CONVERGENCE ANALYSIS OF A COLOCATED FINITE VOLUME SCHEME FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS ON GENERAL 2 OR 3D MESHES

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Abstract. We study a colocated cell centered finite volume method for the approximation of the incompressible Navier-Stokes equations posed on a 2D or 3D finite domain. The discrete unknowns are the components of the velocity and the pressures, all of them colocated at the center of the cells of a unique mesh; hence the need for a stabilization technique, which we choose of the Brezzi-Pitkäranta type. The scheme features two essential properties: the discrete gradient is the transposed of the divergence terms and the discrete trilinear form associated to nonlinear advective terms vanishes on discrete divergence free velocity fields. As a consequence, the scheme is proved to be unconditionally stable and convergent for the Stokes problem, the steady and the transient Navier-Stokes equations. In this latter case, for a given sequence of approximate solutions computed on meshes the size of which tends to zero, we prove, up to a subsequence, the L^2-convergence of the components of the velocity, and, in the steady case, the weak L^2-convergence of the pressure. The proof relies on the study of space and time translates of approximate solutions, which allows the application of Kolmogorov's theorem. The limit of this subsequence is then shown to be a weak solution of the Navier-Stokes equations. Numerical examples are performed to obtain numerical convergence rates in both the linear and the nonlinear case.

Key words. Finite Volume, cell centered scheme, colocated discretizations, steady state and transient Navier-Stokes equations, convergence analysis.

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction.

We are interested in this paper in finding an approximation of the fields $\bar u = (\bar u^{(i)})_{i=1,\ldots,d} : \Omega \times [0,T] \to \mathbb{R}^d$, and $\bar p : \Omega \times [0,T] \to \mathbb{R}$, weak solution to the incompressible Navier-Stokes equations which write:

$$
\begin{aligned}
&\partial_t \bar u^{(i)} - \nu \Delta \bar u^{(i)} + \partial_i \bar p + \sum_{j=1}^{d} \bar u^{(j)} \partial_j \bar u^{(i)} = f^{(i)} \text{ in } \Omega \times (0,T), \quad \text{for } i = 1,\ldots,d,\\
&\mathrm{div} \bar u = \sum_{i=1}^{d} \partial_i \bar u^{(i)} = 0 \text{ in } \Omega \times (0,T).
\end{aligned} \tag{1.1}
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with a homogeneous Dirichlet boundary condition for $\bar u$ and the initial condition

$$\bar u^{(i)}(\cdot,0) = \bar u^{(i)}_{\rm ini} \text{ in } \Omega \text{ for } i = 1,\ldots,d. \tag{1.2}$$ nstoconti

In the above equations, $\bar u^{(i)}$, $i = 1,\ldots,d$ denote the components of the velocity of a fluid which flows in a domain Ω during the time $(0,T)$, $\bar p$ denotes the pressure, $\nu > 0$ stands for the viscosity of the fluid. We make the following assumptions:

(1.3) Ω is a polygonal open bounded connected subset of $\mathbb{R}^d$, $d = 2$ or 3, hypomegat

(1.4) $T > 0$ is the finite duration of the flow, hyptimet

(1.5) $\nu \in (0,+\infty)$, hypnut

(1.6) $\bar u_{\rm ini} \in L^2(\Omega)^d$, hypuini

(1.7) $f^{(i)} \in L^2(\Omega \times (0,T))$, for $i = 1,\ldots,d$. hypfgt

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We denote by $x = (x^{(i)})_{i=1,\dots,d}$ any point of Ω, by $|.|$ the Euclidean norm in $\mathbb{R}^d$, *i.e.*: $|x|^2 = \sum_{i=1}^{d}(x^{(i)})^2$ and by $\mathrm{d}x$ the d-dimensional Lebesgue measure $\mathrm{d}x = \mathrm{d}x^{(1)}\dots\mathrm{d}x^{(d)}$.

The weak sense that we consider for the Navier-Stokes equations is the following.

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DEFINITION 1.1 (Weak solution for the transient Navier-Stokes equations). *Under hypotheses (1.3)-(1.7), let the function space $E(\Omega)$ be defined by:*

$$E(\Omega) := \{\bar{v} = (\bar{v}^{(i)})_{i=1,\dots,d} \in H_0^1(\Omega)^d, \mathrm{div}\bar{v} = 0 \textit{ a.e. in } \Omega\}. \tag{1.8}$$

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Then $\bar{u}$ is called a weak solution of (1.1)-(1.2) if $\bar{u} \in L^2(0,T;E(\Omega)) \cap L^\infty(0,T;L^2(\Omega)^d)$ and:

$$\begin{cases} \forall \varphi \in L^2(0,T;E(\Omega)) \cap C_c^\infty(\Omega \times (-\infty,T))^d, \\ \quad -\displaystyle\int_0^T\int_\Omega \bar{u}(x,t)\cdot\partial_t\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \bar{u}_{\mathrm{ini}}(x)\cdot\varphi(x,0)\,\mathrm{d}x \\ \quad +\nu\displaystyle\int_0^T\int_\Omega \nabla\bar{u}(x,t):\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T b(\bar{u}(\cdot,t),\bar{u}(\cdot,t),\varphi(\cdot,t))\,\mathrm{d}t \\ \qquad = \displaystyle\int_0^T\int_\Omega f(x)\cdot\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \end{cases} \tag{1.9}$$

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where, for all $\bar{u},\bar{v} \in H_0^1(\Omega)^d$ and for a.e. $x \in \Omega$, we use the following notation:

$$\nabla\bar{u}(x):\nabla\bar{v}(x) = \sum_{i=1}^{d}\nabla\bar{u}^{(i)}(x)\cdot\nabla\bar{v}^{(i)}(x)$$

and where the trilinear form $b(.,.,.)$ is defined, for all $\bar{u},\bar{v},\bar{w} \in (H_0^1(\Omega))^d$, by

$$b(\bar{u},\bar{v},\bar{w}) = \sum_{k=1}^{d}\sum_{i=1}^{d}\int_\Omega \bar{u}^{(i)}(x)\partial_i\bar{v}^{(k)}(x)\bar{w}^{(k)}(x)\,\mathrm{d}x. \tag{1.10}$$

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REMARK 1.1. *From (1.9), we get that a weak solution u of (1.1)-(1.2) in the sense of Definition 1.1 satisfies $\partial_t\bar{u} \in L^{4/d}(0,T;E(\Omega)')$, and is therefore a weak solution in the classical sense, such that $\bar{u}(\cdot,0)$ is the orthogonal L^2-projection of $\bar{u}_{\mathrm{ini}}$ on $\{\bar{v} \in L^2(\Omega)^d, \mathrm{div}\bar{v} = 0, \mathrm{trace}(\bar{v}\cdot n_{\partial\Omega},\partial\Omega) = 0\}$ (see for example [36] or [7]).*

Numerical schemes for the Stokes equations and the Navier-Stokes equations have been extensively studied: see [23, 33, 34, 35, 25, 24] and references therein. Among different schemes, finite element schemes and finite volume schemes are frequently used for mathematical or engineering studies. An advantage of finite volume schemes is that the unknowns are approximated by piecewise constant functions: this makes it easy to take into account additional nonlinear phenomena or the coupling with algebraic or differential equations, for instance in the case of reactive flows; in particular, one can find in [33] the presentation of the classical finite volume scheme on rectangular meshes, which has been the basis of many industrial applications. However, the use of rectangular grids makes an important limitation to the type of domain which can be gridded and more recently, finite volume schemes for the Navier-Stokes equations on triangular grids have been presented: see for example [26] where the vorticity formulation is used and [6] where primal variables are used with a Chorin type projection method to ensure the divergence condition. Proofs of convergence for finite volume type schemes for the Stokes and steady-state Navier-Stokes equations are have recently been given for staggered grids [9], [26], [13], [14], [4], following the pioneering work of Nicolaides *et al.* [31], [32].

In this paper, we propose the mathematical and numerical analysis of a discretization method which uses the primitive variables, that is the velocity and the pressure, both approximated by piecewise constant functions on the cells of a 2D or 3D mesh. We emphasize that the approximate velocities and pressures are colocated, and therefore, no dual grid is needed. The only requirement on the mesh is a geometrical assumption needed for the consistency of the approximate diffusion flux (see [15] and section (2) for a precise definition of the admissible discretizations).

As far as we know, this work is a first proof of the convergence, of a finite volume scheme which is of large interest in industry. Indeed, industrial CFD codes (see e.g. [28], [1]) use colocated cell centered finite volume schemes; leaving aside implementation considerations, the principle of these schemes seems to differ from the present scheme only by the stabilization choice. The main reasons why this scheme is so popular in industry are:

- a colocated arrangement of the unknowns,
- a very cheap assembling step, (no numerical integration to perform)
- an easy coupling with other systems of equations.

The finite volume scheme studied here is based on three basic ingredients. First, a stabilization technique *à la* Brezzi-Pikäranta [8] is used to cope with the instability of colocated velocity/pressure approximation spaces. Second, the discretization of the pressure gradient in the momentum balance equation is performed to ensure, by construction, that it is the transpose of the divergence term of the continuity constraint. Finally, the contribution of the discrete nonlinear advection term to the kinetic energy balance vanishes for discrete divergence free velocity fields, as in the continuous case. These features appear to be essential in the proof of convergence.

We are then able to prove the stability of the scheme and the convergence of discrete solutions towards a solution of the continuous problem when the size of the mesh tends to zero, for the steady linear case (generalized Stokes problem), the stationary and the transient Navier-Stokes equations, in 2D and 3D. Our results are valid for general meshes, do not require any assumption on the regularity of the continuous solution nor, in the nonlinear case, any small data condition. We emphasize that the convergence of the fully discrete (time and space) approximation is proven here, using an original estimate on the time translates, which yields, combined with a classical estimate on the space translates, a sufficient relative compactness property.

An error analysis is performed in the steady linear case, under regularity assumptions on the solution. An error bound of order 0.5 with respect to the step size is obtained in the discrete H^1 norm and the L^2 norm for respectively the velocity and the pressure. Of course, this is probably not a sharp estimate, as can be seen from the numerical results shown in Section 5. Indeed, a better rate of convergence can be proved under additional assumptions on the mesh [20].

This paper is organized as follows. In section 2, we introduce the discretization tools together with some discrete functional analysis tools. Section 3 is devoted to the linear steady problem (Stokes problem), for which the finite volume scheme is given and convergence analysis and error estimates are detailed. The complete finite volume scheme for the nonlinear case is presented in section 4, in both the steady and transient cases. We then develop the analysis of its convergence to a weak solution of the continuous problem. We give some numerical results in section 5, and finally conclude with some remarks on open problems (section 6).

2. Spatial discretization and discrete functional analysis.

2.1. Admissible discretization of Ω. We first recall the notion of admissible discretization for a finite volume method, which is given in [15].

DEFINITION 2.1 (Admissible discretization, steady case). *Let Ω be an open bounded polygonal (polyhedral if $d = 3$) subset of $\mathbb{R}^d$, and $\partial\Omega = \overline{\Omega}\setminus\Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

- *$\mathcal{M}$ is a finite family of non empty open polygonal convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}_K > 0$ denote the area of K.*
- *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^d$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is a non empty open subset of E. We then denote by $\mathrm{m}_\sigma > 0$ the (d-1)-dimensional measure of σ. We assume that,for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. It then results from the previous hypotheses that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$.*
- *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$. The coordinates of x_K are denoted by $x_K^{(i)}$, $i = 1, \ldots, d$. The family $\mathcal{P}$ is such that, for all $K \in \mathcal{M}$, $x_K \in K$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K, L) \in \mathcal{M}^2$ with $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) going through x_K and x_L is orthogonal to $K|L$. For all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{E}_K$, let z_σ be the orthogonal projection of x_K on σ. We suppose that $z_\sigma \in \sigma$.*

An example of two neighbouring control volumes K and L of $\mathcal{M}$ is depicted in Figure 2.1.

