ON THE 3-DISTORTION OF A PATH

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Abstract. We prove that, for embeddings of a path of length n in $\mathbb{R}^2$, the 3-distortion is an $\Omega(n^{1/2})$, and that, when embedded in $\mathbb{R}^d$, the 3-distortion is an $O(n^{1/(d-1)})$.

The general context of this paper is the study of the distortion that appears when a metric space is embedded into a Euclidean space. Such a study plays an important role in algorithmic geometry and its applications. In particular, significant memory gains can be achieved when a metric space is embedded into a low dimensional Euclidean space, and, therefore, the study of such embeddings is directly connected with the construction of efficient computer representations of (finite) metric spaces. The price to pay for such memory gains is the inevitable deformations that result from the embedding, and it is therefore quite important to control them, typically to understand their asymptotic behaviour when the size of the metric space increases.

A standard parameter for controlling the deformation is the distortion, that takes into account pairs of points and compares their distances in the source and the target spaces—see precise definition below. The distortion is rather well understood, and, in particular, precise bounds for its values in the case of general finite metric spaces are known [2].

Now, other parameters may be associated with an embedding naturally. Typically, for each k, one can introduce the notion of a *k-distortion* by taking into account k-tuples of points rather than just pairs, and measuring the way the volume of the associated polytope is changed. This is what U. Feige does in [1] in order to construct an algorithm minimizing the bandwidth of a graph, *i.e.*, finding a numbering $v_1, ..., v_n$ of the vertices for which the supremum of $|i-j|$ over all pairs (i, j) such that (v_i, v_j) is an edge is as small as possible. The idea of [1] is to consider volume-respecting embeddings of the graph into a Euclidean space. The point is to show that, among all projections of such an embedding on a line, a positive proportion has a minimal bandwidth of the expected size, and the main step is to investigate the k-distorsion.

Owing to the above applications and connections, understanding k-distortion for every k seems to be a quite natural goal. Now, in contrast to the case $k = 2$, very little is known so far about k-distortion for $k \geq 3$. The aim of this paper is to establish some results about 3-distortion, in the most simple case of a metric space consisting of equidistant points on a line. So, we denote by Π_n the set $\{0, 1, ..., n\}$ equipped with the distance $d(i, j) = |i-j|$. Then, for each $d \geq 2$, there exists a real parameter $\delta_3(\Pi_n, \mathbb{R}^d) \geq 1$ that measures the

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deformation of triangles when Π_n is embedded in $\mathbb{R}^d$. The intuition is that, the bigger δ_3, the flatter the triangles—the precise definition is given in Section 1 below.

As $\mathbb{R}^d$ isometrically embeds in $\mathbb{R}^{d+1}$, the inequality $\delta_3(\Pi_n, \mathbb{R}^{d+1}) \leq \delta_3(\Pi_n, \mathbb{R}^d)$ immediately follows from the precise definition, implying in particular $\delta_3(\Pi_n, \mathbb{R}^d) \leq \delta_3(\Pi_n, \mathbb{R}^2)$ for $d \geq 3$. The meaning is that, when we have more space, we can more easily embed with small distortion. For $d = 2$ (the planar case), hence for every d, it is easy to see that $\delta_3(\Pi_n, \mathbb{R}^d)$ is at most linear in n, so the question is to compare $\delta_3(\Pi_n, \mathbb{R}^d)$ with the polynomial functions n^α, $0 < \alpha < 1$. What we do below is to prove one lower bound result for $d = 2$, and one upper bound result for $d \geq 2$:

Proposition 1. *The 3-distortion $\delta_3(\Pi_n, \mathbb{R}^2)$ is an $\Omega(n^{1/2})$.*

Proposition 2. *For each fixed d, the 3-distortion $\delta_3(\Pi_n, \mathbb{R}^d)$ is an $O(n^{1/(d-1)})$.*

The results are likely not to be optimal: we conjecture that $\delta_3(\Pi_n, \mathbb{R}^2)$ might be an $\Omega(n)$, and that $\delta_3(\Pi_n, \mathbb{R}^d)$ might be lower than polynomial, typically polylogarithmic, for $d \geq 3$. This would mean that the behaviour of the 3-distortion radically differs from the standard distortion which is polynomial in n for each dimension d.

