

Radiation Hydrodynamics Scaling Laws in High Energy Density Physics and Laboratory Astrophysics

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Abstract. In this paper, radiating fluids scaling laws are studied. We focus on optically thin and optically thick regimes which are relevant for both astrophysics and laboratory experiments. By using homothetic Lie groups, we obtain the scaling laws, the similarity properties and the number of free parameters which allow to rescale experiments in the two astrophysical situations.

1. Introduction

High-Energy-Density Physics is a new way for astrophysicists to explore phenomena usually occurring in the Universe. The use of powerful facilities, enables us to bring the matter up to extreme states of density and temperature in laboratory [1]. The astrophysical relevance of these experiments can be checked from scaling laws provided the physical system under study satisfies similarity properties. Thus, scaling laws and similarity properties must be examined with rigorous formalism. Several studies have been published about similarity and scaling laws. For instance, in [2], [3], purely hydrodynamics and MHD scaling laws are respectively considered and in [2], the Birkhoff polytropic symmetries [4] are recovered. Moreover, in [5] optically thin radiative hydrodynamic scaling laws have been considered and in [6], the author not only studied similarity in case of optically thin plasma too but, also, discussed non-LTE situations through a microscopic approach. All these works have been carried out in an astrophysical context and were mainly based upon dimensional arguments. Scaling laws were also obtained for the Inertial Confinement Fusion (ICF) [7], [8] in order to determine the minimum energy required for ignition. These are very interesting too because they can be used as non trivial tests for numerical simulations. In this paper, we study the radiating fluid similarity problem in two different regimes that can be (or will be) achieved in laboratory with current or future facilities. In each case, we derive the corresponding scaling laws and in order to get rigorous and exact relations, our approach is based on the Lie groups [9]. In the first part, we describe this method and remind its fundamental concepts. The second part deals with the optically thin radiating fluids, which are a major topic in astrophysics. Comparisons with other results obtained earlier are carried out. Finally we consider the equilibrium diffusion approximation including radiative pressure and energy. For each approximation, connections with astrophysical objects are provided and we emphasize the number of free parameters left to rescale an experiment.

2. Lie groups, similarity and scaling laws

The invariant transformation group theory elaborated by Sophus Lie is a very powerful tool of theoretical physics to study the symmetry properties of partial differential equations (PDE) and to perform their analytical integration. Among all Lie groups, one of them, namely, the one-parameter homothetic group (HG) is frequently used, first because of its simplicity and, then, because it provides more general self-similar solutions than those derived from dimensional analysis. This property arises because the HG is a sub-group of scaling transformations. Now, remembering the philosophy of Laboratory Astrophysics (*i.e.* to recreate systems having astronomical size on short scales), it seems natural to use the HG in order to study similarity properties, scaling laws and even self-similarity. Here, we will focus on the first two points only. Group invariance of PDE together with their solutions implies that the initial conditions (IC) be preserved from the laboratory system to the astrophysical one. This intuitive but constraining condition is discussed in details in [2], [3]. From this property, experimental data and IC in laboratory provide, first, information about astrophysical environments and, second, a transposition to astronomical objects. Moreover, the invariance of equations by HG implies that the Rankine-Hugoniot relations are also invariant and, therefore, we make sure that small scale shocks correspond to the homothetic structures of astrophysical shocks.

3. Similarity and scaling laws of optically thin radiating fluids

When the cooling (or heating) characteristic time of a plasma gets close to its dynamical time, this should be considered in the modeling. Concerning optically thin plasmas, *i.e.* $\lambda_p \gg L$ (λ_p is the mean free path of photons and L is the characteristic plasmas of the system), a simple modeling of radiating losses (or heating) can be done simply by introducing a loss (or gain) of entropy. Thus, the plasma is described by the following equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot [\rho \vec{v}] = 0, \quad \rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P_{th}, \quad \frac{dP_{th}}{dt} - \gamma \frac{P_{th}}{\rho} \frac{d\rho}{dt} = -(\gamma - 1)\mathcal{L}(\rho, T), \quad dM = \rho \cdot dV, \quad (1)$$

