Polynomial time algorithms for constant capacitated lot sizing problems with equal length step-wise linear costs

Ayşe Akbalık[(1)] and Christophe Rapine[(2)]

(1) LTM – CNRS – UJF, 17, ave. des Martyrs
38054 Grenoble Cedex, France
ayse.akbalik-rapine@cea.fr
(2) G-SCOP, INPG – CNRS – UJF
46 avenue Félix Viallet, 38031 Grenoble, France
christophe.rapine@g-scop.inpg.fr

Abstract

This paper presents polynomial time algorithms for three extensions of the classical capacitated lot sizing problem (CLSP). We consider a constant batch size production with a fixed cost associated to each batch, additionally to the production setup cost and a unit holding cost. The production cost can then be considered as a step-wise function where the step length corresponds to the batch size. We can no more use the efficient methods proposed for the CLSP with concave costs in order to solve the CLSP with step-wise costs. We propose several properties of optimal solutions. Based on these properties, three polynomial time algorithms are provided under the assumptions of constant production capacity and constant batch sizes, as well as linear and non-increasing production costs over time. The uncapacitated case is solved in time complexity $\mathcal{O}(T^3)$. For the constant capacitated case, with a production capacity multiple of the batch size, the algorithm has a time complexity in $\mathcal{O}(T^4)$. The final algorithm concerns the general constant capacitated case, and has a time complexity in $\mathcal{O}(T^6)$.

keywords: Single-item capacitated lot sizing problem, step-wise costs, polynomial time dynamic programming algorithm.

1 Introduction

We study an extension of the well known single-item capacitated lot sizing problem (CLSP). CLSP deals with defining the optimal production quantities so as to satisfy the customer demands while minimizing the total production and storage costs and respecting the production capacity. The production cost function of the classical lot sizing problem is composed of a fixed cost, which is independent of the quantity, a unit production and a unit holding cost per period, function of the quantity produced and stored, respectively. Single-item LSP arises in many production and inventory planning problems in practical situations and is used to solve more complex systems. Many extensions of the single-item LSP have been studied in the literature. Despite the simplicity of its description, the computational complexity of LSP depends on many parameters. Even for the special case in which demands are stationary (which means constant) each period, all storage costs are zero, and the production cost functions are either concave with arbitrary capacity limits or convex with additionnal unit setup costs, LSP has been proven to be NP-hard by Florian et al. (1980).

In this paper, we consider a constant batch production which generates a fixed cost per batch in addition to the previous cost assumptions for the classical CLSP. A fixed cost per batch is very common in the real cases where a machine is to be setup for a batch production. The aim is again to satisfy the deterministic and discrete customer demands over a finite horizon without backlogging. The two fixed production costs can be aggregated which makes the new production cost discontinuous with a step-wise structure (see Figure 1). In literature such a cost structure is called *stair-case*, *multiple setup* or *step-wise cost*. Our problem can be stated as "*Single-item capacitated lot sizing problem with step-wise production costs*", for short CLSP-SW. As one can remark, the only difference between CLSP-SW and the classical CLSP is the production cost function. However the discontinuities in the function do not allow the use of the methods proposed for the classical case.

This problem can also be encoutered in other industrial applications, for instance production and transportation planning problem in a serial supply chain. Consider the situation of a manufacturer producing a single-item with a limited capacity and shipping the finished products to a warehouse, where a fixed cost is paid for each vehicle shipped. The deterministic and discrete demands are known at the warehouse level over a finite horizon. The aim is to satisfy demands without backlogging so as to minimize the total production, transportation and storage costs. Due to space constraints, storage is only possible in the warehouse. Since shipments to the warehouse occur immediately after production, in the same period, the fixed transportation cost per vehicle can be aggregated into the production setup cost, resulting in a step-wise production cost function identical to a CLSP-SW problem. In recent years, firms have paid more attention to the coordination of different activities in their supply chains for potential cost savings. In this setting, transportation and multi-echelon storage decisions can be taken into account while planning the production activities in a simultaneous manner. Although such approaches may lead to significant cost reduction, the integration of various activities constraints into the same optimization model can make the search of the optimal solution harder. In the literature many theoretical studies have been developed on the integrated models, covering different coordinated levels and different solving techniques (see Geunes and Pardalos (2003) for instance). There is also an increasing number of real-case studies showing the cost saving benefits of an integrated approach (see Gnoni et al. (2003) and Matta and Miller (2004)). A part of the literature review presented in section 2 on CLSP uses a transportation costs terminology rather than batch production.

Our focus is on the non-trivial polynomial cases for CLSP-SW making some assumptions on the production capacity, on the costs and also on the batch sizes. We provide a review of the relevant literature in Section 2. In Section 3, we present a mathematical formulation of the problem. In Section 4 we give some dominance properties, either arising from the literature or introduced for our problem. A polynomial time algorithm in time complexity $\mathcal{O}(T^3)$ is presented in Section 5 for the uncapacitated production case. Two polynomial time algorithms are given for the constant capacitated case in Section 6. The first deals with the case where the production capacity is a multiple of the batch size and has a time complexity in $\mathcal{O}(T^4)$. The second, dealing with arbitrary production capacity, has a time complexity in $\mathcal{O}(T^6)$. In Section 7 we summarize our results and present some concluding remarks.

2 State of the art

The following state of the art is organized into two parts. Firstly, we introduce relevant studies on the classical LSP, especially on the single-item CLSP solved by dynamic programming. Secondly, we present more specifically the literature on the single-item CLSP with piece-wise cost structure, arising from the addition of the batch production.

The seminal papers of Manne (1958) and Wagner and Whitin (1958) can be cited as the first studies on the LSP. The model they propose is for the uncapacitated single-item case and is used by many researchers to solve more complex production and inventory problems. Florian and Klein (1971) characterize the extreme points of the feasible domain of the capacitated lot sizing problem (CLSP) with the assumption of concave cost functions. They use the notion of production sequence S_{uv} with zero entering inventory levels ($s_u = s_v = 0$) in order to efficiently decompose the problem and apply dynamic programming. This paper will be detailed later in Section 4 in order to describe dominances. Baker et al. (1978) study CLSP where they consider unit production and holding costs which are linear function of the amount produced and stored. They propose an optimal solution property which will be stated in Section 4. This property has been extended to the CLSP with non-increasing unit production cost by Bitran and Yanasse (1982).

The first complexity studies on LSP began with the paper by Florian et al. (1980). The authors provide a complexity classification of the problems for different values of the parameters. They also provide a polynomial time algorithm in $\mathcal{O}(T^4)$ for the constant capacitated CLSP with concave costs. Hoesel and Wagelmans (1996) improve the algorithm complexity to $\mathcal{O}(T^3)$, under the linear holding cost assumption. Bitran and Yanasse (1982) classify some lot sizing problems based upon their complexities and demonstrate different classes of LSP complexity. They introduce some notations in the literature to make classification easier.

Several papers propose fully polynomial time approximation schemes (FPTAS) for the single-item CLSP. Gavish and Johnson (1990) present algorithms which approximate the optimal production schedule with an error of ε. Hoesel and Wagelmans (2001) also give an FPTAS for the single-item CLSP with concave backlogging and production cost functions and arbitrary holding cost. Another study on FPTAS is presented by Chubanov et al. (2006) for the monotone cost structures.

Other studies concern multi-echelon lot sizing problems arising in the supply chain. In their paper, Hoesel et al. (2005) consider a serial supply chain of L levels, including a capacitated manufacturer, $L-2$ intermediate warehouses and a retailer. They propose non-trivial polynomial time algorithms under various cost structures. Kaminsky and Simchi-Levi (2003) study a system composed of two stages of capacitated production and fixed cost transportation activity between them. They present polynomial time algorithms under various transportation cost structures and capacity assumptions. An important difference between our model and theirs is the transportation cost structure which has only one fixed component in their model, independently of the quantity shipped. Hence this setup cost is charged by transportation period, not by vehicle.

We will now present the studies on the LSP with piece-wise costs. As far as we know, one of the first studies on the integration of fixed transportation costs into the inventory control policy was performed by Lippman (1969). The author takes into account a fixed transportation cost associated to each vehicle sent, and a variable transportation cost which is assumed concave. Holding and production costs are assumed to be non-decreasing functions of the amount stored and produced. Under these assumptions, the author gives some optimal solution properties which will be detailed in Section 4. We can point out as a significant difference with the model of Lippman that we consider a production cost having both fixed and variable components. Swoveland (1975) considers a single product, multi-period production planning model where production and holding-backorder cost functions are assumed to be piece-wise concave. Some optimal schedule properties and a dynamic programming algorithm are given using production (inventory) breakpoints which are the endpoints of the intervals over which the production (inventory) cost functions are concave.

Akbalik and Pochet (2007) study the same structure analyzed in this paper making the same assumptions on the capacity and cost configurations. Different from this paper in which we propose polynomial time dynamic programming algorithms, Akbalik and Pochet (2007) use the polyhedral approach in order to solve the mixed integer linear program associated to CLSP-SW in an efficient manner. They propose a new class of valid inequalities derived from integer flow cover inequalities by a lifting procedure. They show that the addition of different flow cover inequalities together with the new ones reduces very significantly the total number of nodes explored in a Branch&Bound procedure and the total execution time. Pochet and Wolsey (1993) propose extended formulations for the constant batch production problem. They study some cases where matrices are totally unimodular. For the first case, the production capacity is assumed to be constant and a setup cost is paid for each positive amount produced. For the second case, the production capacity in each period is assumed to be multiple of some batch size and a fixed cost is generated for each batch produced. In another paper, Pochet and Wolsey (1994) give the convex hull of the uncapacitated LSP under Wagner-Within (WW[1]) cost structure using the stock minimal solution (for which it is optimal to produce as late as possible). The complete description of the convex hull necessitates $\mathcal{O}(T^2) \times \mathcal{O}(T)$ constraints and variables.