1. The 3-distortion

Our first task is to make the allusive definitions of the introduction precise.

For (V, ρ) a metric space and f a non-expanding (*i.e.*, 1-Lipschitz) embedding of V into $\mathbb{R}^d$, the distortion $\Delta(f)$ of f is defined to be the supremum of the compression ratio between the distance of two points in (V, ρ) and that of their images in $\mathbb{R}^d$:

$$\Delta(f) = \sup\Big\{\frac{\rho(P,Q)}{\mathrm{Dist}(f(P), f(Q))}; P, Q \in V\Big\}. \tag{1}$$

By construction, $\Delta(f)$ is at least 1, and the larger it is, the bigger the deformation of distances caused by f.

Let us turn to $k = 3$, *i.e.*, let us consider images of triangles. In the denominator of (1), the length of the segment $[f(P), f(Q)]$ is replaced with the area of the triangle $[f(P), f(Q), f(R)]$. As for the numerator, the area makes no sense in the source space (V, ρ), but we observe that, at least in good cases, $\rho(P, Q)$ is the sup of the lengths $\mathrm{Dist}(g(P), g(Q))$ for g a non-expanding embedding of V to $\mathbb{R}^d$ (provided $d \geq 1$). This naturally leads to defining $\rho_3(P, Q, R)$ to be the sup of $\mathrm{Area}([g(P), g(Q), g(R)])$ for g a non-expanding embedding of V to $\mathbb{R}^d$ (provided $d \geq 2$), and to defining the 3-distortion of f to be

$$\Delta_3(f) = \sup\Big\{\frac{\rho_3(P,Q,R)}{\mathrm{Area}([f(P), f(Q), f(R)])}; P, Q, R \in V\Big\}. \tag{2}$$

We shall be interested in the minimal possible value of $\Delta_3(f)$, *i.e.*, in the configurations that minimalize the distortion of triangles. We are thus led to the following notion:

Definition. The 3-*distortion* $\delta_3(V, \mathbb{R}^d)$ is defined to be the infimum of $\Delta_3(f)$ over all non-expanding embeddings f of V into $\mathbb{R}^d$.

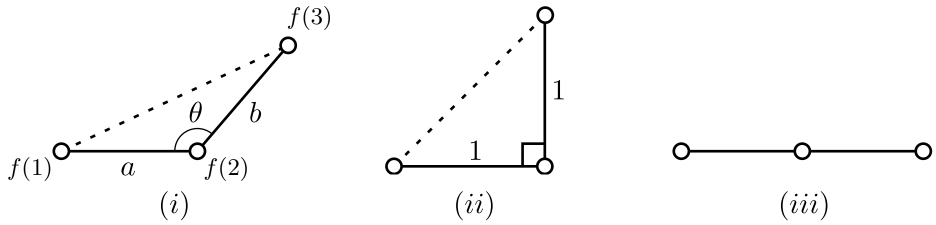

FIGURE 1. The 3-distortion of a non-expanding embedding $f : \Pi_2 \to \mathbb{R}^2$: (i) generic case: the area is $ab\sin(\theta)/2$, whence $\Delta_3(f) = 1/ab\sin\theta$, (ii) an optimal case: $\Delta_3(f) = 1$; (iii) a worst case: the isometrical embedding; then the image of f is a flat triangle of area 0, hence $\Delta_3(f) = \infty$.

The definition for k-tuples would be similar, with volume replacing area.

Figure 1 describes the situation for the graph Π_2. In this (very simple) case, there exist embeddings with 3-distortion equal to 1, namely the ones of Figure 1(ii), and, therefore, we find $\delta_3(\Pi_2) = 1$.

