

SCT and the idempotence condition

Draft

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Abstract

These notes show how to get rid of the idempotence condition ($G = G; G$) in (δ) -Size Change Termination.

Both the “graph algorithm” of the original SCT paper [LJBA01] and the δ SCT criterion of Ben-Amram [BA06] use some kind of idempotence on the potential cycles. This idempotence condition can be replaced by a more semantical condition on the cycles. The new condition is probably not easier to compute. In the case of the original SCT, it is probably not even easier to express. In the case of the δ SCT, however, the new condition is much easier to express (and, hopefully, understand) while the idempotence condition was more contrived.

I mostly explain how things work on the δ SCT, and then briefly explain how to restrict this to basic SCT. This is also likely to work on many extensions of SCT. The main part of the result holds on a key Lemma (Lemma 8).

1 The key Lemma

Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$.

We consider here the $(\overline{\mathbb{Z}}, \min, +)$ semi-ring and denotes \min as \oplus and $+$ as \otimes . These operations are canonically extended to define multiplication of matrices¹ from $\mathcal{M}(\overline{\mathbb{Z}})$.

1.1 Matrices and graphs

Definition 1 (Sign matrix). Let M be a matrix of $\mathcal{M}(\overline{\mathbb{Z}})$. Its *sign matrix*, \overline{M} is the matrix such that $\overline{M}_{i,j}$ is $+\infty$ (resp. 0, -1 , $+1$) if $M_{i,j}$ is $+\infty$ (resp. 0, < 0 , > 0).

Taking the sign matrix corresponds to the abstraction operator α of δ SCT.

Definition 2 (Sign-idempotence). Let M be a square matrix of $\mathcal{M}(\overline{\mathbb{Z}})$. It is *sign-idempotent* if $\overline{M} = \overline{M \otimes M}$, that is M has the same sign as M^2 .

Matrix M is *strongly sign idempotent* (SSI) if for all $k > 0$, $\overline{M} = \overline{M^k}$, that is M has the same sign as all its powers.

Matrix M is *strongly diagonally sign idempotent* (SDSI) if for all $k > 0$, $\overline{M}_{i,i} = \overline{M^k}_{i,i}$, that is the diagonal of M has the same sign as the diagonal of all the powers of M .

Remark 3. Sign idempotence and strong sign idempotence are not equivalent as shown by the following matrix (remember that we are working in the $(\overline{\mathbb{Z}}, \min, +)$ semi-ring and not in the usual $(\mathbb{Z}, +, \times)$ ring):

$$M = \begin{bmatrix} 1 & -5 & -1 \\ 6 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \quad M^2 = \begin{bmatrix} 1 & -4 & -3 \\ 4 & 1 & 3 \\ 3 & -3 & 1 \end{bmatrix} \quad M^3 = \begin{bmatrix} -1 & -4 & -2 \\ 5 & -1 & 3 \\ 3 & -2 & -1 \end{bmatrix}$$

Definition 4 (Constraint graph). Let M be a square matrix of dimension n . Its *constraint graph* is a weighted directed graph G such that:

¹That is, given two matrices A and B , $(A \oplus B)_{i,j} = A_{i,j} \oplus B_{i,j} = \min(A_{i,j}, B_{i,j})$ and $(A \otimes B)_{i,j} = \bigoplus_k A_{i,k} \otimes B_{k,j} = \min_k(A_{i,k} + B_{k,j})$. Similarly, if X is a vector and M a matrix, then $(X \otimes M)_j = \bigoplus_k X_k \otimes M_{k,j} = \min_k(X_k + M_{k,j})$.

- There are n vertices $X_i, 1 \leq i \leq n$ plus an extra vertex Y .
- If $M_{i,j} \neq +\infty$, there is an edge of weight $M_{i,j}$ between X_i and X_j .
- There is an edge of weight 0 between Y and X_i , for all i .

Definition 5 (*l-weight*). Let G be a directed weighted graph. The *l-weight between a and b* is the minimum weight of all paths of length l between a and b and $+\infty$ if there is no such path.

Let $\omega_{a,b}(l)$ be the l -weight between a and b and $\overline{\omega_{a,b}(l)}$ be its sign (in $\{-1, 0, +1, +\infty\}$ as earlier).