There are other studies focusing on the mixed integer linear programming (MILP) formulations to solve the piece-wise linear cost LSP. Diaby and Martel (1993) study an arborescent, multi-echelon distribution system to determine optimal purchasing and shipping quantities. They consider general piecewise linear procurement cost and linear holding costs. They formulate the problem as a MILP and propose a Lagrangean relaxation to solve it. Another study was done on the staircase facility location problem by Holmberg (1994). The author formulates the problem as a MILP and investigates solution methods based on convex piece-wise linearization and Benders' decomposition. Chan et al. (2000) study the less-than-truckload shipments problem integrated with production and inventory activities. The cost function is piece-wise linear and concave. They model the problem as a concave cost multi-commodity network flow problem. They formulate it as a MILP using a set-partitioning approach, and characterize structural properties. Chan et al. (2002) study a special class of piece-wise linear ordering cost LSP, which they refer to as *modified all-unit discount* cost function. This function arises when the transportation is done with less-than-truckload carriers. It is a non-decreasing function of the amount shipped and the marginal cost is non-increasing. They prove this problem to be NP-hard and give some performance guarantee on

[1]For WW cost structure, producing and storing one unit in period t costs more than producing it later.

the optimal cost.

We now present the studies using a dynamic programming approach. Lee et al. (2003) study an integrated inventory replenishment and outbound dispatch scheduling problem. They consider a structure composed of a manufacturer supplying a warehouse, which in turn delivers to downstream distribution centers. In the first echelon transportation problem there is only one fixed cost, but between the warehouse and DCs, each vehicle generates a fixed transportation cost. The authors consider also pre-shipping and late-shipping penalties in their model. They propose a network approach to solve the problem and propose polynomial time algorithms using some optimal solution properties.

Li et al. (2004) study two variants of the dynamic economic lot sizing problem. In the first, the production is restricted to a multiple of a fixed batch size, backlogging is allowed and all cost parameters are time varying. A polynomial time algorithm in $\mathcal{O}(Tlog(T))$ is proposed for the latter, and a $\mathcal{O}(T)$ time algorithm is proposed for additional non-speculative motive conditions. In the last setting, a general form of product order cost structure, which includes a fixed charge for each order, a variable cost and a freight cost for each truck sent with a truckload discount structure is studied. A polynomial time algorithm in $\mathcal{O}(T^3log(T))$ is proposed for the latter, and a $\mathcal{O}(T^3)$ time algorithm is proposed for additional non-speculative motive conditions.

Jin and Muriel (2005) study a system composed of one warehouse receiving a single product from a supplier and replenishing the inventory of n retailers with direct shipments. The process of ordering from the supplier and shipping to the retailers generates full truckload transportation costs with cargo capacity constraints. Giving some optimal solution properties, they study decentralized and centralized systems. In the decentralized system, each retailer and the warehouse decide how and when to replenish independently. Authors propose an algorithm in $\mathcal{O}(nT^2)$ with n being the number of retailers. For the centralized system and single retailer case they propose an algorithm with complexity in $\mathcal{O}(T^3)$. They use Lagrangean decomposition to solve the multi-retailer model.

Shaw and Wagelmans (1998) provide a pseudo-polynomial time dynamic programming algorithm with a complexity in $\mathcal{O}(T^2\overline{q}\overline{d})$ for the CLSP with piece-wise linear cost. In the equation, $\overline{q}$ is the number of pieces in the production cost function and $\overline{d}$ is the average demand. For the classical case where the production cost has only one setup component, the complexity becomes $\mathcal{O}(T^2\overline{d})$ with $\overline{q} = 1$. This is a significant improvement compared to the first pseudo-polynomial dynamic programming algorithm proposed by Florian et al. (1980) with a complexity in $\mathcal{O}(T^2\overline{P}\overline{d})$, where $\overline{P}$ refers to the average production capacity. The improvement is achieved via a clever algorithmic computation of the recursive steps. A detailed review can be found in Brahimi et al. (2006) on the single-item LSP. The reader can refer to Pochet and L.A.Wolsey (2006) for a comprehensive literature survey, particularly for a detailed information on the mixed integer programming approach to solve production planning problems. Among all these studies from literature, anyone makes the same assumptions as ours. We describe in the following section the problem we study in detail.

3 Problem formulation

The problem consists in scheduling a single-item production to meet discrete and deterministic demands d over a finite time horizon $\{1, \dots, T\}$. We consider a production capacity P_t (eventually $P_t = +\infty$) at the plant where a batch production takes place. The batch size is denoted by B. The aim is thus to propose a production planning which satisfies demands without backlogging, with a minimum cost of production and storage.

A production schedule x, or a *planning*, is a vector of size T corresponding to the amount x_t to produce each period. Hence a planning x is feasible iff (i) $0 \leq x_t \leq P_t$ and (ii) $\sum_{u=1}^{t} x_u \geq \sum_{u=1}^{t} d_u$ for all $t \in \{1, \dots, T\}$. We consider the following general cost structures:

Production

Production cost $p_t(x)$ of producing the amount x at period t includes a discrete and a continuous part. The discrete part is composed of a setup cost p_t^f paid whatever the amount $x > 0$ and of a fixed cost per batch p_t^b paid for each batch produced. We call *unit production cost* the continuous part p_t^u of the production cost, which is assumed to be concave with x.

$$p_t(x) = \mathbf{1}_{\{x>0\}}\ p_t^f + p_t^b \lceil \frac{x}{B_t} \rceil + p_t^u(x)$$

We also assume that B_t is lower than production capacity P_t. Although natural, this assumption can be raised for stationnary (constant) capacities. In this case, if $B \geq P$, one knows that the problem becomes polynomial (see Florian et al. (1980)).

Figure 1 gives the shape of production costs for the classical CLSP and CLSP-SW studied in this paper. Notice that p_t appears as a function on $[0, B_t]$ duplicated with a vertical step of p_t^b every B_t units. The resulting function is not concave even if p_t^u is, but *step-wise concave*, possibly with discontinuity every B_t units. When the unit production cost p_t^u is linear, p_t is said to be *step-wise linear*. In what follows, we first restrict our attention to step-wise concave costs to derive dominant properties. Then we propose polynomial time algorithms for the case of step-wise linear production costs.

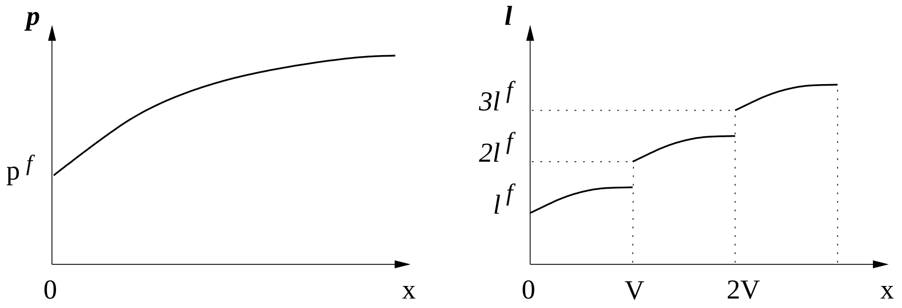

Figure 1: Shape of the costs for classical CLSP and CLSP-SW for a given period.

Storage

To a planning x, one can associate the auxiliary vector $s = (s_1, \ldots, s_T)$ where s_t denotes the inventory level at the beginning of period t. The classical material balance equation writes down as $s_{t+1} = s_t + x_t - d_t$. We consider wlog that initial and final inventory levels are zero ($s_1 = s_{T+1} = 0$). Thus, condition (ii) can be replaced by (ii') $s_t \geq 0$. There is no capacity limit on the inventory level. The cost to keep quantity s in stock from period $t-1$ to t is $h_t(s)$. Wlog we assume $h \geq 0$.

The problem is then to find a production schedule $(x_1, \ldots, x_T)$ minimizing :

$$\sum_{t=1}^{T} p_t(x_t) + h_t(s_t)$$

respecting the production capacity :

$$0 \leq x_t \leq P_t \quad \forall t = 1, \ldots, T$$

and the classical inventory flow conservation :

$$\begin{aligned} s_{t+1} &= s_t + x_t - d_t \quad && \forall t = 1, \ldots, T \\ s_1 &= s_{T+1} = 0 && \\ s_t &\geq 0 && \forall t = 1, \ldots, T \end{aligned}$$

We will now introduce some dominance properties for the purposes of developing polynomial time algorithms under various assumptions.

4 Dominance properties

Dynamic programming approaches for LSP are based on decomposition properties on the inventory level in order to apply sub-optimality principle of Bellman. Indeed, if in a certain optimal production schedule the inventory level has a value of s in period t, the problem can be decomposed into two parts: Finding the best planning between 1 and t with a final inventory level s, and finding the best planning between t and T with an initial inventory level s. In this setting, *regeneration points* introduced by Manne and Veinott (1967) play a central part (see this definition in Florian and Klein (1971)). For a given planning, a period t is said to be a regeneration point if its initial inventory is zero, i.e. $s_t = 0$. Florian and Klein (1971) define a subplan as a sub-sequence of production between two consecutive regeneration points.

