Sharp estimates for the convergence of the density of the Euler scheme in small time

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Abstract

In this work, we approximate a diffusion process by its Euler scheme and we study the convergence of the density of the marginal laws. We improve previous estimates especially for small time.

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1 Introduction

Let us consider a d-dimensional diffusion process $(X_s)_{0\leq s\leq T}$ and a q-dimensional Brownian motion $(W_s)_{0\leq s\leq T}$. X satisfies the following SDE

$$dX_s^i = b_i(s,X_s)ds + \sum_{j=1}^q \sigma_{ij}(s,X_s)dW_s^j, \quad X_0^i = x^i, \forall i \in \{1,\cdots,d\}. \tag{1.1}$$

We approximate X by its Euler scheme with N ($N \geq 1$) time steps, say X^N, defined as follows. We consider the regular grid $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0,T]$, i.e. $t_k = k\frac{T}{N}$. We put $X_0^N = x$ and for all $i \in \{1,\cdots,d\}$ we define

$$X_u^{N,i} = X_{t_k}^{N,i} + b_i(t_k, X_{t_k}^N)(u - t_k^N) + \sum_{j=1}^q \sigma_{ij}(t_k, X_{t_k}^N)(W_u^j - W_{t_k}^j), \text{ for } u \in [t_k, t_{k+1}[. \tag{1.2}$$

The continuous Euler scheme is an Itô process verifying

$$X_u^N = x + \int_0^u b(\varphi(s), X_{\varphi(s)}^N)ds + \int_0^u \sigma(\varphi(s), X_{\varphi(s)}^N)dW_s$$

where $\varphi(u) := \sup\{t_k : t_k \leq u\}$. If σ is uniformly elliptic, the Markov process X admits a transition probability density $p(0,x;s,y)$. Concerning X^N (which is not Markovian except at times $(t_k)_k$), X_s^N has a probability density $p^N(0,x;s,y)$, for any $s > 0$. We aim at proving sharp estimates of the difference $p(0,x;s,y) - p^N(0,x;s,y)$.

It is well known (see Bally and Talay (1996), Konakov and Mammen (2002), Guyon (2006)) that this difference is of order $\frac{1}{N}$. However, the known upper bounds of this difference are too rough for small values of s. In this work, we provide tight upper bounds of $|p(0,x;s,y) - p^N(0,x;s,y)|$ in s (see Theorem 3), so that we can estimate quantities like

$$\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] \text{ or } \mathbb{E}\left[\int_0^T f(X_{\varphi(s)}^N)ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right] \tag{1.3}$$

(without any regularity assumptions on f) more accurately than before (see Theorem 5). For other applications, see Labart (2007). Unlike previous references, we allow b and σ to be time-dependent and assume they are only C^3 in space. Besides, we use Malliavin's calculus tools.

Background results

The difference $p(0,x;s,y) - p^N(0,x;s,y)$ has been studied a lot. We can found several results in the literature on expansions w.r.t. N. First, we mention a result from Bally and Talay (1996) (Corollary 2.7). The authors assume

Hypothesis 1 *σ is elliptic (with σ only depending on x) and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order greater or equal to 1 are bounded.*

By using Malliavin's calculus, they show that

$$p(0,x;T,y) - p^N(0,x;T,y) = \frac{1}{N}\pi_T(x,y) + \frac{1}{N^2}R_T^N(x,y), \tag{1.4}$$

with $|\pi_T(x,y)| + |R_T^N(x,y)| \leq \frac{K(T)}{T^q}\exp(-c\frac{|x-y|^2}{T})$, where $c > 0$, $q > 0$ and $K(\cdot)$ is a non decreasing function. We point out that q is unknown, which doesn't enable to deduce the behavior of $p - p^N$ when $T \to 0$.

Besides that, Konakov and Mammen (2002) have proposed an analytical approach based on the so-called parametrix method to bound $p(0,x;1,y) - p^N(0,x;1,y)$ from above. They assume

Hypothesis 2 *σ is elliptic and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order are bounded.*

For each pair (x,y) they get an expansion of arbitrary order j of $p^N(0,x;1,y)$. The coefficients of the expansion depend on N

$$p(0,x;1,y) - p^N(0,x;1,y) = \sum_{i=1}^{j-1} \frac{1}{N^i}\pi_{N,i}(0,x;1,y) + O(\frac{1}{N^j}). \tag{1.5}$$

The coefficients have Gaussian tails : for each i they find constants $c_1 > 0$, $c_2 > 0$ s.t. for all $N \geq 1$ and all $x, y \in \mathbb{R}^d$, $|\pi_{N,i}(0,x;1,y)| \leq c_1 \exp(-c_2|x-y|^2)$. To do so, they use upper bounds for the partial derivatives of p (coming from Friedman (1964)) and prove analogous results on the derivatives of p^N. Strong though this result may be, nothing is said when replacing 1 by t, for $t \to 0$. That's why we present now the work of Guyon (2006).

