

On the martingale problem associated to the $2D$ and $3D$ Stochastic Navier-Stokes equations

Giuseppe Da Prato
Scuola Normale Superiore di Pisa,
Piazza dei Cavalieri 7, 56126 Pisa, Italy

Arnaud Debussche
Ecole Normale Supérieure de Cachan,
antenne de Bretagne,
Campus de Ker Lann, 35170 Bruz, France

Abstract

We consider the martingale problem associated to the Navier-Stokes in dimension 2 or 3. Existence is well known and it has been recently shown that markovian transition semi group associated to these equations can be constructed. We study the Kolmogorov operator associated to these equations. It can be defined formally as a differential operator on an infinite dimensional Hilbert space. It can be also defined in an abstract way as the infinitesimal generator of the transition semi group. We explicit cores for these abstract operators and identify them with the concrete differential operators on these cores. In dimension 2, the core is explicit and we can use a classical argument to prove uniqueness for the martingale problem. In dimension 3, we are only able to exhibit a core which is defined abstractly and does not allow to prove uniqueness for the martingale problem. Instead, we exhibit a core for a modified Kolmogorov operator which enables us to prove uniqueness for the martingale problem up to the time the solutions are regular.

2000 Mathematics Subject Classification AMS:

Key words: Stochastic Navier–Stokes, Kolmogorov equations, martingale problems, weak uniqueness.

1 Introduction

We consider the stochastic Navier–Stokes on a bounded domain \mathcal{O} of \mathbb{R}^d , $d = 2$ or 3 , with Dirichlet boundary conditions: the unknowns are the velocity $X(t, \xi)$ and the pressure $p(t, \xi)$ defined for $t > 0$ and $\xi \in \overline{\mathcal{O}}$:

$$\begin{cases} dX(t, \xi) = [\Delta X(t, \xi) - (X(t, \xi) \cdot \nabla)X(t, \xi)]dt - \nabla p(t, \xi)dt + f(\xi)dt + \sqrt{Q} dW, \\ \operatorname{div} X(t, \xi) = 0, \end{cases} \quad (1.1)$$

with Dirichlet boundary conditions

$$X(t, \xi) = 0, \quad t > 0, \quad \xi \in \partial\mathcal{O},$$

and supplemented with the initial condition

$$X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}.$$

We have taken the viscosity equal to 1 since it plays no particular role in this work. The understanding of the stochastic Navier-Stokes equations have progressed considerably recently. In dimension two, impressive progresses have been obtained and difficult ergodic properties have been proved (see [1], [8], [10], [13], [14], [15], [16], [17], [18], [19]). In dimension three, the theory is not so advanced. Uniqueness is still an open problem. However, Markov solutions have been constructed and ergodic properties have been proved recently (see [2], [3], [4], [7], [9], [11], [23], [22], [24]).

In this article, our aim is to try to improve the understanding of the martingale problems associated to these equations. Let us first set some notations. Let

$$H = \{x \in (L^2(\mathcal{O}))^d : \operatorname{div} x = 0 \text{ in } \mathcal{O}, x \cdot n = 0 \text{ on } \partial\mathcal{O}\},$$

where n is the outward normal to $\partial\mathcal{O}$, and $V = (H_0^1(\mathcal{O}))^d \cap H$. The norm and inner product in H will be denoted by $|\cdot|$ and (\cdot, \cdot) respectively. Moreover W is a cylindrical Wiener process on H and the covariance of the noise Q is trace class and non degenerate (see (1.3) and (1.4) below for more precise assumptions).

We also denote by A the Stokes operator in H :

$$A = P\Delta, \quad D(A) = (H^2(\mathcal{O}))^d \cap V,$$

where P is the orthogonal projection of $(L^2(\mathcal{O}))^3$ onto H and by b the operator

$$b(x, y) = -P((x \cdot \nabla)y), \quad b(x) = b(x, x), \quad x, y \in V.$$

With these notations we rewrite the equations as

$$\begin{cases} dX = (AX + b(X))dt + \sqrt{Q} dW, \\ X(0) = x. \end{cases} \quad (1.2)$$

We assume that

$$\text{Tr} (-A)^{1+g}Q < \infty, \quad \text{for some } g > 0 \quad (1.3)$$

and

$$|Q^{-1/2}x| \leq c|(-A)^r x|, \quad \text{for some } r \in (1, 3/2). \quad (1.4)$$

In dimension $d = 3$, it is well known that there exists a solution to the martingale problem but weak or strong uniqueness is an open problem (see [9] for a survey). However, it has been proved in [4], [7] (see also [11]) that the above assumptions allow to construct a transition semigroup $(P_t)_{t \geq 0}$ associated to a Markov family of solutions

$$((X(t, x)_{t \geq 0}, \Omega_x, \mathcal{F}_x, \mathbb{P}_x)$$

for $x \in D(A)$. Moreover for sufficiently regular φ defined on $D(A)$, $P_t\varphi$ is a solution of the Kolmogorov equation associated to (1.2)

$$\begin{cases} \frac{du}{dt} = Lu, \quad t > 0, \quad x \in D(A), \\ u(0, x) = \varphi(x), \quad x \in D(A), \end{cases} \quad (1.5)$$

where the Kolmogorov operator L is defined by

$$L\varphi(x) = \frac{1}{2}\text{Tr} \{QD^2\varphi(x)\} + (Ax + b(x), D\varphi(x))$$

for sufficiently smooth functions φ on $D(A)$.

In all the article, we choose one Markov family $((X(t, x)_{t \geq 0}, \Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ as the one constructed in [7].

The fundamental idea in [4] is to introduce a modified semigroup $(S_t)_{t \geq 0}$ defined by

$$S_t\varphi(x) = \mathbb{E}(e^{-K \int_0^t |AX(s, x)|^2 ds} \varphi(X(t, x))). \quad (1.6)$$

It can be seen that for K large enough, this semigroup has very nice smoothing properties and various estimates can be proved. Note that, thanks to

Feynman-Kac formula, this semigroup is formally associated to the following equation

$$\begin{cases} \frac{dv}{dt} = Nv, & t > 0, x \in D(A), \\ v(0, x) = \varphi(x), & x \in D(A), \end{cases} \quad (1.7)$$

where N is defined

$$N\varphi(x) = \frac{1}{2}\text{Tr} \{QD^2\varphi(x)\} + (Ax + b(x), D\varphi(x)) - K|Ax|^2\varphi(x),$$

for sufficiently smooth functions φ on $D(A)$.