Definition 1 (Subplan) *Given a planning, a sub-sequence of production on periods $u, \cdots, v-1$ is a subplan, denoted $S_{(uv)}$, if $s_u = s_v = 0$, and $s_t > 0$ for all $t = u+1, \cdots, v-1$.*

Notice that at least 2 regeneration points exist in any planning, since $s_1 = s_{T+1} = 0$. Florian and Klein (1971) use the notion of subplans to decompose the time horizon into sub-sequences which start and end with zero stock. The optimum can then be computed in $\mathcal{O}(T^2)$ as a shortest path problem, given the costs of the $\mathcal{O}(T^2)$ possible subplans. To efficiently compute the optimal planning on a subplan, Florian and Klein (1971) introduce the following definition:

Definition 2 (Capacity constrained subplan) *A subplan is capacity constrained if it contains at most one period with a production neither null nor at production capacity.*

Assuming concave unit production costs, Florian and Klein (1971) prove the following characterization property:

Property 1 (Florian and Klein (1971)) *When cost functions are concave, a planning belongs to the set of extreme points of solutions if and only if it can be decomposed into capacity constrained subplans.*

This property can be seen as a generalization of the ZIO dominance property used by Wagner and Whitin (1958) to design their algorithm. Recall that ZIO (*Zero Inventory Ordering*) policies consist in ordering only when inventory level drops to zero. A similar property has also been stated by Lippman (1969) as *regeneration point property*. Notice that Florian and Klein (1971) consider no step-wise production cost, contrary to Lippman who in turn considers no production setup cost. We now demonstrate with a simple example that Property 1 does not hold when considering step-wise production costs.

Example 1 *Consider a time horizon of* 3 *periods with demands* $(1, 2, 3)$ *to satisfy. Production capacity is* 3 *and batch size is* 2 *units. Set-up costs are set to* 2 *for production, and* 5 *per batch. A unit holding cost of* 0.5 *is paid per product and per unit of time. Other costs are null.*

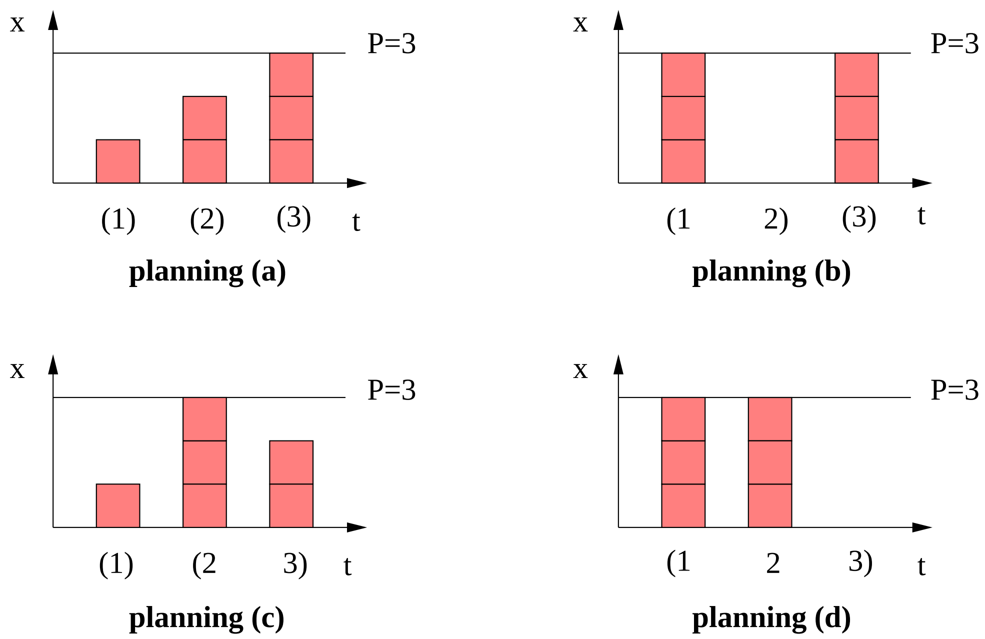

Figure 2: Best capacity constrained planning for each subplan decomposition in Example 1.

Clearly there exist 4 *possible subplan decompositions: (1)(2)(3), (1-2)(3), (1)(2-3), (1-2-3). In the first one, each period forms a subplan; in the second, the first two periods* 1 *and* 2 *constitute a subplan, etc. The best capacity constrained plannings associated to each decomposition are given in Figure 2. In fact there is only one possible capacity constrained schedule for each decomposition. Their costs are respectively: (a) 26, (b) 25, (c) 26.5, (d) 26.5. Hence the best planning respecting the capacity constrained property for the instance is (b). However the optimal planning consists of producing* 2 *units at each period, resulting in a cost of* 22. *This example shows that capacity constrained subplans are not dominant with step-wise linear production costs.*

Notice that in the example, the optimal solution has only one subplan, but with 3 periods with production not at full capacity. With step-wise production costs, Property 1 is not verified any more due to the fact that it can be preferable to produce only full batch sizes to save fixed cost per batch instead of saturating the production capacity. For CLSP-SW we introduce the following definitions on production levels:

Definition 3 (Saturated and fractional periods) *For a given planning, a period t is said to be:*

- *P-saturated if production is at full capacity ($x_t = P_t$)*
- *B-saturated if production corresponds to the maximum number of full batches, without any fractional batch ($x_t = \lfloor P_t/B_t \rfloor B_t$)*
- *Fractional, if there exists a fractional batch ($x_t \bmod B \neq 0$ and $x_t < P_t$)*
- *FBS (Fully Batch Size) finally if the period is neither P-saturated, B-saturated, null (zero production) nor fractional. This corresponds to a production of only full batches ($x_t \bmod B_t = 0$).*

Let us remark that in a fractional period an empty space remains in a batch as well as an unused production capacity. Hence, at least one more unit can be produced with no additional fixed cost. This situation should be quite rare in an optimal schedule, which is expressed formally in Property 2. First, we introduce the following definition which generalizes Definition 2:

Definition 4 (Batch Constrained Subplan) *A subplan is batch constrained if it contains at most one fractional period.*

The following property generalizes the property of Lippman (1969) which assumes only fixed transportation costs (in CLSP-SW, fixed transportation cost corresponds to the fixed cost per batch). It stands that between two consecutive regeneration points, there is at most one fractional batch production.

Property 2 (Decomposition Property) *When production costs are step-wise concave, there exists an optimal planning which can be decomposed into batch constrained subplans.*

Proof. We use an interchange argument to prove the property. Consider an optimal schedule π, containing a subplan $S_{(uv)}$ with two fractional periods t' and t'', $t' < t''$. By the definition of a subplan, the inventory levels s_t at the beginning of each period $t \in [t'+1, t'']$ is strictly positive. We will compare the cost of anticipating a part of the production of t'' at t' with the cost of delaying a part of the production of t' until t''.

Let us denote by $\Delta p_t(x)$ the marginal cost of producing one extra unit at period t in surplus of an amount x, i.e. $\Delta p_t(x) = p_t(x+1) - p_t(x)$. In a similar way we denote $\Delta h_{ab}(s)$ to be the holding cost incurred by one additional unit in stock from the end of period a till the beginning of period b. While p and h remain concave (i.e. while no new fixed cost appears or disappears), these functions are non-increasing. Now let us introduce $\Delta f(q) = \Delta p_{t'}(x_{t'}+q) + \Delta h_{t't''}(s_{t'+1}+q) - \Delta p_{t''}(x_{t''}-q-1)$. From the preceding statements, $\Delta f(q)$ is a non-increasing function of q while no setup costs appear. It represents exactly the overcost induced by anticipating one unit from t'' to t', knowing that q units have already been anticipated compared to planning π.

- If $\Delta f(0) < 0$, then it is more economical to transfer one unit from t'' to t', keeping it in stock between the two periods. Such a planning is feasible since t' cannot be P-saturated (t' is a fractional period). Its cost has decreased by at least $-\Delta f(0)$, which contradicts the optimality of π.
- If $\Delta f(0) > 0$, then it is more economical to delay one extra unit from period t' to t''. Such planning is feasible since the inventory level is strictly positive between t' and t'', and t'' cannot be a P-saturated period (t'' is considered a fractional period). The resulting schedule cost has decreased by at least $\Delta f(0)$ (possibly one setup inventory cost may disappear between t' and t''), which contradicts the optimality of π.

Thus, we must have $\Delta f(0) = 0$. Both transformations do not affect the cost of the schedule. We chose to apply the first transformation by anticipating one production unit at period t'. However, since Δf is non-increasing, $\Delta f(q) \leq 0$ for $q \geq 0$, and thus we can anticipate a second, a third, ...unit, until a setup cost appears or the planning becomes infeasible. Since the inventory level is strictly positive between t' and t'', such a setup appears when the fractional batch at period t' becomes full. Thus, we can anticipate up to the amount $q = \min\{B_{t'} - (x_{t'} \bmod B_{t'}), x_{t''}, P_{t'} - x_{t'}\}$. At this time, either t' or t'' is not any more a fractional period in the new planning. Repeating this process, we can then transform π into another optimal schedule that verifies the property. □

We now introduce another dominance property inspired by Baker et al. (1978) who states that there exists an optimal planning such that for each period t, $s_t(P_t - x_t)x_t = 0$. This property means that if the inventory level at the beginning of period t is strictly positive, the production is either null or at maximum capacity. Baker demonstrates this property with time varying production capacity and setup costs, but stationary linear production and holding costs. Bitran and Yanasse (1982) have extended this result to linear functions non-increasing over time. We generalize this dominance to step-wise linear functions (linearity is based on the quantities) non-increasing over time:

Property 3 *If the unit production cost p_t^u is linear (function of the quantity produced) and non-increasing over time, then there exists an optimal schedule which verifies for each period t*

$$s_t(P_t - x_t)(x_t \bmod B_t) = 0$$

As a consequence, if the inventory level is strictly positive, period t cannot be a fractional period. Hence a fractional period can only appear as the first period of a subplan.

Proof. We will show how to transform an arbitrary planning into a planning of lower cost which verifies the property. Let π be an optimal planning, and consider t to be the latest period such that $s_t > 0$, $x_t \bmod B \neq 0$, and $x_t < P_t$. Let us write $q = x_t \bmod B_t$ the fractional batch size.

Consider one unit in stock at the beginning of period t. This unit has been produced at the first preceeding period t' such that $x_{t'} > 0$ (we can assume that the demand is satisfied using the first-in-first-out principle, i.e. production at a period is assigned to the first unsatisfied demand). Notice that, inventory level can only decrease between t' and t.