The coefficient $M_{i,j}^k$ is the k -weight between X_i and X_j in the constraint graph of M .

1.2 The key Lemma

Lemma 6. *Let M be a square matrix. There exists $k > 0$ such that M^k is strongly diagonally sign idempotent.*

Proof.

Permuting the quantifiers. Firstly, if for all i , there exists k_i such that for all $j > 0$, $\overline{M^{k_i}_{i,i}} = \overline{M^{j k_i}_{i,i}}$, then choosing $k = \prod k_i$ ensure that M^k is strongly diagonally sign idempotent.

Cycles in the graph. Then, we find the k_i by reasoning on the constraint graph of M . If there is no cycle from X_i to itself, then $\overline{M^j_{i,i}} = +\infty$ for all j and we can choose $k_i = 1$.

If there are cycles from X_i to itself, then we look at the sign of their weight. If there exist a cycle of negative weight and length l_- , then we choose $k_i = l_-$ because all cycles of length multiple of l_- can follow this cycle several time and will also have negative weight. Otherwise, if there is a cycle of weight 0 and length l_0 , we choose $k_i = l_0$ and use the same reasoning. Lastly, if all cycles from X_i to itself have strictly positive weight, let k_i be the length of any of those cycles. \square

Remember that deciding the existence of cycles of negative weight, as well as finding the minimum weight of a cycle, can be done in polynomial time via Bellman-Ford's algorithm.

Lemma 7. *Let M be a strongly diagonally sign idempotent square matrix. There exists $k > 0$ such that M^k is strongly sign idempotent.*

Again, the proof will be done on the constraint graph of M by looking on the l -weights.

Proof. Again, we can permute the quantifiers, that is if for all i, j , there exists a $k_{i,j}$ such that $\overline{M^{k_{i,j}}_{i,j}} = \overline{M^{n k_{i,j}}_{i,j}}$ for all $n > 0$ then we can choose $k = \prod k_{i,j}$.

For all pairs of vertices X_i, X_j , we consider all the triples (v, C, p) such that p is a path from X_i to X_j going through v and C is a cycle from v to itself. Since we're now only considering the SDSI case, and given the order in which we consider the different cases, we can without loss of generality consider that C has length 1.

Now, depending on the signs of the weights of both C and p , there can be several cases:

1. There exists (v, C, p) with $\omega(C) < 0$.

In this case, if $k \geq k_0$ is large enough, there exist a path of length k and negative weight that loops sufficiently many times through C to "cancel" the weight of p (because the order on \mathbb{Z} is archimedean). We can choose $k_{i,j} = k_0$.

2. The above does not hold, but there exists (v, C, p) with $\omega(C) = 0$ and $\omega(p) < 0$.

If $k \geq k_1$ the length of p , there exist a path of negative weight and length k . Again, $k_{i,j} = k_1$.

3. None of the above hold, but there exists (v, C, p) with $\omega(C) = \omega(p) = 0$.

The same reasoning yields paths of weight 0.

4. None of the above hold, but there exists (v, C, p) anyway.

In that case, either $\omega(C) = 0$ and $\omega(p) > 0$, in which case all paths will have strictly positive weight, or $\omega(C) > 0$ in which case all sufficiently long paths will have strictly positive weight (because they'll have to go through C sufficiently many time to "cancel" the weight of p). This yields to paths of strictly positive weight.

5. None of the above holds, that is there is no (v, C, p) .

In this case, either there is no path from X_i to X_j , and the corresponding sign is $+\infty$ for all powers, or there are paths but none of them is adjacent to a cycle, and there is a maximum length of a path from X_i to X_j and all further power will have sign $+\infty$.

□

By combining the two Lemmas, we immediately get the key Lemma:

Lemma 8 (Key Lemma). *Let M be a square matrix. There exists k such that M^k is strongly sign idempotent.*

This Lemma allows to get rid of the idempotence condition by, basically, stating than idempotence will eventually be reached simply by repeating the same thing over and over again.

Proof. By Lemma 6, there exists i such that $M^i = M^i$ is strongly diagonally sign idempotent. Then, by Lemma 7, there exists j such that M^{ij} is strongly sign idempotent. Hence, by choosing $k = i \times j$, $M^k = M^{ij} = M^{ij}$ is strongly sign idempotent. □

Remark 9. This Lemma was originally present in the early versions of my RCG paper. The current proof (splitting SDSI and SSI) is an improvement of my proof done by A. Ben Amram.