In [4], [7], this semigroup is defined only on the Galerkin approximations of (1.2). Let P_m denote the projector associated to the first m eigenvalues of A . We consider the following equation in $P_m H$

$$\begin{cases} dX_m = (AX_m + b_m(X_m))dt + \sqrt{Q_m} dW \\ X_m(0) = P_m x, \end{cases} \quad (1.8)$$

where $b_m(x) = P_m b(P_m x)$, $Q_m = P_m Q P_m$. This defines, with obvious notations, $(P_t^m)_{t \geq 0}$ and $(S_t^m)_{t \geq 0}$. The following formula holds by a standard argument:

$$P_t^m \varphi = S_t^m \varphi + K \int_0^t S_{t-s}^m (|A \cdot|^2 P_s^m \varphi ds), \quad \varphi \in C_b(P_m H).$$

Various estimates are proved on $(S_t^m)_{t \geq 0}$ and transferred to $(P_t^m)_{t \geq 0}$ thanks to this identity. A compactness argument allows to construct $(P_t)_{t \geq 0}$. Moreover, a subsequence m_k can be constructed such that for any $x \in D(A)$, $(X_{m_k}(\cdot, x))_{t \geq 0}$ converges in law to $(X(\cdot, x))_{t \geq 0}$.

Note also that similar arguments as in [4] may be used to prove that for smooth φ , $(S_t \varphi)_{t \geq 0}$ is a strict solution to (1.7).

In dimension 2 this result also holds with exactly the same proofs since all arguments for $d = 3$ are still valid. Note that it is well known that for $d = 2$ conditions (1.3)-(1.4) imply that, for $x \in H$, there exists a unique strong solution to (1.2) and the proof of the above facts can be simplified.

In the following, we give some properties of the generator of $(P_t)_{t \geq 0}$ and $(S_t)_{t \geq 0}$. For $d = 2$, we explicit a core, identify the abstract generator with the differential operator L on this core and prove existence and uniqueness for the corresponding martingale problem. (See [21] for a similar result). Again, this follows from strong uniqueness but we think that it is interesting to have a direct proof of this fact. Moreover, it can be very useful to have a

better knowledge of the Kolmogorov generator and we think that this work is a contribution in this direction. In dimension 3, we are not able to prove this. We explain the difficulties encountered. We hope that this article will help the reader to get a better insight into the problem of weak uniqueness for the three dimensional Navier-Stokes equations. Nonetheless, we explicit a core for the generator of the transformed semigroup $(S_t)_{t \geq 0}$, identify it with the differential operator N on this core and prove uniqueness for the stopped martingale problem. In other words, we prove weak uniqueness up to the time solutions are smooth. Again, this could be proved directly thanks to local strong uniqueness.

2 The generators

The space of continuous functions on $D(A)$ is denoted by $C_b(D(A))$. Its norm is denoted by $\|\cdot\|_0$. For $k \in \mathbb{N}$, $C^k(D(A))$ is the space of C^k functions on $D(A)$. We need several other function spaces on $D(A)$.

Let us introduce the set $\mathcal{E}_1 \subset C_b(D(A))$ of C^3 functions on $D(A)$ such that there exists a constant c satisfying

- $|(-A)^{-1}Df(x)|_H \leq c(|Ax|^2 + 1)$
- $|(-A)^{-1}D^2f(x)(-A)^{-1}|_{\mathcal{L}(H)} \leq c(|Ax|^4 + 1)$
- $|(-A)^{-1/2}D^2f(x)(-A)^{-1/2}|_{\mathcal{L}(H)} \leq c(|Ax|^6 + 1)$
- $\|D^3f(x)((-A)^{-1}, (-A)^{-1}, (-A)^{-1})\| \leq c(|Ax|^6 + 1)$
- $\|D^3f(x)((-A)^{-\gamma}, (-A)^{-\gamma}, (-A)^{-\gamma})\| \leq c(|Ax|^8 + 1)$
- $|Df(x)|_H \leq c(|Ax|^4 + 1)$

where $\gamma \in (1/2, 1]$ and

$$\mathcal{E}_2 = \left\{ f \in C_b(D(A)), \sup_{x, y \in D(A)} \frac{|f(x) - f(y)|}{|A(x - y)|(1 + |Ax|^2 + |Ay|^2)} < +\infty \right\}.$$

Note that we identify the gradient and the differential of a real valued function. Also, the second differential is identified with a function with values in $\mathcal{L}(H)$. The third differential is a trilinear operator on $D(A)$ and the norm $\|\cdot\|$ above is the norm of such operators.

Slightly improving the arguments in [4], it can be proved¹ that P_t maps \mathcal{E}_i into itself and that there exists a constant $c > 0$ such that

$$\|P_t f\|_{\mathcal{E}_i} \leq c \|f\|_{\mathcal{E}_i}. \quad (2.1)$$

Moreover, for $f \in \mathcal{E}_1$, $P_t f$ is a strict solution of (1.5) in the sense that it is satisfied for any $x \in D(A)$ and $t \geq 0$. Again, the result of [4] has to be slightly improved to get this result. In fact, using an interpolation argument, Proposition 5.9 and the various other estimates in [4], it is easy to deduce that, for any $x \in D(A)$, $LP_t f(x)$ is continuous on $[0, T]$.

For $f \in \mathcal{E}_2$, $P_t f$ is still a solution of (1.5) but in the mild sense. We define the Ornstein-Uhlenbeck semigroup associated to the linear equation

$$R_t \varphi(x) = \varphi(e^{tA}x + \int_0^t e^{A(t-s)} \sqrt{Q} dW(s), \quad t \geq 0, \quad \varphi \in C_b(D(A)).$$

Then it is shown in [4] that

$$P_t f(x) = R_t f(x) + \int_0^t R_{t-s}(b, DP_s f) ds, \quad t \geq 0, \quad f \in \mathcal{E}_2. \quad (2.2)$$

For any $\lambda > 0$ we set

$$F_\lambda f = \int_0^\infty e^{-\lambda t} P_t f dt, \quad f \in C_b(D(A)).$$

Then since $\|P_t f\|_0 \leq \|f\|_0$, we have

$$\|F_\lambda f\|_0 \leq \frac{1}{\lambda} \|f\|_0.$$

Moreover, since P_t is Feller, we have by dominated convergence

$$F_\lambda f \in C_b(D(A)).$$

It can be easily deduced that

$$F_\lambda f - F_\mu f = (\mu - \lambda) F_\lambda F_\mu f, \quad \mu, \lambda > 0,$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda F_\lambda f(x) = \lim_{\lambda \rightarrow \infty} \int_0^\infty e^{-\tau} P_{\tau/\lambda}(x) d\tau = f(x), \quad x \in D(A).$$

¹In fact, only Lemma 5.3 has to be improved. In this Lemma, the term L_1 can in fact be estimated in a single step by using Proposition 3.5 of [7] instead of Proposition 5.1 of [4].

