A SYMMETRY PROPERTY OF SOME HARMONIC ALGEBRAIC CURVES

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Abstract. The aim of this note is to give a surprising symmetry property of some harmonic algebraic curves: when all the roots z_i of a complex polynomial P lie on the unit circle $\mathcal{U}$, the points of $\mathcal{U}$ different from the z_i, and such that $\text{Arg}(P(z)) = \theta$, form a regular n-gon, where n is the degree of P.

Let $\mathbf{z} = \{z_1, \ldots, z_n\}$ be a multiset of n points in the complex plane $\mathbb{C}$ and P the monic polynomial with root set $\mathbf{z}$:

$$P(z) = \prod_{i=1}^{n}(z - z_i).$$

For θ a fixed real number of your choice, consider

$$C_\theta(P) = \{z \in \mathbb{C} \ : \ \text{Im}(e^{-i\theta}P(z)) = 0\}.$$

The set $C_\theta(P)$ coincides up to $\mathbf{z}$, to the set $\{z \in \mathbb{C} : \ \text{Arg}(P(z)) = \theta[\pi]\}$. These curves arise in the Gauss approach to the Fundamental Theorem of Algebra (see e.g. Stillwell [3], and Martin & al. [1]). In their paper Martin & al. [1] and then Savitt [2] initiated the study of the combinatorial topology of the families $C_\theta(P)$. The idea are the following ones: the curves $C_\theta(P)$ have $2n$ asymptotes at angles $(\pi k + \theta)/n$, for $k \in \{0, \ldots, 2n-1\}$, and form in the generic case n non intersecting curves. This induces a matching: k and k' are matched if and only if the asymptotes $(\pi k + \theta)/n$ and $(\pi k' + \theta)/n$ lie on the same connected component in $C_\theta(P)$. The papers [1] and [2] aim at studying these matchings, and also the properties of the so-called necklaces, formed by the families of matchings obtained when θ traverses the set $[0, \pi]$.

Let us now state and prove our result. The set $\mathbf{z}$ is clearly included in $C_\theta(P)$. It turns out that when $\mathbf{z}$ is included in the unit circle $\mathcal{U} = \{z \ : \ |z| = 1\}$, the set $C_\theta(P) \cap \mathcal{U}$ presents a quite surprising symmetry – illustrated at Figure 1 – that can be stated as follows.

Proposition 1. *If $\mathbf{z}$ is a subset of $\mathcal{U}$, then*

$$C_\theta(P) \cap \mathcal{U} = \mathbf{z} \cup G(\mathbf{z})$$

where $G(\mathbf{z})$ is the regular n-gon on $\mathcal{U}$, with set of vertices $\left\{e^{i(\Omega + 2k\pi/n)}, k = 1, \ldots, n\right\}$, for

$$\Omega := \frac{2\theta - \sum_{j=1}^{n} \text{Arg}(z_j)}{n} - \pi.$$

There exists a purely geometric proof of this Proposition using that the measure of a central angle is twice that of the inscribed angle intercepting the same arc; we provide below a more compact analytic proof.

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Proof. We will only consider $z \notin \mathbf{z}$. We have the equivalence:

$$z \in C_\theta(P) \setminus \mathbf{z} \iff z \notin \mathbf{z},\ \sum_{i=1}^{n} \operatorname{Arg}(z - z_i) = \theta \quad [\pi],$$

where $\operatorname{Arg}(z) \in \mathbb{R}/2\pi\mathbb{Z}$ stands for (any chosen determination of) the argument of $z \neq 0$. Now for any ν and ψ real numbers,

$$e^{i\nu} - e^{i\psi} = e^{i\frac{\nu+\psi}{2}}\left(e^{i\frac{\nu-\psi}{2}} - e^{i\frac{-\nu+\psi}{2}}\right) = 2i\sin((\nu-\psi)/2)e^{i\frac{\nu+\psi}{2}}.$$

Thus

$$\operatorname{Arg}(e^{i\nu} - e^{i\psi}) = \frac{\nu+\psi}{2} + \frac{\pi}{2} + \pi \times \operatorname{sgn}\big(\sin((\nu-\psi)/2)\big) \quad [2\pi]$$

Hence, $z \in C_\theta(P) \setminus \mathbf{z}$ is equivalent to:

$$z \notin \mathbf{z},\ \sum_{j=1}^{n} \left(\frac{\operatorname{Arg}(z) + \operatorname{Arg}(z_j)}{2} + \frac{\pi}{2}\right) = \theta \quad [\pi],$$

which leads to the conclusion at once. □

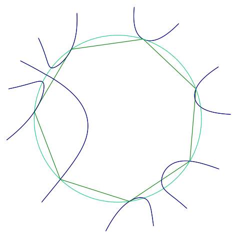

FIGURE 1. An example where $n = 7$, $\theta = 0$ and the roots z_i randomly chosen.

Note. If z_i is a root of multiplicity k of P, and if z_i belongs to $G(\mathbf{z})$, then in the neighborhood of z_i, $C_\theta(P)$ has k tangents, one of them coinciding with the tangent of the circle at z_i. Moreover, it is simple to check that if z_i is not on $G(\mathbf{z})$, then the tangents of $C_\theta(P)$ at z_i are not tangent to $\mathcal{U}$.

References

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[3] J. Stillwell, *Mathematics and its history. Undergraduate Texts in Mathematics.* Springer-Verlag, New York, 1989.

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