Guyon (2006) improves (1.4) and (1.5) in the following way.

Definition 1. *Let $\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z}$ be the set of all measurable functions $\pi : \mathbb{R}^d \times (0,1] \times \mathbb{R}^d \to \mathbb{R}$ s.t.*

- *for all $t \in (0,1], \pi(\cdot;t,\cdot)$ is infinitely differentiable,*
- *for all $\alpha, \beta \in \mathbb{N}^d$, there exist two constants $c_1 \geq 0$ and $c_2 > 0$ s.t. for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_y^\beta \pi(x;t,y)| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp(-c_2|x-y|^2/t).$$

Under Hypothesis 2 and for $T = 1$, the author has proved the following expansions

$$p^N - p = \frac{\pi}{N} + \frac{\pi_N}{N^2}, \tag{1.6}$$

$$p^N - p = \sum_{i=1}^{j-1} \frac{\pi_{N,i}}{N^i} + \sum_{i=2}^{j} \left(t - \frac{\lfloor Nt \rfloor}{N}\right)^i \pi'_{N,i} + \frac{\pi''_{N,j}}{N^j}, \tag{1.7}$$

where $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and $(\pi_N, N \geq 1)$ is a bounded sequence in $\mathcal{G}_4(\mathbb{R}^d)$. For each $i \geq 1$, $(\pi_{N,i}, N \geq 1)$ is a bounded family in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$, and $(\pi'_{N,i}, N \geq 1), (\pi''_{N,i}, N \geq 1)$ are two bounded families in $\mathcal{G}_{2i}(\mathbb{R}^d)$. These expansions can be seen as improvements of (1.4) and (1.5) : it also allows infinite differentiations w.r.t. x and y and makes precise the way the coefficients explode when t tends to 0.

As a consequence (see Guyon (2006, Corollary 22)), one gets

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{c_1}{N s^{\frac{d+2}{2}}} e^{-c_2 \frac{|x-y|^2}{s}}, \tag{1.8}$$

for two positive constants c_1 and c_2, and for any x, y and $s \leq 1$. This result should be compared with the one of Theorem 3 (when $T = 1$), in which the upper bound is tighter (s has a smaller power).

2 Main Results

Before stating the main result of the paper, we introduce the following notation

Definition 2. *$C_b^{k,l}$ denotes the set of continuously differentiable bounded functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).*

The main result of the paper, whose proof is postponed to Section 4, is established under the following Hypothesis

Hypothesis 3 *σ is uniformly elliptic, b and σ are in $C_b^{1,3}$ and $\partial_t \sigma$ is in $C_b^{0,1}$.*

Theorem 3. *Assume Hypothesis 3. Then, there exist a constant $c > 0$ and a non decreasing function K, depending on the dimension d and on the upper bounds of σ, b and their derivatives s.t. $\forall (s,x,y) \in]0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, one has*

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{K(T)T}{N s^{\frac{d+1}{2}}} \exp\left(-\frac{c|x-y|^2}{s}\right).$$

Corollary 4. *Assume Hypothesis 3. From the last inequality and Aronson's inequality (A.1), we deduce*

$$\left| \frac{p(0,x;T,x) - p^N(0,x;T,x)}{p(0,x;T,x)} \right| \leq \frac{K(T)}{N} \sqrt{T}. \tag{2.1}$$

This inequality yields $p(0,x;T,x) \sim p^N(0,x;T,x)$ when $T \to 0$.

Theorem 3 enables to bound quantities like in (1.3) in the following way

Theorem 5. *Assume Hypothesis 3. For any function f such that $|f(x)| \leq c_1 e^{c_2|x|}$, it holds*

$$\begin{aligned} \left|\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]\right| &\leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N}, \\ \left|\mathbb{E}\left[\int_0^T f(X_{\varphi(s)}^N) ds\right] - \mathbb{E}\left[\int_0^T f(X_s) ds\right]\right| &\leq c_1 e^{c_2|x|} K(T) \frac{T}{N}. \end{aligned}$$

Had we used the results stated by Guyon (2006) (and more precisely the one recalled in (1.8)), we would have obtained $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] = O(\frac{1}{N})$. Intuitively, this result is not optimal: the right hand side doesn't tend to 0 when T goes to 0 while it should. Analogously, regarding $\mathbb{E}\left[\int_0^T f(X_{\varphi(s)}^N) ds\right] - \mathbb{E}\left[\int_0^T f(X_{\varphi(s)}) ds\right]$, we would obtain $O(\frac{T \ln N}{N})$ instead of $O(\frac{T}{N})$.