It follows classically (see for instance [20]) that there exists a unique maximal dissipative operator \bar{L} on $C_b(D(A))$ with domain $D(\bar{L})$ such that

$$F_\lambda f = (\lambda - \bar{L})^{-1} f.$$

We recall the following well known characterization of $D(\bar{L})$: $f \in D(\bar{L})$ if and only if

- (i) $f \in C_b(D(A))$,
- (ii) $\frac{1}{t} \|P_t f - f\|_0$ is bounded for $t \in [0, 1]$,
- (iii) $\frac{1}{t} (P_t f(x) - f(x))$ has a limit for any $x \in D(A)$.

Moreover, we have in this case

$$\bar{L}f(x) = \lim_{t \rightarrow 0} \frac{1}{t} (P_t f(x) - f(x)).$$

Recall also that

$$(\lambda - \bar{L})^{-1} f = \int_0^\infty e^{-\lambda t} P_t f dt, \quad f \in C_b(D(A)).$$

By (2.1) we deduce that

$$\|(\lambda - \bar{L})^{-1} f\|_{\mathcal{E}_i} \leq \frac{c}{\lambda} \|f\|_{\mathcal{E}_i}. \quad (2.3)$$

Similarly, we may define, for $k \geq 0$, \mathcal{E}_3^k as the space C^3 functions on $D(A)$ such that there exists a constant c satisfying

- $|(-A)^{-1} Df(x)|_H \leq c(|Ax|^k + 1)$
- $|(-A)^{-1} D^2 f(x)(-A)^{-1}|_{\mathcal{L}(H)} \leq c(|Ax|^k + 1)$
- $|(-A)^{-1/2} D^2 f(x)(-A)^{-1/2}|_{\mathcal{L}(H)} \leq c(|Ax|^k + 1)$
- $\|D^3 f(x)((-A)^{-1}, (-A)^{-1}, (-A)^{-1})\| \leq c(|Ax|^k + 1)$
- $\|D^3 f(x)((-A)^{-\gamma}, (-A)^{-\gamma}, (-A)^{-\gamma})\| \leq c(|Ax|^k + 1)$
- $|Df(x)|_H \leq c(|Ax|^k + 1)$

where $\gamma \in (1/2, 1]$. By the various estimates given in [4], it is easy to check that, provided K is chosen large enough, S_t maps \mathcal{E}_3^k into itself and there exists a constant $c > 0$ such that

$$\|S_t f\|_{\mathcal{E}_3^k} \leq c \|f\|_{\mathcal{E}_3^k}. \quad (2.4)$$

Moreover, for $f \in \mathcal{E}_3^k$, $S_t f$ is a strict solution of (1.7) in the sense that it is satisfied for any $x \in D(A)$ and $t \geq 0$.

For any $\lambda > 0$ we set

$$\tilde{F}_\lambda f = \int_0^\infty e^{-\lambda t} S_t f dt, \quad f \in C_b(D(A)).$$

and prove that there exists a unique maximal dissipative operator \bar{N} on $C_b(D(A))$ with domain $D(\bar{N})$ such that

$$\tilde{F}_\lambda f = (\lambda - \bar{N})^{-1} f,$$

and $f \in D(\bar{N})$ if and only if

- (i) $f \in C_b(D(A))$,
- (ii) $\frac{1}{t} \|S_t f - f\|_0$ is bounded for $t \in [0, 1]$,
- (iii) $\frac{1}{t} (S_t f(x) - f(x))$ has a limit for any $x \in D(A)$.

Finally, by (2.4), we see that

$$\|(\lambda - \bar{N})^{-1} f\|_{\mathcal{E}_3^k} \leq \frac{c}{\lambda} \|f\|_{\mathcal{E}_3^k}. \quad (2.5)$$

3 Construction of cores and identification of the generators

In this section, we analyse the generators defined in the preceding section. We start with the following definition.

Definition 3.1 *Let K be an operator with domain $D(K)$. A set $\mathcal{D} \subset D(K)$ is a π -core for K if for any $\varphi \in D(K)$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in \mathcal{D} which π -converges² to φ and such that $(K\varphi_n)_{n \in \mathbb{N}}$ π -converges to $K\varphi$.*

²Recall that the π -convergence - also called b.p. convergence - is defined by : $(f_n)_{n \in \mathbb{N}}$ π -converges to f iff $f_n(x) \rightarrow f(x)$ for any $x \in D(A)$ and $\sup_{n \in \mathbb{N}} \|f_n\|_0 < \infty$.

Let us set $\mathcal{G}_1 = (\lambda - \bar{L})^{-1}\mathcal{E}_1$ for some $\lambda > 0$. Clearly for any $\varphi \in \mathcal{G}_1$ we have $\varphi \in D(\bar{L})$ and by (2.3), $\varphi \in \mathcal{E}_1$. Moreover,

$$P_t\varphi(x) - \varphi(x) = \int_0^t LP_s\varphi(x)ds,$$

since $(P_t\varphi)_{t \geq 0}$ is a strict solution of the Kolmogorov equation. By (2.1) and the definition of \mathcal{E}_1 , for any $x \in D(A)$ we have

$$|LP_s\varphi(x)| \leq c(1 + |Ax|^6)\|P_s\varphi\|_{\mathcal{E}_1} \leq c(1 + |Ax|^6)\|\varphi\|_{\mathcal{E}_1}.$$

Moreover, since

$$t \rightarrow LP_t\varphi(x)$$

is continuous, we have

$$\lim_{t \rightarrow 0} \frac{1}{t} (P_t\varphi(x) - \varphi(x)) = L\varphi(x).$$

We deduce that

$$\bar{L}\varphi(x) = L\varphi(x), \quad x \in D(A).$$

Since \mathcal{E}_1 is π -dense in $C_b(D(A))$, we deduce that \mathcal{G}_1 is a π -core for \bar{L} .

Also $\mathcal{E}_1 \subset \mathcal{E}_2$ so that

$$\mathcal{G}_1 \subset \mathcal{G}_2 = (\lambda - \bar{L})^{-1}\mathcal{E}_2$$

and \mathcal{G}_2 is also a π -core for \bar{L} .

These results hold both in dimension 2 or 3. The problem is that these cores are abstract and strongly depend on the semigroup $(P_t)_{t \geq 0}$. In dimension 3, this is a real problem since we do not know if the transition semigroup is unique. If we were able to construct a core in terms of the differential operator L , this would certainly imply uniqueness of this transition semigroup.

In dimension 2, we are able to construct such a core. Of course, in this case, uniqueness is well known. However, we think that it is important to have explicit cores. This gives many informations on the transition semigroup $(P_t)_{t \geq 0}$.