The following notations are used. The size of the discretization is defined by:

$$\mathrm{size}(\mathcal{D}) = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}.$$

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{M}$, we denote by $\mathcal{N}_K$ the subset of $\mathcal{M}$ of the neighbouring control volumes. For all $K \in \mathcal{M}$ and $L \in \mathcal{N}_K$, we set $\mathbf{n}_{KL} = \mathbf{n}_{K,K|L}$, we denote by $d_{K|L}$ the Euclidean distance between x_K and x_L.

We shall measure the regularity of the mesh through the function $\mathrm{regul}(\mathcal{D})$ defined by

$$\begin{aligned} \mathrm{regul}(\mathcal{D}) = \inf\ & \Big\{\frac{d_{K,\sigma}}{\mathrm{diam}(K)},\ K \in \mathcal{M},\ \sigma \in \mathcal{E}_K\Big\} \\ & \cup \Big\{\frac{d_{K,K|L}}{d_{K|L}},\ K \in \mathcal{M},\ L \in \mathcal{N}_K\Big\} \cup \Big\{\frac{1}{\mathrm{card}(\mathcal{E}_K)},\ K \in \mathcal{M}\Big\}. \end{aligned} \tag{2.1}$$

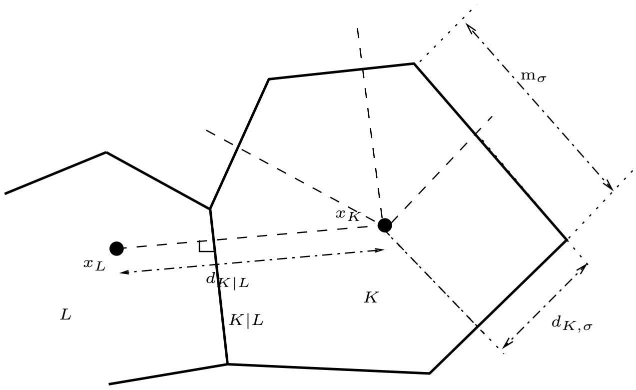

FIG. 2.1. *Notations for an admissible mesh*

2.2. Discrete functional properties. Finite volume schemes are discrete balance equations with an adequate approximation of the fluxes, see e.g. [15]. Recent works dealing with cell centered finite volume methods for elliptic problems [21], [16], [14] introduce an equivalent variational formulation in adequate functional spaces. Here we shall follow this latter path, also introducing discrete analogues of the continuous Laplace, gradient, divergence and transport operators, each of them featuring properties similar to their continuous counterparts.

DEFINITION 2.2. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}_*$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be an admissible finite volume discretization of Ω in the sense of definition 2.1. We denote by $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $w \in H_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by w_K the constant value of w in K. The space $H_{\mathcal{D}}(\Omega)$ is embedded with the following Euclidean structure: For $(v, w) \in (H_{\mathcal{D}}(\Omega))^2$, we first define the following inner product (corresponding to Neumann boundary conditions)*

$$\langle v, w\rangle_{\mathcal{D}} = \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \frac{m_{K|L}}{d_{K|L}} (v_L - v_K)(w_L - w_K). \tag{2.2}$$

We then define another inner product (corresponding to Dirichlet boundary conditions)

$$[v, w]_{\mathcal{D}} = \langle v, w\rangle_{\mathcal{D}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \frac{m_\sigma}{d_{K,\sigma}} v_K w_K. \tag{2.3}$$

Next, we define a seminorm and a norm in $H_{\mathcal{D}}(\Omega)$ (thanks to the discrete Poincaré inequality (2.4) given below) by

$$|w|_{\mathcal{D}} = (\langle w, w\rangle_{\mathcal{D}})^{1/2}, \qquad \|w\|_{\mathcal{D}} = ([w, w]_{\mathcal{D}})^{1/2}.$$

We define the interpolation operator $P_{\mathcal{D}} : C(\Omega) \to H_{\mathcal{D}}(\Omega)$ by $(P_{\mathcal{D}}\varphi)_K = \varphi(x_K)$, for all $K \in \mathcal{M}$, for all $\varphi \in C(\Omega)$.

Similarly, for $u = (u^{(i)})_{i=1,\dots,d} \in (H_{\mathcal{D}}(\Omega))^d$, $v = (v^{(i)})_{i=1,\dots,d} \in (H_{\mathcal{D}}(\Omega))^d$ and $w = (w^{(i)})_{i=1,\dots,d} \in (H_{\mathcal{D}}(\Omega))^d$, we define:

$$\|u\|_{\mathcal{D}} = \left(\sum_{i=1}^{d} [u^{(i)}, u^{(i)}]_{\mathcal{D}}\right)^{1/2}, \qquad [v, w]_{\mathcal{D}} = \sum_{i=1}^{d} [v^{(i)}, w^{(i)}]_{\mathcal{D}},$$

and $P_{\mathcal{D}} \;:\; C(\Omega)^d \to H_{\mathcal{D}}(\Omega)^d$ *by* $(P_{\mathcal{D}}\varphi)_K = \varphi(x_K)$*, for all* $K \in \mathcal{M}$*, for all* $\varphi \in C(\Omega)^d$*.*

The discrete Poincaré inequalities (see [15]) write:

$$\|w\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\|w\|_{\mathcal{D}},\ \forall w \in H_{\mathcal{D}}(\Omega), \tag{2.4}$$

and there exists $C_\Omega > 0$, only depending on Ω, such that

$$\|w\|^2_{L^2(\Omega)} \le C_\Omega |w|^2_{\mathcal{D}},\ \forall w \in H_{\mathcal{D}}(\Omega) \text{ with } \int_\Omega w(x)\mathrm{d}x = 0. \tag{2.5}$$

We define a discrete divergence operator $\mathrm{div}_{\mathcal{D}} \;: (H_{\mathcal{D}}(\Omega))^d \to H_{\mathcal{D}}(\Omega)$, by:

$$\mathrm{div}_{\mathcal{D}}(u)(x) = \frac{1}{\mathrm{m}_K} \sum_{L\in\mathcal{N}_K} A_{KL}\cdot(u_K + u_L), \quad \text{for a.e. } x \in K, \forall K \in \mathcal{M}, \tag{2.6}$$

with

$$A_{KL} = \frac{\mathrm{m}_{K|L}}{d_{K|L}}\frac{x_L - x_K}{2} = \frac{1}{2}\,\mathrm{m}_{K|L}\,\mathbf{n}_{KL},\ \forall K \in \mathcal{M},\ \forall L \in \mathcal{N}_K. \tag{2.7}$$

We then set $E_{\mathcal{D}}(\Omega) = \{u \in (H_{\mathcal{D}}(\Omega))^d, \mathrm{div}_{\mathcal{D}}(u) = 0\}$.

REMARK 2.1. *Any definition of* A_{KL} *such that* $A_{KL} = \mathrm{m}_{K|L}a_{KL}\mathbf{n}_{KL}$ *with* $a_{KL} \ge 0$ *and* $a_{KL} + a_{LK} = 1$*, combined with the definition* $\mathrm{div}_{\mathcal{D}}(u)(x) = \frac{1}{\mathrm{m}_K}\sum_{L\in\mathcal{N}_K}(A_{KL}\cdot u_K - A_{LK}\cdot u_L)$*, produces the same results of convergence as those which are proven in this paper. On particular meshes, one can prove a better error estimate, choosing* $a_{KL} = d(x_L, K|L)/d_{KL}$ *(see [20]). Nevertheless, in the general framework of this paper, other choices do not improve the convergence result and the error estimate. Therefore, we set in this paper* $a_{KL} = 1/2$*, which corresponds to (2.7). The advantage of this choice is that it leads to simpler notations and shorter equations.*

The adjoint of this discrete divergence defines a discrete gradient $\nabla_{\mathcal{D}} \;: H_{\mathcal{D}}(\Omega) \to (H_{\mathcal{D}}(\Omega))^d$:

$$(\nabla_{\mathcal{D}} u)_K = \frac{1}{\mathrm{m}_K}\sum_{L\in\mathcal{N}_K} A_{KL}(u_L - u_K),\ \forall K \in \mathcal{M},\ \forall u \in H_{\mathcal{D}}(\Omega). \tag{2.8}$$

This operator $\nabla_{\mathcal{D}}$ then satisfies the following property.

PROPOSITION 2.3. *Let* $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ *be a sequence of admissible discretizations of* Ω *in the sense of Definition 2.1, such that* $\lim_{m\to\infty} \mathrm{size}(\mathcal{D}^{(m)}) = 0$*. Let us assume that there exists* $C > 0$ *and* $\alpha \in [0, 2)$ *and a sequence* $(u^{(m)})_{m\in\mathbb{N}}$ *such that* $u^{(m)} \in H_{\mathcal{D}^{(m)}}(\Omega)$ *and* $|u^{(m)}|^2_{\mathcal{D}_m} \le C\ \mathrm{size}(\mathcal{D}^{(m)})^{-\alpha}$*, for all* $m \in \mathbb{N}$*.*

Then the following property holds:

$$\lim_{m\to+\infty}\int_\Omega \Big(P_{\mathcal{D}_m}\varphi(x)\nabla_{\mathcal{D}_m}u^{(m)}(x) + u^{(m)}(x)\nabla\varphi(x)\Big)\,dx = 0,\ \forall\varphi \in C_c^\infty(\Omega), \tag{2.9}$$

and therefore:

$$\lim_{m\to+\infty}\int_\Omega \nabla_{\mathcal{D}_m}u^{(m)}(x)\cdot P_{\mathcal{D}_m}\psi(x)\mathrm{d}x = 0,\ \forall\psi \in C_c^\infty(\Omega)^d \cap E(\Omega), \tag{2.10}$$

where $E(\Omega)$ *is defined by (1.8).*

Proof. Let us assume the hypotheses of the above lemma, and let $i = 1, \dots, d$ and $\varphi \in C_c^\infty(\Omega)$ be given. Let us study, for $m \in \mathbb{N}$, the term

$$T_1^{(m)} = \int_\Omega \Big(P_{\mathcal{D}_m}\varphi(x) \nabla_{\mathcal{D}_m} u^{(m)}(x) + u^{(m)}(x) \nabla\varphi(x) \Big)\, dx.$$

From (2.7) and (2.8), we get that

$$T_1^{(m)} = \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma = K|L} (u_L^{(m)} - u_K^{(m)}) \mathrm{m}_{K|L}\ R_{KL}^{(m)},$$

where

$$R_{KL}^{(m)} = \left(\frac{1}{2}(\varphi(x_K) + \varphi(x_L)) - \frac{1}{\mathrm{m}_{K|L}} \int_{K|L} \varphi(x) \mathrm{d}\gamma(x) \right) \mathbf{n}_{KL}.$$

Thanks to the Cauchy-Schwarz inequality,

$$|T_1^{(m)}|^2 \le |u^{(m)}|_{\mathcal{D}_m}^2 \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma = K|L} \left| R_{KL}^{(m)} \right|^2 \mathrm{m}_{K|L} d_{KL}.$$