where d/dt is the Lagrangian derivative and ρ , \vec{v} , P_{th} , γ and M are respectively the density, velocity, thermal pressure, polytropic index and the mass of the fluid. The function $\mathcal{L}(\rho, T)$ writes $\mathcal{L}(\rho, T) = \mathcal{Q}_1(\rho, T) + \mathcal{Q}_2(\rho, T)$ where \mathcal{Q}_1 and \mathcal{Q}_2 are energy sources (or losses). In addition, we assumed a polytropic evolution; *i.e.* $P_{th} = (\gamma - 1)\rho e$ where e is the specific internal energy. Finally, an equation of state should be added to close (1): $P_{th} = \varepsilon_0[Z]\rho^\mu T^\nu$ where $\varepsilon_0[Z]$ is a function of the ionization Z . It should be noticed that to satisfy the first thermodynamical principle we should have $\gamma(1 - \nu) = (\mu - \nu)$. Experimentally, heating can represent the laser energy deposition. From an astrophysical viewpoint, this modeling describes interstellar jets, bow shocks, radiating shocks (point C in Drake diagram [10], Fig 7.17) in Polars and supernova remnants. The relation between the typical quantities in astrophysical objects and laboratory experiments (that we note with \sim) are given by: $r = a^{\delta_1} \tilde{r}$, $t = a^{\delta_2} \tilde{t}$, $\vec{v} = a^{\delta_3} \tilde{\vec{v}}$, $M = a^{\delta_4} \tilde{M}$, $\rho = a^{\delta_5} \tilde{\rho}$, $P_{th} = a^{\delta_6} \tilde{P}_{th}$, $\mathcal{Q}_1 = a^{\delta_7} \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 = a^{\delta_8} \tilde{\mathcal{Q}}_2$, $\varepsilon_0 = a^{\delta_9} \tilde{\varepsilon}_0$, $T = a^{\delta_{10}} \tilde{T}$, $\gamma = a^{\delta_{11}} \tilde{\gamma}$ where a is the group parameter and δ_i are the homothetic exponents. Rescaling ε_0 , and \mathcal{Q}_i can absorb a modification of Z from one system to the other (for example in bremsstrahlung cooling $\mathcal{Q} \propto Z^2$). Up to now, the sources have not been specified but in the applications we will consider power law forms ($\mathcal{Q}_i = \mathcal{Q}_{0,i} \rho^{m_i} P_{th}^{n_i} r^{l_i}$). This type of source is quite suitable for cooling since several processes in the continuum write in this simple form. We can also write $\mathcal{Q}_i \propto \kappa_P \sigma T^4$ where σ is the Stefan-Boltzmann constant and κ_P is the Planck mean opacity that can be modeled by a power law at high temperature. The invariance of equations under the HG provides the group invariants [9] namely: $I_1 = vt/r = St$ (Strouhal number), $I_2 = \gamma$, $I_3 = P_{th}t/\rho vr = Eu \times St = St/[\gamma \mathcal{M}^2]$ (Eu : Euler number, \mathcal{M} : Mach number), $I_4 = \mathcal{Q}_1 t/P_{th} \propto t/t_{\mathcal{Q}_1}$, $I_5 = \mathcal{Q}_2 t/P_{th} \propto t/t_{\mathcal{Q}_2}$, where $t_{\mathcal{Q}_i}$ is the characteristic time of the sources \mathcal{Q}_i and $I_6 = M/[\rho r^{1+d}]$ (mass conservation). As expected, the invariants of this group are

Table 1. *Scaling for optically thin plasmas for power law models of sources (second column). Plane ($d=0$) radiative shock problem for magnetic cataclysmic variables: the third column corresponds to Bremsstrahlung Cooling (BC) [$\Lambda \propto \rho^2 T^{1/2}$] which can be Chevalier-Imamura unstable [11] and the fourth column is obtained for BC plus cyclotronic cooling (CC) [$\Lambda \propto \rho^{0.15} T^{2.5}$] and $\alpha = P_{th} \rho^{-\gamma}$.*

