Bohlin-Arnold-Vassiliev's duality and conserved quantities

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Bohlin-Arnold-Vassiliev's duality transformation establishes a correspondence between motions in different central potentials. It offers a very direct way to construct the dynamical conserved quantities associated to the isotropic harmonic oscillator (Fradkin-Jauch-Hill tensor) and to the Kepler's problem (Laplace-Runge-Lenz vector).

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I. INTRODUCTION

The study of the relation between the classical motions of the harmonic oscillator and the Kepler problem has a long history initiated in Hooke and Newton's works[1]. In 1911, Bohlin showed that this relation can be formulated in terms of conformal mapping[2]. This conformal transform is at the core of the Levi-Civita's regularization scheme[3] of which tri-dimensional generalization has been achieved by Kustaanheimo and Stiefel[4]. In fact, as noted by Needham[5,6] two years before Bohlin's paper, Kasner[7] had established a more general duality law relating pairs of power law potentials (but the result has only been published in 1913). This relation has been rediscovered and generalized by Arnold and Vassiliev[8,9] in 1989 and quite simultaneously by Hojman and al.[10]. In fact, the generalization of Kasner's result had already been obtained by Collas[11] and implicitly enters into the frame of the coupling constant metamorphosis of Hietarinta and al.[12,13,14]. Even if we limit ourselves to the classical aspects (The study of this correspondence in the quantum mechanical frame has an interesting parallel history that we won't deal with here), numerous articles have been published on this subject during the last fifteen years[1,15,16,17,18,19,20]. In all what follows, we stay in a purely Newtonian frame. On the basis of simple but general arguments concerning the complex representation of 2D-motions, we show that the Bohlin-Arnold-Vassiliev's dual correspondence appears very naturally as the relevant transformation of motions among all combinations of analytic change of coordinates and Euler-Sundman reparameterization. We consider the effect of this duality on conserved quantities. In the specific cases of Hooke's and Kepler's problems, we recover in a very simple and direct way the correspondence established by Nersessian and al.[28,29,30] between the associated additional dynamical conserved quantities. This permits to point out the essentially obvious character of Fradkin-Jauch-Hill tensor's and Laplace-Runge-Lenz vector's conservation.

II. MOTIONS TRANSFORMATIONS

A. Complex formulation

Let's consider a planar motion $\overrightarrow{r}(t) = \binom{x(t)}{y(t)}_{(O,\overrightarrow{u_x},\overrightarrow{u_y})}$ for a particle of mass m submitted to a potential $U(\overrightarrow{r})$, eventually singular at the origin. We will adopt a complex formulation and represent the position by its corresponding affix $z(t) = x(t) + iy(t)$, the potential being viewed as a real valued function, $U(z, \overline{z})$. The gradient of such a real valued function of z and $\overline{z}$, is given by:

$$\overrightarrow{\nabla} U(\overrightarrow{r}) = 2\frac{\partial U(z,\overline{z})}{\partial \overline{z}} \tag{1}$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \overline{\frac{\partial}{\partial z}}$. In particular, for a central potential: $U(z,\overline{z}) = U(|z|) = U(r)$ and $\frac{\partial U(r)}{\partial z} = \frac{dU(r)}{dr}\frac{z}{r}$. The equation of motion takes the form:

$$\ddot{z} + \frac{2}{m}\frac{\partial U(z,\overline{z})}{\partial \overline{z}} = 0 \tag{2}$$

where the dot represents the time t derivative.