If we transfer this production unit from t' to t, then the resulting planning remains feasible: stock levels are positive or null, and production capacity at t is necessarily respected (t was not P-saturated). When we compare costs, no additional setup cost appears, since a setup for production and for batch is already paid at t for q. The variable part can only decrease due to the linear and non-increasing cost assumptions over time. Finally, holding cost also decreases, as one unit less is stored between t' and t.

Hence, we have obtained a better planning. We can then repeat this transformation, till the planning becomes infeasible or a setup cost appears. This corresponds to delay an amount of $\min\{B_t - q, P_t - q, s_t\}$ at period t. In each case the new schedule verifies the property for period t.

Delaying the production of a part of s_t at period t may create new periods which do not satisfy Property 3. However such periods can only appear before t. We can therefore repeat this process on the new latest fractional period with positive stock, till the beginning of the planning is reached. It shows that we can transform any schedule to verify the property on a sub-sequence $1, \ldots, t$ without modifying the end of the schedule on $t+1, \ldots, T$. We will use this fact for proof of Property 4. □

Notice that we only require the unit production costs p_t^u to be non-increasing over time, without making any assumption on the fixed cost per batch. Indeed no new setup cost is involved in the transformation. As a corollary, if the fixed costs per batch are non-increasing over time and assuming that the batch sizes are stationary, we have the following property:

Property 4 *If unit production cost p_t^u is step-wise linear and non-increasing over time and $B_t = B$ is stationary, then in addition to Property 3, there exists an optimal schedule which verifies for each period t,*

$$s_t \geq B \Rightarrow x_t(P_t - x_t)(\lfloor P_t/B \rfloor B - x_t) = 0$$

The property implies that if the inventory level is higher than the batch size, then the production can only be null, P-saturated or B-saturated.

Proof. We consider an optimal planning verifying the Property 3. Again, we use a simple interchange argument. Consider the latest period t of a schedule π which does not verify the implication: $s_t \geq B$ while the production at period t is neither null, nor saturated. As inventory level is strictly positive, t must be an FBS period due to Property 3. Let t' be the last period preceeding t where production occurs ($x_{t'} > 0$). We now show that we can delay a full batch from t' to t without increasing the cost of the planning.

First notice that at least B units are produced at time t'. Indeed, if inventory level $s_{t'}$ is strictly positive, Property 3 again imposes that at least a full batch size is produced. Otherwise $s_{t'} = 0$ and we necessarily have $x_{t'} = d_{t'} + \cdots + d_t + s_t \geq s_t \geq B$.

We can therefore delay B units from t' to t maintaining the feasibility of the schedule: inventory levels between t' and t are greater than s_t and hence greater than B, and an additional batch can always be produced in an FBS period without violating production capacity constraint. This interchange can only decrease the cost of the schedule, due to the linear and non-increasing over time assumptions: it is always more economical to delay the production of a full batch size to a period whose setup production cost is already paid.

Notice that since exactly a quantity B has been delayed, period t remains an FBS period, and hence still fulfills Property 3. It may be a different matter for period t' if it was P-saturated in the planning. However, in this case we can modify the new schedule only on sub-sequence $1, \ldots, t'$ such that it verifies Property 3 (cf the proof of this property).

Repeating the transformation leads to either a B-saturated period at t, or to a drop in the inventory level s_t below B, verifying the property at t. We then iterate the same process on previous periods till the beginning of the schedule is reached. □

Sub-intervals

In order to efficiently detect FBS periods in a subplan, we introduce the notion of ***sub-interval***. Let us assume that the batch size is stationary, $B_t = B$. As a result of Property 4, the periods with an inventory level lower than B play a particular role. This role is analogous to that of regeneration points when no fixed costs per batch are involved. Indeed only 3 production possibilities exist for other periods with an inventory level greater or equal than B, namely 0, P_t and $\lfloor P_t/B \rfloor B$. We define, similarly to Definition 1, a sub-interval as follows:

Definition 5 (Sub-interval) *Given a planning, a sub-sequence of production on periods $k, \cdots, f-1$ is a sub-interval, denoted $S_{[kf]}$, if s_k and s_f are strictly lower than B, and $s_t \geq B$ for all $t = k+1, \cdots, f-1$.*

Notice that a sub-interval is necessarily included in a subplan. We denote a sub-interval $S_{(uf]}$ when inventory level is null at u, i.e. u is the begining of the subplan. The same notation stands for $S_{[kv)}$ with v the end of a subplan.

We assume throughout the paper the additional assumptions of Property 4:

- $B_t = B$ is stationary.
- p_t is a step-wise linear function of the quantity produced, non-increasing over time. Fixed production cost p_t^f can take arbitrary positive values.
- h_t is a non-decreasing concave function of the quantity stored, for each period t.

Instead of making the last two assumptions, one can also say that the variable part of the production cost is based on Wagner-Whitin (WW) cost structure. For WW cost structure, producing and storing one unit in period t costs more than producing it later. This assumption is often referred to as *the absence of speculative motive for early production*. Under these assumptions, we have the following corollary of Property 4, underlying the similarity between subplan and sub-interval:

Corollary 1 (Fractional & FBS periods) *Fractional periods can only take place at the beginning of a subplan, while FBS periods can only take place at the beginning of a sub-interval.*

In next section, under these assumptions, we present a polynomial time algorithm to find an optimal planning for the uncapacitated case. Then, assuming a stationary capacity $P_t = P$, we propose two other polynomial time algorithms in Section 6.

5 Polynomial algorithm for uncapacitated production

In this section we consider an uncapacitated production, $P_t = +\infty$. With step-wise linear production costs, the best ZIO policy found by Wagner and Whitin (1958) is no more necessarily optimal. We illustrate in the following example the fact that ZIO policies are no more dominant with step-wise cost structure.

Example 2 *Consider* 3 *periods with demands* $(4, 10, 7)$ *to meet (see Figure 3). The batch size is* 3 *units and generates a fixed cost of* $p^b = 4$*. Production set-up cost is* $p^f = 2$*, and holding cost is linear, with a unit cost of* $h = 0.5$*. All other cost parameters are null. The best ZIO planning (computed using Wagner-Whitin's algorithm) is* $(4, 17, 0)$*, with a cost of* 39.5*. However the optimal cost (computed using a MILP formulation with ILOG Cplex Solver) is* 35.5*, realized by the planning* $(6, 9, 6)$*.*

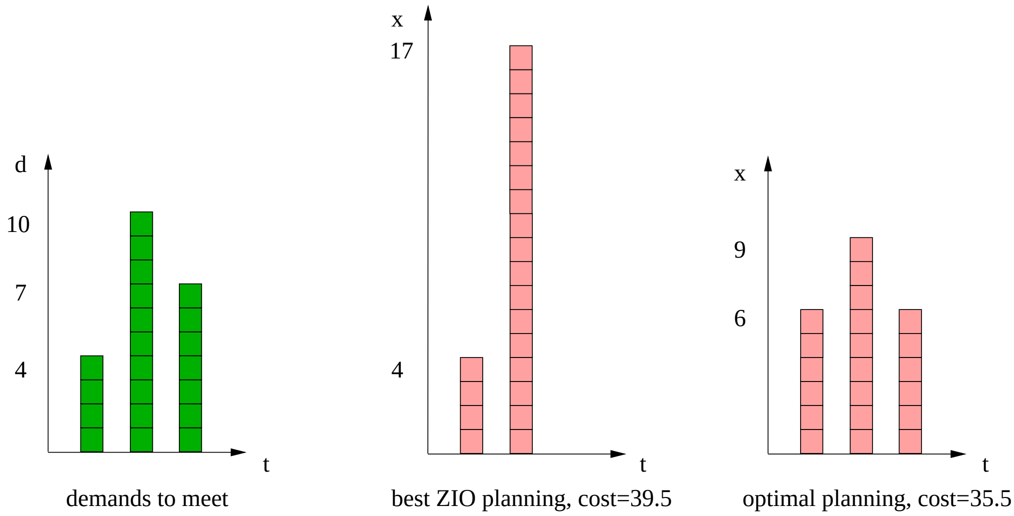

Figure 3: Non-optimality of ZIO policies in Example 2.

The search for an optimal schedule will be based on the decomposition of the time horizon into batch constrained subplans. We will use the notion of sub-interval introduced in Section 4 in order to compute efficiently such an optimal schedule. We first rewrite the different properties presented in Section 4 for the uncapacitated case.

5.1 Structure of an optimal solution

With infinite production capacity, Properties 3 and 4 can be rewritten simply as follow:

Property 5 *If production is uncapacitated, then there exists an optimal schedule, such that, for any* t

$$s_t(x_t \bmod B) = 0 \text{ and } s_t \geq B \Rightarrow x_t = 0$$

Indeed, saturated periods are not possible any more with $P_t = +\infty$. Hence we have a quite simple situation, where production can only occur at the beginning of a sub-interval, i.e. when inventory level is lower than B. This property generalizes the dominance of ZIO policies for concave cost functions, for which production can only occur at the beginning of a subplan, cf Figure 4. Notice that regeneration points still have a particular role, since they are the only periods when a fractional production can happen. As a consequence, we can compute easily the inventory level at the beginning of each sub-interval:

Property 6 *The inventory level at the beginning of a sub-interval* $S_{[kf]}$*, included in a subplan* $S_{(uv)}$*, is given by*

$$\widehat{s}_k^v = \begin{cases} 0 & \text{if } k = u \text{ (k is the beginning of the subplan)} \\ (\sum_{t=k}^{v-1} d_t) \bmod B & \text{otherwise} \end{cases}$$

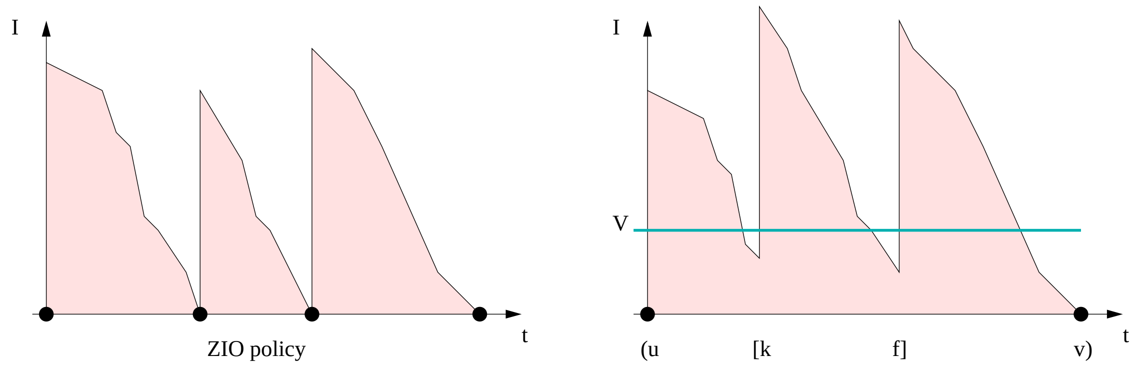

Figure 4: Inventory levels in an optimal solution.
Comparison of inventory level evolution of an optimal policy with concave cost functions (left) and with step-wise linear cost function (right). Regeneration points are marked with black dots on the time axis.

