

Note on 2-rational fields

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Résumé. Nous déterminons le groupe de Galois de la pro-2-extension 2-ramifiée maximale d'un corps de nombres 2-rationnel.

Abstract. We compute the Galois group of the maximal 2-ramified and complexified pro-2-extension of any 2-rational number field.

Nota. This short Note is motivated by the paper "Galois 2-extensions unramified outside 2" of J. Jossey and, at this occasion, we bring into focus some classical techniques of abelian ℓ -ramification which, unfortunately, are often ignored, especially those developed by J-F. Jaulent with the ℓ -adic class field theory, and by G. Gras in his book on class field theory, and which considerably simplify proofs in such subjects; for instance, the main Theorem 2, due to J-F. Jaulent, generalizes the purpose of Jossey's paper in such a way.

1 Introduction and history

The notions of ℓ -rational field and ℓ -regular field (for a prime number ℓ and a number field K), independently introduced by A. Movahhedi and T. Nguyen Quang Do in [MN], and by G. Gras and J-F. Jaulent in [GJ], coincide as soon as K contains the maximal real subfield of the field of ℓ th roots of unity, thus especially for $\ell = 2$.

- The ℓ -regularity expresses the triviality of the regular ℓ -kernel of K (*i.e.* the kernel, in the ℓ -part of the universal group $K_2(K)$, of Hilbert symbols attached to the non-complex places not dividing ℓ).
- The ℓ -rationality traduces the pro- ℓ -freeness of the Galois group $\mathcal{G}_K := \text{Gal}(M_K/K)$ of the maximal pro- ℓ -extension ℓ -ramified ∞ -split M_K of K (*i.e.* unramified at the finite places¹ not dividing ℓ and totally split at the infinite places).

More precisely, let c_K be the number of complex places of K ; let μ_K (resp. μ_{K_1}) be the ℓ -group of roots of unity in K (resp. in the localization K_1); and let

$$V_K := \{x \in K^\times \mid x \in K_1^{\times \ell} \forall 1 \mid \ell \ \& \ v_{\mathfrak{p}}(x) \equiv 0 \pmod{\ell} \ \forall \mathfrak{p} \nmid \ell \infty\}$$

be the group of ℓ -hyperprimary elements in K^\times . Then, with these notations, from [JN, Th.1.2] or [G₃, IV.3.5, III.4.2.3], the ℓ -rationality of K may be expressed as follows:

¹According to the conventions of the ℓ -adic class field theory (*cf.* [G₃, Ja]), we never speak of ramification at infinity but of *complexification* of real places.

Theorem and definition 0. *The following conditions are equivalent:*

- (i) *The Galois group \mathcal{G}_K is a free pro- ℓ -group on $1 + c_K$ generators.*
- (ii) *The abelianization \mathcal{G}_K^{ab} of \mathcal{G}_K is a free \mathbb{Z}_ℓ -module of dimension $1 + c_K$.*
- (iii) *The field K satisfies the Leopoldt conjecture (for the prime ℓ) and the torsion submodule \mathcal{T}_K of \mathcal{G}_K^{ab} is trivial.*
- (iv) *One has the equalities: $V_K = K^{\times\ell} \quad \& \quad \text{rk}_\ell(\boldsymbol{\mu}_K) = \sum_{i|\ell} \text{rk}_\ell(\boldsymbol{\mu}_{K_i})$.*

When any of these conditions is realized, the number field K is said to be ℓ -rational.

Remark. In case $\ell = 2$, it follows from the condition (iv) above that a 2-rational field has a *single* place above 2.

The premises of the notion of ℓ -regularity go back to the works of G. Gras, mainly to his note on the K_2 of number fields [G₂, II, § 2; III, §§ 1, 2], whereas the notion of ℓ -rationality appears (in a hidden form) in the work of H. Miki [Mi] concerning the study of a sufficient condition for the Leopoldt conjecture, as well as those of K. Wingberg [W₁, W₂], concerning the same condition.

Movahhedi's thesis and the above papers [GJ, MN] characterised the going up for ℓ -rationality in any ℓ -extension in terms of ℓ -primitivity of the ramification (a definition given in [G₂, III, § 1] from the use of the Log function defined in [G₁]), a property which was unknown in the preceding approaches.

For instance, this gives immediately that if K is a ℓ -extension of \mathbb{Q} , a N.S.C. for K to be ℓ -rational is that K/\mathbb{Q} be ℓ -ramified, or that K/\mathbb{Q} be $\{p, \ell\}$ -ramified, where $p \neq \ell$ is a prime $\equiv 1 \pmod{\ell}$ such that $p \not\equiv \pm 1 \pmod{8}$ if $\ell = 2$ and $p \not\equiv 1 \pmod{\ell^2}$ if $\ell \neq 2$ (cf. [G₃, IV.3.5.1] giving Jossey's examples [Jo]).

A synthesis of these results is given in [JN] and a systematic exposition is developed in the book of G. Gras ([G₃, III, § 4, (b); IV, § 3, (b); App., § 2]).

Various generalizations of these notions have been studied by O. Sauzet and J-F. Jaulent (cf. [JS₁, JS₂]), especially in the case $\ell = 2$ which is, as usual, the most tricky; in particular, they introduce the notion of 2-birational fields.

Very recently, J. Jossey [Jo] introduced a notion of ℓ -rationality which is incompatible with the classical one (for $\ell = 2$, as soon as K contains real embeddings) and is unlucky since it does not apply to the field of rationals \mathbb{Q} .