Multiplying both sides by $\dot{\overline{z}}$ and taking the real part of the resulting quantity, we obtain a total derivative:

$$\frac{\dot{\overline{z}}\ddot{z}+\dot{z}\ddot{\overline{z}}}{2}++\frac{1}{m}\left(\dot{\overline{z}}\frac{\partial U(z,\overline{z})}{\partial \overline{z}}+\dot{z}\frac{\partial U(z,\overline{z})}{\partial z}\right)=0 \tag{3}$$

Integrating this identity we recover the conservation of energy:

$$E=\frac{1}{2}m\left|\dot{z}\right|^2+U(z,\overline{z})=\frac{1}{2}m\left\|\dot{\overrightarrow{r}}(t)\right\|^2+U(\overrightarrow{r}) \tag{4}$$

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors in the $(O,\overrightarrow{u_x},\overrightarrow{u_y})$ plane, the complex affix of $\overrightarrow{A}\times\overrightarrow{B}$ is given by the real quantity $\mathrm{Im}\left(\overline{A}B\right)$, A and B being the complex affixes of $\overrightarrow{A}$ and $\overrightarrow{B}$ respectively (Note that this real quantity doesn't correspond to a vector in the $(O,\overrightarrow{u_x},\overrightarrow{u_y})$ plane but which is orthogonal to this last).

The angular momentum $\overrightarrow{L}(t)=m\overrightarrow{r}(t)\times\dot{\overrightarrow{r}}(t)=L(t)\overrightarrow{u_z}$ admits then as complex correspondent $L(t)=m\,\mathrm{Im}\left(\overline{z}(t)\dot{z}(t)\right)=\frac{m}{2i}\left(\overline{z}\dot{z}-\dot{\overline{z}}z\right)$.

Moreover:

$$\dot{L}(t)=\frac{m}{2i}\left(\overline{z}\ddot{z}-\ddot{\overline{z}}z\right)=i\left(\overline{z}\frac{\partial U(z,\overline{z})}{\partial \overline{z}}-\frac{\partial U(z,\overline{z})}{\partial z}z\right) \tag{5}$$

which implies that for a central potential $\dot{L}(t)=0$.

B. Conformal change of coordinates

Consider first an arbitrary conformal change of coordinates:

$$z=f(w),\ \overline{z}=\overline{f(w)}=\overline{f}(\overline{w}) \tag{6}$$

Then :

$$\frac{\partial}{\partial \overline{z}}=\left(\frac{\partial \overline{z}}{\partial \overline{w}}\right)^{-1}\frac{\partial}{\partial \overline{w}}=\frac{1}{\overline{f}^{(1)}(\overline{w})}\frac{\partial}{\partial \overline{w}}=\frac{1}{\overline{f^{(1)}(w)}}\frac{\partial}{\partial \overline{w}} \tag{7}$$

where $f^{(n)}(z)=\frac{\partial^n f(z)}{\partial z^n}$.

After substitution, Eq.(2) takes the following form:

$$\ddot{w}+(\dot{w})^2\log\left(f^{(1)}(w)\right)^{(1)}+\frac{2}{m}\frac{1}{\left|f^{(1)}(w)\right|^2}\frac{\partial\widetilde{U}(w,\overline{w})}{\partial \overline{w}}=0 \tag{8}$$

where $\widetilde{U}(w,\overline{w})=U\left(f(w),\overline{f}(\overline{w})\right)$.

C. Euler-Sundman's reparameterization

The previous equation (8) incorporates now a term of first order in time derivative, which has an artificial character. In order to suppress it, we can envisage to change the parameterization for the motion $w(t)$. A global change of parameterization will be inefficient. Rather, we choose a change of parameterization having a local structure, that is an Euler-Sundman's reparameterization of the type:

$$dt=\left|g(w)\right|^2 ds \tag{9}$$

where $g(w)$ is an, a priori, arbitrary analytic function. This form ensures in particular that the correspondence between the initial and the new (fictitious) time is one-to-one and increasing.