As underlined by the notation, this inventory level only depends on the next regeneration point v.

Proof. By definition of a subplan $S_{(uv)}$, the inventory level at the beginning of each period, except u and v, is strictly positive. Property 5 implies that there exists an optimal planning $\widehat{\pi}$ where only FBS production can take place inside the subplan. If we write inventory flow conservation in $\mathbb{Z}_B$, we have $\widehat{s}_{t+1} = \widehat{s}_t - d_t\ [B]$ since $\widehat{x}_t = 0 \bmod B$. Summing up these equations between periods k and v we immediately get $\widehat{s}_k = \widehat{s}_v + \sum_{t=k}^{v-1} d_t\ [B]$. As $\widehat{s}_v = 0$ by definition of a subplan, we know the value of $\widehat{s}_k$ modulo B. By definition of a sub-interval, $\widehat{s}_k$ must be equal to $\widehat{s}_k \bmod B$, which gives the expected formula. □

As a corollary, using again inventory flow conservation, we can deduce the amount of production at the beginning of a sub-interval:

Corollary 2 *The production at the beginning of a sub-interval $S_{[kf]}$, included in a subplan $S_{(uv)}$, is given by $\widehat{x}_k = \sum_{t=k}^{f-1} d_t + \widehat{s}_f^v - \widehat{s}_k^v$. This quantity is a multiple of B if k is not the beginning of the subplan.*

To synthetise the different dominances, we give in Figure 5 the structure of a batch constrained subplan for the uncapacitated production case. We have only sketched the different levels of production possible at each period, without giving a complete list. We develop in the next section

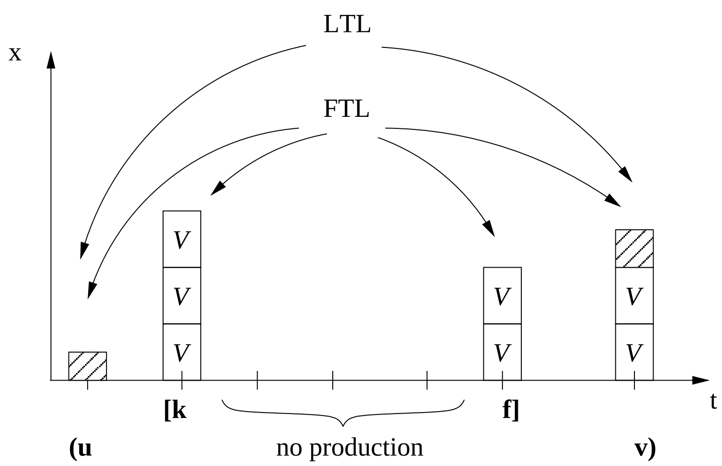

Figure 5: Structure of a batch constrained subplan for $P = +\infty$.
In the figure, (uv) denotes a subplan and $[kf]$ a sub-interval. We have represented the different types of production that are possible at each period.

a dynamic programming algorithm to find an optimal schedule in time complexity $\mathcal{O}(T^3)$, based on previous dominance properties.

5.2 Dynamic Programming Algorithm

The principle of the algorithm, quite classically, is to compute the minimum cost $Z_R^*(u,v)$ associated to each possible batch constrained subplan $S_{(u,v)}$. Using the decomposition property 2, the cost of an optimal schedule can then be computed in $\mathcal{O}(T^2)$ by a shortest-path like dynamic programming algorithm. The computation of the optimum cost of a subplan relies on the notion of sub-interval. Our approach takes advantage of the dominance properties proven in the previous section to focus our search on the periods having an entering inventory level less than the batch size B. The reason is that these periods are the only ones which can have a positive production quantity (Property 5).

Consider that u and v are two consecutive regeneration points, and let $S_{[kf]}$ be a sub-interval inside this subplan. As stated by Property 6 and Corollary 2, the production planning in the sub-interval $S_{[kf]}$ is entirely known for a certain optimal planning, since it depends only on v. Let us denote by $\mathcal{C}^v[k,f]$ the cost of this optimal planning on the sub-interval $S_{[kf]}$, assuming v as the next regeneration point. This cost includes production at period k plus inventory holding till the beginning of period f. The way to compute efficiently in $\mathcal{O}(v^2)$ the costs $\mathcal{C}^v[k,f]$ of all possible sub-intervals for a fixed regeneration point v will be detailed in Algorithm 2.

To compute dynamically $Z_R^*(u,v)$, we introduce $Z_s^*[k,v)$ the cost of an optimal planning between the first period k of a sub-interval and v, the next regeneration point. If we know that the sub-interval ends at period f (possibly $f=v$), since the planning is fixed by v on $S_{[kf]}$, Bellman's sub-optimality principle applies and we have $Z_s^*[k,v) = \mathcal{C}^v[k,f] + Z_s^*[f,v)$. Figure 6 gives a picture of these different notations. Hence we can compute dynamically $Z_s^*[k,v)$ based on the recursive equation:

$$Z_s^*[k,v) = \min\{\ \mathcal{C}^v[k,f] + Z_s^*[f,v) \mid k < f \leq v\ \}$$

Similarly to the computing of $\mathcal{C}^v[k,f]$, we can compute $\mathcal{C}^v(u,k]$, the cost of the first sub-interval of

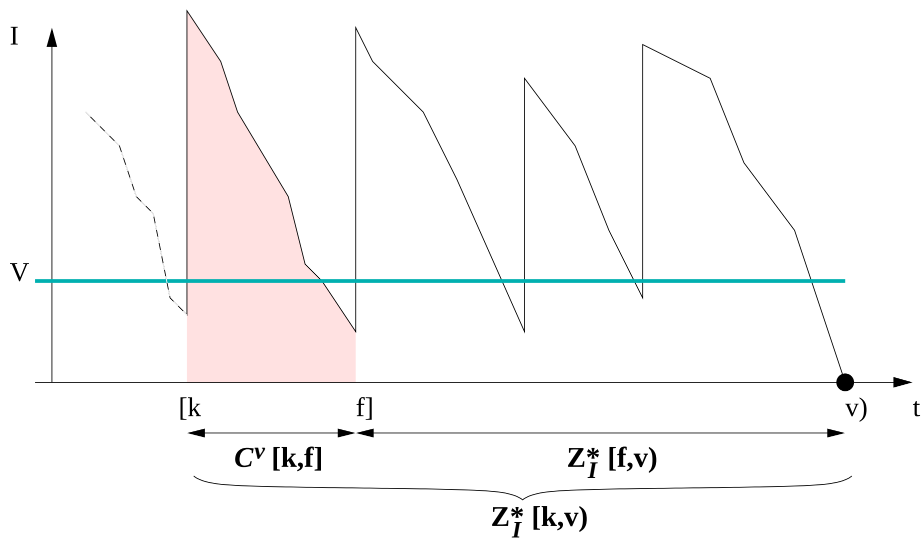

Figure 6: Dynamic computation of $Z_s^*[k,v)$.

the subplan, simply changing the initial inventory level condition (i.e. $\widehat{s}_u^v = 0$). Hence we get the equation $Z_R^*(u,v) = \min\{\ \mathcal{C}^v(u,k] + Z_s^*[k,v) \mid u < k \leq v\ \}$. The different steps of the computation of $Z_R^*(u,v)$ are summarized in pseudo-code in Algorithm 1. In the last step, the cost of an optimal schedule $OPT[u]$ on sub-sequence $u,\ldots,T+1$ is dynamically computed as a shortest path in the graph where nodes are the regeneration points and arcs are weighted with the cost $Z_R^*(u,v)$ to pass from node u to node v.

Algorithm 1 Principle steps of the algorithm for $P = +\infty$

INPUTS: An instance to schedule
OUTPUT: The optimal planning cost
  for $v = 1$ **to** $T + 1$ **do** {Computation of the array Z^*_R of $Z^*_R(u, v)$ costs}
    $C^v \leftarrow$ SubIntervals(v) {Algorithm 2 provides the array of all $\mathcal{C}^v[k, f]$ for $k < f \leq v$}
    for $k = v$ **downto** 1 **do** {Computation of the array Z^*_s of all $Z^*_s[k, v)$ costs}
      $Z^*_s[k] \leftarrow \min_{f=k+1,\dots,v}\{\ C^v[k, f] + Z^*_s[f]\ \}$
    end for
    $C^v_R \leftarrow$ FirstSubIntervals(v) {Provides the array of all $\mathcal{C}^v(u, k]$ for $u < k \leq v$}
    for $u = v$ **downto** 1 **do** {Computation of $Z^*_R[u, v]$ for $u = 1, \dots, v - 1$}
      $Z^*_R[u, v] \leftarrow \min_{k=u+1,\dots,v}\{\ C^v_R[u, k] + Z^*_s[k]\ \}$
    end for
  end for
  for $u = T$ **downto** 1 **do** {Computation of the optimal cost $OPT[u]$ on sub-sequence $u, \dots, T+1$}
    $OPT[u] \leftarrow \min_{v=u+1,\dots,T+1}\{\ Z^*_R[u, v] + OPT[v]\ \}$
  end for
  return $OPT[1]$

Determination of $\mathcal{C}^v[k, f]$.

