VARIATIONAL CHARACTERIZATIONS OF THE EFFECTIVE MULTIPLICATION FACTOR OF A NUCLEAR REACTOR CORE

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ABSTRACT. We prove some inf–sup and sup–inf formulae for the so–called effective multiplication factor arising in the study of reactor analysis. We treat in a same formalism the transport equation and the energy–dependent diffusion equation.
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1. INTRODUCTION

The aim of this paper is to give some variational characterizations of the *effective multiplication factor* arising in nuclear reactor theory. This work follows a very recent paper by M. Mokhtar-Kharroubi [22] devoted to the leading eigenvalue of transport operators.

In practical situations, the *power distribution* in a stable nuclear reactor core is determined as the steady-state solution ϕ of a linear transport equation for the neutron flux. Because of interactions between neutrons and fissile isotopes, a fission chain reaction occurs in the reactor core. Precisely, when an atom undergoes nuclear fission, some neutrons are ejected from the reaction and subsequently shall interact with the surrounding medium. If more fissile fuel is present, some may be absorbed and cause more fissions (see for details Refs. [5, 6, 11, 25]). The linear stationary transport equation is therefore of non-standard type in the sense that the source term is itself a function of the solution. When delayed neutrons are neglected, this equation reads

$$v \cdot \nabla_x \phi(x,v) + \sigma(x,v)\phi(x,v) - \int_V \kappa_s(x,v,v')\phi(x,v')d\mu(v') = \frac{1}{k_{\text{eff}}} \int_V \kappa_f(x,v,v')\phi(x,v')d\mu(v') \quad (1.1)$$

with *free-surface* boundary condition (i.e. the incoming flux is null). Here, the unknown $\phi(x,v)$ is the neutron density at point $x \in \mathcal{D}$ and velocity $v \in V$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^N$ (representing the reactor core) and the velocity space V is a closed subset of $\mathbb{R}^N$, $d\mu(\cdot)$ being a positive Radon measure supported by V. For the usual cases, $d\mu(\cdot)$ is either the Lebesgue measure on $\mathbb{R}^N$ (continuous model) or on spheres (multigroup model). The transfer cross-sections $\kappa_s(\cdot,\cdot,\cdot)$ and $\kappa_f(\cdot,\cdot,\cdot)$ describe respectively the *pure scattering* and the *fission* process. The *nonnegative* bounded function

$\sigma(\cdot,\cdot)$ is the *absorption cross–section* [5, 11, 25]. The positive ratio k_{eff} is called the *criticality eigenvalue* (or the *effective multiplication factor*). It represents the average number of neutrons that go on to cause another fission reaction. The remaining neutrons either fail to induce fission, or never get absorbed and exit the system. Consequently, k_{eff} measures the balance between the number of neutrons in successive generations (where the birth event separating generations is the fission process). The interpretation of the effective multiplication factor k_{eff} is related to the following three cases (see [5, 11, 25]):

• If $k_{\text{eff}} = 1$, there is a perfect balance between production and removal of neutrons. The reactor is then said to be *critical*.
• The reactor is *sub-critical* when $k_{\text{eff}} < 1$. This means that the removal of neutrons (at the boundary or due to absorption by the surrounding media) excesses the fission process and the chain reaction dies out rapidly.
• When $k_{\text{eff}} > 1$, the fission chain reaction grows without bound and the reactor is said to be *super–critical*.

Up to now, in practical applications, the effective multiplication factor k_{eff} was usually given by $r_\sigma\left[(\mathcal{T}-\mathcal{K}_s)^{-1}\mathcal{K}_f\right] = k_{\text{eff}}$, where the precise definition of the operators $\mathcal{T}$, $\mathcal{K}_s$ and $\mathcal{K}_f$ is given subsequently and $r_\sigma[B]$ denotes the spectral radius of any generic bounded operator B: $r_\sigma[B] = \lim_{n\to\infty}\|B^n\|^{1/n}$. Because its requires the computation of the resolvent $(\mathcal{T}-\mathcal{K}_s)^{-1}$, such a characterization makes the analysis of the effective multiplication factor quite difficult to handle. In particular, practical estimates of k_{eff} as well as computational approximations are rather involved and merely rely on (direct or inverse) power method [2]. Our main objective in this paper is to provide tractable variational characterizations of k_{eff} in terms of the different data of the system and suitable test functions. We hope that such a characterization shall be of interest for practical computations or for the homogenization of the above criticality transport equation in periodic media [1, 4, 21]. We also believe that our characterization can be useful in the delicate optimization problem of the assembly distribution in a nuclear reactor (see the recent contribution [28] based upon the homogenization method and where the criticality eigenvalue k_{eff} plays a crucial role).

To be precise, we provide here *variational characterizations* of the effective multiplication factor of the type

$$
\begin{aligned}
\frac{1}{k_{\text{eff}}} &= \min_{\varphi\in W_p^+}\ \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \frac{v\cdot\nabla_x\varphi(x,v)+\sigma(x,v)\varphi(x,v)-\displaystyle\int_V \Sigma_s(x,v,v')\varphi(x,v')d\mu(v')}{\displaystyle\int_V \Sigma_f(x,v,v')\varphi(x,v')d\mu(v')}\\
&= \max_{\varphi\in W_p^+}\ \operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times V} \frac{v\cdot\nabla_x\varphi(x,v)+\sigma(x,v)\varphi(x,v)-\displaystyle\int_V \Sigma_s(x,v,v')\varphi(x,v')d\mu(v')}{\displaystyle\int_V \Sigma_f(x,v,v')\varphi(x,v')d\mu(v')},
\end{aligned}
\tag{1.2}
$$

where W_p^+ is a suitable class of *positive test–functions* in $L^p(\mathcal{D} \times V, dx d\mu(v))$ $(1 \leqslant p < \infty)$. This result (Theorem 3.3) holds true under compactness assumption on the *full collision operator*

$$\mathcal{K} \,:\, \psi \longmapsto \int_V \big(\Sigma_s(x,v,v') + \Sigma_f(x,v,v')\big)\,\psi(x,v')\,d\mu(v')$$

and under *positivity assumptions on the fission cross–section* $\kappa_f(\cdot,\cdot,\cdot)$. The main strategy to derive (1.2) is adapted from [22] where M. Mokhtar-Kharroubi proved similar variational characterizations for the leading eigenvalue of *perturbed* transport operators. Note that the above characterization still holds true for transport equations with general boundary conditions modeled by some nonnegative albedo operator (see Remark 3.6).

At this point, one recalls that, besides the critical eigenvalue k_{eff}, it is also possible to investigate the reactivity of the nuclear reactor core through another physical parameter, namely, the leading eigenvalue $s(\mathcal{A})$ of the operator $\mathcal{A} = \mathcal{T} + \mathcal{K}_s + \mathcal{K}_f$, also associated to positive eigenfunctions. The two parameters k_{eff} and $s(\mathcal{A})$ are related by the following: if $s(\mathcal{A}) < 0$ then the reactor is subcritical (i.e. $k_{\text{eff}} < 1$), while it is super-critical whenever $s(\mathcal{A}) > 0$. The reactor is critical when $s(\mathcal{A}) = 0$. The paper [22] provides a variational characterization of the leading eigenvalue of the transport operator $\mathcal{A}$. However, for practical calculations in nuclear engineering, the critical eigenvalue k_{eff} is a more effective parameter. Actually, the existence of the leading eigenvalue $s(\mathcal{T} + \mathcal{K}_s + \mathcal{K}_f)$ is not always ensured but is related to the size of the domain $\mathcal{D}$ and the possibility of small velocities (for more details on this disappearance phenomenon, see e.g. [19, Chapter 5]). Since the existence of the effective multiplication factor is not restricted by such physical constraints, it appears more efficient to measure the reactivity of nuclear reactor cores by k_{eff}. This is what motivated us to generalize the result of [22] and provide variational characterization of k_{eff}.

In this paper, we also give a characterization of the criticality eigenvalue associated to the *energy-dependent diffusion* model used in nuclear reactor theory [5, 24, 25]. For this description, the critical problem reads

$$-\,\mathrm{div}_x\big(\mathbb{D}(x,\xi)\nabla_x\varrho(x,\xi)\big) + \sigma(x,\xi)\varrho(x,\xi) - \int_E \Sigma_s(x,\xi,\xi')\,\varrho(x,\xi')\,d\xi' = \frac{1}{k_{\text{eff}}}\int_E \Sigma_f(x,\xi,\xi')\,\varrho(x,\xi')\,d\xi', \quad (1.3)$$

complemented by the Dirichlet boundary conditions $\varrho(\cdot,\xi)_{|\partial\mathcal{D}} = 0$ a.e. $\xi \in E$. Here E is the set of admissible energies $\xi = \frac{1}{2}mv^2$ (m being the neutron mass and v the velocity), i.e. E is a subset of $[0,+\infty[$. The diffusion coefficient $\mathbb{D}(\cdot,\cdot)$ is a matrix–valued function over $\mathcal{D} \times E$ and the unknown $\varrho(\cdot,\cdot)$ is *nonnegative*.

The derivation of diffusion-like models for some macroscopic distribution function $\varrho(x,\xi)$ (corresponding to some angular momentum of the solution ϕ to (1.1)) is motivated in nuclear engineering by the necessity to provide simplified models tractable numerically. Such a *energy-dependent diffusion* model can be derived directly from a phenomenological analysis of the scattering models or it can be derived from the above kinetic equation (1.1) through a suitable asymptotic procedure (see [10] and the references therein for more details on that matter).

For this *energy-dependent diffusion* model, we give a variational characterization of k_{eff} in terms of sup-inf and inf-sup criteria in the spirit of (1.2).

Actually, to treat the two above problems (1.1) and (1.3) it is possible to adopt a unified mathematical formalism. Precisely, let us denote by $\mathcal{K}_s$ the integral operator with kernel $\kappa_s(\cdot,\cdot,\cdot)$ and denote by $\mathcal{K}_f$ the integral operator with kernel $\kappa_f(\cdot,\cdot,\cdot)$. Then, problems (1.1) and (1.3) may be written in a *unified abstract way*:

$$\mathcal{T}\phi_{\text{eff}} + \mathcal{K}_s\phi_{\text{eff}} + \frac{1}{k_{\text{eff}}}\mathcal{K}_f\phi_{\text{eff}} = 0, \qquad \phi_{\text{eff}} \geqslant 0,$$

where the unbounded operator $\mathcal{T}$ refers to, according to the model we adopt:

- the transport operator:

$$\mathcal{T}\phi(x,v) = -v\cdot\nabla_x\phi(x,v) - \sigma(x,v)\phi(x,v),$$

associated to the absorbing boundary conditions $\phi_{|\Gamma_-} = 0$.

- the energy–dependent diffusion operator:

$$\mathcal{T}\psi(x,\xi) = \text{div}_x\left(\mathbb{D}(x,\xi)\nabla_x\psi(x,\xi)\right) - \sigma(x,\xi)\psi(x,\xi)$$

with Dirichlet boundary conditions $\psi_{|\,\partial\mathcal{D}}(\cdot,\xi) = 0$ (a.e. $\xi\in E$).

The abstract treatment of the above problem is performed in Section 2 and relies mainly on positivity and compactness arguments. The main *abstract* result of this paper (Theorem 2.15) characterizes the criticality eigenvalue of a large class of (abstract) unbounded operators in L^p-spaces. Besides this main analytical result, we also prove abstract results with their own interest. In particular, we provide in Theorem 2.12 *an approximation resolution for the criticality eigenfunction* ϕ_{eff} which shall be hopefully useful for practical numerical approximations.