Starting from Eq.(8), it is straightforward to show that the equation for the motion $w(s)$ is now given by:

$$w'' + (w')^2 \left(\log \frac{f^{(1)}(w)}{g(w)}\right)^{(1)} - \overline{\left(\log g(w)\right)^{(1)}} |w'|^2 + \frac{2}{m} \frac{|g(w)|^4}{\left|f^{(1)}(w)\right|^2} \frac{\partial \widetilde{U}(w,\overline{w})}{\partial \overline{w}} = 0 \tag{10}$$

A priori, the structure of this last equation is still more complicated than the previous one, since it incorporates two kinds of terms depending on first derivative in time: the first one depending on $(w')^2$ as in Eq.(8)) and the second one, depending on $|w'|^2$. But the gain becomes readily clear. Indeed, the freedom in the choice of $g(w)$ allows to expect to suppress the first one. As for the problem induced by the second, as we just go to see, it is solved by the use of energy conservation identity.

D. Energy conservation

Expressed in terms of w and s, the energy conservation equation (4) gives:

$$|w'|^2 = \frac{2}{m} \left(E - \widetilde{U}(w,\overline{w})\right) \frac{|g(w)|^4}{\left|f^{(1)}(w)\right|^2} \tag{11}$$

We are then in position to replace in Eq.(10) the contribution containing $|w'|^2$ by another one, depending only of the position. We obtain:

$$w'' + (w')^2 \left(\log\left(\frac{f^{(1)}(w)}{g(w)}\right)\right)^{(1)} + \frac{2}{m} \frac{|g(w)|^4}{\left|f^{(1)}(w)\right|^2} \left(\frac{\partial \widetilde{U}(w,\overline{w})}{\partial \overline{w}} - \overline{\left(\log g(w)\right)^{(1)}} \left(E - \widetilde{U}(w,\overline{w})\right)\right) = 0 \tag{12}$$

E. Reparameterization's choice

In order to eliminate the contribution containing $(w')^2$, we have to fix $g(w)$ such that $g(w) = C f^{(1)}(w)\,,\ C \in \mathbb{C}$. The simplest choice, which we'll adopt in the sequel, is to take $C = 1$ and:

$$g(w) = f^{(1)}(w) \tag{13}$$

Then Eq.(**??**) becomes simply :

$$w'' + \frac{2}{m} f^{(1)}(w) \frac{\partial}{\partial \overline{w}} \left(\overline{f^{(1)}(w)} \left(\widetilde{U}(w,\overline{w}) - E\right)\right) = 0 \tag{14}$$

Defining :

$$V_f(w,\overline{w}) = \left|f^{(1)}(w)\right|^2 \left(\widetilde{U}(w,\overline{w}) - E\right) + V_0 \tag{15}$$

where V_0 is an arbitrary real constant, we finally obtain :

$$w'' + \frac{2}{m} \frac{\partial V_f(w,\overline{w})}{\partial \overline{w}} = 0 \tag{16}$$

In other words, $w(s)$ is the motion of a particle of mass m in the potential $V_f(w,\overline{w})$ (if we choose as supplementary additive term an arbitrary holomorphic function $V_0(w)$, we are led to the same equation but the corresponding potential $V_f(w,\overline{w})$ is no more real valued).

As for the energy of this dual system, it's given by :

$$\widetilde{E} = \frac{m}{2}\left|w'\right|^2 + V_f(w,\overline{w}) = V_0 \tag{17}$$

Relation (15) becomes finally:

$$\left|f^{(1)}(w)\right|^2 = \frac{\widetilde{E} - V_f(w,\overline{w})}{E - \widetilde{U}(w,\overline{w})} = \frac{\widetilde{E} - V_f(w,\overline{w})}{E - U(f(w), \overline{f}(\overline{w}))} \tag{18}$$

We then obtain a functional equation to which is submitted the transformation f linking the motions in the potentials U and V respectively.

In conclusion, among all the transformations of a 2D-motion in a potential U combining a conformal (i.e. analytical) change of coordinates $z = f(w)$ and an Euler-Sundman reparameterization $dt = |g(w)|^2 ds$, those for which the transformed motion is an autonomous 2D-motion submitted to a potential V, necessarily satisfy Eq.(13) and Eq.(18).