In the general case, we always have $\Delta_3(f) \geq 1$ by construction, and, the flatter the triangles in the image of f, the larger $\Delta_3(f)$. For instance, when f is an isometrical embedding of Π_n in $\mathbb{R}^d$, all triangles are flat, as in Figure 1(iii), and the distortion $\Delta_3(f)$ is infinite. Thus the 3-distortion is a measure of the inevitable flattening of triangles that occurs when a (large) metric space is embedded in some fixed Euclidean space: then, it is impossible that all triples of vertices are embedded so as to form a rectangular triangle as in Figure 1(ii), and the question is to evaluate how far from that one must lie. The reader can check that, even in the case of embeddings of Π_3 into $\mathbb{R}^2$, it is not so easy to prove that the minimal 3-distortion is $2/\sqrt{3} = 1,1547...$, corresponding to a U-shape with length 1 edges and $2\pi/3$ angles, and obtaining an exact value in the general case of Π_n seems out of reach. This contributes to making asymptotic bounds desirable.

In the specific case of the space Π_n, *i.e.*, of n equidistant points at distance 1 on the real line, the definition of 3-distortion can be given a more simple form. Indeed, if g is a non-expanding embedding of Π_n into $\mathbb{R}^d$, we have $\mathrm{Dist}(g(i), g(j)) \leq |i - j|$ and therefore, for $i < j < k$, we find $\mathrm{Area}([g(i), g(j), g(k)]) \leq (j-i)(k-j)/2$; on the other hand, provided $d \geq 2$, we can always find g such that the latter inequality is an equality as in Figure 1(ii). Hence, for $0 \leq i < j < k \leq n$, we have

$$\rho_3(i, j, k) = (j - i)(k - j)/2.$$

So, for f a non-expanding embedding of Π_n into $\mathbb{R}^d$, (2) takes the form

$$\Delta_3(f) = \sup\left\{\frac{(j-i)(k-j)/2}{\mathrm{Area}([f(i), f(j), f(k)])}; 0 \leq i < j < k \leq n\right\}. \tag{3}$$

In the sequel, we shall forget about embeddings and only work inside the target space $\mathbb{R}^d$.

Definition. A finite sequence of points $(M_0, \ldots, M_n)$ in $\mathbb{R}^d$ is said to be *tame* if, for each i, we have $\mathrm{Dist}(M_i, M_{i+1}) \leq 1$. In this case, we put

$$\Delta_3(M_0, \ldots, M_n) = \sup\left\{\frac{(j-i)(k-j)/2}{\mathrm{Area}([M_i, M_j, M_k])}; 0 \leq i < j < k \leq n\right\}. \tag{4}$$

If f is an embedding of Π_n into $\mathbb{R}^d$, then the sequence $(f(0), \dots, f(n))$ is tame, and, conversely, each tame sequence determines a unique embedding. Now, translating (3) gives (4) for $M_i = f(i)$ and the notation is consistent. Then the 3-distortion of Π_n can be expressed in terms of tame sequences of points: for all n, d, we have

$$\delta_3(\Pi_n, \mathbb{R}^d) = \inf\{\Delta_3(M_0, \dots, M_n); (M_0, \dots, M_n) \text{ a tame sequence in } \mathbb{R}^d\}. \tag{5}$$

Thus, from now on, our aim is to study the possible values of the quantity $\delta_3(\Pi_n, \mathbb{R}^d)$ of (5).

2. A lower bound in the planar case

In order to prove Proposition 1, we shall consider an arbitrary tame sequence in $\mathbb{R}^2$, and prove that some triangle is much distorted, *i.e.*, flattened. To this end we observe that points in convex position provide a triangle with large 3-distortion.

Say that a sequence $(P_0, \dots, P_{m-1})$ of points in the plane is *convex* if the boundary of the convex hull of $\{P_0, \dots, P_{m-1}\}$ is exactly the polygon with vertices $P_0, \dots, P_{m-1}$ in this order.