One has $\sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma = K|L} \mathrm{m}_{K|L} d_{KL} \le d\mathrm{m}(\Omega)$. Thanks to the existence of $C_\varphi > 0$ which only depends on φ such that $|R_{KL}^{(m)}| \le C_\varphi \mathrm{size}(\mathcal{D}^{(m)})$ and since $\alpha < 2$, we then get that

$$\lim_{m\to\infty} T_1^{(m)} = 0,$$

which yields (2.9). ∎

Proposition 2.4 (Discrete Rellich theorem). *Let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of definition 2.1, such that $\lim_{m\to\infty} \mathrm{size}(\mathcal{D}^{(m)}) = 0$. Let us assume that there exists $C > 0$ and a sequence $(u^{(m)})_{m\in\mathbb{N}}$ such that $u^{(m)} \in H_{\mathcal{D}^{(m)}}(\Omega)$ and $\|u^{(m)}\|_{\mathcal{D}_m} \le C$ for all $m \in \mathbb{N}$.*

Then, there exists $\bar{u} \in H_0^1(\Omega)$ and a subsequence of $(u^{(m)})_{m\in\mathbb{N}}$, again denoted $(u^{(m)})_{m\in\mathbb{N}}$, such that:

1. *the sequence $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $\bar{u}$ as $m \to +\infty$,*
2. *for all $\varphi \in C_c^\infty(\Omega)$, we have*

$$\lim_{m\to+\infty} [u^{(m)}, P_{\mathcal{D}_m}\varphi]_{\mathcal{D}_m} = \int_\Omega \nabla\bar{u}(x) \cdot \nabla\varphi(x) \mathrm{d}x, \tag{2.11}$$

3. *$\nabla_{\mathcal{D}_m} u^{(m)}$ weakly converges to $\nabla\bar{u}$ in $L^2(\Omega)^d$ as $m \to +\infty$ and (2.9) holds.*

Proof. The proof of the first two items is given in [15] (see proof of Theorem 91. pp 773–774). Since we have $|u^{(m)}|_{\mathcal{D}_m} \le \|u^{(m)}\|_{\mathcal{D}_m}$, we can apply proposition 2.3, which gives the third item. ∎

Remark 2.2. *Following [12], if we denote*

$$\mathcal{D}_{K,\sigma} = \{t x_K + (1-t) y,\ t \in (0,1),\ y \in \sigma\},\ \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}_K,$$

we may alternatively define a discrete gradient $\tilde{\nabla}_{\mathcal{D}}\ : H_{\mathcal{D}}(\Omega) \to (L^2(\Omega))^d$, by:

for all $K \in \mathcal{M}$,
$\tilde{\nabla}_{\mathcal{D}} u(x) = \frac{d}{d_{KL}}(u_L - u_K)\mathbf{n}_{KL}$, *for a.e.* $x \in \mathcal{D}_{K,K|L} \cup \mathcal{D}_{L,K|L},\ \forall L \in \mathcal{N}_K$,
$\tilde{\nabla}_{\mathcal{D}} u(x) = \dfrac{d}{d_{K,\sigma}}(0 - u_K)\mathbf{n}_{K,\sigma}$, *for a.e.* $x \in \mathcal{D}_{K,\sigma},\ \forall \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$.

A result similar to that of Proposition 2.4 holds with this definition of a discrete gradient, and in fact, it can be shown that the weak convergence of $\tilde{\nabla}_{\mathcal{D}_m} u^{(m)}$ is equivalent to the weak convergence of $\nabla_{\mathcal{D}_m} u^{(m)}$.

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3. Approximation of the linear steady problem.

3.1. The Stokes problem. We first study the following linear steady problem: find an approximation of $\bar{u}$ and $\bar{p}$, weak solution to the generalized Stokes equations, which write:

$$\begin{aligned} &\eta\bar{u} - \nu\Delta\bar{u} + \nabla\bar{p} = f \text{ in } \Omega \\ &\mathrm{div}\bar{u} = 0 \text{ in } \Omega, \end{aligned} \tag{3.1}$$ stocont

For this problem, the following assumptions are made:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^d,\ d = 2 \text{ or } 3 \tag{3.2}$$ hypomega

$$\nu \in (0, +\infty),\ \eta \in [0, +\infty), \tag{3.3}$$ hypnu

$$f \in L^2(\Omega)^d. \tag{3.4}$$ hypfg

We then consider the following weak sense for problem (3.1).

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Definition 3.1 (Weak solution for the steady Stokes equations).
Under hypotheses (3.2)-(3.4), let $E(\Omega)$ be defined by (1.8). Then $(\bar{u}, \bar{p})$ is called a weak solution of (3.1) (see e.g. [36] or [7]) if

$$\left\{\begin{array}{l} \bar{u} \in E(\Omega),\ \bar{p} \in L^2(\Omega) \text{ with } \int_\Omega \bar{p}(x)\mathrm{d}x = 0, \\ \eta \displaystyle\int_\Omega \bar{u}(x)\cdot\bar{v}(x)\mathrm{d}x + \nu \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\mathrm{d}x - \\ \displaystyle\int_\Omega \bar{p}(x)\mathrm{div}\bar{v}(x)\mathrm{d}x = \int_\Omega f(x)\cdot\bar{v}(x)\mathrm{d}x,\ \forall \bar{v} \in H_0^1(\Omega)^d. \end{array}\right. \tag{3.5}$$ stocontf

The existence and uniqueness of the weak solution of (3.1) in the sense of the above definition is a classical result (again, see e.g. [36] or [7]).

3.2. The finite volume scheme. Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. It is then natural to write an approximate problem to the Stokes problem (3.5) in the following way.

$$\left\{\begin{array}{l} u \in E_{\mathcal{D}}(\Omega),\ p \in H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0 \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} \\ \qquad - \displaystyle\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x \qquad \forall v \in H_{\mathcal{D}}(\Omega)^d \end{array}\right. \tag{3.6}$$ schvf0

As we use a colocated approximation for the velocity and the pressure fields, the scheme must be stabilized. Using a non-consistant stabilization *à la* Brezzi-Pitkäranta

[8], we then look for (u,p) such that

$$(3.7)\quad \begin{cases} (u,p)\in H_{\mathcal{D}}(\Omega)^d\times H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0 \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} \\ \qquad - \displaystyle\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x \qquad \forall v\in H_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)q(x)\mathrm{d}x = -\lambda\ \mathrm{size}(\mathcal{D})^\alpha\ \langle p,q\rangle_{\mathcal{D}} \qquad \forall q\in H_{\mathcal{D}}(\Omega) \end{cases}$$

where $\lambda>0$ and $\alpha\in(0,2)$ are adjustable parameters of the scheme which will have to be tuned in order to make a balance between accuracy and stability.

System (3.7) is equivalent to finding the family of vectors $(u_K)_{K\in\mathcal{M}}\subset\mathbb{R}^d$, and scalars $(p_K)_{K\in\mathcal{M}}\subset\mathbb{R}$ solution of the system of equations obtained by writing for each control volume K of $\mathcal{M}$:

$$(3.8)\quad \begin{cases} \eta\ \mathrm{m}_K\ u_K - \nu \displaystyle\sum_{L\in\mathcal{N}_K}\frac{\mathrm{m}_{K|L}}{d_{K|L}}(u_L-u_K) - \nu \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\frac{\mathrm{m}_\sigma}{d_{K,\sigma}}(0-u_K) \\ \qquad\qquad + \displaystyle\sum_{L\in\mathcal{N}_K} A_{KL}\ (p_L-p_K) = \int_K f(x)\mathrm{d}x \\ \displaystyle\sum_{L\in\mathcal{N}_K} A_{KL}\cdot(u_K+u_L) - \lambda\ \mathrm{size}(\mathcal{D})^\alpha \sum_{L\in\mathcal{N}_K}\frac{\mathrm{m}_{K|L}}{d_{K|L}}(p_L-p_K) = 0 \end{cases}$$

supplemented by the relation

$$(3.9)\qquad \sum_{K\in\mathcal{M}} \mathrm{m}_K\ p_K = 0$$

Defining $p_\sigma=(p_K+p_L)/2$ if $\sigma=K|L$, and $p_\sigma=p_K$ if $\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}_K$, and using the fact that $\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma\mathbf{n}_{K,\sigma}=0$, one notices that: $\sum_{L\in\mathcal{N}_K}A_{KL}\ (p_L-p_K)$ is in fact equal to $\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma p_\sigma\mathbf{n}_{K,\sigma}$, thus yielding a conservative form, which shows that (3.8) is indeed a finite volume scheme.

The existence of a solution to (3.7) will be proven below.

3.3. Study of the scheme in the linear case. We first prove a stability estimate for the velocity.

PROPOSITION 3.2 (Discrete H^1 estimate on velocities). *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 2.1. Let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$ be given. Let $(u,p)\in H_{\mathcal{D}}(\Omega)^d\times H_{\mathcal{D}}(\Omega)$ be a solution to (3.7). Then the following inequalities hold:*

$$(3.10)\qquad \nu\|u\|_{\mathcal{D}} \le \mathrm{diam}(\Omega)\|f\|_{(L^2(\Omega))^d},$$

and

$$(3.11)\qquad \nu\ \lambda\ \mathrm{size}(\mathcal{D})^\alpha\ |p|^2_{\mathcal{D}} \le \mathrm{diam}(\Omega)^2\|f\|^2_{(L^2(\Omega))^d}.$$

Proof. We apply (3.7) setting $v=u$. We get

$$\eta\int_\Omega u(x)^2 \mathrm{d}x + \nu\|u\|_{\mathcal{D}}^2 - \int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(u)(x)\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x.$$

Since $\eta \geq 0$, the second equation of (3.7) with $q=p$ and Young's inequality yield that:

$$\eta\int_\Omega u(x)^2 \mathrm{d}x + \nu\|u\|_{\mathcal{D}}^2 + \lambda\ \mathrm{size}(\mathcal{D})^\alpha\ |p|_{\mathcal{D}}^2 \leq \frac{\mathrm{diam}(\Omega)^2}{2\nu}\|f\|_{(L^2(\Omega))^d}^2 + \frac{\nu}{2\mathrm{diam}(\Omega)^2}\|u\|_{(L^2(\Omega))^d}^2.$$

Using the Poincaré inequality (2.4) gives

$$\nu\|u\|_{\mathcal{D}}^2 + \lambda\ \mathrm{size}(\mathcal{D})^\alpha\ |p|_{\mathcal{D}}^2 \leq \frac{\mathrm{diam}(\Omega)^2}{2\nu}\|f\|_{(L^2(\Omega))^d}^2 + \frac{\nu}{2}\|u\|_{\mathcal{D}}^2,$$

which leads to (3.10) and (3.11). ☐

We can now state the existence and the uniqueness of a discrete solution to (3.7).

Corollary 3.3. **[Existence and uniqueness of a solution to the finite volume scheme]** *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given. Then there exists a unique solution to (3.7).*

Proof. System (3.7) is a linear system. Assume that $f=0$. From propositions 3.2 and using (2.5), we get that $u=0$ and $p=0$. This proves that the linear system (3.7) is invertible. ☐

We then prove the following strong estimate on the pressures.