The Lemma can also be directly formulated as:

$$\forall M, \exists k > 0 / \forall i, j, \forall n, m > 0, \overline{M^{kn}}_{i,j} = \overline{M^{km}}_{i,j}$$

1.3 Solving systems

Lemma 10. *The system $X \leq X \otimes M$ has a solution if and only if for all $k \geq 0$, there is no strictly negative coefficient in the diagonal of M^k . In that case, it admits a strictly positive solution.*

It is possible to decide in polynomial time whether such a system admits a solution.

Proof. The matrix inequality corresponds to the set of inequalities $\{X_j \leq \min_i (X_i + M_{i,j})\}$ which can, without modifying the set of solutions, be expressed as $\{X_j \leq X_i + M_{i,j}\}$.

If there is no strictly negative coefficient in the diagonal of M^k , that means that the constraints graph G has no cycle of strictly negative weight. In this case, we can choose for X_i the value of the shortest (that is, smallest weight) path to reach it from Y . This is well defined because there is no cycle of strictly negative weight and provides a solution for the system because $X_j \leq X_i + M_{i,j}$ holds by definition of shortest paths.

Conversely, if there is a path of strictly negative weight, then it is easy to see that by adding the inequations corresponding to the edges in this path one will eventually reach an inequation $X_i < X_i$ and the system has no solution.

If there is a solution, then $X + (1, \dots, 1)$ is also a solution. Hence, there exists a solution where all values are positive.

The system admits a solution if and only if the constraint graph has no cycle of strictly negative weight. This can be decided in polynomial time by Bellman-Ford's algorithm. □

2 SCT

I reuse here all the definitions from the δ SCT paper: graphs, multigraphs, threads, Annotated Control Graphs (ACG). To each δ SCT graph can be associated a matrix over \mathbb{Z} (the coefficient $M_{i,j}$ being the weight of the arc from x_i to y_j , or $+\infty$ if no such arc exists).

2.1 Size-Change Graphs and matrices

Definition 11 (Size Change graphs). Same as in the usual SCT/ δ SCT papers. A Size Change Graph \mathcal{G} is associated with each call (in a functional program) or each instruction (in an imperative program). Edges of Size Change Graphs are labelled by integers (*à la* δ SCT).

To each SCT graph is associated a corresponding SCT matrix. If the graph has A input nodes and B output nodes, then the matrix will have A rows and B columns. Coefficient i, j is equal to the label on the edge between input node i and output node j (or $+\infty$ if there is no such edge).

Definition 12 (Annotated Control Graphs). Let p be a program. Its *Annotated Control Graph* (ACG) is the control flow graph G where each edge has been labelled with the corresponding SCT matrix (or graph).

SCT graph composition becomes SCT matrix multiplication. If \mathcal{G} and \mathcal{G}' are two SCT graphs with corresponding SCT matrices M and M' , then the matrix corresponding to $\mathcal{G}; \mathcal{G}'$ is $M \otimes M'$.

Multipaths can thus be condensed as a single matrix.

Definition 13 (SCT criterion (Ben-Amram)). An ACG satisfies the δ SCT condition if every infinite multipath contains at least one thread of infinite descent.

Definition 14. Let \mathcal{G} be a Size Change Graph. Its skeleton is the bipartite graph obtained by removing all weights.

A Size Change Graph \mathcal{G} is cyclic if and only if the skeletons of \mathcal{G} and $\mathcal{G}; \mathcal{G}$ are the same.

To each matrix $M \in \mathcal{M}_{m,n}(\overline{\mathbb{Z}})$, we associate a boolean matrix $\underline{M} \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $\underline{M}_{i,j} = 0$ if $M_{i,j} = +\infty$ and $\underline{M}_{i,j} = 1$ otherwise. Notice that \bullet is a morphism, that is $\underline{M \otimes N} = \underline{M} \times \underline{N}$ (where \times is here the canonical extension of the product of the boolean algebra to boolean matrices).