Lemma 3. *Assume that $(P_0, \dots, P_{m-1})$ is a convex sequence with $m \geq 3$. Then there exists i such that the 3-distortion of the triangle $P_iP_{i+1}P_{i+2}$—where indices are taken modulo m—is at least $m/(2\pi)$.*

Proof. The sum of angles $\angle P_0P_1P_2 + \angle P_1P_2P_3 + \dots + \angle P_{m-1}P_0P_1$ is $(m-2)2\pi$. As all angles are positive and less than π, one of them is at least $\frac{m-2}{m}\pi$. The 3-distortion of the corresponding triangle is then at least $m/(2\pi)$. □

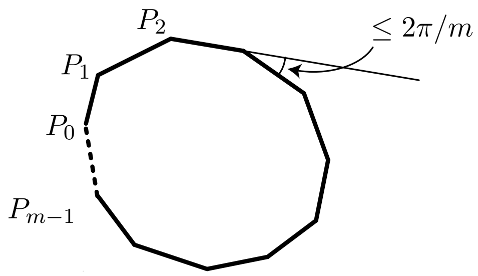

FIGURE 2. Convex sequence of points

Lemma 4. *Assume that $(M_0, \dots, M_n)$ is a tame sequence in $\mathbb{R}^2$, and that δ is an integer greater than or equal to $\Delta_3(M_0, \dots, M_n)$. Then the sequence $(M_0, M_\delta, M_{2\delta}, \dots, M_{\lfloor \frac{n}{\delta} \rfloor \delta})$ is convex.*

Proof. Let $\delta_0 := \Delta_3(M_0, \dots, M_n)$. For all $i < j$, we have $\mathrm{Dist}(M_i, M_j) \leq |j - i|$. Since for $k > j$ the area of the triangle $[M_i, M_j, M_k]$ is at least $\frac{(k-j)(j-i)}{2\delta_0}$, hence *a fortiori* $\frac{(k-j)(j-i)}{2\delta}$, the distance between the point M_k and the line (M_iM_j) is at least $\frac{k-j}{\delta}$ (Figure 2). Therefore, for $k \geq j + \delta$, the points M_k and M_{k+1} lie on the same side of the line (M_iM_j): otherwise, the distance between M_k and M_{k+1} would be at least $2\,\frac{k-j}{\delta}$, contrary to the

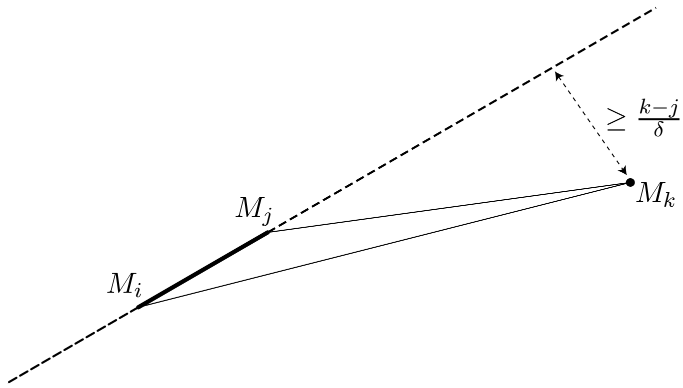

FIGURE 3. Minimal distance from the point M_k to the line (M_iM_j)

tameness hypothesis. Hence, for $k \geq j + \delta$, the point M_k lies on the same side of the line (M_iM_j) as $M_{j+\delta}$.

For a contradiction, assume that, for some i, the sequence $(M_{i\delta}, M_{(i+1)\delta}, M_{(i+2)\delta}, M_{(i+3)\delta})$ is not convex. Then either the four points are not in convex position, or they are in convex position but they do not appear in the right order on the border of their convex hull.

In the first case (Figure 4), one point lies in the convex hull of the three others. But this contradicts the hypothesis that adjacent points lie on the same side of each line $(M_{j\delta}M_{(j+1)\delta})$.