We detail how to compute the cost $\mathcal{C}^v[k, f]$ of each sub-interval $S_{[k,f]}$ when the next regeneration point v is fixed. Algorithm 2 SubIntervals is based on the following remark: ***In a dominant planning, constituted of batch constrained subplans, the inventory level at the end of a period t in a sub-interval ($k \leq t < f$) is independent of the production periods anterior to t.*** Indeed, since production can take place only at the first period of a sub-interval, the stock level at the end of t, s_{t+1}, must be equal to $\sum_{j=t+1}^{f-1} d_j + \widehat{s}^v_f$ whatever the production periods anterior to t. This property allows us to compute efficiently the inventory holding costs of sub-intervals. Let $H(t)$ be the total holding cost between the end of period t till the beginning of period f of a planning with no production on sub-sequence $t + 1, \dots, f - 1$, ending with inventory level $\widehat{s}^v_f$. Imagine that the value of $H(t + 1)$ has already been determined. From what precedes, to compute $H(t)$ knowing $H(t + 1)$, we only need to add the cost of keeping in stock a quantity $\tilde{s}_{t+1} = \sum_{j=t+1}^{f-1} d_j + \widehat{s}^v_f$ between periods t and $t + 1$. Hence we have for $t \leq f - 1$:

$$H(t) = h_t(\tilde{s}_{t+1}) + H(t + 1), \text{ with } \tilde{s}_t = \sum_{j=t}^{f-1} d_j + \widehat{s}^v_f = \tilde{s}_{t+1} + d_t$$

with the initial conditions $H(f) = 0$ and $\tilde{s}_f = \widehat{s}^v_f$. Memorizing the inventory level $\tilde{s}_t$, the computation of $H(t)$ can be done in time $\mathcal{O}(1)$. We assume that arithmetic operations to determine $\bar{p}(x)$ or $h(x)$ for a quantity x can be done in constant time. The cost $\mathcal{C}^v[k, f]$ of the sub-interval can then be decomposed into the production cost $\bar{p}_k(\widehat{x}_k)$ at period k, plus the holding costs $H(k)$. Since $\widehat{x}_k = \widehat{x}_{k+1} + d_k + \widehat{s}^v_{k+1} - \widehat{s}^v_k$ and $\widehat{s}^v_k = (\widehat{s}^v_{k+1} + d_k) \bmod B$, memorizing these values from the previous step allows again to compute cost $\mathcal{C}^v[k, f]$ in constant time knowing the value of $H(k)$. Figure 7 illustrates the computation of $\mathcal{C}^v[k, f]$. Algorithm FirstSubInterval for computing the cost of all first intervals of the potential subplans (i.e. all $S_{(u,k]}$ for $u < k \leq v$) is similar, except that inventory level $\widehat{s}^v_u$ is null by definition, which changes only the expression of $\widehat{x}_u$.

Complexity Analysis.

Finally we provide a complexity study in time and space of each step of Algorithms 1 and 2. For a given v, the time and space complexities to determine the cost of all sub-intervals $\mathcal{C}^v[k, f]$ and $\mathcal{C}^v(u, k]$ are in $\mathcal{O}(v^2)$. Indeed, by maintaining the different values $\widehat{s}^v, \widehat{x}, \tilde{s}$ of the schedule, the determination of $H(k)$ and $\mathcal{C}^v[k, f]$ only requires $\mathcal{O}(1)$ arithmetic computations.

The quantities $Z^*_s[k, v)$ and $Z^*_R(u, v)$ are then dynamically computed in time $\mathcal{O}(v^2)$ and space $\mathcal{O}(v)$ for all $k, u < v$: for a given k or u, the computation consists in the search for next sub-interval period f among $k + 1, \dots, v$ minimizing the cost. Hence the overall computing of $Z^*_R(u, v)$ for all potential subplans can be done in time $\mathcal{O}(T^3)$ and in space $\mathcal{O}(T^2)$. The last step being clearly negligible in comparison, in total the algorithm complexity is in $\mathcal{O}(T^3)$ time and $\mathcal{O}(T^2)$ space.

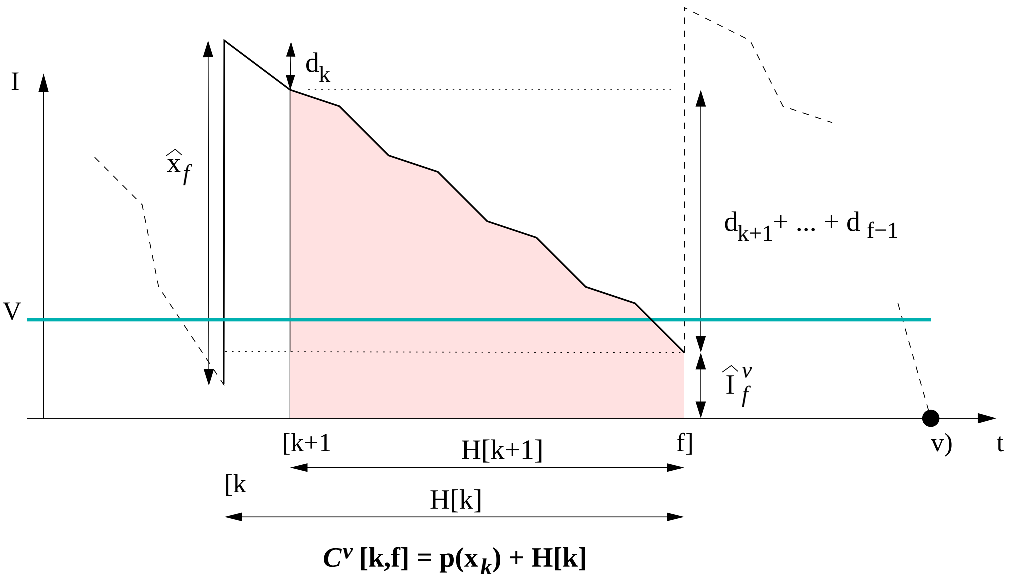

Figure 7: Illustration of $\mathcal{C}^v[k, f]$ computing.

Algorithm 2 SubIntervals

INPUTS: v, the next regeneration point
OUTPUT: C, a 2 dimensional array containing costs $\mathcal{C}^v[k, f]$ for each sub-interval $S_{[k,f]}$

```
Initialize ŝ^v_f ← 0
for f = v downto 1 do
  Initialize H[f] ← 0, s̃ ← ŝ^v_f , x̂_k, ŝ^v_k {s̃ is the current inventory level}
  for k = f − 1 downto 1 do
    H[k] ← h_k(s̃) + H[k + 1] and C[k, f] ← p̄_k(x̂_k) + H[k]
    Update s̃ ← s̃ + d_k, x̂_k, ŝ^v_k
  end for
  Update ŝ^v_f ← (ŝ^v_f + d_f) mod B
end for
return C
```

6 Polynomial time algorithms for constant capacitated production case

We have proposed in Section 5 a polynomial time algorithm in $\mathcal{O}(T^3)$ for the uncapacitated production case. We consider in this section a constant production capacity ($P_t = P$). Compared to the uncapacitated case, difficulties arise as production periods are now possible inside a sub-interval. Figure 8 gives the production structure of a batch constrained subplan, based on Properties 3 and 4: inside a sub-interval, only saturated periods can be encountered. The entering inventory level $\widehat{s}^v_k$ and the production $\widehat{x}_k$ at the beginning of a sub-interval now depend on the number of P-saturated and B-saturated periods in the subplan. We will use the following notations:

$\rho_{[a,b]}$: number of P-saturated periods on sub-sequence $a, \ldots, b-1$ for a given schedule.

$\vartheta_{[a,b]}$: number of B-saturated periods on sub-sequence $a, \ldots, b-1$ for a given schedule.

We also denote by $Q = \lfloor P/B \rfloor B$ the production quantity of a B-saturated period. We first present a polynomial time algorithm in $\mathcal{O}(T^4)$ for the special case when the production capacity P is a multiple of the batch size B. We then propose a polynomial time algorithm in $\mathcal{O}(T^6)$ for the general case with no assumption on the value of P. Similar to the previous section, these algorithms are based on the decomposition of the planning into batch constrained subplans, decomposed in their turn into sub-intervals. The computation of the cost of sub-intervals constitutes hence the heart of our approach. We use the same idea of splitting this cost into the production cost at period k

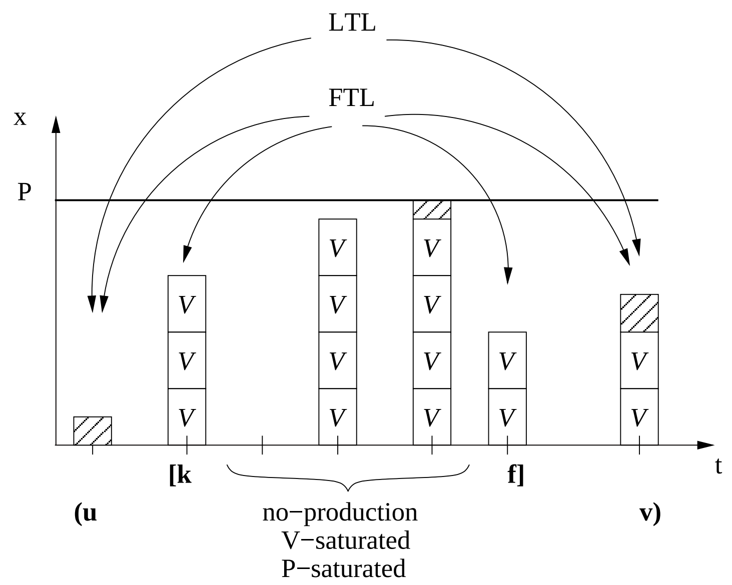

Figure 8: Structure of a batch constrained subplan with stationary production capacity P.
In the figure, (uv) denotes a subplan and $[kf]$ a sub-interval. We have represented the different types of production that are possible at each period.

plus the cost inside the sub-interval, where dominant schedules have a very simple structure, with only a constant number of potential production levels.