III. BOHLIN-ARNOLD-VASSILIEV'S DUALITY

A. Arnold-Vassiliev's potentials

We consider the case where the initial potential $U(\overrightarrow{r})$ takes the form $U(\overrightarrow{r}) = U(z,\overline{z}) = A\overline{u(z)}u(z) = A|u(z)|^2$, where $u(z)$ is an analytical function on $\mathbb{C}^*$ and $A \in \mathbb{R}$. We'll call such a potential, an Arnold-Vassiliev's potential[9]. It is easy to show that the only central Arnold-Vassiliev's potentials are the power law potentials, corresponding to $u(z) \sim z^{\frac{\nu}{2}}$, $\nu \in \mathbb{R}$, that is:

$$U(z,\overline{z}) = A|z|^{\nu}, \quad \nu, A \in \mathbb{R} \tag{19}$$

If $\widetilde{u}(w) = (u \circ f)(w)$, we have, by the transformation associated to f, an associated dual potential of the form (see Eq.(18)):

$$\widetilde{E} - V(w,\overline{w}) = \left|f^{(1)}(w)\right|^2 \left(E - A|\widetilde{u}(w)|^2\right) \tag{20}$$

Let's ask for which type of conformal transformation f, the corresponding image potential V is also an Arnold-Vassiliev's potential $V(w,\overline{w}) = B|v(w)|^2$.

Inserting in Eq.(20) and deriving successively with respect to w and $\overline{w}$, we obtain:

$$\frac{AE}{B\widetilde{E}}\frac{|v(w)|^2}{|\widetilde{u}(w)|^2} = \left(\frac{\widetilde{E} - B|v(w)|^2}{E - A|\widetilde{u}(w)|^2}\right)^2 = \left|f^{(1)}(w)\right|^4 \tag{21}$$

If we choose B and $\widetilde{E}$ such that $\frac{AE}{B\widetilde{E}} = 1$, this gives the following simple constraint :

$$\frac{|v(w)|}{|\widetilde{u}(w)|} = \left|f^{(1)}(w)\right|^2 \tag{22}$$

which is clearly satisfied when:

$$v(w) = f^{(1)}(w) = \frac{1}{u(f(w))} = \frac{1}{u(z)} \tag{23}$$

that is when $w = f^{-1}(z) = \int u(z)dz$.

Then Eq.(21) gives:

$$1 = \frac{\widetilde{E} - B\left|f^{(1)}(w)\right|^2}{E\left|f^{(1)}(w)\right|^2 - A} \tag{24}$$

which, jointly with $\frac{AE}{BE} = 1$, implies:

$$\widetilde{E} = -A, B = -E \tag{25}$$

The transformations satisfying Eq.(23) and Eq.(25) let stable the set of Arnold-Vassiliev's potentials. Since they are involutions, they can be seen as duality transformations:

$$\left(E, A\left|u(z)\right|^2\right) \overset{w=\int u(z)dz}{\longrightarrow} \left(-A, \frac{-E}{\left|u(z)\right|^2}\right) \overset{\zeta=\int v(w)dw=z}{\longrightarrow} \left(E, A\left|u(z)\right|^2\right) \tag{26}$$

The Arnold-Vassiliev's potentials appear then as the more natural class of potentials for which the above combination of an analytical change of coordinates and an Euler-Sundman reparameterization is a dual correspondence.

B. Power-law potentials

Let's consider the particular case of power-law potentials: $u(z) = z^{\frac{\nu}{2}}$, $U(z, \overline{z}) = k\left|z\right|^{\nu}$, at fixed energy E. Eq.(2) becomes:

$$\ddot{z} + \frac{\nu k}{m}\left|z\right|^{\nu-2} z = 0 \tag{27}$$

The preceding results (see Eq.(26)) give:

$$w = f^{-1}(z) = \frac{1}{1+\frac{\nu}{2}} z^{\frac{\nu}{2}+1}, \quad z = f(w) = \left(1+\frac{\nu}{2}\right)^{\frac{1}{1+\frac{\nu}{2}}} w^{\frac{2}{\nu+2}} \tag{28}$$