In the second case (Figure 5), the points are in convex position, but the segment $[M_{(i+1)\delta}, M_{(i+2)\delta}]$ crosses the line $(M_{i\delta}M_{(i+3)\delta})$. Then there exists j with $(i+1)\delta \leq j \leq (i+2)\delta$ such that the distance from M_j to $(M_{i\delta}M_{(i+3)\delta})$ is at most $1/2$. The area of the triangle $[M_{i\delta}, M_j, M_{(i+3)\delta}]$ is therefore at most $3\delta/4$. On the other side, by definition of δ_0, this area is at least $((i+3)\delta - j)(j - i\delta)/2\delta_0$, hence *a fortiori* $((i+3)\delta - j)(j - i\delta)/2\delta$. Since $(i+1)\delta \leq j \leq (i+2)\delta$, the latter quantity is at least δ, a contradiction. □

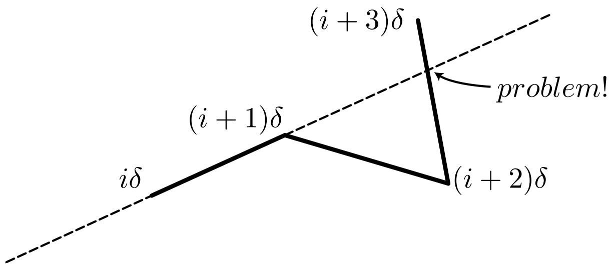

FIGURE 4. Four points not in convex position: a problem arises between $(i+2)\delta$ and $(i+3)\delta$

Proof of Proposition 1. Let $(M_0, \ldots, M_n)$ be a tame sequence in $\mathbb{R}^2$, and let δ be $\lceil \Delta_3(M_0, \ldots, M_n) \rceil$. If we have $\lfloor \frac{n}{\delta} \rfloor < 2$, then we have $\delta \geq n/2$, hence $\delta \in \Omega(n^{1/2})$ *a fortiori*. Assume now

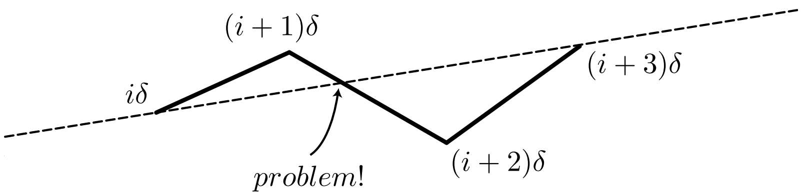

FIGURE 5. Four points not in ordered convex position: a problem arises between $(i+1)\delta$ and $(i+2)\delta$

$\lfloor\frac{n}{\delta}\rfloor \geq 2$. Then by Lemma 4, the sequence $(M_0, M_\delta, \ldots, M_{\lfloor\frac{n}{\delta}\rfloor\delta})$ is convex, and by Lemma 3 there is a triangle whose distortion is least $\lfloor\frac{n}{\delta}\rfloor/2\pi$. By definition, this quantity is at most δ, hence we have $\delta \in \Omega(\frac{n}{\delta})$. So in any case, $\delta_3(\Pi_n, \mathbb{R}^2)$ lies in $\Omega(n^{1/2})$. □

Remark. The proof of Lemma 4 gives many constraints for the sequence $(M_0, \ldots, M_n)$. Here we use these constraints to construct a convex subsequence of size $\sqrt{n}$, but it is likely that larger subsequences with properties slightly weaker than convexity could be constructed as well. So we think that the result of Proposition 1 is not optimal.

3. Construction of a d-dimensional embedding

Now we turn to dimension d and we wish to establish the lower bound result stated as Proposition 2. Our aim is to construct for each n a tame sequence of length n in $\mathbb{R}^d$ with a small 3-distortion, *i.e.*, such that all extracted triangles are not too much flattened.