Proof of Theorem 5. Writing $\mathbb{E}[f(X^N_T)]-\mathbb{E}[f(X_T)]$ as $\int_{\mathbb{R}^d} f(y)(p^N(0,x;T,y)-p(0,x;T,y))dy$ and using Theorem 3 yield the first result. Concerning the second result, we split $\mathbb{E}\left[\int_0^T(f(X^N_{\varphi(s)})-f(X_s))ds\right]$ in two terms $\mathbb{E}\left[\int_0^T(f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]$ and $\mathbb{E}\left[\int_0^T(f(X_{\varphi(s)})-f(X_s))ds\right]$. First, using Theorem 3 leads to

$$\left|\mathbb{E}\left[\int_0^T(f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]\right| = \left|\int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y)(p^N(0,x;\varphi(s),y)-p(0,x;\varphi(s),y))\right|,$$
$$\leq \frac{K(T)T}{N}c_1 e^{c_2|x|}\int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}},$$

where we use the easy inequality $\int_{\mathbb{R}^d} e^{c_2|y|}\frac{e^{\frac{-c|x-y|^2}{s}}}{s^{d/2}}dy \leq K(T)e^{c_2|x|}$. Since $\varphi(s)\geq s-\frac{T}{N}$, we get $\left|\mathbb{E}\left[\int_0^T(f(X^N_{\varphi(s)})-f(X_{\varphi(s)}))ds\right]\right| \leq \frac{K(T)T^{3/2}}{N}c_1e^{c_2|x|}$. Second, we write

$$\left|\mathbb{E}\left[\int_0^T(f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1e^{c_2|x|}\frac{T}{N}+\int_{\mathbb{R}^d} dy\int_{\frac{T}{N}}^T ds c_1 e^{c_2|y|}\int_{\varphi(s)}^s du|\partial_u p(0,x;u,y)|.$$

Then, Proposition 13 yields $\left|\mathbb{E}\left[\int_0^T(f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1e^{c_2|x|}\left(\frac{T}{N}+C\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds\right)$. Moreover, $\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds = \sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\ln(\frac{s}{t_k})ds = \frac{T}{N}\sum_{k=1}^{N-1}((k+1)\ln(\frac{k+1}{k})-1)\leq C\frac{T}{N}$, using a second order Taylor expansion. This gives $\left|\mathbb{E}\left[\int_0^T(f(X_{\varphi(s)})-f(X_s))ds\right]\right| \leq c_1e^{c_2|x|}K(T)\frac{T}{N}$. ∎

In the next section, we give results related to Malliavin's calculus, that will be useful for the proof of Theorem 3.

3 Basic results on Malliavin's calculus

We refer the reader to Nualart (2006), for more details. Fix a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ and let $(W_t)_{t\geq 0}$ be a q-dimensional Brownian motion. For $h(\cdot)\in H=\mathbb{L}^2([0,T],\mathbb{R}^q)$, $W(h)$ is the Wiener stochastic integral $\int_0^T h(t)dW_t$. Let $\mathcal{S}$ denote the class of random variables of the form $F=f(W(h_1),\cdots,W(h_n))$ where f is a C^∞ function with derivatives having a polynomial growth, $(h_1,\cdots,h_n)\in H^n$ and $n\geq 1$. For $F\in\mathcal{S}$, we define its derivative $\mathcal{D}F=(\mathcal{D}_tF:=(\mathcal{D}^1_tF,\cdots,\mathcal{D}^q_tF))_{t\in[0,T]}$ as the H valued random variable given by

$$\mathcal{D}_tF=\sum_{i=1}^n \partial_{x_i}f(W(h_1),\cdots,W(h_n))h_i(t).$$

The operator $\mathcal{D}$ is closable as an operator from $\mathbb{L}^p(\Omega)$ to $\mathbb{L}^p(\Omega;H)$, for $p\geq 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p}=[\mathbb{E}|F|^p+\mathbb{E}(\|\mathcal{D}F\|^p_H)]^{1/p}$. We can define the iteration of the operator $\mathcal{D}$, in such a way that for a smooth random variable F, the derivative $\mathcal{D}^kF$ is a random variable with values on $H^{\otimes k}$. As in the case $k=1$, the operator $\mathcal{D}^k$ is closable from $\mathcal{S}\subset\mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(\Omega;H^{\otimes k}),p\geq 1$. If we define the norm

$$\|F\|_{k,p}=[\mathbb{E}|F|^p+\sum_{j=1}^k\mathbb{E}(\left\|\mathcal{D}^jF\right\|^p_{H^{\otimes j}})]^{1/p},$$

we denote its domain by $\mathbb{D}^{k,p}$. Finally, set $\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}$, and $\mathbb{D}^{\infty} = \cap_{k,p\geq 1}\mathbb{D}^{k,p}$. One has the following chain rule property