Proposition 3.4 (L^2 estimate on pressures). *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 2.1 and let $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$. *Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given. Let $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ be a solution to (3.7). Then there exists C_1, only depending on d, Ω, η, ν, λ, α and θ, and not on* $\mathrm{size}(\mathcal{D})$*, such that the following inequality holds:*

$$\|p\|_{L^2(\Omega)} \leq C_1\|f\|_{(L^2(\Omega))^d}. \tag{3.12}$$

Proof. We first apply a result by Nečas [29]: thanks to $\int_\Omega p(x)\mathrm{d}x = 0$, there exists $C_2 > 0$, which only depends on d and Ω, and $\bar{v} \in H_0^1(\Omega)^d$ such that $\mathrm{div}\bar{v}(x) = p(x)$ for a.e. $x\in\Omega$ and

$$\|\bar{v}\|_{H_0^1(\Omega)^d} \leq C_2\|p\|_{L^2(\Omega)}. \tag{3.13}$$

We then set

$$v_\sigma^{(i)} = \frac{1}{\mathrm{m}_\sigma}\int_\sigma \bar{v}^{(i)}(x)\mathrm{d}\gamma(x),\ \forall\sigma\in\mathcal{E},\ \forall i=1,\ldots,d.$$

(note that $v_\sigma^{(i)} = 0$ for all $\sigma\in\mathcal{E}_{\mathrm{ext}}$ and $i=1,\ldots,d$) and we define $v\in H_{\mathcal{D}}(\Omega)^d$ by

$$v_K^{(i)} = \frac{1}{\mathrm{m}_K}\int_K \bar{v}^{(i)}(x)\mathrm{d}x,\ \forall K\in\mathcal{M},\ \forall i=1,\ldots,d.$$

Applying the results given p 777 in [15], we get that there exists $C_3 > 0$, only depending on d and θ, such that

$$(v_K^{(i)} - v_\sigma^{(i)})^2 \le C_3 \frac{\operatorname{diam}(K)}{\mathrm{m}_\sigma} \int_K (\nabla v^{(i)}(x))^2 \mathrm{d}x, \tag{3.14}$$

and

$$\|v\|_{\mathcal{D}} \le C_3 \|\bar{v}\|_{H_0^1(\Omega)^d}. \tag{3.15}$$

We then have

$$\int_\Omega p(x) \mathrm{div}_{\mathcal{D}} v(x) \mathrm{d}x = \sum_{K\in\mathcal{M}} p_K \sum_{L\in\mathcal{N}_K} A_{KL} \cdot (v_K + v_L) = T_2 + T_3,$$

where

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{M}} p_K \sum_{L\in\mathcal{N}_K} 2A_{KL} \cdot v_{K|L} \\ &= \sum_{K\in\mathcal{M}} p_K \sum_{L\in\mathcal{N}_K} \int_{K|L} \bar{v}(x) \cdot \mathbf{n}_{KL} \mathrm{d}\gamma(x) \\ &= \int_\Omega p(x) \mathrm{div} \bar{v}(x) \mathrm{d}x = \|p\|_{L^2(\Omega)}^2, \end{aligned}$$

and

$$\begin{aligned} T_3 &= \sum_{K\in\mathcal{M}} p_K \sum_{L\in\mathcal{N}_K} \mathrm{m}_{K|L} \left(\frac{1}{2}(v_K + v_L) - v_{K|L}\right) \cdot \mathbf{n}_{KL} \\ &= \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \mathrm{m}_{K|L}(p_K - p_L) \left(\frac{1}{2}(v_K + v_L) - v_{K|L}\right) \cdot \mathbf{n}_{KL}. \end{aligned}$$

We then have, thanks to the Cauchy-Schwarz inequality

$$T_3^2 \le |p|_{\mathcal{D}}^2 \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \mathrm{m}_{K|L} d_{KL} \left(\frac{1}{2}(v_K + v_L) - v_{K|L}\right)^2.$$

Applying Inequality (3.14) and thanks to $(\frac{1}{2}(v_K + v_L) - v_{K|L})^2 \le \frac{1}{2}((v_K - v_{K|L})^2 + (v_L - v_{K|L})^2)$, we get that

$$T_3^2 \le |p|_{\mathcal{D}}^2 \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} d_{KL} C_3 \operatorname{size}(\mathcal{D}) \int_{K\cup L} \sum_{i=1}^d (\nabla v^{(i)}(x))^2 \mathrm{d}x.$$

This in turn implies the existence of $C_4 > 0$, only depending on d and θ, such that

$$T_3^2 \le C_4 \operatorname{size}(\mathcal{D})^2 |p|_{\mathcal{D}}^2 \|\bar{v}\|_{H_0^1(\Omega)^d}^2.$$

Thanks to (3.13), we then get, gathering the previous results

$$\int_\Omega p(x) \mathrm{div}_{\mathcal{D}} v(x) \mathrm{d}x \ge \|p\|_{L^2(\Omega)}^2 - C_4 \operatorname{size}(\mathcal{D}) |p|_{\mathcal{D}} C_2 \|p\|_{L^2(\Omega)}. \tag{3.16}$$

We then introduce v as a test function in (3.7). We get

$$\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \eta\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega f(x)\cdot v(x)\mathrm{d}x. \tag{3.17}$$

Applying the discrete Poincaré inequality, (3.15) and (3.16), we get the existence of C_5, only depending on d, Ω, f, η, ν, λ and θ, such that

$$\|p\|^2_{L^2(\Omega)} - C_4\,\mathrm{size}(\mathcal{D})|p|_{\mathcal{D}} C_2 \|p\|_{L^2(\Omega)} \le C_5\left(\|u\|_{\mathcal{D}} + \|f\|_{L^2(\Omega)^d}\right)\|p\|_{L^2(\Omega)}.$$

We now apply (3.10) and (3.11). Since $\mathrm{size}(\mathcal{D})^2 \le \mathrm{size}(\mathcal{D})^\alpha \mathrm{diam}(\Omega)^{2-\alpha}$, the condition $\alpha \le 2$ suffices to produce (3.12) from the above inequality, a factor $1/\lambda$ being introduced in the expression of C_1 (it is therefore not possible to let λ tend to 0 in (3.12)). □

We then have the following result, which states the convergence of the scheme (3.7).

Proposition 3.5 (Convergence in the linear case). *Under hypotheses (3.2)-(3.4), let $(\bar u,\bar p)$ be the unique weak solution of the Stokes problem (3.1) in the sense of definition 3.1. Let $\lambda\in(0,+\infty)$, $\alpha\in(0,2)$ and $\theta>0$ be given and let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 2.1 such that* $\mathrm{regul}(\mathcal{D})\ge\theta$. *Let $(u,p)\in H_{\mathcal{D}}(\Omega)^d\times H_{\mathcal{D}}(\Omega)$ be the unique solution to (3.7).*

Then u converges to $\bar u$ in $(L^2(\Omega))^d$ and p weakly converges to $\bar p$ in $L^2(\Omega)$ as $\mathrm{size}(\mathcal{D})$ *tends to* 0.

Proof. Under the hypotheses of the above proposition, let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of definition 2.1, such that $\lim_{m\to\infty}\mathrm{size}(\mathcal{D}^{(m)})=0$ and such that $\mathrm{regul}(\mathcal{D}^{(m)})\ge\theta$, for all $m\in\mathbb{N}$.
Let $(u^{(m)},p^{(m)})\in H_{\mathcal{D}^{(m)}}(\Omega)^d\times H_{\mathcal{D}^{(m)}}(\Omega)$ be given by (3.7) for all $m\in\mathbb{N}$. Let us prove the existence of a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ such that the corresponding sequence $(u^{(m)})_{m\in\mathbb{N}}$ converges in $(L^2(\Omega))^2$ to $\bar u$ and the sequence $(p^{(m)})_{m\in\mathbb{N}}$ weakly converges in $(L^2(\Omega))^2$ to $\bar p$, as $m\to\infty$. Then the proof is complete thanks to the uniqueness of $(\bar u,\bar p)$.

Using (3.10), we obtain (see [18], [15]) an estimate on the translates of $u^{(m)}$: for all $m\in\mathbb{N}$, there exists $C_6>0$, only depending on Ω, ν, f and g such that

$$\begin{array}{l}\int_\Omega (u^{(m,k)}(x+\xi)-u^{(m,k)}(x))^2\mathrm{d}x \le C_6|\xi|(|\xi|+4\mathrm{size}(\mathcal{D}^{(m)})),\\ \text{for } k=1,\ldots,d,\ \forall\xi\in\mathbb{R}^d,\end{array} \tag{3.18}$$

where $u^{(m,k)}$ denotes the k-th component of $u^{(m)}$. We may then apply Kolmogorov's theorem, and obtain the existence of a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ and of $\bar u\in H^1_0(\Omega)^2$ such that $(u^{(m)})_{m\in\mathbb{N}}$ converges to $\bar u$ in $L^2(\Omega)^2$. Thanks to proposition 3.4, we extract from this subsequence another one (still denoted $u^{(m)}$) such that $(p^{(m)})_{m\in\mathbb{N}}$ weakly converges to some function $\bar p$ in $L^2(\Omega)$. In order to conclude the proof of the convergence of the scheme, there only remains to prove that $(\bar u,\bar p)$ is the solution of (3.5), thanks to the uniqueness of this solution.

Let $\varphi\in(C_c^\infty(\Omega))^d$. Let $m\in\mathbb{N}$ such that $\mathcal{D}^{(m)}$ belongs to the above extracted subsequence and let $(u^{(m)},p^{(m)})$ be the solution to (3.7) with $\mathcal{D}=\mathcal{D}^{(m)}$. We suppose that m is large enough and thus $\mathrm{size}(\mathcal{D}^{(m)})$ is small enough to ensure for all $K\in\mathcal{M}$ such that $K\cap$ support$(\varphi)\ne\emptyset$, then $\partial K\cap\partial\Omega=\emptyset$ holds. Let us take $v=P_{\mathcal{D}^{(m)}}\varphi$ in (3.7). Applying proposition 2.4, we get

$$\lim_{n\to\infty}[u^{(m)},P_{\mathcal{D}^{(m)}}\varphi]_{\mathcal{D}^{(m)}} = \int_\Omega \nabla\bar u(x):\nabla\varphi(x)\mathrm{d}x.$$

Moreover, it is clear that

$$\lim_{n\to\infty}\int_\Omega f(x)\cdot P_{\mathcal{D}^{(m)}}\varphi(x)\mathrm{d}x = \int_\Omega f(x)\cdot\varphi(x)\mathrm{d}x,$$

and

$$\lim_{n\to\infty}\eta\int_\Omega u^{(m)}(x)\cdot P_{\mathcal{D}^{(m)}}\varphi(x)\mathrm{d}x = \eta\int_\Omega \bar{u}(x)\cdot\varphi(x)\mathrm{d}x.$$

Thanks to the weak convergence of the sequence of approximate pressures, to (3.11) and to the hypothesis $\alpha < 2$, we now apply proposition 2.3, which gives

$$\lim_{n\to\infty}\int_\Omega p^{(m)}(x)\mathrm{div}_{\mathcal{D}^{(m)}}(P_{\mathcal{D}^{(m)}}\varphi)(x)\mathrm{d}x = \int_\Omega \bar{p}(x)\mathrm{div}\varphi(x)\mathrm{d}x. \tag{3.19}$$