A natural idea would be to construct the nth sequence $(M_{0,n}, \ldots, M_{n,n})$ by taking more and more points on a single curve Γ of length 1, and rescaling. But then a small 3-distortion would require a complicated curve Γ. Indeed, assume that Γ is an immersion of class C^2. As Γ is compact, the infimum r_Γ of the radii of the osculating circles of Γ is reached at some point, and therefore it is non-zero. For any n, there exists i such that the curvilinear distance between $M_{i,n}$ and $M_{i+2,n}$ is lower than $2/n$ before rescaling. Then the distances between $M_{i,n}$ and $M_{i+1,n}$, and between $M_{i+1,n}$ and $M_{i+2,n}$ are lower than $2/n$ too. Therefore the sine of the angle between the lines $(M_{i,n}M_{i+1,n})$ and $(M_{i+1,n}M_{i+2,n})$ is at most r_Γ/n, and the distortion of the triangle $M_{i,n}M_{i+1,n}M_{i+2,n}$ is at least n/r_Γ. This leads to a 3-distortion in $\Omega(n)$ for $(M_{0,n}, \ldots, M_{n,n})$. So, in order to construct sequences of points with small 3-distortion, we have either to use curves depending on n, or to use a non-C^2 curve (typically a fractal curve). In the following construction we choose the first option.

Proof of Proposition 2. For simplicity, we assume $n = m^{d-1}$ for some m. We recursively construct a family of curves $\Gamma_{m,d}$ in $\mathbb{R}^d$, and, on each of them, we mark $m^{d-1}+1$ points $P_{m,d,0}, \ldots, P_{m,d,m^{d-1}}$ in such a way that $\Delta_3(P_{m,d,0}, \ldots, P_{m,d,m^{d-1}})$ lies in $O(m)$ for each fixed d.

When $m+1$ points lie at mutual distance 1 on an arc of circle, the 3-distortion is in $\Theta(m)$. The idea of our construction is to use this fact and to recursively put circles one above the others.

Let Γ_0 be the sixth of a circle whose radius r will be chosen later. On Γ_0 we put points $P_0, \ldots, P_m$ with regular angular distance $\frac{\pi}{3m}$. Then we replace the arc between P_i and P_{i+1}

with a coplanar arc of radius $2r$ lying between the original arc and the chord connecting P_i to P_{i+1}. We rescale the figure so that the curvilinear coordinate of P_i becomes i for each i. We let $\Gamma_{m,2}$ be the resulting curve (oriented from P_0 to P_m) and $P_{m,2,0}, \dots, P_{m,2,m}$ be the marked points on $\Gamma_{m,2}$.

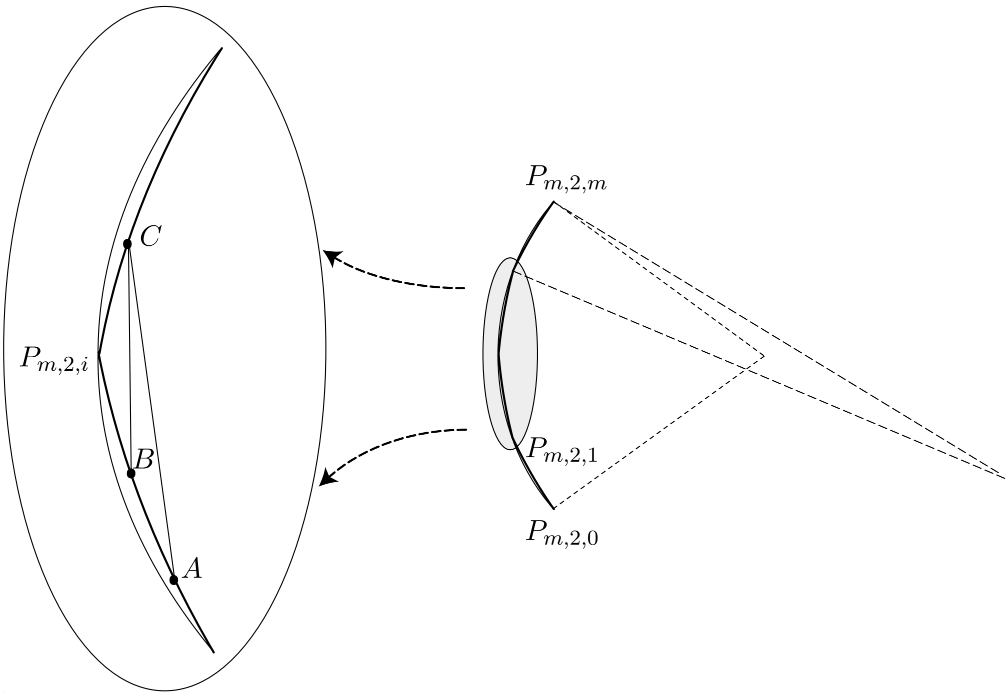

FIGURE 6. On the right: the curve $\Gamma_{m,2}$ and the points $P_{m,2,0}, \dots P_{m,2,m}$. On the left: three points A, B, C with at least one $P_{m,2,i}$ between them yield an angle $\angle ABC \leq \pi(1 - \frac{1}{6m})$