Proposition 6. *Fix $p \geq 1$. For $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$, and $F = (F_1, \cdots, F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \geq 0$, one has $\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F$, with the notation*

$$\mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_t F_1 \\ \vdots \\ \mathcal{D}_t F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce the Skorohod integral δ, defined as the adjoint operator of $\mathcal{D}$.

Proposition 7. *δ is a linear operator on $\mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ with values in $\mathbb{L}^2(\Omega)$ s.t.*

- *the domain of δ (denoted by $Dom(\delta)$) is the set of processes $u \in \mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ s.t. $|\mathbb{E}(\int_0^T \mathcal{D}_t F \cdot u_t dt)| \leq c(u)|F|_{\mathbb{L}^2}$ for any $F \in \mathbb{D}^{1,2}$.*
- *If u belongs to $Dom(\delta)$, then $\delta(u)$ is the one element of $\mathbb{L}^2(\Omega)$ characterized by the integration by parts formula*

$$\forall F \in \mathbb{D}^{1,2},\ \ \mathbb{E}(F\delta(u)) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right).$$

Remark 8. *If u is an adapted process belonging to $\mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincide : $\delta(u) = \int_0^T u_t dW_t$, and the preceding integration by parts formula becomes*

$$\forall F \in \mathbb{D}^{1,2},\ \ \mathbb{E}\left(F\int_0^T u_t dW_t\right) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right). \tag{3.1}$$

This equality is also called the duality formula.

This duality formula is the corner stone to establish general integration by parts formula of the form

$$\mathbb{E}[\partial^\alpha g(F)G] = \mathbb{E}[g(F)H_\alpha(F,G)]$$

for any non degenerate random variables F. We only give the formulation in the case of interest $F = X_t^N$.

Proposition 9. *We assume that σ is uniformly elliptic and b and σ are in $C_b^{0,3}$. For all $p > 1$, for all multi-index α s.t. $|\alpha| \leq 2$, for all $t \in]0,T]$, all $u, r, s \in [0,T]$ and for any functions f and g in $C_b^{|\alpha|}$, there exist a random variable $H_\alpha \in \mathbb{L}^p$ and a function $K(T)$ (uniform in N, x, s, u, r, t, f and g) s.t.*

$$\mathbb{E}[\partial_x^\alpha f(X_t^N)g(X_u^N, X_r^N, X_s^N)] = \mathbb{E}[f(X_t^N)H_\alpha], \tag{3.2}$$

with

$$|H_\alpha|_{\mathbb{L}_p} \leq \frac{K(T)}{t^{\frac{|\alpha|}{2}}}\|g\|_{C_b^{|\alpha|}}. \tag{3.3}$$

These results are given in the article of Kusuoka and Stroock (1984): (3.3) is owed to Theorem 1.20 and Corollary 3.7.
Another consequence of the duality formula is the derivation of an upper bound for p^N.

Proposition 10. *Assume σ is uniformly elliptic and b and σ are in $C_b^{0,2}$. Then, for any $x, y \in \mathbb{R}^d$, $s \in]0,T]$, one has*

$$p^N(0,x;s,y) \leq \frac{K(T)}{s^{d/2}} e^{-c\frac{|x-y|^2}{s}}, \tag{3.4}$$

for a positive constant c and a non decreasing function K, both depending on d and on the upper bounds for b, σ and their derivatives.

Although this upper bound seems to be quite standard, to our knowledge such a result has not appeared in the literature before, except in the case of time homogeneous coefficients (see Konakov and Mammen (2002), proof of Theorem 1.1).