physical ratio	ratio (scaling factor)	BC	BC + CC
r/\tilde{r}	a^{δ_1}	$a^{\delta_6-2\delta_5}$	$a^{-3\delta_5/40}$
$\rho/\tilde{\rho}$	a^{δ_5}	a^{δ_5}	a^{δ_5}
P/\tilde{P}	a^{δ_6}	a^{δ_6}	$a^{77\delta_5/40}$
t/\tilde{t}	$a^{\delta_1+(\delta_5-\delta_6)/2}$	$a^{(\delta_6-3\delta_5)/2}$	$a^{-43\delta_5/80}$
v/\tilde{v}	$a^{(\delta_6-\delta_5)/2}$	$a^{(\delta_6-\delta_5)/2}$	$a^{37\delta_5/80}$
T/\tilde{T}	$a^{(\delta_6-\delta_9-\mu\delta_5)/\nu}$	$a^{(\delta_6-\delta_5)}$	$a^{37\delta_5/40}$
M/\tilde{M}	$a^{\delta_5+(1+d)\delta_1}$	$a^{\delta_6-\delta_5}$	$a^{37\delta_5/40}$
$\alpha/\tilde{\alpha}$	$a^{\delta_6-\gamma\delta_5}$	$a^{\delta_6-\gamma\delta_5}$	$a^{(77/40-\gamma)\delta_5}$
$\mathcal{Q}_{0,1}/\tilde{\mathcal{Q}}_{0,1}$	$a^{(3/2-n_1)\delta_6-(m_1+1/2)\delta_5-(l_1+1)\delta_1}$	1	1
$\mathcal{Q}_{0,2}/\tilde{\mathcal{Q}}_{0,2}$	$a^{(3/2-n_2)\delta_6-(m_2+1/2)\delta_5-(l_2+1)\delta_1}$	0	1

identical to the dimensionless numbers derived in similarity studies [6]. However, our approach is more general since we have local dimensionless quantities in contrast to global dimensionless numbers obtained thanks to the dimensional analysis. Thus, in our extension, the physical fields are conserved. Table 1 shows scaling laws for polars. Generally, we have four free parameters (δ_1 , δ_5 , δ_6 and δ_9) and, if we preserve ionization, only two (resp. one) exponent(s) remain(s) for a single source (resp. two sources). Moreover, if we set $\delta_5 = 0$, $\delta_1 = 1$, $\delta_6 = 2$ and $\mathcal{Q}_2 = 0$, we get the similarity considerations of [12]. Thus, with the same formalism, we can study similarity properties, scaling laws, and include the specific case presented in [12].

4. Similarity and scaling laws of optically thick radiating fluids

Many systems, as well in laboratory as in astrophysics, are optically thick to radiation. For instance, the many classes of stars are more or less affected by radiation. Radiative pressure implies that there is an upper limit to the mass of a star (Eddington limit). Generally, including radiative flux in laboratory experiments is enough and that is why, researches about scaling laws in this regime have been carried out in ICF. Here, we add the energy and pressure of radiation (see [13]) in the diffusion approximation at ETL. In experiments, LTE is usually satisfied [6] and it will be achieved on LMJ and NIF. In Astrophysics, radiation pressure and energy play a key role in stars, supernovae, in evaporation phenomena, in clumps [14]... The plasma evolution is then governed by the equations ([10], pp 270-271):

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla}[P_{th} + P_{rad}], \quad \frac{d}{dt}(\rho e + E_{rad}) - \frac{\rho e + P_{th} + E_{rad} + P_{rad}}{\rho} \frac{d\rho}{dt} = -\vec{\nabla} \cdot \vec{F}_{rad} - \mathcal{Q}, \quad (2)$$

where \vec{F}_{rad} , E_{rad} , P_{rad} and \mathcal{Q} are respectively the radiative flux, radiative energy density, radiative pressure and the energy source term. In the application, we will consider that $E_{rad} = a_R T^4$, $P_{rad} = E_{rad}/3$, $\vec{F}_{rad} = -\kappa_{rad} \vec{\nabla} T$ where κ_{rad} is the radiative conductivity given by $\kappa_{rad} = \kappa_0 \rho^m T^n$ [we still have $P_{th} = \varepsilon_0 \rho^\mu T^\nu$]. In addition to the optically thin case we add the radiative relations: $\vec{F}_{rad} = a^{\delta_{12}} \tilde{\vec{F}}_{rad}$; $\kappa_{rad} = a^{\delta_{13}} \tilde{\kappa}_{rad}$; $\kappa_0 = a^{\delta_{14}} \tilde{\kappa}_0$; $E_{rad} = a^{\delta_{15}} \tilde{E}_{rad}$; $P_{rad} = a^{\delta_{16}} \tilde{P}_{rad}$; $\mathcal{Q} = a^{\delta_{17}} \tilde{\mathcal{Q}}$. As before I_1 , I_2 , I_3 , I_4 (or I_5) and I_6 are five invariants. The additional