6.1 Case with P multiple of B

In this case P-saturated and B-saturated periods coincide and we have $Q = P$. As a consequence, all production quantities inside a subplan are multiple of the batch size. The expression of the inventory level $\widehat{s}_k^v$ remains hence unchanged compared to the uncapacitated case: s_k has to represent exactly the total demand until the next regeneration point modulo B. However the production quantity at period k depends on the number of B-saturated periods in the sub-interval. Using the inventory flow conservation on $S_{[kf]}$, we have:

$$x_k + \vartheta_{[k+1,f]}Q = \sum_{t=k}^{f-1} d_t + s_f - s_k$$

Let us remark that the number $\vartheta_{[k+1,f]}$ of B-saturated periods is (almost) fixed for given inventory levels s_f and s_k. Indeed let us denote by (q, r) the quotient and the rest of the Euclidean division of the right side of the previous equation by Q. If $x_k < Q$, we have by definition $(\vartheta_{[k+1,f]}, x_k) = (q, r)$. Otherwise we have $x_k = Q$, $r = 0$ and $\vartheta_{[k+1,f]} = q - 1$. Hence the number $\widehat{\vartheta}_{[kf]}$ in a dominant schedule is given by $\lfloor (\sum_{t=k}^{f-1} d_t + \widehat{s}_f^v - \widehat{s}_k^v)/Q \rfloor$. We can state the following proposition:

Proposition 1 *The production at the beginning of a sub-interval $S_{[kf]}$, included in a subplan $S_{(uv)}$, depends only on periods f and v, and is given by*

$$\widehat{x}_k = \begin{cases} (\sum_{t=k}^{f-1} d_t + \widehat{s}_f^v - \widehat{s}_k^v) \bmod Q & \textit{if this quantity is not null (FBS period)} \\ 0 \textit{ or } Q & \textit{otherwise (no production or B-saturated period)} \end{cases}$$

$$\textit{with} \quad \widehat{s}_k^v = \begin{cases} 0 & \textit{if } k = u \textit{ (k is the beginning of the subplan)} \\ (\sum_{t=k}^{v-1} d_t) \bmod B & \textit{otherwise} \end{cases}$$

In addition we must have $\widehat{x}_k + \widehat{s}_k^v \geq d_k$ for the planning to be feasible.

Algorithm 1 to compute the cost of subplans remains unchanged, since the entering inventory of a sub-interval is still entirely determined by the next regeneration point. On the contrary, we need a new way to compute the cost $\mathcal{C}^v[k, f]$ of sub-intervals. Notice that with the possibility of having production inside the sub-interval, the inventory level at a period t ($k < t \leq f \leq v$) now depends on the number of saturated periods between t and f. Let us denote by $\tilde{s}_t(\vartheta)$ the entering inventory level at t if exactly ϑ B-saturated periods occur in $t, \ldots, f-1$. Inventory flow conservation gives:

$$\tilde{s}_t(\vartheta) = \sum_{i=t}^{f-1} d_i + \widehat{s}_f^v - \vartheta Q$$

Let us introduce $G(t, \vartheta)$ the minimum cost of a planning between the end of period t till the beginning of period f, such that exactly ϑ B-saturated periods occur in sub-sequence $t+1, \ldots, f-1$, with no other production, and the final inventory level at f is $\widehat{s}_f^v$. In addition we require that the inventory level at each period is greater than B. Due to our dominances we clearly have

$$\mathcal{C}^v[k, f] = \min \begin{cases} \bar{p}(\widehat{x}_k) + G(k, \widehat{\vartheta}_{[k,f]}) & \text{if } \widehat{x}_k < Q \\ \bar{p}(Q) + G(k, \widehat{\vartheta}_{[k,f]} - 1) & \text{if } \widehat{x}_k = Q \end{cases}$$

For fixed f, the computation of all $\mathcal{C}^v[k, f]$ can be achieved in linear time by a backward iteration, since quantities $\widehat{x}_k$ and $\vartheta_{[kf]}$ can be clearly deduced from their values at $k+1$ in constant time. This stands, of course, if $G(t, \vartheta)$ are precomputed first for all values $t = 1, \ldots, f-1$ and $\vartheta = 0, \ldots, f-t-1$. Since only 2 production levels are possible (0 or Q), we can express recursively this quantity with the simple equation:

$$G(t, \vartheta) = h_t(\tilde{s}_{t+1}(\vartheta)) + \min \begin{cases} G(t+1, \vartheta) \text{ if } \tilde{s}_{t+1}(\vartheta) \geq d_{t+1} \\ // \textit{ no production at } t+1 \\ \bar{p}_{t+1}(Q) + G(t+1, \vartheta - 1) \text{ if } \tilde{s}_{t+1}(\vartheta - 1) \geq d_{t+1} \\ // \ t+1 \textit{ is B-saturated} \\ +\infty \text{ if } s_{t+1}(\vartheta) < B \text{ and t+1<f} \\ // \textit{ not valid} \end{cases}$$

As initial conditions we have $G(f, 0) = 0$ and $G(f, \vartheta) = +\infty$ for $\vartheta > 0$. Since $\tilde{s}_t$ can be deduced in constant time from $\tilde{s}_{t+1}$, for given f and v, one can compute G for all t and ρ with a complexity in $\mathcal{O}(T^2)$. It results that for a given regeneration point v, the computation of the cost of all the sub-intervals before v take a time complexity in $\mathcal{O}(T^3)$ for a space in $\mathcal{O}(T^2)$. The overall complexity of Algorithm 1 becomes hence in time complexity $\mathcal{O}(T^4)$ and space complexity $\mathcal{O}(T^2)$.

6.2 General Case

In this case, the periods inside a sub-interval can be either with no production, B-saturated or P-saturated. These latter periods ($P \neq Q$) are not multiple of batch size B, and thus the entering inventory level of the sub-interval will depend on them. Let us denote by $\rho_{[kv)}$ the number of P-saturated periods on the sub-sequence $k, \ldots, v-1$. The inventory flow conservation can be written now as:

$$\widehat{s}_k^v(\rho_{[kv)}) = \begin{cases} 0 & \text{if } k = u \text{ (} k \text{ is the beginning of the subplan)} \\ (\sum_{t=k}^{v-1} d_t - \rho_{[kv)} P) \bmod B & \text{otherwise} \end{cases}$$

Consider a sub-interval $S_{[kf]}$ included in a subplan $S_{(uv)}$. Since inventory level at period k is function of the number of P-saturated periods till next regeneration point, the cost $\mathcal{C}^v[k, f]$ is no more entirely determined by v. For given $\rho_{[k,v)}$ and $\rho_{[fv)}$, we can compute inventory levels s_k and s_f and thus decompose the problem into computing the best planning on the sub-interval respecting those inventory levels. Let us note $\mathcal{C}^v[k, f](\rho_{[k,f]}, \rho_{[f,v)})$ the minimum cost of a schedule on the sub-interval that admits exactly $\rho_{[k,f]}$ P-saturated periods on $S_{[kf]}$ and $\rho_{[f,v)}$ P-saturated periods on sub-sequence $f, \ldots, v-1$. In such a schedule, production at period k corresponds to the total demand in sub-interval $S_{[k,f]}$ which is not satisfied by saturated periods (neither P-saturated nor B-saturated), knowing that the entering inventory is $\widehat{s}_k^v(\rho_{[kf]} + \rho_{[f,v)})$ and the final inventory at f is $\widehat{s}_f^v(\rho_{[f,v)})$. With material balance constraint we have only 3 possibilities for x_k:

$$\widehat{x}_k = \begin{cases} P & (k \text{ is P-saturated), or} \\ Q & (k \text{ is B-saturated), or} \\ \left(\sum_{t=k}^{f-1} d_t + \widehat{s}_f^v(\rho_{[f,v)}) - \widehat{s}_k^v(\rho_{[kf]} + \rho_{[f,v)}) - \rho_{[kf]} P\right) \bmod Q \end{cases}$$

with the additionnal constraints to fullfill the demand at k, and to be a multiple of B if not the beginning of the subplan. We denote for short by $\widehat{\vartheta}_k$ the quotient in the Euclidian division of $\sum_{t=k}^{f-1} d_t + \widehat{s}_f^v(\rho_{[f,v)}) - \widehat{s}_k^v(\rho_{[kv)]}) - \rho_{[kf]}P$ by Q. Based on the 3 potential productions at k, we then have the formula:

$$\mathcal{C}^v[k,f](\rho_{[k,f]},\rho_{[f,v]}) = \bar{p}_k(\widehat{x}_k) + \min \begin{cases} F(k,\widehat{\vartheta}-1,\rho_{[kf]}-1) & \text{if } \widehat{x}_k = P \\ F(k,\widehat{\vartheta}-1,\rho_{[kf]}) & \text{if } \widehat{x}_k = Q \\ F(k,\widehat{\vartheta},\rho_{[kf]}) & \text{otherwise} \end{cases}$$

where $F(t,\vartheta,\rho)$ is the minimum cost of a planning from the end of period t till the beginning of period f, with no fractional nor FBS periods, but with exactly ϑ B-saturated periods and ρ P-saturated periods. The inventory level at each period must be greater than B and the final inventory at f must be $\widehat{s}_f^v(\rho_{[f,v)})$. Notice that the inventory level at period t is hence equal to the quantity $\tilde{s}_t(\vartheta,\rho) = \sum_{i=t}^{f-1} d_i + \widehat{s}_f^v(\rho_{[f,v)}) - \vartheta Q - \rho P$. We have the following recursive expression for F:

$$F(t,\vartheta,\rho) = h_t(s_{t+1}(\vartheta,\rho)) + \min \begin{cases} F(t+1,\vartheta,\rho) \text{ if } \tilde{s}_{t+1}(\vartheta,\rho) \geq d_{t+1} \\ \bar{p}_{t+1}(Q) + F(t+1,\vartheta-1,\rho) \text{ if } \tilde{s}_{t+1}(\vartheta-1,\rho) \geq d_{t+1} \\ \bar{p}_{t+1}(P) + F(t+1,\vartheta,\rho-1) \text{ if } \tilde{s}_{t+1}(\vartheta,\rho-1) \geq d_{t+1} \\ +\infty \ \text{ if } \tilde{s}_{t+1}(\vartheta,\rho) < B \text{ if } t+1 < f \end{cases}$$