By the duality transformation considered above, the corresponding image potential (see Eq.(**??**), Eq.(25) and note that $A = k$) is then given by:

$$v(w) = \left(1+\frac{\nu}{2}\right)^{-\frac{\nu}{\nu+2}} w^{-\frac{\nu}{\nu+2}}, \quad V(w, \overline{w}) = B\left|v(w)\right|^2 = \widetilde{k}\left|w\right|^{\mu} \tag{29}$$

where:

$$\mu = -\frac{\nu}{1+\frac{\nu}{2}}, \ \widetilde{k} = -E\left(1+\frac{\mu}{2}\right)^{\mu} \tag{30}$$

The energy for the dual motion is (see Eq.(25)) $\widetilde{E} = -k$ and the equation of the dual image motion takes the form:

$$w'' + \frac{\mu\widetilde{k}}{m}\left|w\right|^{\mu-2} w = 0 \tag{31}$$

We recover for the dual image motion, a motion of energy $(-A)$, submitted to a power law potential of characteristic exponent μ, such that:

$$\left(1+\frac{\nu}{2}\right)\left(1+\frac{\mu}{2}\right) = 1 \tag{32}$$

and for which the coupling constant is proportional to the opposite of the initial energy E (and real if $\nu > -2$).

The relation between the two motions will be called "Bohlin-Arnold-Vassiliev's duality".

In the special case $\nu = 2$, we obtain $\mu = -1$. In other words, the dual motion of the plane harmonic oscillator (Hooke potential $\frac{1}{2}kr^2$, energy E) is nothing but the Kepler motion (Newton potential $-\frac{E}{2}\frac{1}{\rho}$, energy $-\frac{k}{2}$). We recover the usual Levi-Civita regularizing transformation with the associated change of variables:

$$z = \sqrt{2w}, \; w = \frac{1}{2}z^2 \tag{33}$$

and reparameterization:

$$ds = 2\rho dt = r^2 dt \tag{34}$$

where $r = |z|$ and $\rho = |w|$.

IV. DUALITY AND CONSERVED QUANTITIES

For a general 2D-central potential we have two real conserved quantities: the energy E, associated to translational invariance in time, and the angular momentum L, associated to rotational invariance. In the case of power-law potentials, the effect of the duality transformation on the energy is given by Eq.(30). Concerning the angular momentum, it is easy to show that (see Eq.(13)):

$$L = \frac{m}{2i}\left(\overline{z}\dot{z} - \dot{\overline{z}}z\right) = \frac{m}{2i}\left(\frac{w'}{\overline{\left(\log f\left(w\right)\right)^{(1)}}} - \frac{\overline{w}'}{\left(\log f\left(w\right)\right)^{(1)}}\right)$$

which gives for power law potentials (see Eq.(28)):

$$L = \left(1 + \frac{\nu}{2}\right)\widetilde{L}$$

where $\widetilde{L}$ is the angular momentum of the $w(s)$ motion.

Among all the central potentials, only the Kepler system and the isotropic harmonic oscillator possess true additional conserved quantities which are the Laplace-Runge-Lenz vector[21] and the Fradkin-Jauch-Hill tensor[22,23] respectively. With all the other central potentials are associated only piecewise conserved vectors[24,25].

This peculiarity of Kepler and harmonic potentials is at the origin of the existence of closed orbits for all the bound states[26,27].

In complex representation the existence of a specific conserved quantity for the harmonic oscillator becomes a matter of course. Indeed, if we consider the equation for a plane motion in a central potential $U(r)$:

$$\ddot{z} + \frac{1}{m}\frac{U^{(1)}(r)}{r}z = 0 \tag{35}$$

We see immediately that among all these potentials, the only one for which the corresponding equation is not $\overline{z}$ dependent is the linear one, that is the one associated to the Hooke potential (i.e. isotropic harmonic oscillator):

$$\ddot{z} + \frac{k}{m}z = 0 \tag{36}$$

This characteristic feature allows the existence of an immediate integrating factor $\dot{z}$, which after integration leads to the following complex conserved quantity:

$$\mathcal{T} = m\frac{\left(\dot{z}\right)^2}{2} + k\frac{z^2}{2} = cste \tag{37}$$