The outline of the paper is as follows. In Section 2, we describe the unified and abstract framework which allows us to treat in a same formalism Problems (1.1) and (1.3) with the aim of establishing general inf–sup and sup–inf formulae for the criticality eigenvalue of a class of unbounded operator. In Section 3, we are concerned with the characterization of the effective multiplication factor k_{eff} associated to the transport problem (1.1). In Section 4, we investigate the effective multiplication factor associated to the energy-dependent diffusion model (1.3).

2. Abstract variational characterization

This section is devoted to several abstract variational characterizations of the criticality eigenvalue. It is this abstract material that shall allow us to treat in the same formalism Problems (1.1) and (1.3).

2.1. **Setting of the problem and existence result.** Let us introduce the functional framework we shall use in the sequel. Given a measure space $(\mathbf{\Omega},\nu)$ and a fixed $1\leqslant p<\infty$, define

$$X_p = L^p(\mathbf{\Omega}, d\nu)$$

and denote by X_q its dual space, i.e. $X_q = L^q(\mathbf{\Omega}, d\nu)$ $(1/p + 1/q = 1)$. We first recall several definitions and facts about *positive operators*. Though the various concepts we shall deal with could be defined in general complex Banach lattices, we restrict ourselves to operators in X_p $(1\leqslant p<\infty)$:

Definition 2.1. *A bounded operator B in X_p is said to be* ***irreducible*** *if, for every nonnegative $\varphi \in X_p \setminus \{0\}$ and nonnegative $\psi \in X_q \setminus \{0\}$, there exists $n \in \mathbb{N}$ such that*

$$\langle B^n \varphi, \psi \rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X_p and the dual space X_q.

Let us denote the set of *quasi-interiors* elements of X_p by X_p^+, i.e.

$$X_p^+ = \{ f \in X_p \ ; \ f(\omega) > 0 \ d\nu - \text{a.e. } \omega \in \boldsymbol{\Omega} \}.$$

Notice that, if $f \in X_p^+$, then $\langle f, \psi \rangle > 0$ for any nonnegative $\psi \in X_q \setminus \{0\}$.

Definition 2.2. *A bounded operator B in X_p will be said to be* ***positivity improving*** *if its maps nonnegative $f \in X_p \setminus \{0\}$ into X_p^+, i.e.*

$$f \in X_p \setminus \{0\}, f \geqslant 0 \Longrightarrow Bf \in X_p^+.$$

Remark 2.3. *Notice that, given a bounded operator B in X_p, if some power of B is positivity improving, then B is irreducible. This provides a practical criterion of irreducibility.*

Recall also several fact about power-compact operators.

Definition 2.4. *A bounded operator B in a Banach space X is said to be* ***power-compact*** *if there exists $n \in \mathbb{N}$ such that B^n is a compact operator in X.*

The following fundamental result is due to B. De Pagter [8].

Theorem 2.5. *Let B be a bounded operator in a Banach space X. If B is irreducible and power-compact then $r_\sigma(B) > 0$ where $r_\sigma(B)$ denotes the spectral radius of B.*

Let $\mathcal{T}$ be a given densely defined unbounded operator

$$\mathcal{T} \ : \ \mathscr{D}(\mathcal{T}) \subset X_p \longrightarrow X_p$$

such that

$$s(\mathcal{T}) < 0 \qquad \text{and} \qquad (0 - \mathcal{T})^{-1}(X_p^+) \subset X_p^+. \tag{2.1}$$

Let $\mathcal{K}_s$ and $\mathcal{K}_f$ be two *nonnegative bounded* operators in X_p. We are interested in the abstract critical problem:

$$(\mathcal{T} + \mathcal{K}_s + \frac{1}{k_{\text{eff}}}\mathcal{K}_f)\phi_{\text{eff}} = 0, \qquad \phi_{\text{eff}} \in \mathscr{D}(\mathcal{T}) \,, \ \phi_{\text{eff}} \geqslant 0, \ \phi_{\text{eff}} \neq 0. \tag{2.2}$$

Since $s(\mathcal{T}) < 0$, Problem (2.2) is equivalent to

$$(0 - \mathcal{T})^{-1}(\mathcal{K}_s + \frac{1}{k_{\text{eff}}}\mathcal{K}_f)\phi_{\text{eff}} = \phi_{\text{eff}} \,, \qquad \phi_{\text{eff}} \geqslant 0, \ \phi_{\text{eff}} \neq 0.$$

Let us introduce the family of operators indexed by the positive parameter γ:

$$\mathcal{K}(\gamma) = \mathcal{K}_s + \frac{1}{\gamma}\mathcal{K}_f \qquad \gamma > 0.$$

Therefore, solving (2.2) is equivalent to prove the existence (and uniqueness) of $k_{\text{eff}} > 0$ such that 1 is an eigenvalue of $(0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})$ associated to a *nonnegative eigenfunction*. Such an existence and uniqueness result can be found in [19, Theorem 5.30] (see also [18]). We set $\mathcal{K} = \mathcal{K}_s + \mathcal{K}_f$

Theorem 2.6. *Assume that $(0-\mathcal{T})^{-1}\mathcal{K}$ is power-compact and that $(0-T)^{-1}\mathcal{K}_f$ is irreducible. Then, the spectral problem (2.2) admits a unique solution $k_{\text{eff}}>0$ associated with a nonnegative eigenfunction ϕ_{eff} if and only if*

$$\lim_{\gamma\to 0} r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]>1 \quad \textit{and} \quad r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}_s]<1. \tag{2.3}$$

Remark 2.7. *Notice that our assumptions differs slightly from that of* [19]*. Actually, in* [19]*, it is assumed that $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ is power-compact and irreducible for any $\gamma>0$. Our assumption implies those of* [19]*. Indeed, in this case, there is an integer $N\in\mathbb{N}$ such that $\left[(0-\mathcal{T})^{-1}\mathcal{K}\right]^N$ is compact. Since, for any $\gamma>0$, $\mathcal{K}(\gamma)\leqslant\max\{1,1/\gamma\}\mathcal{K}$, one gets by a domination argument that $\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)\right]^N$ is compact for any $\gamma>0$. This means that the power at which $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ becomes compact is independent of $\gamma>0$. In the same way, since $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)\geqslant\frac{1}{\gamma}(0-\mathcal{T})^{-1}\mathcal{K}_f$ for any $\gamma>0$, our assumption implies the irreducibility of $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ for any $\gamma>0$. Notice also that the result still holds if $(0-\mathcal{T})^{-1}\mathcal{K}_s$ is irreducible.*

Remark 2.8. *Under the assumptions of the previous Theorem, we point out that the mapping $\gamma>0\mapsto r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]$ is continuous (see* [15, Remark 3.3, p. 208]*). By analyticity arguments (Gohberg-Shmulyan theorem), it is also strictly decreasing. Thus, k_{eff} is **characterized** by*

$$r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})]=1.$$

Let us now give some variational characterizations of the criticality eigenvalue k_{eff} appearing in Theorem 2.6.

2.2. **Abstract variational characterization of k_{eff}.** From now on, Assumption (2.3) is assumed to be fulfilled. Let

$$W_p^+:=\mathscr{D}(\mathcal{T})\cap X_p^+. \tag{2.4}$$

We start with the following characterization of k_{eff} in terms of *super-solution* to the spectral problem (2.2).

Proposition 2.9. *Assume that $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ is power-compact and irreducible for any $\gamma>0$. For any $\varphi\in W_p^+$, let*

$$\tau_+(\varphi):=\sup\{\gamma>0 \textit{ such that } (\mathcal{T}+\mathcal{K}(\gamma))\,\varphi \textit{ is nonnegative}\}$$

with the convention $\sup\varnothing=0$. Then

$$k_{\text{eff}}=\sup_{\varphi\in W_p^+}\tau_+(\varphi).$$

Proof. Let $\varphi\in X_p$ be a nonnegative eigenfunction of $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})$ associated with the spectral radius $r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})]=1$, i.e.

$$(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi=\varphi.$$

Clearly, $\varphi\in\mathscr{D}(\mathcal{T})$. Moreover, since $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})$ is power-compact and irreducible, a well-known consequence of Krein-Rutman Theorem is that

$$\varphi(\omega)>0 \quad \text{a.e. } \omega\in\boldsymbol{\Omega}$$

i.e. $\varphi \in W_p^+$. Now

$$(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi = 0$$

is nonnegative so that $\tau_+(\varphi) \geqslant k_{\text{eff}}$ and consequently

$$k_{\text{eff}} \leqslant \sup_{\varphi \in W_p^+} \tau_+(\varphi).$$

Assume now that $k_{\text{eff}} < \sup_{\varphi \in W_p^+} \tau_+(\varphi)$ et let $\psi \in W_p^+$ be such that $\tau_+(\psi) > k_{\text{eff}}$. Denote $\gamma = \tau_+(\psi)$. By definition $(\mathcal{T} + \mathcal{K}(\gamma))\,\psi \geqslant 0$, i.e. $(0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)\,\psi \geqslant \psi$. Thus, for any $n \in \mathbb{N}$, $[(0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)]^n\,\psi \geqslant \psi$ so

$$r_\sigma[(0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)] \geqslant 1.$$

Since, $r_\sigma[(0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})] = 1$, Remark 2.8 ensures that $\gamma \leqslant k_{\text{eff}}$, which contradicts the choice of $\tau_+(\psi) > k_{\text{eff}}$. □

The following illustrates the fact that the extremal value to the above variational result is reached only by the nonnegative solution to the spectral problem (2.2).

Corollary 2.10. *Assume that* $(0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)$ *is power-compact and irreducible for any* $\gamma > 0$. *Then, for any nonnegative* $\varphi \in \mathscr{D}(\mathcal{T}) \setminus \{0\}$

$$\tau_+(\varphi) = k_{\text{eff}} \qquad \textit{if and only if} \qquad (\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi = 0.$$

Proof. Let $\varphi \in \mathscr{D}(\mathcal{T}) \setminus \{0\}$, $\varphi \geqslant 0$, be such that $\tau_+(\varphi) = k_{\text{eff}}$. Then $(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi \geqslant 0$ i.e.

$$(0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi \geqslant \varphi. \tag{2.5}$$

Suppose that

$$\varphi \neq (0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi, \tag{2.6}$$

and let $\psi^\star \in X_q$ be a *nonnegative* eigenfunction of the dual operator $((0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}}))^\star$ associated to $r_\sigma[((0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}}))^\star] = r_\sigma[(0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})] = 1$. Then, by (2.5) and (2.6)

$$\begin{aligned}\langle \varphi, \psi^\star \rangle &< \left\langle (0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi, \psi^\star \right\rangle \\ &= \left\langle \varphi, ((0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}}))^\star\,\psi^\star \right\rangle = r_\sigma[((0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}}))^\star]\,\langle \varphi, \psi^\star \rangle \\ &= \langle \varphi, \psi^\star \rangle\,,\end{aligned}$$

which leads to a contradiction. Hence, $\varphi = (0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi$, i.e.

$$(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\varphi = 0.$$

Conversely, if φ is a nonnegative eigenfunction of $\mathcal{T} + \mathcal{K}(k_{\text{eff}})$ associated to the *null eigenvalue*, then $\tau_+(\varphi) \geqslant k_{\text{eff}}$ and the identity $\tau_+(\varphi) = k_{\text{eff}}$ follows from Proposition 2.9. □

Remark 2.11. *A careful reading of the proof here above shows that, if* $\varphi \in X_p$ *is such that* $(0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\,\varphi \geqslant \varphi \geqslant 0$, *then* $\varphi \in \mathscr{D}(\mathcal{T})$ *and* $(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi = 0$.