Theorem 3.2 *Let us set*

$$\mathcal{H} = \{f \in \mathcal{E}_1 : Lf \in \mathcal{E}_1\}$$

then, in dimension $d = 2$, $\mathcal{H} \subset D(\bar{L})$ and it is a π -core for \bar{L} . Moreover, for any $f \in \mathcal{H}$, we have

$$\bar{L}f = Lf.$$

The crucial point is to prove the following result.

Proposition 3.3 *Let $d = 2$. For any $f \in \mathcal{H}$ we have*

$$P_{t_1}f - P_{t_2}f = \int_{t_1}^{t_2} P_s Lf ds, \quad 0 \leq t_1 \leq t_2.$$

Proof. Let $f \in \mathcal{H}$. By Itô formula applied to the Galerkin equation (1.8), we have for $\epsilon > 0$

$$\begin{aligned} & d \left(e^{-\epsilon \int_0^t |(-A)^{1/2} X_m(s,x)|^6 ds} f(X_m(t,x)) \right) \\ &= \left(-\epsilon |(-A)^{1/2} X_m(t,x)|^6 f(X_m(t,x)) + L_m f_m(X(t,x)) \right) e^{-\epsilon \int_0^t |(-A)^{1/2} X_m(s,x)|^6 ds} dt \\ &+ e^{-\epsilon \int_0^t |(-A)^{1/2} X_m(s,x)|^6 ds} (Df_m(X_m(t,x)), \sqrt{Q_m} dW) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left(e^{-\epsilon \int_0^{t_2} |(-A)^{1/2} X_m(s,x)|^6 ds} f(X_m(t_2,x)) \right) \\ & - \left(e^{-\epsilon \int_0^{t_1} |(-A)^{1/2} X_m(s,x)|^6 ds} f(X_m(t_1,x)) \right) \\ &= \mathbb{E} \left(\int_{t_1}^{t_2} \left(-\epsilon |(-A)^{1/2} X_m(s,x)|^6 f(X_m(s,x)) \right. \right. \\ & \left. \left. + L_m f(X_m(s,x)) \right) e^{-\epsilon \int_0^s |(-A)^{1/2} X_m(\sigma,x)|^6 d\sigma} ds \right). \end{aligned} \tag{3.1}$$

We have denoted by L_m the Kolmogorov operator associated to (1.8). Since $f \in \mathcal{H}$, we have

$$|Lf_m(x)| \leq c(1 + |Ax|^6).$$

By Proposition 5.4 and Lemma 5.3, the right hand side of (3.1) is uniformly integrable on $\Omega \times [t_1, t_2]$ with respect to m . Thus, we can take the limit $m \rightarrow \infty$ in (3.1) and obtain

$$\begin{aligned} & \mathbb{E}_x \left(e^{-\epsilon \int_0^{t_2} |(-A)^{1/2} X(s,x)|^6 ds} f(X(t_2,x)) \right) \\ & - \mathbb{E}_x \left(e^{-\epsilon \int_0^{t_1} |(-A)^{1/2} X(s,x)|^6 ds} f(X(t_1,x)) \right) \\ &= \mathbb{E}_x \left(\int_{t_1}^{t_2} \left(-\epsilon |(-A)^{1/2} X(s,x)|^6 f(X(s,x)) \right. \right. \\ & \left. \left. + Lf(X(s,x)) \right) e^{-\epsilon \int_0^s |(-A)^{1/2} X(\sigma,x)|^6 d\sigma} ds \right). \end{aligned} \tag{3.2}$$

It is easy to prove by dominated convergence that

$$\mathbb{E}_x \left(e^{-\epsilon \int_0^{t_i} |X(s,x)|_1^6 ds} f(X(t_i, x)) \right) \rightarrow P_{t_i} f(x),$$

$$\mathbb{E}_x \int_{t_1}^{t_2} Lf(X(s, x)) e^{-\epsilon \int_0^s |X(\sigma, x)|_1^6 d\sigma} ds \rightarrow \mathbb{E}_x \int_{t_1}^{t_2} P_s Lf(x),$$

when $\epsilon \rightarrow 0$. Indeed by Lemma 5.3 below, we have

$$\int_0^{t_i} |X(s, x)|_1^6 ds < \infty \quad \mathbb{P}\text{-a.s.}$$

Moreover

$$\left| \mathbb{E}_x \int_{t_1}^{t_2} \left(\epsilon |X(s, x)|_1^6 f(X(s, x)) e^{-\epsilon \int_0^s |X(\sigma, x)|_1^6 d\sigma} ds \right) \right|$$

$$\leq \|f\|_0 \mathbb{E}_x \left(e^{-\epsilon \int_0^{t_1} |X(\sigma, x)|_1^6 d\sigma} - e^{-\epsilon \int_0^{t_2} |X(\sigma, x)|_1^6 d\sigma} \right) \rightarrow 0,$$

as $\epsilon \rightarrow 0$. The result follows. \square

It is now easy to conclude the proof of Theorem 3.2. Indeed, by Proposition 3.3, for $f \in \mathcal{H}$ we have, since $\|P_s Lf\|_0 \leq \|Lf\|_0$,

$$\|P_t f - f\|_0 \leq t \|Lf\|_0.$$

Moreover, since $s \mapsto P_s Lf(x)$, is continuous for any $x \in D(A)$

$$\frac{1}{t} (P_t f(x) - f(x)) \rightarrow Lf(x), \text{ as } t \rightarrow 0.$$

It follows that $f \in D(\bar{L})$ and $\bar{L}f = Lf$. Finally

$$\mathcal{G}_1 \subset \mathcal{H}$$

and since \mathcal{G}_1 is a π -core we deduce that \mathcal{H} is also a π -core.

Remark 3.4 We do not use that $P_t f$ is a strict solution of the Kolmogorov equation to prove that $\mathcal{H} \subset D(\bar{L})$ and $\bar{L}f = Lf$. But we do not know if there is a direct proof of the fact that \mathcal{H} is a π -core. We have used that $\mathcal{G}_1 \subset \mathcal{H}$ and that \mathcal{G}_1 is a π -core. The proof of $\mathcal{G}_1 \subset \mathcal{H}$ requires (2.1) which is almost as strong as the construction of a strict solution. \square

Remark 3.5 For $d = 3$, using Lemma 3.1 in [4], it is easy to prove a formula similar to (3.2) with $|(-A)^{1/2}X(s, x)|^6$ replaced by $|AX(s, x)|^4$ in the exponential terms. The problem is that

$$\lim_{\epsilon \rightarrow 0} e^{-\epsilon \int_0^t |AX(s, x)|^4 ds} = \mathbb{1}_{[0, \tau^*(x))}(t),$$

where $\tau^*(x)$ is the life time of the solution in $D(A)$. Thus we are not able to prove Proposition 3.3 in this case.