We then have:

$$\mathcal{T} = (T_{11} - T_{22}) + iT_{12} \tag{38}$$

where T_{ij} is the well known Fradkin-Jauch-Hill's tensor[22,23]:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix} = \begin{pmatrix} \frac{m}{2}\left(\dot{x}\right)^2 + \frac{k}{2}x^2 & m\frac{\dot{x}\dot{y}}{2} + k\frac{xy}{2} \\ m\frac{\dot{x}\dot{y}}{2} + k\frac{xy}{2} & \frac{m}{2}\left(\dot{y}\right)^2 + \frac{k}{2}y^2 \end{pmatrix} \tag{39}$$

As for the energy, it is nothing else but the trace of this tensor:

$$E = T_{11} + T_{22} = \frac{1}{2}m\left|\dot{z}\right|^2 + \frac{1}{2}k\left|z\right|^2 \tag{40}$$

In complex representation, the existence of Fradkin-Jauch-Hill's tensor appears therefore as a trivial consequence of the linearity of the equation of motion.

Let's interest ourselves to the interpretation of $\mathcal{T}$ for the associated dual Kepler motion. In terms of w variable and s parameter (see Eq.(33) and Eq.(34)), $\mathcal{T}$ takes the form :

$$\mathcal{T} = \frac{m}{4w}\left(\dot{w}\right)^2 + kw = m\overline{w}\left(w'\right)^2 + kw \tag{41}$$

Since the energy of the dual motion is given by $\widetilde{E} = -\frac{k}{2} = \frac{1}{2}m\left|w'\right|^2 - \frac{E}{2}\frac{1}{\rho}$, we can write:

$$\mathcal{T} = mw'\left(\overline{w}w' - w\overline{w}'\right) + E\frac{w}{\rho} = 2imw'\widetilde{L} - 2\widetilde{k}\frac{w}{\rho} \tag{42}$$

where $\widetilde{k} = -\frac{E}{2}$.

If we note $\overrightarrow{\rho} = \xi\overrightarrow{u}_x + \eta\overrightarrow{u}_y$ the vector of affix w, $i\widetilde{L}w'$ is the affix of $\overrightarrow{\widetilde{L}} \times \overrightarrow{\rho}'$. We then obtain:

$$\mathcal{T} = 2\widetilde{k}\mathcal{A} = -E\mathcal{A} \tag{43}$$

where:

$$\mathcal{A} = \frac{m}{\widetilde{k}}iw'\widetilde{L} - \frac{w}{\rho} \tag{44}$$

We immediately recognize the affix of the well-known Laplace-Runge-Lenz vector[21] for the Kepler motion w:

$$\overrightarrow{A} = \frac{m}{\widetilde{k}}\overrightarrow{\widetilde{L}} \times \overrightarrow{\rho}' - \frac{\overrightarrow{\rho}}{\rho} \tag{45}$$

By Eq.(43) we then have a very direct interpretation of the Laplace-Runge-Lenz vector as the ratio of two conserved quantities associated to the dual motion of isotropic harmonic oscillator:

$$\mathcal{A} = -\frac{\mathcal{T}}{E} \tag{46}$$

This ensures immediately the conservative character of $\mathcal{A}$.

The complex Newtonian frame adopted here permits to bring out the obvious character of the conservation Laplace-Runge-Lenz vector as a consequence, via the dual transform, of the triviality of the Fradkin-Jauch-Hill tensor's conservation. It is interesting to note that the elementary approach developed here can be extended to a much more general set of planar motions conserving only the angular momentum direction[31]. In this case, the duality connects different classes of motions. For those presenting closed orbits which satisfy generalized Gorringe-Leach equations, there exist additional conserved quantities which are linked by a relation similar to Eq.(43).

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