For these reasons, to avoid any confusion, we propose to speak, in his context, of *2-superrational* fields. More precisely:

Definition 1. Let K be a number field with r_K real places and c_K complex places, M'_K the maximal 2-ramified pro-2-extension of K , and M_K the maximal subextension of M'_K totally split at the infinite places. We say that K is:

- (i) *2-superrational*, if $\mathcal{G}'_K := \text{Gal}(M'_K/K)$ is pro-2-free;
- (ii) *2-rational*, if its quotient $\mathcal{G}_K := \text{Gal}(M_K/K)$ is pro-2-free.

The purpose of the next section is to determine the structure of the Galois group \mathcal{G}'_K when the number field K is 2-rational.

Our proof relies on the functorial properties of ℓ -ramification theory.

2 Main Theorem: description of $\mathcal{G}'_K = \text{Gal}(M'_K/K)$

Our very simple result has the following statement:

Theorem 2. *Let K be a 2-rational number field having r_K real places and c_K complex places. The Galois group $\mathcal{G}'_K := \text{Gal}(M'_K/K)$ of the maximal 2-ramified pro-2-extension M'_K of K is the pro-2-free product*

$$\mathcal{G}'_K \simeq \mathbb{Z}_2^{\otimes(1+c_K)} \otimes (\mathbb{Z}/2\mathbb{Z})^{\otimes r_K}$$

of $(1 + c_K)$ copies of the procyclic group \mathbb{Z}_2 and of r_K copies of $\mathbb{Z}/2\mathbb{Z}$.

Corollary 3. *The 2-rational number fields which are 2-superrational are the totally imaginary ones.*

Proof. Consider the quadratic extension $L = K[i]$ generated by the 4th roots of unity. It is 2-ramified over K , thus thanks to the going up theorem of [GJ, MN] (cf. e.g. [JN, Th. 3.5] or [G₃, IV.3.4.3, (iii)]), it is 2-rational, then 2-superrational since it is totally imaginary. In other words, the Galois group $\mathcal{G}_L = \mathcal{G}'_L$ of the maximal 2-ramified pro-2-extension M_L of L is pro-2-free.

Since the quadratic extension L/K is 2-ramified, M_L is also the maximal 2-ramified pro-2-extension M'_K of K ; the Galois group \mathcal{G}'_K is *potentially free* since it contains the pro-2-free open subgroup \mathcal{G}_L of index 2 in \mathcal{G}'_K .

As in [Jo], the results of W. Herfort and P. Zalesskii (cf. [HZ, Th.0.2]) give the existence of a finite family $(\mathcal{F}_i)_{i=0, \dots, k}$ of free pro-2-groups on respectively d_0, \dots, d_k generators (where k is the number of conjugacy classes of subgroups of order 2 in \mathcal{G}'_K), such that:

$$\mathcal{G}'_K \simeq \mathcal{F}_0 \otimes \left(\bigotimes_{i=1}^k (\mathcal{F}_i \times \mathbb{Z}/2\mathbb{Z}) \right).$$

In particular, the abelianisation $\mathcal{G}'_K{}^{ab}$ of \mathcal{G}'_K admits the direct decomposition:

$$\mathcal{G}'_K{}^{ab} \simeq \mathbb{Z}_2^{d_0} \oplus \left(\bigoplus_{i=1}^k (\mathbb{Z}_2^{d_i} \oplus \mathbb{Z}/2\mathbb{Z}) \right) \simeq \mathbb{Z}_2^{d_0+d_1+\dots+d_k} \oplus (\mathbb{Z}/2\mathbb{Z})^k.$$

Since the 2-rational field K satisfies the Leopoldt conjecture, we get $\sum_{i=0}^k d_i = 1 + c_K$ as well as the isomorphism $\mathcal{T}'_K := \text{tor}_{\mathbb{Z}_2}(\mathcal{G}'_K{}^{ab}) \simeq (\mathbb{Z}/2\mathbb{Z})^k$. Moreover $\mathcal{T}_K := \text{tor}_{\mathbb{Z}_2}(\mathcal{G}_K{}^{ab}) = 1$, so that \mathcal{T}'_K is generated by the decomposition groups of the real places of K which are deployed, a key argument of class field theory (cf. [Ja] or [G₃, III.4.1.5]) giving $k = r_K$.

Now the pro-2-decomposition of \mathcal{G}'_K clearly shows that the minimal number of generators $d(\mathcal{G}'_K)$ and of relations $r(\mathcal{G}'_K)$, defining \mathcal{G}'_K as a pro-2-group, are:

$$d(\mathcal{G}'_K) = k + \sum_{i=0}^k d_i = r_K + 1 + c_K \text{ and } r(\mathcal{G}'_K) = \sum_{i=1}^k (1 + d_i) = d(\mathcal{G}'_K) - d_0.$$

It is well-known (cf. e.g. [G₃, App., Th.2.2, (i)]) that one has²:

$$d(\mathcal{G}'_K) - r(\mathcal{G}'_K) = \dim_{\mathbb{F}_2}(H^1(\mathcal{G}'_K, \mathbb{F}_2)) - \dim_{\mathbb{F}_2}(H^2(\mathcal{G}'_K, \mathbb{F}_2)) = 1 + c_K.$$

Thus we obtain $d_0 = 1 + c_K$, giving $d_i = 0$ for $1 \leq i \leq k$, then the expected result. \square

²This argument is equivalent to the use of the formulas of Šafarevič (cf. [Sa] or [NSW, Th.8.7.3])

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