We have the following result on the operator \bar{N} .

Theorem 3.6 *Let $d = 2$ or 3 and $k \in \mathbb{N}$, define*

$$\widetilde{\mathcal{H}}_k = \{f \in \mathcal{E}_3^k : Nf \in \mathcal{E}_3^k\}.$$

Then $\widetilde{\mathcal{H}}_k \subset D(\bar{N})$ and it is π -core for \bar{N} . Moreover, for any $f \in \widetilde{H}_k$ we have

$$\bar{N}f = Nf.$$

The proof follows the same line as above. Indeed, it is easy to use similar arguments as in [4] and prove that for $f \in \mathcal{E}_3^k$, $(S_t f)_{t \geq 0}$ is a strict solution to (1.7). Arguing as above, we deduce that $(\lambda - \bar{N})^{-1} \mathcal{E}_3^k$ is a π -core for \bar{N} .

Moreover, applying Itô formula to the Galerkin approximations and letting $m \rightarrow \infty$ along the subsequence m_k - thanks to Lemma 3.1 of [4] to get uniform integrability - we prove, for $f \in \widetilde{\mathcal{H}}_k$,

$$\begin{aligned} & \mathbb{E}_x(e^{-K \int_0^{t_2} |AX(s, x)|^2 ds} f(X(t_2, x))) - \mathbb{E}_x(e^{-K \int_0^{t_1} |AX(s, x)|^2 ds} f(X(t_1, x))) \\ &= \mathbb{E}_x \left(\int_{t_1}^{t_2} (-K |AX(s, x)|^2 f(X(s, x)) + Lf(X(s, x))) e^{-K \int_0^s |AX(\sigma, x)|^2 d\sigma} ds \right). \end{aligned}$$

We rewrite this as

$$S_{t_2} f(x) - S_{t_1} f(x) = \int_{t_1}^{t_2} S_s Nf(x),$$

and deduce as above that $f \in D(\bar{N})$ and $\bar{N}f = Nf$. Finally, since $(\lambda - \bar{N})^{-1} \mathcal{E}_3^k \subset \widetilde{H}_k$, we know that $\widetilde{\mathcal{H}}_k$ is also a π -core.

4 Uniqueness for the martingale problem

Let us study the following martingale problem.

Definition 4.1 We say that a probability measure \mathbb{P}_x on $C([0, T]; D((-A)^{-\epsilon})$, $\epsilon > 0$ is a solution of the martingale problem associated to (1.2) if

$$\mathbb{P}_x(\eta(t) \in D(A)) = 1, \quad t \geq 0, \quad \mathbb{P}_x(\eta(0) = x) = 1$$

and for any $f \in \mathcal{H}$

$$f(\eta(t)) - \int_0^t Lf(\eta(s))ds,$$

is a martingale with respect to the natural filtration.

Remark 4.2 In general, it is proved the existence of a solution to a different martingale problem where f is required to be in a smaller class. In particular, it is required that $f \in C_b(D((-A)^{-\epsilon}))$ for some $\epsilon > 0$. However, in all concrete construction of solutions, it can be shown that a solution of our martingale problem is in fact obtained. \square

Theorem 4.3 Let $d = 2$, then for any $x \in D(A)$, there exists a unique solution to the martingale problem.

Proof. By a similar proof as for Proposition 3.3, we know that there exists a solution to the martingale problem

Uniqueness follows from a classical argument. Let $f \in \mathcal{E}_1$ and, for $\lambda > 0$ set $\varphi = (\lambda - \bar{L})^{-1}f$. Then $\varphi \in \mathcal{G}_1 \subset \mathcal{H}$ and

$$\varphi(\eta(t)) - \varphi(x) - \int_0^t L\varphi(\eta(s))ds$$

is a martingale. Thus, for any solution \tilde{P}_x of the martingale problem,

$$\tilde{\mathbb{E}}_x \left(\varphi(\eta(t)) - \varphi(x) - \int_0^t L\varphi(\eta(s))ds \right) = \varphi(x).$$

We multiply by $\lambda e^{-\lambda t}$, integrate over $[0, \infty)$ and obtain, since $\bar{L}\varphi = L\varphi$,

$$\tilde{\mathbb{E}}_x \int_0^\infty e^{-\lambda t} f(\eta(t))dt = \varphi(x) = (\lambda - \bar{L})^{-1}f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)dt.$$

By inversion of Laplace transform we deduce

$$\tilde{\mathbb{E}}_x(f(\eta(t))) = P_t f(x).$$

Thus the law at a fixed time t is uniquely defined. A standard argument allows to prove that this implies uniqueness for the martingale problem. \square

For $d = 3$ the proof of uniqueness still works. The problem is that we cannot prove existence of a solution of the martingale problem. More precisely, we cannot prove Proposition 3.3.

We can prove existence and uniqueness in $d = 3$ for the the martingale problem where \mathcal{H} is replaced by \mathcal{G}_1 , but since the definition of \mathcal{G}_1 depends on the semigroup, this does not give any real information.

We have the following weaker result on a stopped martingale problem.

Definition 4.4 *We say that a probability measure \mathbb{P}_x on $C([0, T]; D(A))$ is a solution of the stopped martingale problem associated to (1.2) if*

$$\mathbb{P}_x(\eta(0) = 1) = 1,$$

and for any $f \in \widetilde{\mathcal{H}}_k$

$$f(\eta(t \wedge \tau^*)) - \int_0^{t \wedge \tau^*} Lf(\eta(s))ds,$$

is a martingale with respect to the natural filtration and

$$\eta(t) = \eta(\tau^*), \quad t \geq \tau^*.$$

The stopping time τ^* is defined by

$$\tau^* = \lim_{R \rightarrow \infty} \tau_R, \quad \tau_R = \inf\{t \in [0, T], |A\eta(t)| \geq R\}.$$

Theorem 4.5 *For any $x \in D(A)$, there exists a unique solution to the stopped martingale problem.*

Proof. Existence of a solution for this martingale problem is classical. A possible proof follows the same line as the proofs of Proposition 3.3 and Theorem 3.6, see also Remark 3.5 (see also [9] for more details). In fact, we may choose the Markov family $((X(t, x))_{t \geq 0}, \Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ constructed in [7]. It is easy to see that $X(t, x)$ is continuous up to τ^* . We slightly change notation and set $X(t, x) = X(t \wedge \tau^*, x)$.