Proof. The inequality (1.32) of Kusuoka and Stroock (1984, Theorem 1.31) gives $p^N(0,x;s,y) \leq \frac{K(T)}{s^{d/2}}$ for any x and y. This implies the required upper bound when $|x-y| \leq \sqrt{s}$. Let us now consider the case $|x-y| > \sqrt{s}$. Using the same notations as in Kusuoka and Stroock (1984), we denote $\psi(y) = \rho(\frac{|y-x|}{r})$ where $r > 0$ and ρ is a C_b^∞ function such that $\mathbf{1}_{\{[3/4,\infty[\}} \leq \rho \leq \mathbf{1}_{\{[1/2,\infty[\}}$. Then, combining inequality (1.33) of Kusuoka and Stroock (1984, Theorem 1.31) and Corollary 3.7 leads to

$$\sup_{|y-x|\geq r} p^N(0,x;s,y) \leq K(T)\frac{e^{-c\frac{r^2}{s}}}{s^{d/2}}\left(1+\sqrt{\frac{s}{r^2}}\right),$$

where we use $|\psi(X_s^N)|_{1,q} \leq K(T)e^{-c\frac{r^2}{s}}\left(1+\sqrt{\frac{s}{r^2}}\right)$. This easily completes the proof in the case $|x-y| \geq \sqrt{s}$. ∎

4 Proof of Theorem 3

In the following, $K(\cdot)$ denotes a generic non decreasing function (which may depend on d, b and σ). To prove Theorem 3, we take advantage of Propositions 9 and 10. The scheme of the proof is the following

- Use a PDE and Itô's calculus to write the difference $p^N(0,x;s,y) - p(0,x;s,y)$

$$= \int_0^s \mathbb{E}\left[\sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N))\partial_{x_i} p(r, X_r^N; s, y)\right.$$
$$\left.+\frac{1}{2}\sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N))\partial^2_{x_i x_j} p(r, X_r^N; s, y)\right] dr := E_1 + E_2. \tag{4.1}$$

- Prove the intermediate result $\forall (r,x,y) \in [0,s[\times \mathbb{R}^d \times \mathbb{R}^d$ and $c > 0$

$$\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right] \leq K(T)\left(\frac{s-r}{s}\right)^{\frac{d}{2}} \exp\left(-c'\frac{|x-y|^2}{s}\right), \tag{4.2}$$

  where $c' > 0$.

- Use Malliavin's calculus, Proposition 10 and the intermediate result, to show that each term E_1 and E_2 (see (4.1)) is bounded by $\frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{s})$.

Definition 11. *We say that a term $E(x,s,y)$ satisfies property $\mathcal{P}$ if $\forall (x,s,y) \in \mathbb{R}^d \times]0,T] \times \mathbb{R}^d$*

$$|E(x,s,y)| \leq \frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right). \quad (\mathcal{P})$$

4.1 Proof of equality (4.1)

First, the transition density function $(r,x) \longmapsto p(r,x;s,y)$ satisfies the PDE

$$(\partial_r + \mathcal{L}_{(r,x)})p(r,x;s,y) = 0, \ \ \forall r \in [0,s[, \forall x \in \mathbb{R}^d,$$

where $\mathcal{L}_{(r,x)}$ is defined by $\mathcal{L}_{(r,x)} = \sum_{i,j} a_{ij}(r,x)\partial^2_{x_i x_j} + \sum_i b_i(r,x)\partial_{x_i}$, and $a_{ij}(r,x) = \frac{1}{2}[\sigma\sigma^*]_{ij}(r,x)$. The function, as well as its first derivatives, are uniformly bounded by a constant depending on ϵ for $|s-r| \geq \epsilon$ (see Appendix A).
Second, since $p^N(0,x;s,y)$ is a continuous function in s and y (convolution of Gaussian densities), we observe that

$$p^N(0,x;s,y) - p(0,x;s,y) = \lim_{\epsilon \longrightarrow 0} \mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon};s,y) - p(0,x;s,y)].$$

Then, for any $\epsilon > 0$, Itô's formula leads to

$$\mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon};s,y) - p(0,x;s,y)] = \mathbb{E}\left[\int_0^{s-\epsilon} \partial_r p(r, X^N_r;s,y)dr\right]$$
$$+\mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i=1}^d b_i(\varphi(r), X^N_{\varphi(r)})\partial_{x_i}p(r,X^N_r;s,y)dr + \frac{1}{2}\int_0^{s-\epsilon}\sum_{i,j=1}^d a_{ij}(\varphi(r), X^N_{\varphi(r)})\partial^2_{x_i x_j}p(r,X^N_r;s,y)dr\right].$$

From the PDE, the above equality becomes

$$\begin{aligned}\mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon};s,y) - p(0,x;s,y)] &= \mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r,X^N_r))\partial_{x_i}p(r,X^N_r;s,y)dr\right] \\ &+ \frac{1}{2}\mathbb{E}\left[\int_0^{s-\epsilon}\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r,X^N_r))\partial^2_{x_i x_j}p(r,X^N_r;s,y)dr\right], \\ &:= \int_0^{s-\epsilon}\mathbb{E}[\phi(r)]dr,\end{aligned}$$

where $\phi(r) = \sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r,X^N_r))\partial_{x_i}p(r,X^N_r;s,y) + \frac{1}{2}\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r,X^N_r))\partial^2_{x_i x_j}p(r,X^N_r;s,y)$. To get (4.1), it remains to prove that $\mathbb{E}(\phi(r))$ is integrable over $[0,s]$. We check it by looking at the rest of the proof.

