check that the operators $(L+i)^{-1}[L,[L,\chi]]$ and $(L+i)^{-1}[L,\chi]$ can be extended to bounded operators on $\mathcal{E}^{\mathrm{mod}}$. Note also that all commutators can be multiplied on the left and on the right by a cut-off function $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\widetilde{\chi}\chi = \chi$ without changing them.

By Theorem 1.3, the left hand side of (3.24) is equal to

$$L\gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + LE_2(t),$$

in $\mathcal{E}^{\mathrm{mod},-1}$. By the same theorem, the right hand side of (3.24) is equal to $\alpha + \widetilde{E}_2(t)$ where α is a constant in time and

$$\|\widetilde{E}_2(t)\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}(L+i)u\big\|_{\mathcal{E}^{\mathrm{mod}}}.$$

It follows $\alpha = L\gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix}$ and thus the first part of the remark. □

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