Uniqueness follows from a similar argument as in Theorem 4.3. For $\epsilon > 0$, we define $(S^\epsilon(t))_{t \geq 0}$ similarly as $(S_t)_{t \geq 0}$ but we replace $e^{-K \int_0^t |A\eta(s)|^2 ds}$ by $e^{-\epsilon \int_0^t |A\eta(s)|^4 ds}$ in (1.6). Proceeding as above, we then define $N_\epsilon, \bar{N}_\epsilon, \widetilde{\mathcal{H}}_k^\epsilon$, and prove that $\widetilde{\mathcal{H}}_k^\epsilon$ is a π -core for \bar{N}_ϵ and $N_\epsilon \varphi = \bar{N}_\epsilon \varphi$ for $\varphi \in \widetilde{\mathcal{H}}_k^\epsilon$.

Let $\tilde{\mathbb{P}}_x$ be a solution to the martingale problem and $f \in \mathcal{E}_3^k$. For $\lambda, \epsilon > 0$, we set $\varphi = (\lambda - \bar{N}_\epsilon)^{-1}$, then $\varphi \in \widetilde{\mathcal{H}}_k^\epsilon$.

By Itô formula - note that in Definition 4.4 it is required that the measure is supported by $C([0, T]; D(A))$ - we prove that

$$\begin{aligned} & e^{-\epsilon \int_0^t |A\eta(s)|^4 ds} \varphi(\eta(t)) - \int_0^t (-\epsilon |A\eta(s)|^4 \varphi(\eta(s)) + L\varphi(\eta(s))) e^{-\epsilon \int_0^s |A\eta(\sigma)|^4 d\sigma} ds \\ &= e^{-\epsilon \int_0^t |A\eta(s)|^4 ds} \varphi(\eta(t)) - \int_0^t N_\epsilon \varphi(\eta(s)) e^{-\epsilon \int_0^s |A\eta(\sigma)|^4 d\sigma} ds \end{aligned}$$

is also a martingale. We have used :

$$e^{-\epsilon \int_0^t |A\eta(s)|^4 ds} = 0, \quad t \geq \tau^*.$$

Thus:

$$\tilde{\mathbb{E}}_x \left(e^{-\epsilon \int_0^t |A\eta(s)|^4 ds} \varphi(\eta(t)) - \int_0^t N_\epsilon \varphi(\eta(s)) e^{-\epsilon \int_0^s |A\eta(\sigma)|^4 d\sigma} ds \right) = \varphi(x).$$

We multiply by $e^{-\lambda t}$ and integrate over $[0, \infty)$ and obtain, since $\bar{N}_\epsilon \varphi = N_\epsilon \varphi$,

$$\begin{aligned} & \tilde{\mathbb{E}}_x \left(\int_0^\infty e^{-\lambda t - \epsilon \int_0^t |A\eta(s)|^4 ds} f(\eta(t)) dt \right) \\ &= \varphi(x) = (\lambda - \bar{N}_\epsilon)^{-1} f(x) = \int_0^\infty e^{-\lambda t} S_t^\epsilon f(x) dt. \end{aligned}$$

By dominated convergence, we may let $\epsilon \rightarrow 0$ and obtain

$$\tilde{\mathbb{E}}_x \left(\int_0^\infty e^{-\lambda t} \mathbb{1}_{t \leq \tau^*} f(\eta(t)) dt \right) = \int_0^\infty e^{-\lambda t} S_t^0 f(x) dt,$$

where $S_t^0 f(x) = \lim_{\epsilon \rightarrow 0} S_t^\epsilon f(x) = \mathbb{E}_x(\mathbb{1}_{t \leq \tau^*} f(X(t, x)))$. The conclusion follows. \square

5 Technical results

In all this section, we assume that $d = 2$. Also, for $s \in \mathbb{R}$, we set $|\cdot|_s = |(-A)^s \cdot|$.

Lemma 5.1 *There exists c depending on T, Q, A such that*

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \in [0, T]} |X(t, x)|^2 + \int_0^T |X(s, x)|_1^2 ds \right) \leq c(1 + |x|^2), \\ & \mathbb{E} \left(\sup_{t \in [0, T]} |X(t, x)|^4 + \int_0^T |X(s, x)|^2 |X(s, x)|_1^2 ds \right) \leq c(1 + |x|^4). \end{aligned}$$

Proof. We first apply Itô's formula to $\frac{1}{2} |x|^2$ (as usual the computation is formal and it should be justified by Galerkin approximations):

$$\frac{1}{2} d|X(t, x)|^2 + |X(t, x)|_1^2 dt = (X(t, x), \sqrt{Q}dW) + \frac{1}{2} \text{Tr } Q dt.$$

We deduce, thanks to a classical martingale inequality,

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{2} \sup_{t \in [0, T]} |X(t, x)|^2 + \int_0^T |X(s, x)|_1^2 ds \right) \\ & \leq \mathbb{E} \left(\sup_{t \in [0, T]} \left| \int_0^t (X(s, x), \sqrt{Q}dW(s)) \right| \right) + \frac{1}{2} (|x|^2 + \text{Tr } Q T) \\ & \leq 2\mathbb{E} \left(\left(\int_0^T |\sqrt{Q}X(s, x)|^2 ds \right)^{1/2} \right) + \frac{1}{2} (|x|^2 + \text{Tr } Q T) \\ & \leq \frac{1}{2} \mathbb{E} \int_0^T |X(s, x)|^2 ds + C + \frac{1}{2} |x|^2, \end{aligned}$$

where C depends on T, Q, A . It follows that

$$\mathbb{E} \left(\sup_{t \in [0, T]} |X(t, x)|^2 + \int_0^T |X(s, x)|_1^2 ds \right) \leq C + |x|^2. \quad (5.1)$$

We now apply Itô's formula to $\frac{1}{4} |x|^4$,

$$\begin{aligned} & \frac{1}{4} d|X(t, x)|^4 + |X(t, x)|^2 |X(t, x)|_1^2 dt = |X(t, x)|^2 (X(t, x), \sqrt{Q}dW) \\ & + \left(\frac{1}{2} \text{Tr } Q |X(t, x)|^2 + |\sqrt{Q}X(t, x)|^2 \right) dt \\ & \leq |X(t, x)|^2 (X(t, x), \sqrt{Q}dW) + c |X(t, x)|^2 dt. \end{aligned}$$