Let us denote by $\phi_{\rm eff}$ the unique critical eigenfunction with unit norm, i.e. $\phi_{\rm eff}\in W_p^+$ satisfies

$$(\mathcal{T}+\mathcal{K}(k_{\rm eff}))\phi_{\rm eff}=\phi_{\rm eff}, \qquad \|\phi_{\rm eff}\|=1.$$

Then one can prove the following approximation resolution for the criticality eigenfunction $\phi_{\rm eff}$ whose proof is inspired by [20, Theorem 7]. Such a result shall be hopefully useful for practical numerical approximation of the critical mode $\phi_{\rm eff}$ of the reactor.

Theorem 2.12. *Let $(\varphi_k)_k\in\mathcal{D}(\mathcal{T})\cap X_p^+$ be such that $\gamma_k:=\tau_+(\varphi_k)\to k_{\rm eff}$. We assume φ_k to be normalized by*

$$\| \left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N \varphi_k\|=1 \qquad (k\in\mathbb{N}) \tag{2.7}$$

where N is the integer given by Remark 2.7. Let us assume that 1 is a simple eigenvalue of $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})$. Moreover, in the case $p=1$, let us assume that the dual operator $\left[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\right]^\star$ admits an eigenfunction associated to its spectral radius which is bounded away from zero. Then,

$$\lim_{k\to\infty}\|\varphi_k-\phi_{\rm eff}\|=0$$

where $\phi_{\rm eff}\in\mathcal{D}(\mathcal{T})\cap X_p^+$ is the unique positive eigenfunction of $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})$ associated to 1 and with unit norm .

Proof. According to the definition of $\gamma_k:=\tau_+(\varphi_k)$, $(\mathcal{T}+\mathcal{K}(\gamma_k))\varphi_k$ is nonnegative. Therefore,

$$\varphi_k\leqslant(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\varphi_k \tag{2.8}$$

and, iterating up to N

$$\varphi_k\leqslant\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N\varphi_k \qquad (k\in\mathbb{N}). \tag{2.9}$$

This shows, according to (2.7), that $\|\varphi_k\|\leqslant 1$. Now, if $1<p<\infty$ there exists a subsequence $(\psi_k)_k$ which converges weakly to some $\psi\in X_p$. If $p=1$, the fact that $\gamma_k\to k_{\rm eff}$ combined with the compactness of $\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N$ lead to the relative compactness of

$$\left\{\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N\varphi_k\right\}_k$$

in X_1. In particular, this sequence is equi–integrable and by domination (2.9), $(\varphi_k)_k$ is also equi–integrable. We can extract a subsequence $(\psi_k)_k$ converging weakly to some $\psi\in X_1$. In both cases, the compactness of $\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N$ together with $\gamma_k\to k_{\rm eff}$ yield to the strong convergence

$$\left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N\psi_k\to\left[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\right]^N\psi$$

so that

$$\|\left[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\right]^N\psi\|=1.$$

In particular $\psi\neq 0$, and, taking the weak limit in (2.8), $\psi\leqslant(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\psi$. Now, according to Remark 2.11, this last inequality is actually an equality, i. e.

$$\psi=(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\psi.$$

Iterating again, one gets

$$\|\psi\|=\|\left[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\rm eff})\right]^N\psi\|=1.$$

Now, since 1 is a simple eigenvalue of $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})$, the set of eigenfunctions of $(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})$ with unit norm reduces to a singleton. This shows that ψ is the (unique) weak limit of any subsequence of $(\varphi_k)_k$ so that the whole sequence $(\varphi_k)_k$ *converges weakly* to $\psi \in X_p$. The remainder of the proof consists in showing that the convergence actually holds in the *strong sense*.

Let us consider first the case $1 < p < \infty$. To show now that $\|\varphi_k - \psi\| \to 0$, it suffices to prove that $\|\varphi_k\| \to \|\psi\|$. A consequence of the weak convergence leads to

$$1 = \|\psi\| \leqslant \liminf_{k\to\infty} \|\varphi_k\|.$$

Since $\|\varphi_k\| \leqslant 1$ for any $k \in \mathbb{N}$, this proves the Theorem for $1 < p < \infty$.

Let us now assume $p = 1$. Then $\varphi_k \to \psi$ strongly in X_1 if and only if the convergence holds in measure, i. e., for any $\Xi \subset \mathbf{\Omega}$ with finite $d\nu$–measure and every $\epsilon > 0$

$$\lim_{k\to\infty} d\nu\{\omega \in \Xi\,;\, |\psi_k(\omega) - \psi(\omega)| > \epsilon\} = 0.$$

Arguing by contradiction, assume there exist $\Xi \subset \mathbf{\Omega}$ with finite $d\nu$–measure, a subsequence still denoted $(\psi_k)_k$ and some $\delta > 0$ and some $\epsilon_0 > 0$ such that

$$d\nu\{\omega \in \Xi\,;\, |\psi_k(\omega) - \psi(\omega)| > \epsilon_0\} \geqslant \delta \qquad \text{for all } m \in \mathbb{N}. \tag{2.10}$$

Setting

$$\overline{\psi}_k = \left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N \psi_k$$

one has

$$\lim_{k\to\infty} \|\overline{\psi}_k - \psi\| = 0$$

since $\psi = \left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N \psi$. Consequently,

$$\lim_{k\to\infty} d\nu\{\omega \in \Xi\,;\, |\overline{\psi}_k(\omega) - \psi(\omega)| > \epsilon_0/2\} = 0$$

and, one deduces immediately from (2.10) that

$$d\nu\{\omega \in \Xi\,;\, |\psi_k(\omega) - \psi(\omega)| > \epsilon_0/2\} \geqslant \delta/2 \qquad \text{for large } k. \tag{2.11}$$

Now, let $\psi^\star \in L^\infty(\mathbf{\Omega})$ be a positive eigenfunction of $\left[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\right]^\star$ associated to the eigenvalue 1, with $\psi^\star$ bounded away from zero. One has

$$\begin{aligned}\langle \overline{\psi}_k - \psi_k, \psi^\star\rangle &\geqslant \int_{\{\omega\in\Xi\,;\,|\overline{\psi}_k-\psi_k|>\epsilon_0/2\}} \left(\overline{\psi}_k(\omega) - \psi_k(\omega)\right)\psi^\star(\omega)\,d\nu(\omega)\\ &\geqslant \inf_w \psi^\star(\omega) \times \epsilon_0/2 \times \delta/2 = \eta > 0\end{aligned}$$

or else

$$\left\langle \left[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)\right]^N \psi_k, \psi^\star\right\rangle - \langle \psi_k, \psi^\star\rangle \geqslant \eta \qquad \text{for large } k.$$

Equivalently

$$\left\langle \psi_k, \{[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)]^\star\}^N \psi^\star\right\rangle - \langle \psi_k, \psi^\star\rangle \geqslant \eta \qquad \text{for large } k.$$

But, $\{[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)]^\star\}^N \psi^\star$ converges strongly to $\{[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})]^\star\}^N \psi^\star$ in $L^\infty(\mathbf{\Omega})$, which, combined with the weak convergence $\psi_k \rightharpoonup \psi$, implies

$$\left\langle \psi_k, \{[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma_k)]^\star\}^N \psi^\star\right\rangle \to \left\langle \psi, \{[(0-\mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})]^\star\}^N \psi^\star\right\rangle = \langle \psi, \psi^\star\rangle\,.$$

Now, the contradiction follows from the fact that $\langle \psi_k, \psi^\star \rangle \to \langle \psi, \psi^\star \rangle$. □

The following characterizes k_{eff} in terms of *sub-solution* to the spectral problem (2.2).

Proposition 2.13. *Assume that* $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ *is power-compact and irreducible for any* $\gamma > 0$. *For any* $\varphi \in W_p^+$, *define*

$$\tau_-(\varphi) := \inf\{\gamma > 0\,;\, -(\mathcal{T}+\mathcal{K}(\gamma))\varphi \in X_p^+\}$$

with the convention $\inf \varnothing = +\infty$. *Then*

$$k_{\text{eff}} = \inf_{\varphi \in W_p^+} \tau_-(\varphi).$$

Proof. Let $\varphi \in W_p^+$ be such that $\tau_-(\varphi) < +\infty$, and let $\gamma > \tau_-(\varphi)$. Then

$$-(\mathcal{T}+\mathcal{K}(\gamma))\,\varphi \in X_p^+. \tag{2.12}$$

Since $(0-\mathcal{T})^{-1}(X_p^+) \subset X_p^+$ (see Eq. (2.1)), one gets that

$$\varphi - (0-\mathcal{T})^{-1}\mathcal{K}(\gamma)\,\varphi \in W_p^+. \tag{2.13}$$

Now, let $\psi^\star \in X_q$ be a nonnegative eigenfunction of the dual operator $((0-\mathcal{T})^{-1}\mathcal{K}(\gamma))^\star$ associated with the spectral radius $r_\sigma[((0-\mathcal{T})^{-1}\mathcal{K}(\gamma))^\star] = r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]$. Note that $r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)] > 0$ according to Theorem 2.5. From Krein-Rutman theorem, $\psi^\star(\omega) > 0$ $d\nu$ - a.e. $\omega \in \mathbf{\Omega}$. Thus, by (2.13)

$$\left\langle \varphi - (0-\mathcal{T})^{-1}\mathcal{K}(\gamma)\,\varphi, \psi^\star \right\rangle > 0$$

i.e.

$$\begin{aligned}\langle \varphi, \psi^\star \rangle &> \left\langle (0-\mathcal{T})^{-1}\mathcal{K}(\gamma)\,\varphi, \psi^\star \right\rangle \\ &= \left\langle \varphi, ((0-\mathcal{T})^{-1}\mathcal{K}(\gamma))^\star\,\psi^\star \right\rangle = r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]\,\langle \varphi, \psi^\star \rangle.\end{aligned}$$

Since $\langle \varphi, \psi^\star \rangle \neq 0$, we get $r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)] < 1$. By Remark 2.8, this means that $\gamma > k_{\text{eff}}$. Since $\gamma > \tau_-(\varphi)$ is arbitrary, we obtain

$$k_{\text{eff}} \leqslant \inf_{\varphi \in W_p^+} \tau_-(\varphi).$$

Conversely, let $\gamma > k_{\text{eff}}$. Then, $r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)] < 1$, and

$$\begin{aligned}[I-(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^{-1} &= \sum_{k=0}^{\infty}[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^k \\ &\geqslant [(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^n \quad \forall\, n \in \mathbb{N}.\end{aligned}$$

Given $\psi \in X_p^+$, set $\tilde{\psi} = (0-\mathcal{T})^{-1}\,\psi$. Then, $\tilde{\psi} \in W_p^+$. Define

$$\varphi = [I-(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^{-1}\,\tilde{\psi}. \tag{2.14}$$

Clearly, $\varphi \in D(\mathcal{T})$ and

$$\varphi \geqslant [(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^n\,\tilde{\psi} \quad (n \in \mathbb{N}).$$

Let $\varphi^\star \in X_q \setminus \{0\}$ be *nonnegative*. Since $(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)$ is *irreducible*, there exists $n \in \mathbb{N}$ such that

$$\langle \varphi, \varphi^\star \rangle \geqslant \left\langle [(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]^n\,\tilde{\psi}, \varphi^\star \right\rangle > 0$$

Hence, $\langle \varphi, \varphi^\star \rangle > 0$ for any nonnegative $\varphi^\star \in X_q \setminus \{0\}$, so that $\varphi \in X_p^+$. Moreover, by (2.14),

$$-(\mathcal{T} + \mathcal{K}(\gamma))\,\varphi = (0 - \mathcal{T})\,\tilde{\psi} = \psi \in X_p^+.$$

Hence, $\gamma \geqslant \tau_-(\varphi)$ which proves that $k_{\text{eff}} \geqslant \inf_{\varphi \in W_p^+} \tau_-(\varphi)$. □

The following result shows that only the solution of the criticality problem (2.2) realizes the above variational characterization.