Table 2. *Scaling laws of optically thick plasma (Column 1) and ideal gas (Column 2).*

physical ratio	ratio (scaling factor)	Ideal gas
r/\tilde{r}	$a^{\delta_{14} + ([n-5]/[4-\nu])\delta_9 + ([m+1/2] + \mu[n-5]/[4-\nu])\delta_5}$	$a^{([m+1/2] + [n-5]/3)\delta_5}$
t/\tilde{t}	$a^{\delta_{14} + ([n-7]/[4-\nu])\delta_9 + (m+1 + \mu[n-7]/[4-\nu])\delta_5}$	$a^{(m+1 + [n-7]/3)\delta_5}$
v/\tilde{v}	$a^{(2/[4-\nu])\delta_9 + ([4\mu + \nu - 4]/[8 - 2\nu])\delta_5}$	$a^{\delta_5/6}$
$\rho/\tilde{\rho}$	a^{δ_5}	a^{δ_5}
P_{th}/\tilde{P}_{th}	$a^{(4/[4-\nu])\delta_9 + (4\mu/[4-\nu])\delta_5}$	$a^{(4/3)\delta_5}$
T/\tilde{T}	$a^{(1/[4-\nu])\delta_9 + (\mu/[4-\nu])\delta_5}$	$a^{\delta_5/3}$
E_{rad}/\tilde{E}_{rad}	$a^{(4/[4-\nu])\delta_9 + (4\mu/[4-\nu])\delta_5}$	$a^{(4/3)\delta_5}$
F_{rad}/\tilde{F}_{rad}	$a^{(6/[4-\nu])\delta_9 + ([12\mu - 4 + \nu]/[8 - 2\nu])\delta_5}$	$a^{(3/2)\delta_5}$
P_{rad}/\tilde{P}_{rad}	$a^{(4/[4-\nu])\delta_9 + (4\mu/[4-\nu])\delta_5}$	$a^{(4/3)\delta_5}$
$\alpha/\tilde{\alpha}$	$a^{(4/[4-\nu])\delta_9 + ([4\mu]/[4-\nu] - \gamma)\delta_5}$	$a^{(4/3 - \gamma)\delta_5}$
Q_{rad}/\tilde{Q}_{rad}	$a^{-\delta_{14} + ([11-n]/[4-\nu])\delta_9 + (\mu[11-n]/[4-\nu] - [m+1])\delta_5}$	no source
$\kappa_0/\tilde{\kappa}_0$	$a^{\delta_{14}}$	1
$\varepsilon_0/\tilde{\varepsilon}_0$	a^{δ_9}	1

ones are $I_7 = P_{rad}t/(\rho vr) = Eu_{rad} \times St$ (Eu_{rad} : Radiative Euler number), $I_8 = E_{rad}/P_{th} \propto 1/R$ (R : Mihalas numbers), $I_9 = tF_{rad}/(P_{th}r) = 1/Bo$ (Bo : Boltzmann number). We recover the standard dimensionless numbers [13] which describe these radiating fluids. The scaling laws are presented in table 2. We see that for an ideal gas with an ionization conservation state, we have a single (δ_5) parameter to rescale experiments, but if the ionization is not preserved, we have at least three free parameters (δ_5 , δ_9 , δ_{14}). Finally, we find that α (entropy) is conserved for $\gamma = 4/3$, which corresponds to a dominant photon regime. Notice that if we set $P_{rad} = E_{rad} = 0$, we find an extended scaling laws version of [7].

5. Conclusion

We presented the general scaling laws for two radiative regimes of major interest in laboratory and astrophysics situations. The number of free parameters depends on the structure of the model: the more phenomena we add, the more difficult it is to rescale an experiment. However, requiring a *partial* similarity ('almost' equivalent regime) allows to add free parameters and study 'almost' astrophysical situations.

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