4.2 Proof of the intermediate result (4.2)

We prove inequality (4.2). $\mathbb{E}[\exp(-c\frac{|y-X_r^N|^2}{s-r})] = \int_{\mathbb{R}^d}\exp(-c\frac{|y-z|^2}{s-r})p^N(0,x;r,z)dz$. Using Proposition 10, we get

$$\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right] \leq \frac{K(T)}{r^{\frac{d}{2}}}\int_{\mathbb{R}^d}\exp\left(-c\frac{|y-z|^2}{s-r}\right)\exp\left(-c'\frac{|x-z|^2}{r}\right)dz$$
$$\leq K(T)\Pi_{i=1}^d\int_{\mathbb{R}}\frac{1}{\sqrt{r}}\exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right)\exp\left(-c'\frac{|x_i-z_i|^2}{r}\right)dz_i,$$

and $\displaystyle\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\frac{(s-r)}{2c}}}\exp(-c\frac{|y_i-z_i|^2}{s-r})\frac{1}{\sqrt{2\pi\frac{r}{2c'}}}\exp(-c'\frac{|x_i-z_i|^2}{r})dz_i$ is the convolution product of the density of two independant Gaussian random variables $\mathcal{N}(-x_i,\frac{r}{2c'})$ and $\mathcal{N}(y_i,\frac{s-r}{2c})$ computed at 0. Hence, the integral is equal to $\displaystyle\frac{1}{\sqrt{2\pi(\frac{r}{2c'}+\frac{s-r}{2c})}}\exp\left(-\frac{|x_i-y_i|^2}{\frac{r}{c'}+\frac{s-r}{c}}\right)$. Then,

$$\int_{\mathbb{R}}\frac{1}{\sqrt{r}}\exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right)\exp\left(-c'\frac{|x_i-z_i|^2}{r}\right)dz_i \leq C\left(\frac{s-r}{s}\right)^{\frac{1}{2}}\exp\left(-c''\frac{|x_i-y_i|^2}{s}\right)$$

and (4.2) follows.

4.3 Upper bound for E_1

We recall that $E_1=\int_0^s\mathbb{E}\left[\sum_{i=1}^d(b_i(\varphi(r),X^N_{\varphi(r)})-b_i(r,X_r^N))\partial_{x_i}p(r,X_r^N;s,y)\right]dr$. For each i, we apply Itô's formula to $b_i(u,X_u^N)$ between $u=\varphi(r)$ and $u=r$. We get

$$b_i(\varphi(r),X^N_{\varphi(r)})-b_i(r,X_r^N)=\int_{\varphi(r)}^r\alpha_u^i du+\int_{\varphi(r)}^r\sum_{k=1}^q\beta_u^{i,k}dW_u^k, \tag{4.3}$$

where α_u^i depends on $\partial_t b,\partial_x b,\partial_x^2 b,\sigma$, and $\beta_u^i=-\nabla_x b_i(u,X_u^N)\sigma(\varphi(r),X^N_{\varphi(r)})$. Since b,σ belong to $C_b^{1,3}$, α^i and $(\beta^{i,k})_{1\leq k\leq q}$ are uniformly bounded. Using (4.3) and the duality formula (3.1) yield

$$E_1=\sum_{i=1}^d\int_0^s\{\mathbb{E}[\int_{\varphi(r)}^r\partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i du+\mathbb{E}[\int_{\varphi(r)}^r\mathcal{D}_u(\partial_{x_i}p(r,X_r^N;s,y))\cdot\beta_u^i du]\}dr$$
$$:=E_{11}+E_{12}, \tag{4.4}$$

where β_u^i is a row vector of q components. We upper bound E_{11} and E_{12}.