Corollary 2.14. *Assume that $(0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)$ is power-compact and irreducible for any $\gamma > 0$. Then, for any $\varphi \in W_p^+$*

$$\tau_-(\varphi) = k_{\text{eff}} \quad \textit{if and only if} \quad (\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi = 0.$$

Proof. Let $\varphi \in W_p^+$ be such that $\tau_-(\varphi) = k_{\text{eff}}$. Then, $-(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi \geqslant 0$ so

$$\varphi \leqslant (0 - \mathcal{T})^{-1}\mathcal{K}(k_{\text{eff}})\varphi.$$

Arguing as in the proof of Corollary 2.10, one can prove that $\varphi = (0 - \mathcal{T})^{-1}\mathcal{K}(\gamma)\,\varphi$, which means that

$$(\mathcal{T} + \mathcal{K}(k_{\text{eff}}))\,\varphi = 0.$$

The sufficient condition follows directly from Proposition 2.13. □

We are now able to characterize the effective multiplication factor k_{eff} by means of Inf-Sup and Sup-Inf criteria, where we recall that $\mathcal{K} = \mathcal{K}_s + \mathcal{K}_f$.

Theorem 2.15. *Under the assumptions of Theorem 2.6, if $\mathcal{K}_f(X_p^+) \subset X_p^+$ then the criticality eigenvalue k_{eff} is characterized by the following:*

$$\frac{1}{k_{\text{eff}}} = \min_{\varphi \in W_p^+} \operatorname*{ess\,sup}_{\omega \in \boldsymbol{\Omega}} \frac{-(\mathcal{T} + \mathcal{K}_s)\varphi(\omega)}{\mathcal{K}_f\varphi(\omega)} = \max_{\varphi \in W_p^+} \operatorname*{ess\,inf}_{\omega \in \boldsymbol{\Omega}} \frac{-(\mathcal{T} + \mathcal{K}_s)\varphi(\omega)}{\mathcal{K}_f\varphi(\omega)}.$$

Proof. Let $\varphi \in W_p^+$ be given,

$$\begin{aligned}\tau_+(\varphi) &= \sup\{\gamma > 0\,;\ (\mathcal{T} + \mathcal{K}_s)\,\varphi + \frac{1}{\gamma}\mathcal{K}_f\,\varphi \geqslant 0\} = \sup\{\gamma > 0\,;\ -(\mathcal{T} + \mathcal{K}_s)\,\varphi \leqslant \frac{1}{\gamma}\mathcal{K}_f\,\varphi\} \\ &= \sup\{\gamma > 0\,;\ -(\mathcal{T} + \mathcal{K}_s)\,\varphi(\omega) \leqslant \frac{1}{\gamma}\mathcal{K}_f\,\varphi(\omega)\ \ d\nu - \text{a.e. } \omega \in \boldsymbol{\Omega}\}.\end{aligned}$$

Since $\mathcal{K}_f(X_p^+) \subset X_p^+$, one gets $\dfrac{1}{\tau_+(\varphi)} = \operatorname*{ess\,sup}_{\omega \in \boldsymbol{\Omega}} \dfrac{-(\mathcal{T} + \mathcal{K}_s)\,\varphi(\omega)}{\mathcal{K}_f\,\varphi(\omega)}$. By Proposition 2.9,

$$\frac{1}{k_{\text{eff}}} = \inf_{\varphi \in W_p^+} \operatorname*{ess\,sup}_{\omega \in \boldsymbol{\Omega}} \frac{-(\mathcal{T} + \mathcal{K}_s)\,\varphi(\omega)}{\mathcal{K}_f\,\varphi(\omega)}$$

and the infimum is attained for the criticality eigenfunction. Similarly, let $\varphi \in W_p^+$

$$\begin{aligned}\tau_-(\varphi) &= \inf\{\gamma > 0\,;\ -(\mathcal{T} + \mathcal{K}_s)\,\varphi - \frac{1}{\gamma}\mathcal{K}_f\,\varphi \geqslant 0\} = \inf\{\gamma > 0\,;\ -(\mathcal{T} + \mathcal{K}_s)\,\varphi \geqslant \frac{1}{\gamma}\mathcal{K}_f\,\varphi\} \\ &= \inf\{\gamma > 0\,;\ -(\mathcal{T} + \mathcal{K}_s)\,\varphi(\omega) \geqslant \frac{1}{\gamma}\mathcal{K}_f\,\varphi(\omega)\ \ d\nu - \text{a.e. } \omega \in \boldsymbol{\Omega}\}.\end{aligned}$$

So

$$\tau_-(\varphi) = \inf\{\gamma > 0\,;\, \frac{-(\mathcal{T}+\mathcal{K}_s)\,\varphi(\omega)}{\mathcal{K}_f\,\varphi(\omega)} \geqslant \frac{1}{\gamma} \;\; d\nu - \text{a.e. } \omega \in \mathbf{\Omega}\},$$

i.e.

$$\frac{1}{\tau_-(\varphi)} = \operatorname*{ess\,inf}_{\omega\in\mathbf{\Omega}} \frac{-(\mathcal{T}+\mathcal{K}_s)\,\varphi(\omega)}{\mathcal{K}_f\,\varphi(\omega)}.$$

Using Proposition 2.13, one proves that

$$\frac{1}{k_{\text{eff}}} = \sup_{\varphi\in W_p^+} \frac{1}{\tau_-(\varphi)} = \sup_{\varphi\in W_p^+} \operatorname*{ess\,inf}_{\omega\in\mathbf{\Omega}} \frac{-(\mathcal{T}+\mathcal{K}_s)\,\varphi(\omega)}{\mathcal{K}_f\,\varphi(\omega)},$$

which ends the proof. □

2.3. **The class of regular collision operators.** We end this section by recalling the class of regular collision operators introduced in kinetic theory by M. Mokhtar-Kharroubi [19]. This class of operators will also be useful to study diffusion problems of type (1.3). We assume here that the measure space $(\mathbf{\Omega}, d\nu)$ writes as follows:

$$\mathbf{\Omega} = \mathcal{D}\times V, \qquad d\nu(\omega) = dx \otimes d\mu(v), \qquad \omega = (x,v) \in \mathbf{\Omega}$$

where $d\mu$ is a suitable Radon measure over V. Let $\mathcal{K} \in \mathscr{B}(L^p(\mathbf{\Omega}, d\nu))$ be given by

$$\mathcal{K}\;:\;\varphi \longmapsto \mathcal{K}\varphi(x,v) = \int_V k(x,v,v')\varphi(x,v')d\mu(v') \in L^p(\mathbf{\Omega}, d\nu) \tag{2.15}$$

where the kernel $k(\cdot,\cdot,\cdot)$ is measurable. For almost every $x \in \mathcal{D}$, define

$$\widetilde{\mathcal{K}}(x)\;:\;\psi \in L^p(V,d\mu) \longmapsto \int_V k(x,v,v')\psi(v')d\mu(v') \in L^p(V,d\mu)$$

and assume that the mapping $\widetilde{\mathcal{K}}\;:\;x\in\mathcal{D} \mapsto \widetilde{\mathcal{K}}(x) \in \mathscr{B}(L^p(V,d\mu))$ is strongly measurable and bounded, i.e.

$$\operatorname*{ess\,sup}_{x\in\mathcal{D}} \|\widetilde{\mathcal{K}}(x)\|_{\mathscr{B}(L^p(V,d\mu))} < \infty.$$

The class of regular operators in L^p spaces with $1 < p < \infty$ is given by the following (see [19, Definition 4.1]).

Definition 2.16 (Regular operator). *Let $1 < p < \infty$. The operator $\mathcal{K}$ defined by* (2.15) *is said to be regular if :*

(1) For almost every $x \in \mathcal{D}$, $\widetilde{\mathcal{K}}(x) \in \mathscr{B}(L^p(V,d\mu))$ is a compact operator,
(2) $\{\widetilde{\mathcal{K}}(x)\,;\,x\in\mathcal{D}\}$ is relatively compact in $\mathscr{B}(L^p(V,d\mu))$.

In L^1-spaces, the definition differs a bit. We have the following [17]

Definition 2.17. *Let $\mathcal{K}$ be defined by* (2.15)*. Then, $\mathcal{K}$ is said to be a regular operator whenever $\{|k(x,\cdot,v')|\,,\,(x,v')\in\mathcal{D}\times V\}$ is a relatively weakly compact subset of $L^1(V,d\mu)$.*

The main interest of that classes of operators relies to the following (see Ref. [19] for $1 < p < \infty$ and Ref. [17] for a similar result whenever $p = 1$):

Proposition 2.18 (Approximation of regular operators). *Let $1 < p < \infty$ and let $\mathcal{K}$ defined by* (2.15) *be a regular operator in $L^p(\mathcal{D} \times V, dx \otimes d\mu(v))$. Then, $\mathcal{K}$ can be approximated in the norm operator by operators of the form:*

$$\varphi \longmapsto \sum_{i \in I} \alpha_i(x)\beta_i(v) \int_V \theta_i(v')\varphi(x, v')d\mu(v')$$

where I is finite, $\alpha_i \in L^\infty(\mathcal{D})$, $\beta_i \in L^p(V, d\mu)$ and $\theta_i \in L^q(V, d\mu)$, $1/p + 1/q = 1$.

3. The critical transport problem

3.1. **Variational characterization.** This section is devoted to the determination of the effective multiplication factor associated to the transport operator. We adopt the notations of Section 2.3, namely $\mathbf{\Omega} = \mathcal{D} \times V$ and $d\nu(x, v) = dx \otimes d\mu(v)$. Throughout this section, we assume $\mathcal{D}$ to be a ***convex and bounded*** open subset of $\mathbb{R}^N$ while μ is the Lebesgue measure over $\mathbb{R}^N$ or on spheres. In particular, our results cover ***continuous or multi-group neutron transport problems*** but do not apply to transport problems with discrete velocities. Let

$$\Gamma_- := \Big\{(x, v) \in \partial\mathcal{D} \times V\,;\; v \cdot n(x) < 0\Big\}$$

where $n(x)$ denotes the outward unit normal at $x \in \partial\mathcal{D}$. Let $\mathcal{T}$ be the unbounded absorption operator

$$\begin{cases} \mathcal{T} \,:\, \mathscr{D}(\mathcal{T}) \subset \quad X_p \longrightarrow X_p \\ \qquad\qquad\qquad\quad \varphi \longmapsto \mathcal{T}\varphi(x, v) := -v \cdot \nabla_x\varphi(x, v) - \sigma(x, v)\varphi(x, v), \end{cases}$$

with domain

$$\mathscr{D}(\mathcal{T}) = \Big\{\psi \in X_p\,;\;\; v \cdot \nabla_x\psi \in X_p \;\text{ and }\; \psi_{|\Gamma_-} = 0\Big\}.$$

Here, the *nonnegative* function $\sigma(\cdot, \cdot) \in L^\infty(\mathcal{D} \times V)$ is the *collision frequency*. It is assumed to admit a *positive lower bound*

$$\sigma(x, v) \geqslant c > 0 \qquad \text{a.e. } (x, v) \in \mathcal{D} \times V. \tag{3.1}$$

Define the (full) collision operator $\mathcal{K}$ as the *bounded* linear (partial) integral operator

$$\mathcal{K} \,:\, \psi \in X_p \longmapsto \mathcal{K}\psi(x, v) := \int_V \Sigma(x, v, v')\psi(x, v')d\mu(v') \in X_p.$$