The last step is to prove that $\mathrm{div}(\bar{u}) = 0$ a.e. in Ω. Let $\varphi\in C_c^\infty(\Omega)$ and let $m\in\mathbb{N}$ be given. Let us take $q = P_{\mathcal{D}^{(m)}}\varphi$ in (3.7). We get $T_4^{(m)} = -T_5^{(m)}$, where

$$T_4^{(m)} = \int_\Omega \mathrm{div}_{\mathcal{D}^{(m)}}(x)(u^{(m)})P_{\mathcal{D}^{(m)}}\varphi(x)\mathrm{d}x.$$

and

$$T_5^{(m)} = \lambda\ \mathrm{size}(\mathcal{D}^{(m)})^\alpha\langle p^{(m)}, P_{\mathcal{D}^{(m)}}\varphi\rangle_{\mathcal{D}}.$$

On the one hand, the third item of proposition 2.4 produces

$$\lim_{n\to\infty} T_4^{(m)} = \sum_{i=1}^d\int_\Omega \varphi(x)\partial_i\bar{u}^{(i)}\mathrm{d}x.$$

On the other hand, using the Cauchy-Schwarz inequality, we get:

$$T_5^{(m)} \le \lambda\mathrm{size}(\mathcal{D}^{(m)})^\alpha|p^{(m)}|_{\mathcal{D}}|P_{\mathcal{D}^{(m)}}\varphi|_{\mathcal{D}}$$

Therefore, thanks to (3.11) and to the regularity of φ (that implies that $|P_{\mathcal{D}^{(m)}}\varphi|_{\mathcal{D}}$ remains bounded independently on $\mathrm{size}(\mathcal{D}^{(m)})$) we obtain $\lim_{n\to\infty} T_5^{(m)} = 0$. This in turn implies that:

$$\sum_{i=1}^d\int_\Omega \varphi(x)\partial_i\bar{u}^{(i)}(x)\mathrm{d}x = 0,\ \text{for all }\varphi\in C_c^\infty(\Omega), \tag{3.20}$$

which proves that $\bar{u}\in E(\Omega)$. □

Remark 3.1 (Strong convergence of the pressure). *Note that the proof of the strong convergence of p to $\bar{p}$ is a straightforward consequence of the error estimate stated in Proposition 3.6 below, which holds under additional regularity hypotheses.*

3.4. An error estimate. We then have the following result, which states an error estimate for the scheme (3.7).

Proposition 3.6 (Error estimate in the linear case). *Under hypotheses (3.2)-(3.4), we assume that the weak solution $(\bar{u},\bar{p})$ of the Stokes problem (3.1) in the sense of definition (3.1) is such that $(\bar{u},\bar{p})\in H^2(\Omega)^d\times H^1(\Omega)$. Let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$ be given, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition*

2.1 and let $\theta > 0$ such that $\mathrm{regul}(\mathcal{D}^{(m)}) \geq \theta$. *Let* $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ *be the solution to (3.7). Then there exists* C_7*, which only depends on* d, Ω, ν, η *and* θ *such that*

$$\|u - \bar{u}\|^2_{L^2(\Omega)} \leq C_7\, \varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u}), \tag{3.21}$$

$$\lambda\ \mathrm{size}(\mathcal{D})^{\alpha}\ |p|^2_{\mathcal{D}} \leq C_7\, \varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u}) \tag{3.22}$$

$$\|p - \bar{p}\|^2_{L^2(\Omega)} \leq C_7\, \varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u}). \tag{3.23}$$

where

$$\begin{aligned} \varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u}) = \quad & \min\left(\lambda \mathrm{size}(\mathcal{D})^{\alpha}, \tfrac{1}{\lambda}\mathrm{size}(\mathcal{D})^{2-\alpha}\right) \\ & \times \left(\|\bar{p}\|^2_{H^1(\Omega)} + \|\bar{u}\|^2_{H^2(\Omega)}\right). \end{aligned} \tag{3.24}$$

Proof. We define $(\hat{u}, \hat{p}) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ by $\hat{u} = P_{\mathcal{D}}\bar{u}$, which means $\hat{u}_K = \bar{u}(x_k)$ for all $K \in \mathcal{M}$, and $\hat{p}_K = \frac{1}{\mathrm{m}_K}\int_K \bar{p}(x)\mathrm{d}x$ for all $K \in \mathcal{M}$. Integrating the first equation of (3.1) on $K \in \mathcal{M}$ gives

$$\eta \int_K \bar{u}(x)\mathrm{d}x + \sum_{\sigma \in \mathcal{E}_K} \left(\begin{array}{l} -\nu \int_\sigma \nabla \bar{u}(x) : \mathbf{n}_{K,\sigma}\mathrm{d}\gamma(x) + \\ \int_\sigma \bar{p}(x)\mathbf{n}_{K,\sigma}\mathrm{d}\gamma(x) \end{array} \right) = \int_K f(x)\mathrm{d}x. \tag{3.25}$$

We introduce, for $K \in \mathcal{M}$, $\varepsilon^u_K = \hat{u}_K - \frac{1}{\mathrm{m}_K}\int_K \bar{u}(x)\mathrm{d}x$, and, for $L \in \mathcal{N}_K$:
$R_{K,L} = \frac{1}{d_{K|L}}(\hat{u}_L - \hat{u}_K) - \frac{1}{\mathrm{m}_{K|L}}\int_\sigma \nabla\bar{u}(x) : \mathbf{n}_{K,\sigma}\mathrm{d}\gamma(x)$,
and for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$, $R_{K,\sigma} = \frac{1}{d_{K,\sigma}}(0 - \hat{u}_K) - \frac{1}{\mathrm{m}_\sigma}\int_\sigma \nabla\bar{u}(x) : \mathbf{n}_{K,\sigma}\mathrm{d}\gamma(x)$;
moreover, we define for $L \in \mathcal{N}_K$: $\varepsilon^p_{K|L} = \frac{1}{2}(\hat{p}_K + \hat{p}_L) - \frac{1}{\mathrm{m}_{K|L}}\int_{K|L}\bar{p}(x)\mathrm{d}\gamma(x)$, and for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$, $\varepsilon^p_\sigma = \hat{p}_K - \frac{1}{\mathrm{m}_\sigma}\int_\sigma \bar{p}(x)\mathrm{d}\gamma(x)$. Using these notations and the relation $\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma \mathbf{n}_{K,\sigma} = 0$, we get from (3.25)

$$\begin{aligned} &\eta\ \mathrm{m}_K \hat{u}_K - \nu\left(\sum_{L\in\mathcal{N}_K} \frac{\mathrm{m}_{K|L}}{d_{K|L}}(\hat{u}_L - \hat{u}_K) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}}(0 - \hat{u}_K)\right) + \\ &\sum_{L\in\mathcal{N}_K} A_{KL}\ (\hat{p}_L - \hat{p}_K) = \int_K f(x)\mathrm{d}x + R_K, \end{aligned}$$

with

$$R_K = \eta\ \mathrm{m}_K \varepsilon^u_K - \nu\left(\sum_{L\in\mathcal{N}_K}\mathrm{m}_{K|L}R_{K,L} + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}}\mathrm{m}_\sigma R_{K,\sigma}\right) + \sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma\ \varepsilon^p_\sigma \mathbf{n}_{K,\sigma}.$$

We then set $\delta u = \hat{u} - u$ and $\delta p = \hat{p} - p$. We then get, substracting the first relation of the scheme (3.8) to the above equation,

$$\begin{aligned} &\eta \int_\Omega \delta u(x)v(x)\mathrm{d}x + \nu[\delta u, v]_{\mathcal{D}} - \int_\Omega \delta p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \\ &\int_\Omega R(x)v\mathrm{d}x,\ \ \forall v \in H_{\mathcal{D}}(\Omega)^d, \end{aligned} \tag{3.26}$$

and, setting $v = \delta u$ in (3.26),

$$\eta \int_\Omega \delta u(x)^2\mathrm{d}x + \nu\|\delta u\|^2_{\mathcal{D}} - \int_\Omega \delta p(x)\mathrm{div}_{\mathcal{D}}(\delta u)(x)\mathrm{d}x = \int_\Omega R(x)\delta u(x)\mathrm{d}x.$$

We now integrate the second equation of (3.1) on $K \in \mathcal{M}$. This gives

$$\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \bar{u}(x) \cdot \mathbf{n}_{K,\sigma} \mathrm{d}\gamma(x) = 0, \ \forall K \in \mathcal{M}.$$

Using $\bar{u} \in H_0^1(\Omega)$, we then obtain

$$\sum_{L \in \mathcal{N}_K} A_{KL} \cdot (\hat{u}_K + \hat{u}_L) = \sum_{L \in \mathcal{N}_K} \mathrm{m}_{K|L} \varepsilon^u_{K|L}, \ \forall K \in \mathcal{M}$$

with

$$\varepsilon^u_{K|L} = \left(\frac{1}{2}(\hat{u}_K + \hat{u}_L) - \frac{1}{\mathrm{m}_{K|L}} \int_{K|L} \bar{u}(x) \mathrm{d}\gamma(x) \right) \cdot \mathbf{n}_{KL}, \ \forall K \in \mathcal{M}, \ \forall L \in \mathcal{N}_K.$$

We then give, substracting the second relation of the scheme (3.8) to the above equation,

$$\int_\Omega \mathrm{div}_\mathcal{D}(\delta u)(x) \delta p(x) \mathrm{d}x = \lambda \ \mathrm{size}(\mathcal{D})^\alpha \ \langle p, \hat{p} - p \rangle_\mathcal{D} + T_6,$$

with

$$T_6 = \sum_{K|L \in \mathcal{E}_{\mathrm{int}}} \mathrm{m}_{K|L} \varepsilon^u_{K|L} (\delta p_K - \delta p_L),$$

Gathering the above results, we get

$$\begin{aligned} &\eta \textstyle\int_\Omega \delta u(x)^2 \mathrm{d}x + \nu \|\delta u\|^2_\mathcal{D} + \lambda \ \mathrm{size}(\mathcal{D})^\alpha \ |p|^2_\mathcal{D} = \\ &\lambda \ \mathrm{size}(\mathcal{D})^\alpha \ \langle p, \hat{p} \rangle_\mathcal{D} + \textstyle\int_\Omega R(x) \cdot \delta u(x) \mathrm{d}x + T_6. \end{aligned} \tag{3.27}$$

Let us study the terms at the right hand side of the above equation. We have, using the Young inequality,

$$\langle p, \hat{p} \rangle_\mathcal{D} \leq \frac{1}{4} |p|^2_\mathcal{D} + |\hat{p}|^2_\mathcal{D} \leq \frac{1}{4} |p|^2_\mathcal{D} + C_8 \ \|\bar{p}\|^2_{H^1(\Omega)}. \tag{3.28}$$

We then study $\int_\Omega R(x) \cdot \delta u(x) \mathrm{d}x = T_7 + T_8 + T_9$, with

$$T_7 = \eta \int_\Omega \varepsilon^u(x) \cdot \delta u(x) \mathrm{d}x,$$

$$T_8 = \nu \sum_{K \in \mathcal{M}} \left(\sum_{L \in \mathcal{N}_K} \mathrm{m}_{K|L} R_{K,L} + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}} \mathrm{m}_\sigma R_{K,\sigma} \right) \cdot \delta u_K,$$

and

$$T_9 = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \mathrm{m}_\sigma \ \varepsilon^p_\sigma \mathbf{n}_{K,\sigma} \cdot \delta u_K.$$