Bound for $E_{11}=\sum_{i=1}^d\int_0^s\mathbb{E}[\int_{\varphi(r)}^r\partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i du]dr$.
Since $|\sum_{i=1}^d\partial_{x_i}p(r,X_r^N;s,y)\alpha_u^i|\leq|\alpha_u||\partial_x p(r,X_r^N;s,y)|$ and α_u is uniformly bounded in u, we have

$$|E_{11}|\leq C\frac{T}{N}\int_0^s\mathbb{E}|\partial_x p(r,X_r^N;s,y)|dr.$$

Besides that, from Proposition 13, $|\partial_x p(r, X_r^N; s, y)| \le \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp\big(-c\frac{|y-X_r^N|^2}{s-r}\big)$. Then,

$$|E_{11}| \le K(T)\frac{T}{N}\int_0^s \frac{1}{(s-r)^{\frac{d+1}{2}}}\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right]dr.$$

Using the intermediate result (4.2) yields

$$|E_{11}| \le K(T)\frac{T}{N}\int_0^s \frac{1}{\sqrt{s-r}}\frac{1}{s^{\frac{d}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right)dr \le K(T)\frac{T}{N}\frac{1}{s^{\frac{d-1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right)$$

and thus, E_{11} satisfies property $\mathcal{P}$ (see Definition 11).

Bound for $E_{12} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du]dr.$

To rewrite E_{12}, we use the expression of β_u^i and Proposition 6, which gives $\mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i} p(r, X_r^N; s, y))\sigma(\varphi(r), X_{\varphi(r)}^N)$. Then,

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial^2_{x_i x_k} p(r, X_r^N; s, y)[(\sigma\sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k]du. \quad (4.5)$$

Using the integration by parts formula (3.2), we get that

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_k}^i(r, u)]du$$

where e_k is a vector whose k-th component is 1 and other components are 0. From (3.3), we deduce $\mathbb{E}[|H_{e_k}^i(r,u)|^p]^{1/p} \le C\frac{K(T)}{r^{1/2}}$, where C only depends on $|\sigma|_\infty$, $|\partial_x \sigma|_\infty$, $|\partial_x b|_\infty$, $|\partial^2_{xx} b|_\infty$. By the Hölder inequality, it follows that

$$|E_{12}| \le K(T)\int_0^s dr \int_{\varphi(r)}^r \frac{1}{r^{1/2}}\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}}du.$$

Using Proposition 13 leads to $|\partial_x p(r, X_r^N; s, y)| \le \frac{K(T)}{(s-r)^{\frac{d+1}{2}}}\exp(-c\frac{|y-X_r^N|^2}{s-r})$, and combining this inequality with the intermediate result (4.2) yields

$$\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{d/(d+1)} \le \frac{K(T)}{(s-r)^{\frac{d+1}{2}}}\left(\frac{s-r}{s}\right)^{\frac{d^2}{2(d+1)}}\exp\left(-c\frac{|y-x|^2}{s}\right). \quad (4.6)$$

Hence, E_{12} is bounded by

$$\frac{K(T)}{s^{\frac{d^2}{2(d+1)}}}\frac{T}{N}\exp\left(-c\frac{|y-x|^2}{s}\right)\int_0^s \frac{1}{r^{1/2}}\frac{1}{(s-r)^{\frac{d+1}{2}-\frac{d^2}{2(d+1)}}}dr.$$

The above integral equals $s^{\frac{1}{2}-\frac{d+1}{2}+\frac{d^2}{2(d+1)}}B(\frac{1}{2}, \frac{1}{2(d+1)})$ where B is the function Beta. Thus $|E_{12}| \le \frac{K(T)}{s^{d/2}}\frac{T}{N}\exp(-c\frac{|y-x|^2}{s})$, and E_{12} satisfies property $\mathcal{P}$.

4.4 Upper bound for E_2

We recall $E_2 = \dfrac{1}{2}\displaystyle\int_0^s \mathbb{E}[\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r))\partial^2_{x_i x_j} p(r, X^N_r; s, y)]dr$. As we did for E_1, we apply Itô's formula to $a_{ij}(u, X^N_u)$ between $\varphi(r)$ and r. We get $a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r) = \int_{\varphi(r)}^r \gamma^{ij}_u du + \int_{\varphi(r)}^r \delta^{ij}_u dW_u$, where γ^{ij}_u depends on $\sigma, \partial_t\sigma, \partial_x\sigma, b, \partial^2_{xx}\sigma$ and δ^{ij}_u is a row vector of size q, with l-th component $(\delta^{ij}_u)_l = -\sum_{k=1}^d \partial_{x_k} a_{ij}(u, X^N_u)\sigma_{kl}(\varphi(r), X^N_{\varphi(r)})$. Then, the duality formula (3.1) leads to

$$\begin{aligned} E_2 = \sum_{i,j=1}^d \int_0^s \{\mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_i x_j} p(r, X^N_r; s, y)\gamma^{ij}_u du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_i x_j} p(r, X^N_r; s, y)) \cdot \delta^{ij}_u du]\}dr \\ := E_{21} + E_{22}. \end{aligned}$$

Bound for $E_{21} = \sum_{ij=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_i x_j} p(r, X^N_r; s, y)\gamma^{ij}_u du]dr$**.**
As $\sigma, b, \partial_t\sigma, \partial_x\sigma, \partial^2_x\sigma$ are C^1_b in space, γ^{ij}_u has the same smoothness properties as the term $[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x b_i(u, X^N_u))^*]_k$ appearing in (4.5). Thus, E_{21} can be treated as E_{12} and satisfies to the same estimate.