The collision kernel $\Sigma(\cdot, \cdot, \cdot)$ is assumed to be nonnegative. In nuclear reactor theory, in a fissile material, this collision kernel splits as

$$\Sigma(x, v, v') = \Sigma_s(x, v, v') + \Sigma_f(x, v, v')$$

where $\Sigma_s(x, v, v')$ describes the pure scattering phenomena and $\Sigma_f(x, v, v')$ describes the fission processes. Define the corresponding linear operators

$$\mathcal{K}_s \,:\, \psi \in X_p \longmapsto \mathcal{K}_s\psi(x, v) := \int_V \Sigma_s(x, v, v')\psi(x, v')d\mu(v') \in X_p$$

and

$$\mathcal{K}_f \,:\, \psi \in X_p \longmapsto \mathcal{K}_f\psi(x, v) := \int_V \Sigma_f(x, v, v')\psi(x, v')d\mu(v') \in X_p.$$

As we told it in Introduction, we are interested here in the critical problem:

$$v\cdot\nabla_x\varphi(x,v)+\sigma(x,v)\varphi(x,v)-\int_V\Sigma_s(x,v,v')\varphi(x,v')d\mu(v')\\ =\frac{1}{k_{\text{eff}}}\int_V\Sigma_f(x,v,v')\varphi(x,v')d\mu(v'),\quad (3.2)$$

where the eigenfunction φ is *nonnegative* and satisfies the boundary condition $\varphi_{|\Gamma_-}=0$. We recall that the spectral bound of $\mathcal{T}$ is given by [29]

$$s(\mathcal{T})=-\lim_{t\to\infty}\inf_{\substack{(x,v)\in\mathcal{D}\times V\\ t<\tau(x,v)}} t^{-1}\int_0^t\sigma(x+sv,v)ds,$$

with $\tau(x,v):=\inf\{s>0\,;\,x-sv\notin\mathcal{D}\}$. Therefore, by (3.1), we have $s(\mathcal{T})<0$. Moreover,

$$(0-\mathcal{T})^{-1}\varphi(x,v)=\int_0^{\tau(x,v)}\exp\left\{-\int_0^t\sigma(x-vs,v)ds\right\}\varphi(x-vt,v)dt,$$

so that $(0-\mathcal{T})^{-1}$ fulfills (2.1). Let us now recall the irreducibility properties of $(0-\mathcal{T})^{-1}\mathcal{K}_f$ for the continuous and multigroup models. The following result may be found in Ref. [19], Theorem 5.15, Theorem 5.16, (see also [29]).

Theorem 3.1. *Let $\mathcal{D}$ be convex. Then, $(0-\mathcal{T})^{-1}\mathcal{K}_f$ is irreducible in the two following cases:*

(1) V is a closed subset of $\mathbb{R}^N$ equipped with the Lebesgue measure $d\mu$ and there exist $0<c_1<c_2<\infty$, such that $V_0=\{v\in\mathbb{R}^N\,;\,c_1<|v|<c_2\}\subset V$ with

$$\Sigma_f(x,v,v')>0\quad \textit{a.e. } (x,v,v')\in\big(\mathcal{D}\times V\times V_0\big)\cup\big(\mathcal{D}\times V_0\times V\big).\quad (3.3)$$

(2) V is the union of k disjoint spheres $(k\geqslant 1)$,

$$V=\bigcup_{i=1}^k V_i,\qquad V_i=\{v\in\mathbb{R}^N\,;|v|=r_i\},\qquad (r_i>0,\ i=1,\ldots,k)$$

and, on each sphere, $d\mu$ is the surface Lebesgue measure. Moreover, for any $i,j\in\{1,\ldots,k\}$, there exists $\ell\in\{1,\ldots,m\}$ such that

$$\Sigma_f(x,v,v')>0\qquad \textit{a.e. } (x,v,v')\in\big(\mathcal{D}\times V_i\times V_\ell\big)\cup\big(\mathcal{D}\times V_\ell\times V_j\big).\quad (3.4)$$

Remark 3.2. *In the above case (1), corresponding to* ***continuous models****, it is possible to provide different criteria ensuring the irreducibility of $(0-\mathcal{T})^{-1}\mathcal{K}_f$ (see for instance Ref.* [13]*). In the second case (2), which corresponds to* ***multigroup*** *transport equation, several different criteria also exist* [23].

Using the notations of Section 2, we have the following characterization of the effective multiplication factor of the transport operator.

Theorem 3.3. *Let us assume that $\mathcal{K}$ is a regular collision operator and that one of the hypothesis of Theorem 3.1 holds. The critical problem* (3.2) *admits a unique solution k_{eff} if and only if*

$$\lim_{\gamma\to 0} r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)]>1\qquad \textit{and}\qquad r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}_s]<1.$$

Moreover,

$$
\begin{aligned}
\frac{1}{k_{\text{eff}}} &= \min_{\varphi \in W_p^+} \operatorname*{ess\,sup}_{(x,v)\in \mathcal{D}\times V} \frac{v\cdot\nabla_x\varphi(x,v)+\sigma(x,v)\varphi(x,v)-\displaystyle\int_V \Sigma_s(x,v,v')\varphi(x,v')\,d\mu(v')}{\displaystyle\int_V \Sigma_f(x,v,v')\varphi(x,v')\,d\mu(v')} \\
&= \max_{\varphi \in W_p^+} \operatorname*{ess\,inf}_{(x,v)\in \mathcal{D}\times V} \frac{v\cdot\nabla_x\varphi(x,v)+\sigma(x,v)\varphi(x,v)-\displaystyle\int_V \Sigma_s(x,v,v')\varphi(x,v')\,d\mu(v')}{\displaystyle\int_V \Sigma_f(x,v,v')\varphi(x,v')\,d\mu(v')}.
\end{aligned}
\tag{3.5}
$$

Proof. Since $\mathcal{K}$ is a regular collision operator, one deduces from [19, Theorems 4.1 & 4.4] when $1 < p < \infty$ (respectively [17] if $p = 1$) that $(0 - \mathcal{T})^{-1}\mathcal{K}$ is a power-compact operator in X_p $(1 \leqslant p < \infty)$ under our assumptions on the measure μ. Moreover, in the continuous case, thanks to (3.3), for any $\varphi \in X_p^+$

$$
\begin{aligned}
\mathcal{K}_f\varphi(x,v) &= \int_V \Sigma_f(x,v,v')\,\varphi(x,v')d\mu(v') \\
&\geqslant \int_{V_0} \Sigma_f(x,v,v')\,\varphi(x,v')d\mu(v') > 0 \qquad \text{a.e. } (x,v)\in\mathcal{D}\times V,
\end{aligned}
$$

i.e. $\mathcal{K}_f(X_p^+) \subset X_p^+$. Similarly, in the multigroup case, Eq. (3.4) implies $\mathcal{K}_f(X_p^+) \subset X_p^+$. Now, the existence of k_{eff} follows from Theorem 2.6 while (3.5) follows from Theorem 2.15. □

Remark 3.4. *Denote by ϕ_{eff} the nonnegative solution of* (3.2)*, one can check that $\phi_{\text{eff}} \in W_p^+$. Therefore, in* (3.5)*, the supremum and the infimum are reached for $\phi = \phi_{\text{eff}}$.*

Remark 3.5. *Note that it is possible to provide practical criteria that are satisfied in nuclear reactor theory and that ensure the existence of k_{eff}* [4, 28]*. Such criteria usually rely on dissipative properties of the pure scattering operator.*

Remark 3.6. *It is important to point out that the above characterization is not restricted to the case of absorbing conditions but also holds for general boundary conditions modeled by some suitable **nonnegative albedo operator**. Actually, if one considers a transport operator $\mathcal{T}_H$ associated to general nonnegative albedo boundary operator H which relates the incoming and outgoing fluxes in $\mathcal{D}$* [16]*, then the above theorem holds true provided $(0 - \mathcal{T}_H)^{-1}\mathcal{K}$ is a power-compact operator in X_p $(1 \leqslant p < \infty)$ when $\mathcal{K}_s$ and $\mathcal{K}_f$ are regular operators. This is always the case whenever $1 < p < \infty$ by virtue of the velocity averaging lemma* [16]*. The problem is more delicate in a L^1-setting and is related to the geometry of the domain $\mathcal{D}$* [27].

3.2. **Necessary conditions of super-criticality and sub-criticality.** We shall use the result of the previous section to derive necessary conditions ensuring the reactor to be super-critical or sub-critical. Note that, for practical implications, a nuclear reactor can be operative and create energy only when slightly super-critical (i.e. $1 < k_{\text{eff}} < 1 + \delta$ with $\delta > 0$ small enough), in this case, the whole chain fission being controlled by rods of absorbing matter. Throughout this section, we shall assume k_{eff} to exist.

We shall provide lower and upper bounds on the effective multiplicative factor k_{eff} only when the velocity space V is ***bounded away from zero***. Recall that, since V is assumed to be closed, this means that $0 \notin V$ (see also Remark 3.14).

For almost every $x \in \mathcal{D}$, define $\mathcal{K}^\tau(x)$ as the following operator on $L^p(V, d\mu)$:

$$\mathcal{K}^\tau(x) \; : \psi \in L^p(V, d\mu) \mapsto \int_V \frac{\Sigma(x, v, v')\tau(x, v')}{1 + \sigma(x, v)\tau(x, v)} \psi(v') d\mu(v') \in L^p(V, d\mu)$$

where we recall that $\Sigma(x, v, v') = \Sigma_s(x, v, v') + \Sigma_f(x, v, v')$ and $\tau(x, v)$ is the stay time in $\mathcal{D}$. Then, one defines as in [22], the following

$$\overline{\vartheta} := \inf_{\psi \in L^p_+(V, d\mu)} \operatorname*{ess\,sup}_{(x,v) \in \mathcal{D} \times V} \frac{[\mathcal{K}^\tau(x)\psi](v)}{\psi(v)}$$

where $L^p_+(V, d\mu) = \{\psi \in L^p(V, d\mu)\,;\, \psi(v) > 0 \; d\mu - \text{a.e. } v \in V\}$. Then, one has the following estimate:

Proposition 3.7. *Under the assumptions of Theorem 3.3, if $\overline{\vartheta} < 1$, then $k_{\text{eff}} \leqslant \overline{\vartheta}$.*

Proof. Assume $\overline{\vartheta} < 1$. Given $\vartheta \in (\overline{\vartheta}, 1)$, let $\psi_0 \in L^p_+(V, d\mu)$ be such that

$$\operatorname*{ess\,sup}_{(x,v) \in \mathcal{D} \times V} \frac{[\mathcal{K}^\tau(x)\psi_0](v)}{\psi_0(v)} \leqslant \vartheta.$$

Let us consider then the following test-function $\varphi_0(x, v) = \tau(x, v)\psi_0(v)$. Since $0 \notin V$, $\tau(\cdot, \cdot)$ is bounded and such an application φ_0 belongs to W_p^+ since

$$\tau(x + tv, v) = \tau(x, v) + t \qquad \text{a.e. } (x, v) \in \mathcal{D} \times V,\; t \geqslant 0,$$

implies $v \cdot \nabla_x \varphi_0(x, v) = \psi_0(v)$. Then, for any $\gamma > 0$, one sees that

$$\begin{aligned} -(\mathcal{T} + \mathcal{K}(\gamma))\varphi_0(x, v) = \vartheta^{-1}\left(1 + \sigma(x, v)\tau(x, v)\right) \Bigg(& \vartheta\psi_0(v) - [\mathcal{K}^\tau(x)\psi_0](v) \\ & + (1 - \vartheta) \int_V \frac{\Sigma_s(x, v, v')\tau(x, v')}{1 + \sigma(x, v)\tau(x, v)} \psi_0(v') d\mu(v') + \\ & + \frac{\gamma - \vartheta}{\gamma} \int_V \frac{\Sigma_f(x, v, v')\tau(x, v')}{1 + \sigma(x, v)\tau(x, v)} \psi_0(v') d\mu(v') \Bigg). \end{aligned} \tag{3.6}$$

Since $\Sigma_s \geqslant 0$ and $1 - \vartheta \geqslant 0$, one sees that

$$\begin{aligned} -(\mathcal{T} + \mathcal{K}(\gamma))\varphi_0(x, v) \geqslant \vartheta^{-1}\left(1 + \sigma(x, v)\tau(x, v)\right) \Bigg(& \vartheta\psi_0(v) - [\mathcal{K}^\tau(x)\psi_0](v) \\ & + \frac{\gamma - \vartheta}{\gamma} \int_V \frac{\Sigma_f(x, v, v')\tau(x, v')}{1 + \sigma(x, v)\tau(x, v)} \psi_0(v') d\mu(v') \Bigg). \end{aligned}$$

In particular, from the positivity of Σ_f, one sees that, provided $\gamma \geqslant \vartheta$, $-(\mathcal{T} + \mathcal{K}(\gamma))\varphi_0(x, v) \geqslant 0$ for almost every $(x, v) \in \mathcal{D} \times V$. Then, from Proposition 2.13, this means that $\tau_-(\varphi_0) \leqslant \vartheta$ and $k_{\text{eff}} \leqslant \vartheta$. Since $\vartheta > \overline{\vartheta}$ is arbitrary, one gets the result. □

Remark 3.8. *From the above result, one sees that the reactor is sub-critical whenever* $\overline{\vartheta} < 1$. *Note that the fact that* $\overline{\vartheta} < 1$ *implies* $k_{\text{eff}} \leqslant 1$ *is already contained in* [22, Theorem 7].