Thanks to interpolation results proven in [15] and to (2.4), we obtain

$$T_7 \leq C_9 \, \mathrm{size}(\mathcal{D})^2 \|\bar{u}\|^2_{H^2(\Omega)} + \frac{\nu}{4} \|\delta u\|^2_\mathcal{D}, \tag{3.29}$$

$$T_8 \le C_{10}\,\mathrm{size}(\mathcal{D})^2 \|\bar{u}\|^2_{H^2(\Omega)} + \frac{\nu}{4}\|\delta u\|^2_{\mathcal{D}}, \tag{3.30}$$

and

$$T_9 \le C_{11}\,\mathrm{size}(\mathcal{D})^2 \|\bar{p}\|^2_{H^1(\Omega)} + \frac{\nu}{4}\|\delta u\|^2_{\mathcal{D}}. \tag{3.31}$$

We then study T_6. We have $T_6 = T_{10} - T_{11}$ with

$$T_{10} = \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \mathrm{m}_{K|L}\varepsilon^u_{K|L}(\hat{p}_K - \hat{p}_L),$$

which verifies

$$T_{10} \le C_{12}\,\mathrm{size}(\mathcal{D})\left(\|\bar{p}\|^2_{H^1(\Omega)} + \|\bar{u}\|^2_{H^2(\Omega)}\right), \tag{3.32}$$

and

$$T_{11} = \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \mathrm{m}_{K|L}\varepsilon^u_{K|L}(p_K - p_L),$$

which verifies

$$T_{11} \le \frac{1}{4}\lambda\ \mathrm{size}(\mathcal{D})^\alpha\ |p|^2_{\mathcal{D}} + C_{13}\,\frac{1}{\lambda}\mathrm{size}(\mathcal{D})^{2-\alpha}\|\bar{u}\|^2_{H^2(\Omega)}. \tag{3.33}$$

Gathering equations (3.27)-(3.33) gives

$$\|\delta u\|^2_{\mathcal{D}} + \lambda\ \mathrm{size}(\mathcal{D})^\alpha\ |p|^2_{\mathcal{D}} \le C_{14}\,\varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u}),$$

where $\varepsilon(\lambda, \mathrm{size}(\mathcal{D}), \bar{p}, \bar{u})$ is defined by (3.24). This in turn yields (3.21) and (3.22). We then again follow the method used in the proof of Proposition 3.4. Using $\int_\Omega \hat{p}(x)\mathrm{d}x = 0$ and therefore $\int_\Omega \delta p(x)\mathrm{d}x = 0$, let $\bar{v} \in H^1_0(\Omega)^d$ be given such that $\mathrm{div}\bar{v}(x) = \delta p(x)$ for a.e. $x \in \Omega$ and

$$\|\bar{v}\|_{H^1_0(\Omega)^d} \le C_2\,\|\delta p\|_{L^2(\Omega)}. \tag{3.34}$$

We again set

$$v^{(i)}_\sigma = \frac{1}{\mathrm{m}_\sigma}\int_\sigma \bar{v}^{(i)}(x)\mathrm{d}\gamma(x),\ \forall \sigma \in \mathcal{E},\ \forall i = 1, \ldots, d.$$

and we define $v \in H_{\mathcal{D}}(\Omega)^d$ by

$$v^{(i)}_K = \frac{1}{\mathrm{m}_K}\int_K \bar{v}^{(i)}(x)\mathrm{d}x,\ \forall K \in \mathcal{M},\ \forall i = 1, \ldots, d.$$

The same method gives

$$\begin{aligned}\|\delta p\|^2_{L^2(\Omega)} &\le \int_\Omega \delta p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x + C_4\,\mathrm{size}(\mathcal{D})|p|_{\mathcal{D}}\|\bar{v}\|_{H^1_0(\Omega)^d} \\ &\le \int_\Omega \delta p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x + C_{15}\,\mathrm{size}(\mathcal{D})^2|p|^2_{\mathcal{D}} + \frac{1}{4}\|\delta p\|^2_{L^2(\Omega)}.\end{aligned}$$

We now use v as test function in (3.26). We get

$$\int_\Omega \delta p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \eta\int_\Omega \delta u(x)v(x)\mathrm{d}x + \nu[\delta u, v]_{\mathcal{D}} + \int_\Omega R(x)v\mathrm{d}x.$$

Gathering the two above inequalities, (3.29), (3.30), (3.31) and (3.34) produces

$$\begin{aligned}\|\delta p\|^2_{L^2(\Omega)} \leq\ & \frac{1}{2}\|\delta p\|^2_{L^2(\Omega)} + C_{16}\,\text{size}(\mathcal{D})^2\left(\|\bar{p}\|^2_{H^1(\Omega)} + \|\bar{u}\|^2_{H^2(\Omega)}\right)\\ & + C_{17}\|\delta u\|^2_{\mathcal{D}} + C_{15}\,\text{size}(\mathcal{D})^2|p|^2_{\mathcal{D}}.\end{aligned}$$

Applying (3.21) and (3.22) gives (3.23). ☐

REMARK 3.2. *In the above result, it suffices to let $\alpha = 1$ to obtain the proof of an order $1/2$ for the convergence of the scheme. We recall that this result is not sharp, and that the numerical results show a much better order of convergence.*

secfvsnlin

4. The finite volume scheme for the Navier-Stokes equations. Before handling the transient nonlinear case, we first address in the following section the steady-state case.

stnl

4.1. The steady-state case. For the following continuous equations,

$$\begin{aligned}&\eta\bar{u}^{(i)} - \nu\Delta\bar{u}^{(i)} + \partial_i\bar{p} + \sum_{j=1}^{d}\bar{u}^{(j)}\partial_j\bar{u}^{(i)} = f^{(i)} \text{ in } \Omega, \quad \text{for } i = 1,\ldots,d,\\ &\text{div}\bar{u} = \sum_{i=1}^{d}\partial_i\bar{u}^{(i)} = 0 \text{ in } \Omega.\end{aligned} \tag{4.1}$$

nstocontss

with a homogeneous Dirichlet boundary condition, we define the following weak sense.

weaksolss

DEFINITION 4.1 (Weak solution for the steady Navier-Stokes equations). *Under hypotheses (3.2)-(3.4), let $E(\Omega)$ be defined by (1.8). Then $(\bar{u},\bar{p})$ is called a weak solution of (4.1) if*

$$\begin{cases}\bar{u} \in E(\Omega),\ \bar{p} \in L^2(\Omega) \text{ with } \displaystyle\int_\Omega \bar{p}(x)\mathrm{d}x = 0,\\ \eta\displaystyle\int_\Omega \bar{u}(x)\cdot\bar{v}(x)\mathrm{d}x + \nu\int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\mathrm{d}x\\ \quad -\displaystyle\int_\Omega \bar{p}(x)\text{div}\bar{v}(x)\mathrm{d}x + b(\bar{u},\bar{u},\bar{v}) = \int_\Omega f(x)\cdot\bar{v}(x)\mathrm{d}x \qquad \forall\bar{v} \in H^1_0(\Omega)^d,\end{cases} \tag{4.2}$$

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where the trilinear form $b(.,.,.)$ is defined by (1.10).

We now give the finite volume scheme for this problem. Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. We introduce Bernoulli's pressure $p + \frac{1}{2}u^2$ instead of p, again denoted by p, and for any real value $\lambda > 0$ and $\alpha \in (0,2)$, we look for (u,p) such that

$$\begin{cases}(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0,\\ \eta\displaystyle\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} + \frac{1}{2}\int_\Omega u(x)^2\text{div}_{\mathcal{D}}(v)(x)\mathrm{d}x\\ \quad -\displaystyle\int_\Omega p(x)\text{div}_{\mathcal{D}}(v)(x)\mathrm{d}x + b_{\mathcal{D}}(u,u,v) = \int_\Omega f(x)\cdot v(x)\mathrm{d}x \quad \forall v \in H_{\mathcal{D}}(\Omega)^d\\ \displaystyle\int_\Omega \text{div}_{\mathcal{D}}(u)(x)q(x)\mathrm{d}x = -\lambda\ \text{size}(\mathcal{D})^\alpha\ \langle p,q\rangle_{\mathcal{D}} \qquad \forall q \in H_{\mathcal{D}}(\Omega)\end{cases} \tag{4.3}$$

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where, for $u, v, w \in H_{\mathcal{D}}(\Omega)$, we define the following approximation for $b(u, v, w)$

$$b_{\mathcal{D}}(u, v, w) = \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} (A_{KL} \cdot (u_K + u_L))\ ((v_L - v_K) \cdot w_K) \tag{4.4}$$

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System (4.3) is equivalent to finding the family of vectors $(u_K)_{K\in\mathcal{M}} \subset \mathbb{R}^d$, and scalars $(p_K)_{K\in\mathcal{M}} \subset \mathbb{R}$ solution of the system of equations obtained by writing for each control volume K of $\mathcal{M}$:

$$\left\{\begin{array}{l} \eta\ \mathrm{m}_K\ u_K - \nu \displaystyle\sum_{L\in\mathcal{N}_K} \frac{\mathrm{m}_{K|L}}{d_{K|L}}(u_L - u_K) - \nu \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}}(0 - u_K) \\ + \displaystyle\sum_{L\in\mathcal{N}_K} (A_{KL} \cdot (\frac{1}{2}(u_K + u_L)))\ (u_L - u_K) \\ + \displaystyle\sum_{L\in\mathcal{N}_K} A_{KL}\ (p_L - p_K) - \frac{1}{2} \sum_{L\in\mathcal{N}_K} A_{KL}\ (u_L^2 - u_K^2) = \int_K f(x)\mathrm{d}x \\ \displaystyle\sum_{L\in\mathcal{N}_K} A_{KL} \cdot (u_K + u_L) - \lambda\ \mathrm{size}(\mathcal{D})^\alpha \sum_{L\in\mathcal{N}_K} \frac{\mathrm{m}_{K|L}}{d_{K|L}}(p_L - p_K) = 0 \end{array}\right. \tag{4.5}$$

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supplemented by the relation:

$$\sum_{K\in\mathcal{M}} \mathrm{m}_K\ p_K = 0$$

Defining $\tilde{p}_K = p_K - u_K^2/2$ and $\tilde{p}_\sigma = (\tilde{p}_K + \tilde{p}_L)/2$ if $\sigma = K|L$, $\tilde{p}_\sigma = \tilde{p}_K$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, and using the fact that $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma \mathbf{n}_{K,\sigma} = 0$, one again notices that: $\sum_{L\in\mathcal{N}_K} A_{KL}\ (\tilde{p}_L - \tilde{p}_K)$ is in fact equal to $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma \tilde{p}_\sigma \mathbf{n}_{K,\sigma}$, thus yielding a conservative form for the fifth and sixth terms of the left handside of the discrete momentum equation in (4.5). Defining $u_\sigma = (u_K + u_L)/2$ if $\sigma = K|L$, $u_\sigma = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, one obtains that the nonlinear convective term $\sum_{L\in\mathcal{N}_K} (A_{KL} \cdot (\frac{1}{2}(u_K + u_L)))\ (u_L - u_K)$ is equal to $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma(\mathbf{n}_{K,\sigma} \cdot u_\sigma)u_\sigma - \mathrm{m}_K u_K (\mathrm{div}_{\mathcal{D}} u)_K$; one may note that $(\mathrm{div}_{\mathcal{D}} u)_K = \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma \mathbf{n}_{K,\sigma} \cdot u_\sigma$. Hence the nonlinear convective term is the sum of a conservative form and a source term due to the stabilization (this source term vanishes for a discrete divergence free function u).