Bound for $E_{22} = \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_i x_j} p(r, X^N_r; s, y)) \cdot \delta^{ij}_u du]dr$.

To rewrite E_{22}, we use the expression of δ^{ij}_u and Proposition 6, which asserts $\mathcal{D}_u(\partial^2_{x_i x_j} p(r, X^N_r; s, y)) = \nabla_x(\partial^2_{x_i x_j} p(r, X^N_r; s, y))\sigma(\varphi(r), X^N_{\varphi(r)})$. Thus,

$$E_{22} = -\sum_{i,j,k=1}^d \int_0^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial^3_{x_i x_j x_k} p(r, X^N_r; s, y)[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x a_{ij}(u, X^N_u))^*]_k]du.$$

To complete this proof, we split E_{22} in two terms : E^1_{22} (resp E^2_{22}) corresponds to the integral in r from 0 to $\frac{s}{2}$ (resp. from $\frac{s}{2}$ to s).

- On $[0, \frac{s}{2}]$, E^1_{22} is bounded by $C\frac{T}{N}\int_0^{\frac{s}{2}} \mathbb{E}[|\partial^3_{x_i x_j x_k} p(r, X^N_r; s, y)|]dr$. Using Proposition 13 and (4.2), it gives

$$|E^1_{22}| \le \frac{K(T)T}{N}\frac{1}{s^{d/2}}\exp\left(-c\frac{|x-y|^2}{s}\right)\int_0^{\frac{s}{2}} \frac{1}{(s-r)^{3/2}}dr.$$

  Hence, E_{22} satisfies $\mathcal{P}$.

- On $[\frac{s}{2}, s]$, we use the integration by parts formula (3.2) of Proposition 9, with $|\alpha| = 2$.

$$E^2_{22} = -\sum_{i,j,k=1}^d \int_{\frac{s}{2}}^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i} p(r, X^N_r; s, y)H^i_{e_{jk}}]du,$$

  where e_{jk} is a vector full of zeros except the j-th and the k-th components. Using Hölder's inequality and (3.3) (remember that $\sigma \in C^{1,3}_b$), we obtain

$$|E^2_{22}| \le K(T)\frac{T}{N}\int_{\frac{s}{2}}^s \frac{1}{r}\mathbb{E}[|\partial_x p(r, X^N_r; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}}dr. \tag{4.7}$$

By applying (4.6), we get

$$|E_{22}^2| \le K(T)\frac{T}{N}\frac{1}{s^{1+\frac{d^2}{2(d+1)}}}\exp\left(-c\frac{|x-y|^2}{s}\right)\int_{\frac{s}{2}}^{s}\frac{1}{(s-r)^{\frac{2d+1}{2d+2}}}dr,$$

and the result follows.

A Bounds for the transition density function and its derivatives

We bring together classical results related to bounds for the transition probability density of X defined by (1.1).

Proposition 12 (Aronson (1967)). *Assume that the coefficients σ and b are bounded measurable functions and that σ is uniformly elliptic. There exist positive constants K, α_0, α_1 s.t. for any x, y in $\mathbb{R}^d$ and any $0 \le t < s \le T$, one has*

$$\frac{K^{-1}}{(2\pi\alpha_1(s-t))^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2\alpha_1(s-t)}} \le p(t,x;s,y) \le K\frac{1}{(2\pi\alpha_2(s-t))^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2\alpha_2(s-t)}}. \tag{A.1}$$

Proposition 13 (Friedman (1964)). *Assume that the coefficients b and σ are Hölder continuous in time, C_b^2 in space and that σ is uniformly elliptic. Then, $\partial_x^{m+a}\partial_y^b p(t,x;s,y)$ exist and are continuous functions for all $0 \le |a| + |b| \le 2, |m| = 0, 1$. Moreover, there exist two positive constants c and K s.t. for any x, y in $\mathbb{R}^d$ and any $0 \le t < s \le T$, one has*

$$|\partial_x^{m+a}\partial_y^b p(t,x;s,y)| \le \frac{K}{(s-t)^{(|m|+|a|+|b|+d)/2}}\exp\left(-c\frac{|y-x|^2}{s-t}\right).$$

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