The above result provides an upper bound of k_{eff} leading to the sub-criticality of the reactor core. To get a lower bound of k_{eff}, one defines a similar quantity

$$\underline{\vartheta} = \sup_{\psi \in L^p_+(V,d\mu)} \operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times V} \frac{[\mathcal{K}^\tau(x)\psi](v)}{\psi(v)}.$$

Proposition 3.9. *Under the assumptions of Theorem 3.3, if* $\underline{\vartheta} > 1$, *then* $k_{\text{eff}} \geqslant \underline{\vartheta}$. *In particular, for a reactor core to be sub-critical, it is necessary that* $\underline{\vartheta} \leqslant 1$.

Proof. The proof is very similar to that of Prop. 3.7. Namely, assume $\underline{\vartheta} > 1$. For any $\vartheta \in (1, \underline{\vartheta})$, let $\psi_0 \in L^p_+(V, d\mu)$ be such that $[\mathcal{K}^\tau(x)\psi_0](v) \geqslant \vartheta\psi_0(v)$ for almost every $(x, v) \in \mathcal{D} \times V$. Then, the function $\varphi_0(x, v) = \tau(x, v)\psi_0(v)$ belongs to W^+_p and, arguing as in Prop. 3.7, thanks to Eq. (3.6), one sees that, $(\mathcal{T} + \mathcal{K}(\gamma))\varphi$ is nonnegative for any $\gamma \leqslant \vartheta$. Consequently, $\tau_+(\varphi) \geqslant \vartheta$ and Prop. 2.9 implies that $k_{\text{eff}} \geqslant \vartheta$ for any $\vartheta \in (1, \underline{\vartheta})$. □

Remark 3.10. *To the author's knowledge, the identity* $\overline{\vartheta} = \underline{\vartheta}$ *is an open question. Notice however that, according to I. Marek's result, Ref.* [18], *Theorem 3.2, for any* $x \in \mathcal{D}$, *one has the identity*

$$\sup_{\psi \in L^p_+(V,d\mu)} \operatorname*{ess\,inf}_{v\in V} \frac{[\mathcal{K}^\tau(x)\psi](v)}{\psi(v)} = \inf_{\psi \in L^p_+(V,d\mu)} \operatorname*{ess\,sup}_{v\in V} \frac{[\mathcal{K}^\tau(x)\psi](v)}{\psi(v)} = r_\sigma[\mathcal{K}^\tau(x)],$$

where we recall that $\mathcal{K}^\tau(x)$ *is an operator in* $L^p(V, d\mu)$.

In the same spirit, for almost every $x \in \mathcal{D}$, define $\mathcal{K}^\tau_f(x)$ as the following operator on $L^p(V, d\mu)$:

$$\mathcal{K}^\tau_f(x) \;:\psi \in L^p(V,d\mu) \mapsto \int_V \frac{\Sigma_f(x,v,v')\tau(x,v')}{1+\sigma(x,v)\tau(x,v)}\psi(v')d\mu(v') \in L^p(V,d\mu)$$

and let us define, as in [22], the set I_f of all $\beta \geqslant 0$ for which there exists $\psi \in L^p_+(V, d\mu) \setminus \{0\}$ such that

$$[\mathcal{K}^\tau_f(x)\psi](v) \geqslant \beta\psi(v), \qquad \text{for almost every} \qquad (x, v) \in \mathcal{D} \times V.$$

According to [22, Lemma 4] the set I is closed so that, if one defines

$$\beta_f := \sup\{\beta, \beta \in I\}$$

then, there exists $\psi_f \in L^p_+(V, d\mu)\setminus\{0\}$ such that $[\mathcal{K}^\tau_f(x)\psi_f](v) \geqslant \beta_f\psi_f(v)$ for almost every $(x, v) \in \mathcal{D} \times V$. When the velocity space is bounded away from 0 then, β_f provides a lower bound for k_{eff}:

Proposition 3.11. *Under the assumptions of Theorem 3.3, one has* $k_{\text{eff}} \geqslant \beta_f$.

Proof. Set $\varphi_f(x, v) = \tau(x, v)\psi_f(v)$ where $\psi_f \in L^p_+(V, d\mu)$ is defined here above. Arguing as in the proof of Proposition 3.7, one sees that, since $0 \notin V$, $\varphi_f \in W^+_p$. Therefore, Theorem 3.3 ensures that

$$\frac{1}{k_{\text{eff}}} \leqslant \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \frac{v\cdot\nabla_x\varphi_f(x,v) + \sigma(x,v)\varphi_f(x,v) - \displaystyle\int_V \Sigma_s(x,v,v')\varphi_f(x,v')\,d\mu(v')}{\displaystyle\int_V \Sigma_f(x,v,v')\varphi_f(x,v')\,d\mu(v')}.$$

As in the proof of Prop. 3.7 and since Σ_s and φ_s are nonnegative, one gets

$$\frac{1}{k_{\text{eff}}} \leqslant \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \frac{\psi_f(v)}{[\mathcal{K}_f^{\tau}(x)\psi_f](v)} \leqslant \frac{1}{\beta_f}$$

which ends the proof. □

Remark 3.12. *The above Proposition provides a lower bound of the criticality eigenvalue k_{eff} that depends* ***only on the fission collision operator*** *$\mathcal{K}_f$. In particular, a sufficient condition for the reactor to be super-critical is $\beta_f > 1$.*

Proposition 3.13. *Let $v_0 := \inf\{|v|\,;\, v \in V\}$ and let d be the diameter of $\mathcal{D}$. Define*

$$\mathbf{\Lambda}_f := \sup_{\psi\in L^p_+(V,d\mu)} \operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times V} \frac{1}{\psi(v)} \int_V \Sigma_f(x,v,v')\tau(x,v')\psi(v')d\mu(v').$$

Then, under the assumptions of Theorem 3.3,

$$\frac{1}{k_{\text{eff}}} \leqslant \frac{1+\overline{\sigma}\,\frac{d}{v_0}}{\mathbf{\Lambda}_f}, \tag{3.7}$$

where $\overline{\sigma} := \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \sigma(x,v)$. In particular, if V bounded then

$$k_{\text{eff}} \geqslant \frac{1}{1+\overline{\sigma}\,\frac{d}{v_0}} \operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times V} \int_V \Sigma_f(x,v,v')\tau(x,v')d\mu(v'). \tag{3.8}$$

Proof. Let us consider again test-functions of the form $\varphi(x,v) = \tau(x,v)\psi(v)$ where $\psi \in L^p_+(V,d\mu)$. Then, as above, according to (3.5)

$$\begin{aligned}\frac{1}{k_{\text{eff}}} &\leqslant \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \frac{1+\sigma(x,v)\tau(x,v) - \dfrac{1}{\psi(v)}\displaystyle\int_V \Sigma_s(x,v,v')\tau(x,v')\psi(v')d\mu(v')}{\dfrac{1}{\psi(v)}\displaystyle\int_V \Sigma_f(x,v,v')\tau(x,v')\psi(v')d\mu(v')}\\ &\leqslant \frac{1+\operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times V} \sigma(x,v)\tau(x,v)}{\operatorname*{ess\,inf}_{(x,v)\in\mathcal{D}\times V} \dfrac{1}{\psi(v)}\displaystyle\int_V \Sigma_f(x,v,v')\tau(x,v')\psi(v')d\mu(v')}.\end{aligned}$$

Since such an inequality holds for arbitrary $\psi(v) > 0$ and since $\tau(x,v) \leqslant d/|v| \leqslant d/v_0$ for almost every $(x,v) \in \mathcal{D}\times V$, one gets (3.7). To prove (3.8), it suffices to consider the test-function $\psi(v) = 1$ $(v \in V)$, which belongs to $L^p_+(V,d\mu)$ provided V is bounded. □

Remark 3.14. *We dealt in this section with the case of velocities bounded away from zero. For practical use in nuclear engineering, this is no major restriction. However, it should also be possible to derive explicit bounds of k_{eff} when $0 \in V$. In such a case, the exit time $\tau(x,v)$ is not bounded anymore but behave as $\frac{1}{|v|}$ for small $|v|$. Therefore, test-functions of the form $\varphi(x,v) = \tau(x,v)\psi(v)$ belong to W^+_p if and only if $\frac{\psi(v)}{|v|} \in L^p_+(V,d\mu)$.*

4. THE CRITICAL PROBLEM FOR THE ENERGY-DEPENDENT DIFFUSION MODEL

4.1. **Variational characterization.** In this section, we are concerned with the following

$$-\,\mathrm{div}_x\left(\mathbb{D}(x,\xi)\nabla_x\varrho(x,\xi)\right)+\sigma(x,\xi)\varrho(x,\xi)-\int_E\Sigma_s(x,\xi,\xi')\,\varrho(x,\xi')\,d\xi'$$
$$=\frac{1}{k_{\mathrm{eff}}}\int_E\Sigma_f(x,\xi,\xi')\,\varrho(x,\xi')\,d\xi',\quad (4.1)$$

where the unknown $\varrho(\cdot,\cdot)$ is assumed to be *nonnegative* and to satisfy the Dirichlet boundary conditions

$$\varrho_{|\partial\mathcal{D}}(\cdot,\xi)=0\qquad\qquad \text{a.e. } \xi\in E,$$

where $\mathcal{D}$ is C^2 open *bounded and connected* subset of $\mathbb{R}^N$ and E is an interval of $]0,\infty[$. We will assume throughout this section that there exist some constants $\sigma_i>0$ $(i=1,2)$ such that

$$0<\sigma_1\leqslant\sigma(x,\xi)\leqslant\sigma_2<\infty,\ \text{a.e. } (x,\xi)\in\mathcal{D}\times E. \quad (4.2)$$