Let us then study some properties of the trilinear form $b_{\mathcal{D}}$. First note that the quantity $b_{\mathcal{D}}(u, v, w)$ also writes

$$b_{\mathcal{D}}(u, v, w) = \frac{1}{2} \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} (A_{KL} \cdot (u_K + u_L))\ ((v_L - v_K) \cdot (w_L + w_K)) \tag{4.6}$$

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We thus get that, for all $u, v \in H_{\mathcal{D}}(\Omega)^d$,

$$\begin{aligned} b_{\mathcal{D}}(u, v, v) &= \frac{1}{2} \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} (A_{KL} \cdot (u_K + u_L))((v_L)^2 - (v_K)^2) \\ &= -\frac{1}{2} \int_\Omega v(x)^2\ \mathrm{div}_{\mathcal{D}}(u)(x)\ \mathrm{d}x \end{aligned} \tag{4.7}$$

superb

We get in particular, that, for all $u \in E_{\mathcal{D}}(\Omega)$, $b_{\mathcal{D}}(u, u, u) = 0$, which is the discrete equivalent of the continuous property.

REMARK 4.1. **[Upstream weighting versions of the scheme]** *All the results of this paper are available, setting* $F_{KL}(u) = A_{KL} \cdot (u_K + u_L)$ *and considering, for* $u, v, w \in H_{\mathcal{D}}(\Omega)$,

$$b_{\mathcal{D}}^{\rm ups}(u,v,w) = b_{\mathcal{D}}(u,v,w) + \frac{1}{2} \sum_{K|L \in \mathcal{E}_{\rm int}} \Theta_{KL} |F_{KL}(u)| \ (v_L - v_K) \cdot (w_L - w_K),$$

with, for example, $\Theta_{KL} = \max(1 - 2\nu \frac{{\rm m}_{K|L}}{d_{K|L}} / |F_{KL}(u)|, 0)$. *We then get, for all* $u, v \in H_{\mathcal{D}}(\Omega)$, *the inequality*

$$b_{\mathcal{D}}^{\rm ups}(u,v,v) \geq -\frac{1}{2} \int_\Omega v(x)^2 {\rm div}_{\mathcal{D}}(u)(x) {\rm d}x,$$

which is sufficient to get all the estimates of this paper, together with the convergence properties of the scheme. The use of such a local upwinding technique may be useful to avoid the development of nonphysical oscillations only where meshes are too coarse.

The following technical estimates are crucial to prove the convergence properties of the scheme.

LEMMA 4.2 (Estimates on $b_{\mathcal{D}}(.,.,.)$ by discrete Sobolev norms). *Under hypotheses (1.3)-(1.7), let* $\mathcal{D}$ *be an admissible discretization of* $\Omega \times (0,T)$ *in the sense of definition 4.8, and* $\theta > 0$ *such that* ${\rm regul}(\mathcal{D}) \geq \theta$. *Then there exists* $C_{18} > 0$ *and* $C_{19} > 0$, *only depending on* d, θ *and* Ω, *such that*

$$b_{\mathcal{D}}(u,v,w) \leq C_{18} \|u\|_{L^4(\Omega)^d} \ \|v\|_{\mathcal{D}} \ \|w\|_{L^4(\Omega)^d} \leq C_{19} \ \|u\|_{\mathcal{D}} \ \|v\|_{\mathcal{D}} \ \|w\|_{\mathcal{D}}. \tag{4.8}$$

Proof. The quantity $b_{\mathcal{D}}(u,v,w)$ reads

$$b_{\mathcal{D}}(u,v,w) = \frac{1}{4} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} (w_K \cdot (v_L - v_K)) \ \frac{{\rm m}_{K|L}}{d_{K|L}} \ ((x_L - x_K) \cdot (u_K + u_L))$$

Applying the Cauchy-Schwarz inequality twice and using the fact that $(x_L - x_K)^2 = d_{KL}^2$ and that, for any admissible discretization $\sum_{L \in \mathcal{N}_K} \frac{{\rm m}_{K|L}}{d_{K|L}} \ d_{KL}^2 \leq d \ \frac{m_K}{\theta}$ yield:

$$\begin{aligned} b_{\mathcal{D}}(u,v,w)^2 \ &\leq \ C_{20} \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \frac{{\rm m}_{K|L}}{d_{K|L}} (w_K)^2 (x_L - x_K)^2 (2(u_K)^2 + 2(u_L)^2) \right) \\ &\qquad \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \frac{{\rm m}_{K|L}}{d_{K|L}} (v_L - v_K)^2 \right) \\ &\leq \ C_{21} \left(\sum_{K \in \mathcal{M}} m_K |w_K|^4 \right)^{1/2} \left(\sum_{K \in \mathcal{M}} m_K |u_K|^4 \right)^{1/2} \ \|v\|_{\mathcal{D}}^2. \end{aligned}$$

The inequality (4.8) is now a straightforward consequence of the following discrete Sobolev inequality, which holds under the same regularity assumptions on the mesh (see proof in [10] or [15, pp. 790-791]):

$$\|u\|_{L^4(\Omega)} \leq C_{22} \ \|u\|_{\mathcal{D}}. \tag{4.9}$$

∎

Remark 4.2 (Two dimensional case). *In the case $d = 2$, it may be proven setting $\alpha = 2, p = p' = 2$ in the proof p791 of [15], that*

$$\|u\|_{L^4(\Omega)} \le C_{23} \|u\|^{1/2}_{L^2(\Omega)} \|u\|^{1/2}_{\mathcal{D}}$$

and therefore, that there exists $C_{24} > 0$, only depending on d and Ω, such that

$$b_{\mathcal{D}}(u, v, w) \le C_{24} \|v\|_{\mathcal{D}} \left(\|u\|_{\mathcal{D}} \ \|u\|_{L^2(\Omega)} \ \|w\|_{\mathcal{D}} \ \|w\|_{L^2(\Omega)} \right)^{1/2}.$$

This is a discrete analogue to the classical continuous estimate on the trilinear form.

The existence of a solution to the scheme (4.3) is obtained through a so-called "topological degree" argument. For the sake of completeness, we recall this argument (which was first used for numerical schemes in [17]) in the finite dimensional case in the following theorem and refer to [11] for the general case.

Theorem 4.3 (Application of the topological degree, finite dimensional case). *Let V be a finite dimensional vector space on $\mathbb{R}$ and g be a continuous function from V to V. Let us assume that there exists a continuous function F from $V \times [0, 1]$ to V satisfying:*

1. *$F(\cdot, 1) = g$, $F(\cdot, 0)$ is an affine function.*
2. *There exists $R > 0$, such that for any $(v, \rho) \in V \times [0, 1]$, if $F(v, \rho) = 0$, then $\|v\|_V \neq R$.*
3. *The equation $F(v, 0) = 0$ has a solution $v \in V$ such that $\|v\|_V < R$.*

Then there exists at least a solution $v \in V$ such that $g(v) = 0$ and $\|v\|_V < R$.

Here $g(v) = 0$ represents the nonlinear system (4.3), and we are now going to construct the function F and show the required estimates. Note that here, the use of Bernouilli's pressure leads to simpler calculations.

Proposition 4.4 (Discrete $H^1_0(\Omega)$ estimate on the velocities). *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0, T)$ in the sense of definition 4.8. Let $\lambda \in (0, +\infty)$ and $\alpha \in (0, 2)$ be given. Let $\rho \in [0, 1]$ be given and let $(u, p) \in (H_{\mathcal{D}}(\Omega))^d \times H_{\mathcal{D}}(\Omega)$, be a solution to the following system of equations (which reduces to (4.3) as $\rho = 1$ and to (3.7) as $\rho = 0$)*

$$\left\{ \begin{array}{ll} (u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0, & \\ \eta \displaystyle\int_\Omega u(x) \cdot v(x) \mathrm{d}x + \nu[u,v]_{\mathcal{D}} + \frac{\rho}{2} \int_\Omega u(x)^2 \mathrm{div}_{\mathcal{D}}(v)(x) \mathrm{d}x & \\ +\rho\ b_{\mathcal{D}}(u,u,v) - \displaystyle\int_\Omega p(x) \mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x = \int_\Omega f(x) \cdot v(x) \mathrm{d}x & \forall v \in H_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x) q(x) \mathrm{d}x = -\lambda \ \mathrm{size}(\mathcal{D})^\alpha \ \langle p, q \rangle_{\mathcal{D}} & \forall q \in H_{\mathcal{D}}(\Omega) \end{array} \right. \tag{4.10}$$

Then u and p satisfy the following estimates, which are the same inequalities as obtained in the linear case (inequalities (3.10) and (3.11)):

$$\nu \|u\|_{\mathcal{D}} \le \mathrm{diam}(\Omega) \|f\|_{(L^2(\Omega))^d}$$

$$\nu \ \lambda \ \mathrm{size}(\mathcal{D})^\alpha \ |p|^2_{\mathcal{D}} \le \mathrm{diam}(\Omega)^2 \|f\|^2_{(L^2(\Omega))^d}$$

Proof. The proof is similar to that of Proposition 3.2, using the property (4.7) on the discrete trilinear form. □

We are now in position to prove the existence of at least one solution to scheme (4.3).

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PROPOSITION 4.5 (Existence of a discrete solution). *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of definition 4.8. Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given. Then there exists at least one $(u,p) \in (H_{\mathcal{D}}(\Omega))^d \times H_{\mathcal{D}}(\Omega)$, solution to (4.3).*

Proof. Let us define $V = \{(u,p) \in (H_{\mathcal{D}}(\Omega))^d \times H_{\mathcal{D}}(\Omega) \text{ s.t. } \int_\Omega p(x)\mathrm{d}x = 0\}$. Consider the continuous application $F : V \times [0,1] \to V$ such that, for a given $(u,p) \in V$ and $\rho \in [0,1]$, $(\hat{u},\hat{p}) = F(u,p,\rho)$ is defined by

$$\begin{aligned}
\int_\Omega \hat{u}(x)\cdot v(x)\mathrm{d}x \quad &= \eta \int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x \\
&+\rho\left(\frac{1}{2}\int_\Omega u(x)^2\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x + b_{\mathcal{D}}(u,u,v)\right) \\
&-\int_\Omega f(x)\cdot v(x)\mathrm{d}x \qquad\qquad \forall v \in H_{\mathcal{D}}(\Omega)^d \\
\int_\Omega \hat{p}(x)\cdot q(x)\mathrm{d}x \quad &= \int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)q(x)\mathrm{d}x + \lambda\ \mathrm{size}(\mathcal{D})^\alpha\ \langle p,q\rangle_{\mathcal{D}} \qquad \forall q \in H_{\mathcal{D}}(\Omega).
\end{aligned}$$

It is easily checked that the two above relations define a one to one function $F(.,.,.)$. Indeed, the value of $\hat{u}_K^{(i)}$ and $\hat{p}_K$ for a given $K \in \mathcal{M}$ and $i = 1,\ldots,d$ are readily obtained by setting $v^{(i)} = 1_K$, $v^{(j)} = 0$ for $j \neq i$, and $q = 1_K$.

The application $F(.,.,.)$ is continuous, and, for a given (u,p) such that $F(u,p,\rho) = (0,0)$, we can apply proposition 4.4 and (2.5), which prove that (u,p) is bounded independently on ρ. Since $F(u,p,0)$ is an affine function of (u,p) (indeed invertible, see corollary 3.3), we may apply Theorem 4.3 and conclude to the existence of at least one solution (u,p) to (4.3). ☐

We then have the following strong estimate on the pressures.