The main remark for the proof is that, for all triples A, B, C taken in increasing order on $\Gamma_{m,2}$ (not necessarily some $P_{m,2,i}$'s) and not all lying on some arc $(P_{m,2,i}P_{m,2,i+1})$, we have $\angle ABC \leq \pi(1 - \frac{1}{6m})$. By construction, the Euclidean distance between two points of $\Gamma_{m,2}$ is at least $3/\pi$ times their curvilinear distance, and therefore the 3-distortion of the triangle ABC is in $O(m)$.

The idea for the induction is to add a copy of $\Gamma_{m,2}$ between $P_{m,d-1,i}$ and $P_{m,d-1,i+1}$, orthogonally to the hyperplane in which $\Gamma_{m,d-1}$ lies. More precisely, we construct $\Gamma_{m,d}$ and $P_{m,d,0}, \dots, P_{m,d,m^{d-1}}$ from $\Gamma_{m,d-1}$ and $P_{m,d-1,0}, \dots, P_{m,d-1,m^{d-2}}$ so that the following induction hypothesis is preserved:

(i) $\Gamma_{m,d}$ is a curve of length m^{d-1} in $\mathbb{R}^d$ such that two points at curvilinear distance ℓ lie at euclidian distance at least $(2/\sqrt{3})^{-d+2}\pi/3 \times \ell$;

(ii) If A, B, C are three points that do not all lie on some arc $(P_{m,d,i}P_{m,d,i+1})$ for any i, then the 3-distortion of the triangle $[A, B, C]$ is at most $c_d m$, where $c_d = (2/\sqrt{3})^{-d+2} \times 6/\pi$.

The induction hypothesis holds for $d = 2$.

The construction of $\Gamma_{m,d}$ is as follows. We identify $\mathbb{R}^d$ with $\mathbb{R}^{d-1} \times \mathbb{R}$, where $\mathbb{R}^{d-1}$ is the space containing $\Gamma_{m,d-1}$. Next we work in the cylinder $Z_{m,d-1}$ defined by $\Gamma_{m,d-1} \times \mathbb{R}_+$ with the induced metric. Note that this cylinder $Z_{m,d-1}$ is orthogonal to the hyperplane containing $\Gamma_{m,d-1}$. For each i between 0 and $m^{d-2} - 1$, we insert in $Z_{m,d-1}$ a rescaled copy of $\Gamma_{m,2}$ from $P_{m,d-1,i}$ to $P_{m,d-1,i+1}$. In this way, we obtain a curve on which $m^{d-1} + 1$ points are marked: the $P_{m,d-1,i}$'s from $\Gamma_{m,d-1}$ plus $m^{d-2} \times (m-1)$ new points between $P_{m,d-1,i}$ and $P_{m,d-1,i+1}$ for $i = 0, \ldots, m^{d-2} - 1$. We denote them by $P_{m,d,0}, \ldots, P_{m,d,m^{d-1}}$ according to the linear ordering. We then rescale the figure so that the curvilinear distance between consecutive points $P_{m,d,i}$'s is 1. We call $\Gamma_{m,d}$ the resulting curve.