Moreover, we assume the measurable matrix–valued application $\mathbb{D}(\cdot,\cdot)$ satisfies the following (uniform) ellipticity property

$$\operatorname*{ess\,inf}_{(x,\xi)\in\mathcal{D}\times E}\sum_{i,j=1}^N d_{ij}(x,\xi)\eta_i\eta_j\geqslant d_1|\eta|^2\qquad\qquad(\eta\in\mathbb{R}^N)\quad (4.3)$$

and regularity assumption $d_{ij}(\cdot,\xi)\in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$ for almost every $\xi\in E$. We will study Problem (4.1) in a *Hilbert space setting* for simplicity. Namely, set

$$X_2=L^2(\mathcal{D}\times E,dxd\xi).$$

Let us assume the kernels $\Sigma_s(\cdot,\cdot,\cdot)$ and $\Sigma_f(\cdot,\cdot,\cdot)$ to be *nonnegative* and define the *scattering operator*

$$\mathcal{K}_s\ :\ \psi\in X_2\longmapsto\mathcal{K}_s\psi(x,\xi)=\int_E\Sigma_s(x,\xi,\xi')\psi(x,\xi')d\xi'\in X_2,$$

and the *fission operator*

$$\mathcal{K}_f\ :\ \psi\in X_2\longmapsto\mathcal{K}_f\psi(x,\xi)=\int_E\Sigma_f(x,\xi,\xi')\psi(x,\xi')d\xi'\in X_2.$$

We will assume $\mathcal{K}_s$ and $\mathcal{K}_f$ to be *bounded* operators in X_2. Define then the full collision operator

$$\mathcal{K}\ :\ \psi\in X_2\longmapsto\mathcal{K}\psi(x,\xi)=\int_E\Sigma(x,\xi,\xi')\psi(x,\xi')d\xi'\in X_2,$$

where

$$\Sigma(x,\xi,\xi')=\Sigma_s(x,\xi,\xi')+\Sigma_f(x,\xi,\xi')\qquad(x,\xi,\xi')\in\mathcal{D}\times E\times E.$$

Let us introduce the *diffusion operator*

$$\begin{cases}\mathcal{T}\ :\quad \mathscr{D}(\mathcal{T})\subset X_2\longrightarrow X_2\\ \qquad\ \ \varrho\longmapsto\mathcal{T}\varrho(x,\xi)=\mathrm{div}_x(\mathbb{D}(x,\xi)\nabla_x\varrho(x,\xi))-\sigma(x,\xi)\varrho(x,\xi),\end{cases}$$

with domain

$$\mathscr{D}(\mathcal{T}) = \{\psi \in X_2\,;\ \psi(\cdot,\xi) \in H^1_0(\mathcal{D}) \cap H^2(\mathcal{D}) \text{ a.e. } \xi \in E \text{ and } \mathcal{T}\psi \in X_2\}$$

where $H^1_0(\mathcal{D})$ and $H^2(\mathcal{D})$ are the usual Sobolev spaces. With these notations, the spectral problem (4.1) reads

$$(\mathcal{T} + \mathcal{K}_s + \frac{1}{k_{\text{eff}}}\mathcal{K}_f)\varrho_{\text{eff}} = 0, \qquad \varrho_{\text{eff}} \in \mathscr{D}(\mathcal{T})\ ,\ \varrho_{\text{eff}} \geqslant 0,\ \varrho_{\text{eff}} \neq 0.$$

According to the strong maximum principle, it is clear that $s(\mathcal{T}) < 0$ and $(0 - \mathcal{T})^{-1}(X_2^+) \subset X_2^+$. In order to apply Theorem 2.15, one has to make sure that $(0-\mathcal{T})^{-1}\mathcal{K}$ is power-compact and that $(0-\mathcal{T})^{-1}\mathcal{K}_f$ is irreducible. Let us begin with the following compactness result which is similar to the usual *velocity averaging* lemma (see [14] and [19, Chapter 2]) for transport equations and is based on some consequence of the Sobolev embedding Theorem [7].

Theorem 4.1. *If $\mathcal{K} \in \mathscr{B}(X_2)$ is regular then $\mathcal{K}(0-\mathcal{T})^{-1}$ is a compact operator in X_2.*

Proof. By Proposition 2.18, it suffices to prove the result for a collision operator $\mathcal{K}$ such that

$$\mathcal{K}\ :\ \varrho \in X_2 \longmapsto \mathcal{K}\varrho(x,\xi) = \alpha(x)h(\xi)\int_E f(\xi')\varrho(x,\xi')d\xi' \in X_2$$

where

$$\alpha \in L^\infty(\mathcal{D}), \quad h \in L^2(E, d\xi) \text{ and } f \in L^2(E, d\xi).$$

Moreover, by a density argument, one can also assume f and h to be continuous functions with compact support in E. Let us split $\mathcal{K}(0-\mathcal{T})^{-1}$ as:

$$\mathcal{K}(0-\mathcal{T})^{-1} = \Theta\mathcal{M}(0-\mathcal{T})^{-1}$$

where

$$\Theta\ :\ \varrho \in L^2(\mathcal{D}, dx) \longmapsto [\Theta\varrho](x,\xi) = \alpha(x)h(\xi)\varrho(x) \in X_2,$$

and $\mathcal{M}$ is the *averaging operator*

$$\mathcal{M}\ :\ \psi \in X_2 \longmapsto \mathcal{M}\psi(x) = \int_E f(\xi')\psi(x,\xi')d\xi' \in L^2(\mathcal{D}).$$

It is enough to prove that $\mathcal{M}(0-\mathcal{T})^{-1}\ :\ X_2 \to L^2(\mathcal{D})$ is compact. Let $\mathcal{B}$ be a bounded subset of X_2. One has to show that $\{\mathcal{M}g\,;\ g \in (0-\mathcal{T})^{-1}(\mathcal{B})\}$ is a relatively compact subset of $L^2(\mathcal{D})$. For any $\varphi \in \mathcal{B}$, set

$$g(x,\xi) = (0-\mathcal{T})^{-1}\varphi(x,\xi).$$

For almost every $\xi \in E$, $g(\cdot,\xi) \in H^1_0(\mathcal{D})$. One extends g to the whole space $\mathbb{R}^N$ by

$$\tilde{g}(x,\xi) = \begin{cases} g(x,\xi) \text{ if } x \in \mathcal{D} \\ 0 \ \text{ else.} \end{cases}$$

Clearly, for almost every $\xi \in E$, $\tilde{g}(\cdot,\xi) \in H^1(\mathbb{R}^N)$. Consequently, according to [7, Proposition IX.3], for a. e. $\xi \in E$ and any $h \in \mathbb{R}^N$

$$\|\tau_h\tilde{g}(\cdot,\xi) - \tilde{g}(\cdot,\xi)\|_{L^2(\mathcal{D})} \leqslant |h|\|\nabla_x\tilde{g}(\cdot,\xi)\|_{L^2(\mathcal{D})},$$

where $\tau_h f(x) = f(x+h)$ $(x \in \mathcal{D},\ h \in \mathbb{R}^N)$, i.e.

$$\int_{\mathcal{D}} |\tilde{g}(x+h,\xi) - \tilde{g}(x,\xi)|^2 dx \leqslant |h|^2 \int_{\mathcal{D}} |\nabla_x g(x,\xi)|^2 dx. \tag{4.4}$$

Now, recall that

$$-\mathrm{div}_x(\mathbb{D}(x,\xi)\nabla_x g(x,\xi)) + \sigma(x,\xi)g(x,\xi) = \varphi(x,\xi) \qquad (x,\xi) \in \mathcal{D} \times E.$$

Multiplying this identity by $g(x,\xi)$ and integrating by parts yield, thanks to the ellipticity property (4.3),

$$d_1 \int_{\mathcal{D}\times E} |\nabla_x g(x,\xi)|^2 dx d\xi \leqslant \int_{\mathcal{D}\times E} |g(x,\xi)||\varphi(x,\xi)| dx d\xi.$$

In particular, since $\mathcal{B}$ is bounded, by Cauchy-Schwarz inequality, there exists $c > 0$ such that

$$\sup_{g \in (0-\mathcal{T})^{-1}(\mathcal{B})} \int_{\mathcal{D}\times E} |\nabla_x g(x,\xi)|^2 dx d\xi \leqslant c. \tag{4.5}$$

Then, (4.5) together with (4.4) yield

$$\int_{\mathcal{D}\times E} |\tilde{g}(x+h,\xi) - \tilde{g}(x,\xi)|^2 dx d\xi \leqslant c|h|^2.$$

By Hölder's inequality, since f is continuous with compact support

$$\begin{aligned}\int_{\mathcal{D}} |M\tilde{g}(x+h) - M\tilde{g}(x)|^2 dx &= \int_{\mathcal{D}} dx \left| \int_E (\tilde{g}(x+h,\xi) - \tilde{g}(x,\xi)) f(\xi) d\xi \right|^2 \\ &\leqslant C \int_{\mathcal{D}\times E} |\tilde{g}(x+h,\xi) - \tilde{g}(x,\xi)|^2 dx d\xi \\ &\leqslant |h|^2 C,\end{aligned}$$

where $C > 0$ does not depend on g. In particular,

$$\lim_{h\to 0} \sup_{g \in (0-\mathcal{T})^{-1}(\mathcal{B})} \int_{\mathcal{D}} |M\tilde{g}(x+h) - M\tilde{g}(x)|^2 dx = 0.$$

Now, using that $\widetilde{Mg} = M\tilde{g}$ one deduces the conclusion from Riesz-Fréchet-Kolmogorov Theorem [7]. □

We are now in position to prove the main result of this section where the notations of Section 2 are adopted :

Theorem 4.2. *Let $\mathcal{K} \in \mathscr{B}(X_2)$ be regular. Assume there exists an open subset $E_0 \subset E$ such that*

$$\Sigma_f(x,\xi,\xi') > 0 \quad \textit{a.e. } (x,\xi,\xi') \in \mathcal{D} \times E \times E_0. \tag{4.6}$$

Then, the problem (4.1) admits a effective multiplication factor $k_{\mathrm{eff}} > 0$ if, and only if,

$$\lim_{\gamma\to 0} r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}(\gamma)] > 1 \qquad \textit{and} \qquad r_\sigma[(0-\mathcal{T})^{-1}\mathcal{K}_s] < 1.$$

Moreover, $k_{\rm eff}$ is characterized

$$\begin{aligned}\frac{1}{k_{\rm eff}} &= \min_{\varphi\in W_p^+}\ \operatorname*{ess\,sup}_{(x,v)\in\mathcal{D}\times E}\frac{-\mathrm{div}(\mathbb{D}(x,\xi)\nabla_x\varphi(x,\xi))+\sigma(x,\xi)\varphi(x,\xi)-\displaystyle\int_E\Sigma_s(x,\xi,\xi')\varphi(x,\xi')d\xi'}{\displaystyle\int_E\Sigma_f(x,\xi,\xi')\varphi(x,\xi')d\xi'}\\ &= \max_{\varphi\in W_p^+}\ \operatorname*{ess\,inf}_{(x,\xi)\in\mathcal{D}\times V}\frac{-\mathrm{div}(\mathbb{D}(x,\xi)\nabla_x\varphi(x,\xi))+\sigma(x,\xi)\varphi(x,\xi)-\displaystyle\int_E\Sigma_s(x,\xi,\xi')\varphi(x,\xi')d\xi'}{\displaystyle\int_E\Sigma_f(x,\xi,v')\varphi(x,\xi')d\xi'}.\end{aligned}\tag{4.7}$$