Fact 15. If M is the matrix corresponding to a Size Change Graph \mathcal{G} , then \underline{M} corresponds to its skeleton. \mathcal{G} is cyclic if and only if the corresponding boolean matrix is idempotent: $\underline{M} \times \underline{M} = \underline{M}$.

Fact 16. If M is Strongly Sign Idempotent, then \underline{M} is idempotent.

Theorem 17 (Ben-Amram). An ACG with fan-in free Size Change Graphs satisfies the δ SCT condition if and only if for every cyclic multipath \mathcal{M} , $\overline{\mathcal{M}}$ has an in-situ arc with a negative label.

Where $\overline{\mathcal{M}}$ denotes the ‘‘condensed’’ multipath (in a single Size Change graph).

This Theorem can be rephrased with matrices as follows:

Theorem 18 (δ SCT). An ACG does not satisfy the δ SCT condition if there exists a cycle in it whose SCT matrix M is such that:

1. \underline{M} is idempotent: $\underline{M} \times \underline{M} = \underline{M}$;
2. and M is not decreasing: there is no strictly negative number on the diagonal of M .

The first condition is about cyclicity of the multipath. The second about in-situ decreasing arcs.

This Theorem has a idempotence condition (cyclicity) in it. It is similar to the ‘‘graph algorithm’’ of the original SCT paper, but with extra stuff to deal with weights on edges.

2.2 SCT without idempotence

If M is a matrix, by $\lfloor M \rfloor$ we denote its in-situ part, that is the matrix where all non diagonal coefficients have been replaced by $+\infty$.

Lemma 19 (Ben-Amram). If \underline{M} is idempotent, then $M \otimes M = \lfloor M \rfloor \otimes M$.

This is the adaptation to matrices of Lemma 4.2 of δ SCT.

We can now replace the idempotence condition by a more semantical one.

Theorem 20. An ACG with fan-in free SCT matrices does not satisfy the δ SCT condition if and only if there exists a cycle in it whose SCT matrix M is such that the system $X \leq X \otimes M$ has a solution.

Proof. If such a cycle exists, by Lemma 8, there exists k such that M^k is Strongly Sign Idempotent. By Lemma 10, $M_{i,i}^n \geq 0$ for all n, i . Hence M^k is such that:

1. \underline{M}^k is idempotent (because M^k is SSI, Fact 16).
2. There is no strictly negative number on the diagonal of M^k .

Hence the ACG does not satisfies the δ SCT condition.

Conversely, if the ACG does not satisfy the δ SCT condition, then there exists a cycle with SCT matrix M such that \underline{M} is idempotent and M is not decreasing. Then there exists k such that M^k is SSI.

However, since M is fan-in free, $M^k = \lfloor M \rfloor^{k-1} \otimes M$, hence the diagonal of M^k is $\lfloor M \rfloor^k$. Since M is not decreasing, M^k has no strictly negative coefficient on the diagonal.

Then if one takes the cycle corresponding to M k times, the corresponding matrix is M^k and the system $X \leq X \otimes M^k$ admits a solution because M^k has no strictly negative coefficient on the diagonal of its powers. \square

Actually, for the “if” part of the Theorem, fan-in freedom is not necessarily:

Proposition 21. *An ACG does not satisfies the δ SCT condition if there exists a cycle in it whose SCT matrix M is such that the system $X \leq X \otimes M$ has a solution.*

Sketch of proof. The idea is to build a multipath by repeating the cycle infinitely many times. Any infinite thread in it must go infinitely many time through the same variable. Since the diagonal of M has no strictly negative coefficient, the thread cannot be of infinite descent. \square

2.3 Regular SCT

The regular SCT is obtained by considering the 3-valued semi-ring $(\{-1, 0, +\infty\}, \min, +)$ instead of $\bar{\mathbb{Z}}$.

All the rest works the same way with the natural restrictions to the operations.

3 Conclusion

I don't think that the condition $X \leq X \otimes M$ is really interesting for usual SCT where the graph condition $\mathcal{G} = \mathcal{G}$; \mathcal{G} is perfectly OK and easy to express. However, I think that it might be an improvement for the δ SCT since it completely remove the need of cyclicity which is not as straightforward as the SCT graph idempotence. I also think a direct proof can be obtained rather than referring to Ben-Amram's proof. I think such criterion can be adapted for variants of the SCT.

References

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