We deduce

$$\begin{aligned}
& \mathbb{E} \left(\frac{1}{4} \sup_{t \in [0, T]} |X(t, x)|^2 + \int_0^T |X(s, x)|^2 |X(s, x)|_1^2 ds \right) \\
& \leq \mathbb{E} \left(\sup_{t \in [0, T]} \left| \int_0^t |X(s, x)|^2 (X(s, x), \sqrt{Q} dW(s)) \right| \right) \\
& \quad + c \mathbb{E} \int_0^T |X(s, x)|^2 ds + \frac{1}{4} |x|^4 \\
& \leq 2 \mathbb{E} \left(\left(\int_0^T |X(s, x)|^4 |\sqrt{Q} X(s, x)|^2 ds \right)^{1/2} \right) + c(1 + |x|^4) \\
& \leq 2 \mathbb{E} \left(\sup_{t \in [0, T]} |X(s, x)|^2 \left(\int_0^T |\sqrt{Q} X(s, x)|^2 ds \right)^{1/2} \right) + c(1 + |x|^4) \\
& \leq \frac{1}{8} \mathbb{E} \left(\sup_{t \in [0, T]} |X(t, x)|^4 \right) + c \mathbb{E} \left(\int_0^T |\sqrt{Q} X(s, x)|^2 ds \right) + c(1 + |x|^4).
\end{aligned}$$

Since \sqrt{Q} is a bounded operator, using (5.1) we deduce

$$\mathbb{E} \left(\sup_{t \in [0, T]} |X(t, x)|^4 + \int_0^T |X(s, x)|^2 |X(s, x)|_1^2 ds \right) \leq (1 + |x|^4).$$

□

Lemma 5.2 *There exists c depending on T, Q, A such that*

$$\begin{aligned}
& \mathbb{E} \left(\sup_{t \in [0, T]} e^{-c \int_0^t |X(s, x)|^2 |X(s, x)|_1^2 ds} |X(t, x)|_1^2 \right) \\
& \quad + \mathbb{E} \left(\int_0^T e^{-c \int_0^s |X(\sigma, x)|^2 |X(\sigma, x)|_1^2 d\sigma} |X(s, x)|_2^2 ds \right) \leq c(1 + |x|_1^2).
\end{aligned}$$

Proof. We apply Itô's formula to

$$e^{-c \int_0^t |X(s, x)|^2 |X(s, x)|_1^2 ds} |X(t, x)|_1^2,$$

and obtain

$$\begin{aligned}
& \frac{1}{2} d \left(e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} |X(t,x)|_1^2 \right) + e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} |X(t,x)|_2^2 dt \\
&= e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} \left(-c |X(t,x)|^2 |X(t,x)|_1^4 + (b(X(t,x)), AX(t,x)) \right) dt \\
&+ (Ax, \sqrt{Q}dW) - \frac{1}{2} \text{Tr} [AQ] dt.
\end{aligned}$$

We have

$$\begin{aligned}
(b(x), Ax) &\leq |b(x)| |Ax| \\
&\leq \tilde{c} |x|_{L^4} |\nabla x|_{L^4} |Ax| \\
&\leq \tilde{c} |x|^{1/2} |x|_1 |x|_2^{3/2} \\
&\leq \frac{1}{2} |x|_2^2 + \tilde{c} |x|^2 |x|_1^4.
\end{aligned}$$

We deduce that if $c \geq \tilde{c}$,

$$\begin{aligned}
& \frac{1}{2} d \left(e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} |X(t,x)|_1^2 \right) + \frac{1}{2} e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} |X(t,x)|_2^2 dt \\
&\leq e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} (AX(t,x), \sqrt{Q}dW) + c dt
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left(\sup_{t \in [0, T]} e^{-c \int_0^t |X(s,x)|^2 |X(s,x)|_1^2 ds} |X(t,x)|_1^2 \right) \\
&+ \mathbb{E} \left(\int_0^T e^{-c \int_0^s |X(\sigma,x)|^2 |X(\sigma,x)|_1^2 d\sigma} |AX(s,x)|^2 ds \right) \\
&\leq 2 \mathbb{E} \left(\left(\int_0^T e^{-2c \int_0^s |X(\sigma,x)|^2 |X(\sigma,x)|_1^2 d\sigma} |\sqrt{Q}X(s,x)|^2 ds \right)^{1/2} \right) \\
&+ cT + |x|_1^2.
\end{aligned}$$

Since $\text{Tr}(QA) < \infty$, we know that QA is a bounded operator and

$$\begin{aligned} & \mathbb{E} \left(\left(\int_0^T e^{-2c \int_0^s |X(\sigma, x)|^2 |X(\sigma, x)|_1^2 d\sigma} |\sqrt{Q}X(s, x)|^2 ds \right)^{1/2} \right) \\ & \leq c \mathbb{E} \left(\left(\int_0^T |X(s, x)|_1^2 ds \right)^{1/2} \right) \\ & \leq (|x| + 1), \end{aligned}$$

by Lemma 5.1. The result follows. \square

Lemma 5.3 *For any $k \in \mathbb{N}$, there exists c depending on k, T, Q, A such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} e^{-c \int_0^t |X(s, x)|^2 |X(s, x)|_1^2 ds} |X(t, x)|_1^k \right) \leq c(1 + |x|_1^k).$$

The proof of this Lemma follows the same argument as above. It is left to the reader.

Proposition 5.4 *For any $k \in \mathbb{N}$, $\epsilon > 0$, there exists $C(\epsilon, k, T, Q, A)$ such that for any $m \in \mathbb{N}$, $x \in D(A)$, $t \in [0, T]$,*

$$\mathbb{E} \left(e^{-\epsilon \int_0^t |(-A)^{1/2} X_m(s, x)|^6 ds} |AX_m(t, x)|^k \right) \leq C(\epsilon, k, T, Q, A)(1 + |Ax|^k).$$

Proof. Let us set

$$Z(t) = \int_0^t e^{(t-s)A} \sqrt{Q} dW(s), \quad Y(t) = X(t, x) - z(t).$$

Then, by the factorization method (see [5]),

$$\mathbb{E} \left(\sup_{t \in [0, T]} |Z(t)|_{2+\epsilon}^h \right) \leq C, \tag{5.2}$$

for any $\epsilon < g$, and we have

$$\frac{dY}{dt} = AY + b(Y + Z).$$

We take the scalar product with A^2Y :

$$\frac{1}{2} \frac{d}{dt} |Y|_2^2 + |Y|_3^2 = (b(Y + Z), A^2Y) = ((-A)^{1/2}b(Y + Z), A^{3/2}Y).$$

We have

$$\begin{aligned} |(-A)^{1/2}b(Y+Z)| &= |\nabla b(Y+Z)| \\ &\leq c(|Y+Z|_{W^{1,4}}^2 + |Y+Z|_{L^p}|Y+Z|_{W^{2,q}}), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