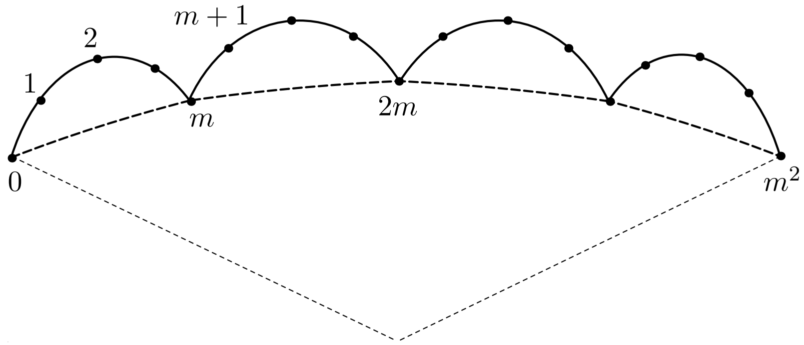

FIGURE 7. The curve $\Gamma_{m,3}$ in the space

It remains to show that the induction hypothesis is preserved.

For (i), we observe that the angle between any chord of $\Gamma_{m,d}$ and the hyperplane containing $\Gamma_{m,d-1}$ is lower than $\pi/6$. Therefore, when going from $\Gamma_{m,d-1}$ to $\Gamma_{m,d}$, no distance is decreased by more than a factor $2/\sqrt{3}$.

For (ii), let A, B, C be three points on $\Gamma_{m,d}$ and let i be such that A lies before $P_{m,d,i}$ and C lies after $P_{m,d,i}$ according to the fixed curvilinear ordering.

First case: There exists j such that A, B, C lie between $P_{m,d,jm}$ and $P_{m,d,(j+1)m}$. This means that A, B, C lie on some copy of $\Gamma_{m,2}$ in $Z_{m,d-1}$ inserted in the last step of the inductive construction. In the case of $\Gamma_{m,2}$, we know that the 3-distortion is at most $c_2 m$. Here there is an additional 3-distortion due to the fact that the copy was made on the cylinder $Z_{m,d-1}$. The projection of $\Gamma_{m,d}$ on $\mathbb{R}^{d-1}$ is $\Gamma_{m,d-1}$, and not a line as in the $d = 2$ case. By induction hypothesis, the distances on $\Gamma_{m,d-1}$ (compared with the Euclidean distances) are not contracted by more than $(2/\sqrt{3})^{-d+1}\pi/3$, hence the distortion of the triangle $[A, B, C]$ is bounded by $(2/\sqrt{3})^{-d+1}\pi/3 \times c_2 m \leq c_d m$.

Second case: There exists j such that A lies before $P_{m,d,jm}$ and C lies after $P_{m,d,jm}$. Then, when A, B, C are projected from $\Gamma_{m,d}$ on $\Gamma_{m,d-1}$ along $Z_{m,d-1}$, the area of the triangle $[A, B, C]$ decreases by a multiplicative factor at most $\sqrt{3}/2$. By the induction hypothesis the projection of the triangle has 3-distortion at most $c_{d-1}m$, therefore the original triangle $[A, B, C]$ has 3-distortion at most $c_d m$. □

Remarks. (i) The choice of the curve $\Gamma_{m,2}$ may look strange, in particular the choice of an arc of radius $2r$ between $P_{m,i}$ and $P_{m,i+1}$ rather than an arc of radius r or a chord. The reason is that, in both cases, the key property, namely that the triangle $[A, B, C]$ has 3-distortion $O(m)$ if A, B, C do not all lie on some arc $(P_{d,m,i}P_{d,m,i+1})$, fails. With arcs of radius r, if we take A, B, C close to some $P_{d,m,i}$, then the 3-distortion of $[A, B, C]$ can be arbitrary large. With chords, if we take A, B strictly between $P_{d,m,i}$ and $P_{d,m,i+1}$ and C just after $P_{d,m,i+1}$, then the 3-distortion is not bounded either.

(ii) Our construction uses $d-1$ pairwise orthogonal directions to draw the curves $\Gamma_{m,d}$ one above the other. We could use other fixed directions as well, the point being that the projections preserve the convexity of the specific patterns we consider. Alternatively we could replace cylinders by cones, as central projection also preserves the needed convexity. But it seems difficult to use more than one cylinder, and therefore more than one curve, for each new dimension, because no projection preserves the needed convexity for several sufficiently distinct directions simultaneously.

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