Proof. From (4.6), $\mathcal{K}_f(X_2^+)\subset X_2^+$. Now, for almost every $\xi\in E$, define T_ξ as the following operator on $L^2(\mathcal{D})$:

$$T_\xi\ :\ \varrho\in\mathscr{D}(T_\xi)\mapsto T_\xi\varrho(x)=\mathrm{div}_x(\mathbb{D}(x,\xi)\nabla_x\varrho(x))-\sigma(x,\xi)\varrho(x),$$

where $\mathscr{D}(T_\xi)=H_0^1(\mathcal{D})\cap H^2(\mathcal{D})$ turns out to be independent of ξ. Since $\mathcal{D}$ is connected, the (elliptic) maximum principle implies that $(0-T_\xi)^{-1}$ is irreducible (see [9, Theorem 3.3.5] or [3, Section 11.2]). Actually, since T_ξ is the generator of a holomorphic semigroup, this implies that $(0-T_\xi)^{-1}$ is positivity improving (see [23, p. 306]), i.e. $(0-T_\xi)^{-1}\varrho(x)>0$ for almost every $x\in\mathcal{D}$ provided $\varrho\in L^2(\mathcal{D})$, $\varrho(x)\geqslant 0$ for almost every $x\in\mathcal{D}$ and $\varrho\neq 0$. Now, let $\psi\in X_2$, $\psi(x,\xi)\geqslant 0$ for almost every $(x,\xi)\in\mathcal{D}\times E$, $\psi\neq 0$. Then, $\mathcal{K}_f\psi\geqslant 0$ and $(0-T_\xi)^{-1}\mathcal{K}_f\psi(x,\xi)>0$ for almost every $(x,\xi)\in\mathcal{D}\times E$. It is easy to see that this exactly means that $(0-\mathcal{T})^{-1}\mathcal{K}_f\psi(x,\xi)>0$ for almost every $(x,\xi)\in\mathcal{D}\times E$ and the irreducibility of $(0-\mathcal{T})^{-1}\mathcal{K}_f$ follows. Since $(0-\mathcal{T})^{-1}\mathcal{K}$ is a compact operator by Theorem 4.1, the conclusion follows from Theorems 2.6 and 2.15. $\square$

4.2. **Explicit bounds.** In this section, we derive explicit bounds for the effective multiplication factor $k_{\rm eff}$. As we did in Section 3.2, the strategy consists in applying Theorem 4.7 to suitable test-functions. We assume the hypothesis of Theorem 4.2 to be met. Moreover, we assume here that the diffusion coefficient $\mathbb{D}(\cdot,\cdot)$ is *degenerate*, i.e.

$$\mathbb{D}(x,\xi)=\mathbb{D}_0(x)d_1(\xi),\qquad\qquad (x,\xi)\in\mathcal{D}\times E,$$

where $\mathbb{D}_0(\cdot)$ is a matrix-valued application satisfying the ellipticity condition (4.3), $\mathbb{D}(\cdot)\in W^{1,2}_{\rm loc}(\mathcal{D})$ and $d_1(\cdot)$ is a *bounded* real-valued application with

$$\operatorname*{ess\,inf}_{\xi\in E}d_1(\xi)>0.$$

Let λ_0 be the principal eigenvalue of the following elliptic problem in $L^2(\mathcal{D})$

$$\begin{cases}\mathrm{div}_x(\mathbb{D}_0(x)\nabla\varrho(x))+\lambda_0\varrho(x)=0, & (x\in\mathcal{D})\\ \varrho_{|\partial\mathcal{D}}(x)=0 & (x\in\partial\mathcal{D}).\end{cases}\tag{4.8}$$

It is well-known [9] that $\lambda_0>0$ and that there exists a *positive* eigenfunction ϱ_0 solution to (4.8). Set $\mathcal{E}^+=\{\psi\in L^2(E,d\xi)\,;\,\psi(\xi)>0\text{ a.e. }\xi\in E\}$. In the spirit of Section 3.2, for almost every $x\in\mathcal{D}$, define $\mathcal{K}_f^{\lambda_0}(x)$ as the following operator on $L^2(E,d\xi)$:

$$\mathcal{K}_f^{\lambda_0}(x)\ :\psi\in L^2(E,d\xi)\mapsto\int_E\frac{\Sigma_f(x,\xi,\xi')}{\lambda_0 d_1(\xi)+\sigma(x,\xi)}\psi(\xi')d\xi'\in L^2(E,d\xi)$$

and let I_f be the set of all $\beta \geqslant 0$ for which there exists $\psi \in \mathcal{E}^+$ such that

$$[\mathcal{K}_f^{\lambda_0}(x)\psi](\xi) \geqslant \beta\psi(\xi), \qquad \text{for almost every} \qquad (x,\xi) \in \mathcal{D} \times E.$$

Proposition 4.3. *Setting $\beta_0 := \sup\{\beta, \beta \in I\}$, one has $k_{\text{eff}} \geqslant \beta_0$. In particular, a necessary condition to the reactor to be sub-critical is $\beta_0 < 1$.*

Proof. As in Section 3.2, the set I is closed. Therefore, there exists $\psi \in \mathcal{E}^+$ such that $[\mathcal{K}_f^{\lambda_0}(x)\psi](\xi) \geqslant \beta_0\psi(\xi)$ for almost every $(x,\xi) \in \mathcal{D} \times E$. Now, set $\varphi_0(x,\xi) = \varrho_0(x)\psi(\xi)$, then, $\varphi \in W_2^+$ and

$$\begin{aligned} -\text{div}_x(\mathbb{D}(x,\xi)\nabla_x\varphi(x,\xi)) &= -d_1(\xi)\psi(\xi)\text{div}_x(\mathbb{D}_0(x)\nabla\varrho_0(x)) \\ &= \lambda_0 d_1(\xi)\psi(\xi)\varrho_0(x) \qquad\qquad (x,\xi) \in \mathcal{D} \times E. \end{aligned}$$

Consequently, thanks to (4.7) one has

$$\begin{aligned} \frac{1}{k_{\text{eff}}} &\leqslant \operatorname*{ess\,sup}_{(x,\xi)\in\mathcal{D}\times E} \frac{[\lambda_0 d_1(\xi) + \sigma(x,\xi)]\varrho_0(x)\psi(\xi) - \varrho_0(x)\displaystyle\int_E \Sigma_s(x,\xi,\xi')\psi(\xi')d\xi'}{\varrho_0(x)\displaystyle\int_E \Sigma_f(x,\xi,\xi')\psi(\xi')d\xi'} \\ &\leqslant \operatorname*{ess\,sup}_{(x,\xi)\in\mathcal{D}\times E} \frac{[\lambda_0 d_1(\xi) + \sigma(x,\xi)]\psi(\xi)}{\displaystyle\int_E \Sigma_f(x,\xi,\xi')\psi(\xi')d\xi'} = \operatorname*{ess\,sup}_{(x,\xi)\in\mathcal{D}\times E} \frac{\psi(\xi)}{[\mathcal{K}_f^{\lambda_0}(x)\psi](\xi)} \end{aligned}$$

which proves that $\frac{1}{k_{\text{eff}}} \leqslant \frac{1}{\beta_0}$. □

In the same spirit, for almost every $x \in \mathcal{D}$, define $\mathcal{K}^{\lambda_0}(x)$ as the operator on $L^2(E, d\xi)$ given by

$$\mathcal{K}^{\lambda_0}(x)\ :\psi \in L^2(E,d\xi) \mapsto \int_E \frac{\Sigma(x,\xi,\xi')}{\lambda_0 d_1(\xi) + \sigma(x,\xi)}\psi(\xi')d\xi' \in L^2(E,d\xi)$$

where we set $\Sigma(x,\xi,\xi') = \Sigma_s(x,\xi,\xi') + \Sigma_f(x,\xi,\xi')$. As in Section 3.2, set

$$\overline{\vartheta} := \inf_{\psi\in\mathcal{E}^+} \operatorname*{ess\,sup}_{(x,\xi)\in\mathcal{D}\times E} \frac{[\mathcal{K}^{\lambda_0}(x)\psi](\xi)}{\psi(\xi)}, \qquad \text{and} \qquad \underline{\vartheta} := \sup_{\psi\in\mathcal{E}^+} \operatorname*{ess\,inf}_{(x,\xi)\in\mathcal{D}\times E} \frac{[\mathcal{K}^{\lambda_0}(x)\psi](\xi)}{\psi(\xi)}.$$

Then, one has the following bounds of k_{eff}, in the spirit of Propositions 3.7 & 3.9.

Proposition 4.4. *Under the assumptions of Theorem 4.2, if $\underline{\vartheta} > 1$, then $k_{\text{eff}} \geqslant \underline{\vartheta}$. On the other hand, if $\overline{\vartheta} > 1$, then $k_{\text{eff}} \leqslant \overline{\vartheta}$.*

Proof. The proof is very similar to that of Prop. 3.7 & 3.9. We only prove the first part of the result, the second part proceeding along the same lines. Assume thus that $\underline{\vartheta} > 1$. For any $\vartheta \in (1, \underline{\vartheta})$, let $\psi_0 \in \mathcal{E}^+$ be such that $\operatorname{ess\,inf}_{(x,\xi)\in\mathcal{D}\times E} \frac{[\mathcal{K}^{\lambda_0}(x)\psi_0](\xi)}{\psi_0(\xi)} \geqslant \vartheta$. Choose then the test-function $\varphi(x,\xi) = \varrho_0(x)\psi_0(\xi)$. Such an application φ belongs to W_2^+ and, as in the above proof, for any $\gamma > 0$, one sees that

$$(\mathcal{T}+\mathcal{K}(\gamma))\varphi(x,\xi)=\frac{\varrho_0(x)}{\vartheta}\left(\lambda_0 d_1(\xi)+\sigma(x,\xi)\right)\bigg(-\vartheta\psi_0(\xi)+[\mathcal{K}^{\lambda_0}(x)\psi_0](\xi)+ \\ (\vartheta-1)[\mathcal{K}_s^{\lambda_0}(x)\psi_0](\xi)+\frac{\vartheta-\gamma}{\gamma}[\mathcal{K}_f^{\lambda_0}(x)\psi_0](\xi)\bigg)$$

where $\mathcal{K}_f^{\lambda_0}(x)$ has been already defined and the definition of $\mathcal{K}_s^{\lambda_0}(x)$ is similar (Σ_s replacing Σ_f). Then, from the positivity of Σ_s and Σ_f, the assumption $\vartheta>1$ implies that $(\mathcal{T}+\mathcal{K}(\gamma))\varphi$ is nonnegative for any $\gamma\leqslant\vartheta$. Consequently, $\tau_+(\varphi)\geqslant\vartheta$ and Prop. 2.9 implies that $k_{\text{eff}}\geqslant\vartheta$. Since $\vartheta\in(1,\underline{\vartheta})$ is arbitrary, one obtains $k_{\text{eff}}\geqslant\underline{\vartheta}$. □

Whenever E is of finite Lebesgue measure, one has the following practical criteria, already stated by C. V. Pao [24, Theorem 5.3] using completely different arguments.

Corollary 4.5. *Assume E to be of finite Lebesgue measure. If*

$$\lambda_0\, d_1(\xi)+\sigma(x,\xi)<\int_E[\Sigma_s(x,\xi,\xi')+\Sigma_f(x,\xi,\xi')]d\xi' \qquad (x,\xi)\in\mathcal{D}\times E, \tag{4.9}$$

then, the reactor core is non super-critical, i.e. $k_{\text{eff}}\geqslant 1$.

Proof. Since E is of finite Lebesgue measure, the constant function $\psi=\mathbf{1}_E$ such that $\psi(\xi)=1$ for any $\xi\in E$ belongs to $\mathcal{E}^+$. Then, assumption (4.9) means exactly that $[\mathcal{K}^{\lambda_0}(x)\mathbf{1}_E](\xi)>\mathbf{1}_E(\xi)$ for almost any $(x,\xi)\in\mathcal{D}\times E$. Therefore, $\underline{\vartheta}>1$ and the conclusion follows from Prop. 4.4. □

Remark 4.6. *Notice that, under the above assumption, one has*

$$k_{\text{eff}}\geqslant\operatorname*{ess\,inf}_{(x,\xi)}\frac{\int_E[\Sigma_s(x,\xi,\xi')+\Sigma_f(x,\xi,\xi')]d\xi'}{\lambda_0 d_1(\xi)+\sigma(x,\xi)}.$$

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