By Gagliardo-Nirenberg inequality

$$|Y+Z|_{W^{1,4}}^2 \leq c|Y+Z|_1 |Y+Z|_2.$$

Setting $\frac{1}{p} = \frac{1}{2} - \frac{s}{2}$ we have by Sobolev's embedding

$$|Y+Z|_{L^p} |Y+Z|_{W^{2,q}} \leq c|Y+Z|_{s/2} |Y+Z|_{3-s/2}.$$

Therefore

$$\begin{aligned} ((-A)^{1/2}b(Y+Z), (-A)^{3/2}Y) &\leq c|Y+Z|_1 |Y+Z|_2 |Y|_3 \\ &\leq c|Y+Z|_{s/2} |Y+Z|_{3-s/2} |Y|_3 \\ &\leq \frac{1}{4} |Y|_3^2 + c|Y+Z|_1^2 |Z|_2^2 + c|Y+Z|_1^2 |Y|_2^2 \\ &\quad + c|Y+Z|_{s/2} |Y|_{3-s/2} |Y|_3 + c|Y+Z|_{s/2}^2 |Z|_{3-s/2}^2. \end{aligned}$$

Since

$$\begin{aligned} |Y+Z|_{s/2} |Y|_{3-s/2} |Y|_3 &\leq |Y+Z|_{s/2} |Y|_1^{s/4} |Y|_3^{2-s/4} \\ &\leq c|Y+Z|_{s/2}^{8/s} |Y|_1^2 + \frac{1}{4} |Y|_3^2, \end{aligned}$$

we finally get

$$\begin{aligned} \frac{d}{dt} |Y|_2^2 &\leq c|Y+Z|_1^2 |Y|_2^2 \\ &\quad + c \left(|Y+Z|_1^2 |Z|_2^2 + |Y+Z|_{s/2}^2 |Z|_{3-s/2}^2 + |Y+Z|_{s/2}^{8/s} |Y|_1^2 \right) \end{aligned}$$

and

$$\begin{aligned} |Y(t)|_2^2 &\leq e^{c \int_0^t |Y+Z|_1^2 ds} \\ &\quad \left(|x|_2^2 + c \int_0^t (|Y+Z|_1^2 |Z|_2^2 + |Y+Z|_{s/2}^2 |Z|_{3-s/2}^2 + |Y+Z|_1^{8/s} |Y|_{1/2}^2) ds \right). \end{aligned}$$

We then write by Hölder and Poincaré inequalities

$$e^{-\epsilon \int_0^t |Y+Z|_1^6 ds} |Y(t)|_2^k \leq c_k e^{-\epsilon \int_0^t |Y+Z|_1^6 ds + c_k \int_0^t |Y+Z|_1^2 ds} \\ \times \left(|x|_2^k + \int_0^t (|Y+Z|_1^4 + |Z|_{2+g}^4 + |Y+Z|_1^{16/s}) ds \right)$$

(we choose $3 - s/2 < 2 + g$ and set $\epsilon = 1 - s/2$).

The conclusion follows from Lemma 5.3 and by the boundedness of $x \mapsto -\epsilon x^6 + cx^4$. \square

References

- [1] J. Bricmont, A. Kupiainen and R. Lefevere, *Exponential mixing for the 2D Navier-Stokes dynamics*, Comm. Math. Phys., **230**, no 1, 87-132, 2002.
- [2] I. Chueshov, S. B. Kuksin, *On the random kick-forced 3D Navier-Stokes equations in a thin domain*, Arch. Ration. Mech. Anal. 188, no. 1, 117–153, 2008.
- [3] I. Chueshov, S. B. Kuksin, *Stochastic 3D Navier-Stokes equations in a thin domain and its α -approximation*, Preprint (2007).
- [4] G. Da Prato, A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations*, J. Math. Pures et Appl., 82, 877-947, 2003.
- [5] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.
- [6] G. Da Prato and J. Zabczyk, *Ergodicity for infinite dimensional systems*, London Mathematical Society Lecture Notes, **229**, Cambridge University Press, 1996.
- [7] A. Debussche and C. Odasso, *Markov solutions for the 3D stochastic Navier–Stokes equations with state dependent noise*, Journal of Evol. Equ., vol 6, no 2, 305-324, 2006.
- [8] W. E, J.C. Mattingly, Y. G. Sinai, *Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation*, Commun. Math. Phys. **224**, 83–106, 2001.
- [9] F. Flandoli, *An introduction to 3D stochastic Fluid Dynamics*, CIME lecture notes 2005.
- [10] F. Flandoli and B. Maslowski, *Ergodicity of the 2D Navier-Stokes equations under random perturbations*, Comm. Math. Phys., **172**, no 1, 119-141, 1995.
- [11] F. Flandoli, M. Romito, *Markov selections for the 3D stochastic Navier-Stokes equation*, Probab. Theory Relat. Fields 140, nr. 3-4 (2008), 407–458.
- [12] B. Goldys, M. Röckner and X. Zhang, *Martingale solutions and Markov selections for stochastic partial differential equations*, Bielefeld, Preprint, 2008.
- [13] M. Hairer, J.C. Mattingly, *Ergodicity of the 2D Navier-Stokes Equations with Degenerate Stochastic Forcing*, Annals of Mathematics, vol. 164 no. 3, 2006.
- [14] M. Hairer, J.C. Mattingly. *Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations*. 2006. Preprint.
- [15] S B Kuksin, *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions*, European Mathematical Society Publishing House, 2006.
- [16] S. Kuksin and A. Shirikyan, *Ergodicity for the randomly forced 2D Navier-Stokes equations*, Math. Phys. Anal. and Geom., **4**, no 2, 147-195, 2001.
- [17] S. Kuksin and A. Shirikyan, *A coupling approach to randomly forced nonlinear PDEs. I.*, Comm. Math. Phys., **221**, 351-366, 2001.
- [18] J.C. Mattingly, *Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics*. Commun. Math. Phys. **230**, no. 3, 421-462, 2002.
- [19] J. C. Mattingly and E. Pardoux *Malliavin calculus for the stochastic 2D Navier-Stokes equation*, Comm. Pure Appl. Math, **59**, no. 12 1742-1790, 2006.

- [20] E. Priola, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, *Studia Math.* 136 (1999), no. 3, 271–295.
- [21] M. Röckner, Z. Sobol, *A new approach to Kolmogorov equations in infinite dimensions and applications to the stochastic 2d Navier Stokes equation* *C. R. Acad. Sci. Paris, Série I*, 4 pp., 2007.
- [22] M. Romito, *Analysis of equilibrium states of Markov solutions to the 3D Navier-Stokes equations driven by additive noise*, to appear on *J. Stat. Phys.*
- [23] C. Odasso, *Exponential mixing for the 3D Navier-Stokes equation*. 2005. To appear in *Commun. Math. Phys.*
- [24] A. Shirikyan, *Qualitative properties of stationary measures for three-dimensional Navier-Stokes equations*, *J. Funct. Anal.* 249 (2007), 284-306.