For given f, v and $\rho[f,v)$, computing F for all possible triplets (t,ϑ,ρ) is achievable in time $\mathcal{O}(T^3)$, since $\tilde{s}_t$ can be deduced in constant time from preceeding values. The time complexity of algorithm SubIntervals becomes in $\mathcal{O}(T^5)$ in order to compute all possible costs $\mathcal{C}^v[k,f](\rho_{[k,f]},\rho_{[f,v]})$ of subintervals for a given regeneration point v. The space complexity is simply in $\mathcal{O}(T^4)$ to store each value.

The recursive formula to compute $Z_s^*[k,v)(\rho)$, which is the optimal planning cost over $k,\ldots,v-1$ with exactly ρ P-saturated periods is given by:

$$Z_s^*[k,v)(\rho) = \min_f \min_{\rho_{[kf]}} \{ \ \mathcal{C}^v[k,f](\rho_{[k,f]},\rho-\rho_{[k,f]}) + Z_s^*[f,v)(\rho-\rho_{[k,f]}) \ \}.$$

The computation of all these costs for a given v can be done in space $\mathcal{O}(T^2)$ and time $\mathcal{O}(T^4)$, if costs $\mathcal{C}^v[k,f]$ are provided. The optimal cost of the subplan $S_{(uv)}$ is then:

$$Z_R^*(u,v) = \min_\rho \{ \ Z_R^*(u,v)(\rho) \ \}$$

wich requires a complexity in $\mathcal{O}(T^2)$ time to determine these values for given u and v. In total, the algorithm has a time complexity in $\mathcal{O}(T^6)$ for a space in $\mathcal{O}(T^4)$.

7 Conclusion and perspectives

In this paper we have presented polynomial time algorithms to solve the single-item lot sizing problem with step-wise costs for the uncapacitated and constant capacitated cases. In order to find polynomially solvable exact algorithms, we made additional assumptions on the production and holding costs. The particularity of our model lies in the production cost structure which is assumed to be equal length step-wise due to the fixed costs per batch. The discontinuities in the cost function makes the lot sizing problem more difficult to solve.

We introduced several dominance properties inspired from those already existing in the literature. Based on these properties, we have proposed three polynomial time algorithms under the assumptions of constant production capacity and constant batch sizes and also linear and non-increasing over time production costs. In Table 1 we give time and space complexities of the three dynamic programming algorithms presented in this paper.

Table 1: Complexities of the polynomial time dynamic programming algorithms.

	Assumptions	Complexity (time)	Complexity (space)
P, B=constant, p_t NI and linear	$P = +\infty$	$\mathcal{O}(T^3)$	$\mathcal{O}(T^2)$
	P mod $B = 0$	$\mathcal{O}(T^4)$	$\mathcal{O}(T^2)$
	P arbitrary	$\mathcal{O}(T^6)$	$\mathcal{O}(T^5)$

Without any additional assumptions on the cost structures, the same problem with constant production capacities, constant batch sizes and concave costs has a complexity still open. It is clearly an interesting question to determine if the problem becomes $\mathcal{NP}$-hard with step-wise concave costs. The problem is also open for step-wise linear costs but with variable capacities.

References

Akbalik, A., Y. Pochet. 2007. Valid inequalities for the single-item capacitated lot sizing problem with step-wise costs. Tech. Rep. 87, CORE, Université Catholique de Louvain.

Baker, K.R., P. Dixon, M.J. Magazine, E.A. Silver. 1978. An algorithm for the dynamic lot-size problem with time-varying production capacity constraints. *Management Science* **24**(16) 1710–1720.

Bitran, G.R., H.H. Yanasse. 1982. Computational complexity of the capacitated lot size problem. *Management Science* **28**(10) 1174–1186.

Brahimi, N., S. Dauzère-Pérès, N.M. Najid, A. Nordli. 2006. Single item lot sizing problems. *European Journal of Operational Research* **168**(1) 1–16.

Chan, L.M.A., A. Muriel, Z.-J. Shen, D. Simchi-Levi. 2002. On the effectiveness of zero-inventory-ordering policies for the economic lot-sizing model with a class of piecewise linear cost structures. *Operations Research* **50**(6) 1058–1067.

Chan, L.M.A., A. Muriel, D. Simchi-Levi. 2000. Production/distribution planning problems with piecewise linear concave costs. Tech. rep., University of Toronto, University of Massachusetts and Northwestern University.

Chubanov, S., M.Y. Kovalyov, E. Pesch. 2006. An FPTAS for a single-item capacitated economic lot-sizing problem with monotone cost structure. *Mathematical Programming* **106**(3) 453–466.

Diaby, M., A. Martel. 1993. Dynamic lot sizing for multi-echelon distribution systems with purchasing and transportation price discounts. *Operations Research* **41**(1) 48–59.

Florian, M., M. Klein. 1971. Deterministic production planning with concave costs and capacity constraints. *Management Science* **18**(1) 12–20.

Florian, M., J.K. Lenstra, A.H.G. Rinnooy Kan. 1980. Deterministic production planning: Algorithms and complexity. *Management Science* **26**(7) 669–679.

Gavish, B., R.E. Johnson. 1990. A fully polynomial approximation scheme for single product scheduling in a finite capacity facility. *Operations Research* **38**(1) 70–83.

Geunes, J., P.M. Pardalos. 2003. Network optimization in supply chain management and financial engineering: an annotated bibliography. *Networks* **42**(2) 66–84.

Gnoni, M.G., R. Iavagnilio, G. Mossa, G. Mummolo, A. Di Leva. 2003. Production planning of a multi-site manufacturing system by hybrid modelling: A case study from the automotive industry. *International Journal of Production Economics* **85** 251–262.

Hoesel, C.P.M. Van, A.P.M. Wagelmans. 1996. An $\mathcal{O}(T^3)$ algorithm for the economic lot-sizing problem with constant capacities. *Management Science* **42**(1) 142–150.

Hoesel, C.P.M. Van, A.P.M. Wagelmans. 2001. Fully polynomial approximation schemes for single item capacitated economic lot-sizing problems. *Mathematics of Operations Research* **26**(2) 339–357.

Hoesel, S. Van, H.E. Romeijn, D.R. Morales, A.P.M. Wagelmans. 2005. Integrated lot sizing in serial supply chains with production capacities. *Management Science* **51**(11) 1706–1719.

Holmberg, K. 1994. Solving the staircase cost facility location problem with decomposition and piecewise linearization. *European Journal of Operational Research* **75** 41–61.

Jin, Y., A. Muriel. 2005. Single-warehouse multi-retailer inventory systems with full truckload shipments. Tech. rep., Department of Mechanical and Industrial Engineering, University of Massachusetts.

Kaminsky, P., D. Simchi-Levi. 2003. Production and distribution lot sizing in a two stage supply chain. *IIE Transactions* **35**(11) 1065–1075.

Lee, C.-Y., S. Çetinkaya, W. Jaruphongsa. 2003. A dynamic model for inventory lot sizing and outbound shipment scheduling at a third-party warehouse. *Operations Research* **51**(5) 735–747.

Li, C.-L., V.N. Hsu, W.-Q. Xiao. 2004. Dynamic lot sizing with batch ordering and truckload discounts. *Operations Research* **52**(4) 639–654.

Lippman, S.A. 1969. Optimal inventory policy with multiple set-up costs. *Management Science* **16**(1) 118–138.

Manne, A., Jr.A.F. Veinott. 1967. *Optimal arbitrary increasing time paths of demand*, chap. 11 in A.S. Manne (ed.), Investments for capacity expansion: Size, location and time phasing. MIT Press, Cambridge, Massachusetts, 178–190.

Manne, A.S. 1958. Programming of economic lot sizes. *Management Science* **4** 115–135.

Matta, R.D., T. Miller. 2004. Production and inter-facility transportation scheduling for a process industry. *European Journal of Operational Research* **158** 72–88.

Pochet, Y., L.A.Wolsey. 2006. *Production Planning by Mixed Integer Programming*. Springer.

Pochet, Y., L.A. Wolsey. 1993. Lot-sizing with constant batches: Formulation and valid inequalities. *Mathematics of Operations Research* **18**(4) 767–785.

Pochet, Y., L.A. Wolsey. 1994. Polyhedra for lot sizing with Wagner-Whitin costs. *Mathematical Programming* **67** 297–324.

Shaw, D.X., A.P.M. Wagelmans. 1998. An algorithm for single-item capacitated economic lot sizing with piecewise linear production costs and general holding costs. *Management Science* **44**(6) 831–838.

Swoveland, C. 1975. A deterministic multi-period production planning model with piecewise concave production and holding-backorder costs. *Management Science* **21**(9) 1007–1013.

Wagner, H.M., T.M. Whitin. 1958. Dynamic version of the economic lot size model. *Management Science* **5**(1) 89–96.