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PROPOSITION 4.6 (L^2 estimate on pressures). *Under hypotheses (3.2)-(3.4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 2.1, and let $\theta > 0$ such that* $\mathrm{regul}(\mathcal{D}) > \theta$. *Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given. Let $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ be a solution to (4.3). Then there exists C_{25}, only depending on d, Ω, η, ν, λ, α and θ, and not on* $\mathrm{size}(\mathcal{D})$, *such that the following inequality holds:*

$$\|p\|_{L^2(\Omega)} \le C_{25}\left(\|f\|_{(L^2(\Omega))^d} + \left(\|f\|_{(L^2(\Omega))^d}\right)^2\right) \tag{4.11}$$

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Proof. We may follow the proof of proposition 3.4 until (3.17), which is changed to:

$$\begin{aligned}
\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x \quad &= \eta \int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega f(x)\cdot v(x)\mathrm{d}x \\
&+\frac{1}{2}\int_\Omega u(x)^2\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x + b_{\mathcal{D}}(u,u,v).
\end{aligned} \tag{4.12}$$

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We again apply the discrete Poincaré inequality (2.4), (3.15), (3.16) and we use (4.8). We get the existence of C_{26}, only depending on d, Ω, f, η, ν, λ and θ, such that

$$\|p\|^2_{L^2(\Omega)} - C_4\,\mathrm{size}(\mathcal{D})|p|_{\mathcal{D}} C_2 \|p\|_{L^2(\Omega)} \le \\ C_{26}\left(\|u\|_{\mathcal{D}} + \|f\|_{L^2(\Omega)^d} + \|u\|^2_{\mathcal{D}}\right)\|p\|_{L^2(\Omega)}$$

We now apply (3.10) and (3.11), which yields the conclusion. □

We now can state the convergence of Scheme (4.3).

THEOREM 4.7 (Convergence of the scheme). *Under hypotheses (3.2)-(3.4), let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of definition 2.1, such that* $\mathrm{size}(\mathcal{D}^{(m)})$ *tends to* 0 *as $m \to \infty$ and such that there exists $\theta > 0$ with* $\mathrm{regul}(\mathcal{D}^{(m)}) \ge \theta$, *for all $m \in \mathbb{N}$. Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given. Let, for all $m \in \mathbb{N}$, $(u^{(m)}, p^{(m)}) \in (H_{\mathcal{D}^{(m)}}(\Omega))^d \times H_{\mathcal{D}^{(m)}}(\Omega)$, be a solution to (4.3) with $\mathcal{D} = \mathcal{D}^{(m)}$. Then there exists a weak solution $(\bar u, \bar p)$ of (4.1) in the sense of definition 4.1 and a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, such that the corresponding subsequence of solutions $(u^{(m)})_{m\in\mathbb{N}}$ converges to $\bar u$ in $L^2(\Omega)$ and $(p^{(m)} - \frac{1}{2}(u^{(m)})^2)_{m\in\mathbb{N}}$ weakly converges to $\bar p$ in $L^2(\Omega)$.*

Proof. Since the same estimates as in the linear case are available in the steady nonlinear case, the proof of proposition 3.5 holds for all the terms of (4.2) which are present in (3.5). We only have to prove that for a given $\varphi \in (C_c^\infty(\Omega))^d$, as $m \to +\infty$:

$$T_{12}^{(m)} = \int_\Omega u^{(m)}(x)^2 \mathrm{div}_{\mathcal{D}^{(m)}}(P_{\mathcal{D}^{(m)}}\varphi)(x)\mathrm{d}x \quad \text{tends to} \quad \int_\Omega \bar u(x)^2 \mathrm{div}\varphi(x)\mathrm{d}x$$

and

$$T_{13}^{(m)} = b_{\mathcal{D}}(u^{(m)}, u^{(m)}, P_{\mathcal{D}^{(m)}}\varphi) \quad \text{tends to} \quad b(\bar u, \bar u, \varphi).$$

Thanks to the convergence in $L^2(\Omega)$ of $(u^{(m)})_{m\in\mathbb{N}}$ to $\bar u$ and to the discrete Sobolev inequalities $\|v\|_{L^q(\Omega)} \le C_{27}\|v\|_{\mathcal{D}^{(m)}}$ for all $v \in H_{\mathcal{D}^{(m)}}(\Omega)$ and all $q \le 6$ (see [15, p. 790]), we get using (3.10) the convergence in $L^2(\Omega)$ of $((u^{(m)})^2)_{m\in\mathbb{N}}$ to $\bar u^2$. We now remark that for $i = 1,\dots,d$, the sequence $(P_{\mathcal{D}^{(m)}}\varphi^{(i)})_{m\in\mathbb{N}}$ satisfies the hypotheses of Proposition 2.4. Hence, $\nabla_{\mathcal{D}^{(m)}} P_{\mathcal{D}^{(m)}}\varphi^{(i)}$ weakly converges to $\nabla\varphi^{(i)}$ in $L^2(\Omega)^d$. One has $\mathrm{div}_{\mathcal{D}} u = \sum_{i=1}^d \nabla^{(i)}_{\mathcal{D}} u^{(i)}$ for all $u \in (H_{\mathcal{D}}(\Omega))^d$ such that $u_K = 0$ if $\mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}} \neq \emptyset$. Hence $\mathrm{div}_{\mathcal{D}^{(m)}}(P_{\mathcal{D}^{(m)}}\varphi)$ weakly converges to $\mathrm{div}\varphi$ in $L^2(\Omega)$, thus providing the limit of $T_{12}^{(m)}$.

Thanks to (4.6), setting for simplicity $\mathcal{D} = \mathcal{D}^{(m)}$, we have:

$$b_{\mathcal{D}}(u,u,P_{\mathcal{D}}\varphi) = T_{14}^{(m)} - T_{15}^{(m)}$$

with:

$$\begin{aligned} T_{14}^{(m)} &= \sum_{K\in\mathcal{M}}\sum_{L\in\mathcal{N}_K} (A_{KL}\cdot u_K)((u_L - u_K)\cdot\varphi(x_K)) \\ &= \sum_{k=1}^d\sum_{i=1}^d \int_\Omega u^{(i)}(x)\nabla^{(i)}_{\mathcal{D}}(u^{(k)})(x) P_{\mathcal{D}}\varphi^{(k)}(x)\mathrm{d}x \\ T_{15}^{(m)} &= \frac{1}{2}\sum_{K|L\in\mathcal{E}_{\mathrm{int}}} (A_{KL}\cdot(u_L - u_K))((u_L - u_K)\cdot(\varphi(x_K) - \varphi(x_L))) \end{aligned}$$

Thanks to the convergence in $L^2(\Omega)$ of $(u^{(m)}P_{\mathcal{D}^{(m)}}\varphi)_{m\in\mathbb{N}}$ to $\bar{u}\varphi$, we get from proposition 2.4 that:

$$\lim_{m\to\infty} T_{14}^{(m)} = \sum_{k=1}^{d}\sum_{i=1}^{d}\int_\Omega \bar{u}^{(i)}(x)\partial_i\bar{u}^{(k)}(x)\bar{\varphi}^{(k)}(x)\mathrm{d}x = b(\bar{u},\bar{u},\varphi).$$

We have:

$$T_{15}^{(m)} = \frac{1}{4}\sum_{K|L\in\mathcal{E}_{\mathrm{int}}} d_{KL}(\frac{\mathrm{m}_{K|L}}{d_{K|L}}\mathbf{n}_{KL}\cdot(u_L-u_K))((u_L-u_K)\cdot(\varphi(x_K)-\varphi(x_L)))$$

and therefore, since $|\varphi(x_K)-\varphi(x_L)|\le d_{KL}C_\varphi\mathrm{size}(\mathcal{D})$ where C_φ is a bound of $\nabla\varphi$ in $L^\infty(\Omega)^d$, and since $d_{KL}\le 2\mathrm{size}(\mathcal{D})$, the following estimate holds:

$$|T_{15}^{(m)}| \le 4\mathrm{size}(\mathcal{D})^2 C_\varphi\|u\|_{\mathcal{D}}^2.$$

Therefore, (3.10) yields:

$$\lim_{m\to\infty} T_{15}^{(m)} = 0,$$

which concludes the proof of convergence. □

4.2. The transient case. We now turn to the study of the finite volume scheme for the transient Navier-Stokes equations, the weak formulation of which is given in (1.1).

We first give the definition of an admissible discretization for a space-time domain.

Definition 4.8 (Admissible discretization, transient case). *Let Ω be an open bounded polygonal (polyhedral if $d=3$) subset of $\mathbb{R}^d$, and $\partial\Omega=\overline{\Omega}\setminus\Omega$ its boundary, and let $T>0$. An admissible finite volume discretization of $\Omega\times(0,T)$, denoted by $\mathcal{D}$, is given by $\mathcal{D}=(\mathcal{M},\mathcal{E},\mathcal{P},N)$, where $(\mathcal{M},\mathcal{E},\mathcal{P})$ is an admissible discretization of Ω in the sense of definition 2.1 and $N\in\mathbb{N}_\star$ is given. We then define $\delta t=T/N$, and we denote by $\mathrm{size}(\mathcal{D})=\max(\mathrm{size}(\mathcal{M},\mathcal{E},\mathcal{P}),\delta t)$ and $\mathrm{regul}(\mathcal{D})=\mathrm{regul}(\mathcal{M},\mathcal{E},\mathcal{P})$.*

Under hypotheses (1.3)-(1.7), let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of definition 4.8 and let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$ be given. We write a Crank–Nicholson scheme for the time discretization, and follow the nonlinear steady–state case for the space discretization; the finite volume scheme for the approximation of the solution (1.1)–(1.2) is then:

$$\begin{aligned}&u_0\in H_{\mathcal{D}}(\Omega)^d,\\ &u_{0,K}=\frac{1}{\mathrm{m}_K}\int_K u_{\mathrm{ini}}(x)\mathrm{d}x,\ \forall K\in\mathcal{M},\end{aligned}\tag{4.13}$$

and, again using Bernoulli's pressure $p+\frac{1}{2}u^2$ instead of p, again denoted by p,

$$\begin{aligned}&(u_{n+1},p_{n+\frac12})\in(H_{\mathcal{D}}(\Omega))^d\times H_{\mathcal{D}}(\Omega),\\ &\textstyle\int_\Omega p_{n+\frac12}(x)\mathrm{d}x=0,\ u_{n+\frac12}=\frac12(u_{n+1}+u_n),\\ &\int_\Omega(u_{n+1}(x)-u_n(x))\cdot v(x)\mathrm{d}x+\nu\delta t[u_{n+\frac12},v]_{\mathcal{D}}\\ &-\delta t\int_\Omega p_{n+\frac12}(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x+\frac{\delta t}{2}\int_\Omega u_{n+\frac12}(x)^2\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x\\ &+\delta t b_{\mathcal{D}}(u_{n+\frac12},u_{n+\frac12},v)=\int_{n\delta t}^{(n+1)\delta t}\int_\Omega f(x,t)\cdot v(x)\mathrm{d}x\mathrm{d}t,\\ &\int_\Omega \mathrm{div}_{\mathcal{D}}(u_{n+\frac12})(x)q(x)\mathrm{d}x=-\lambda\ \mathrm{size}(\mathcal{D})^\alpha\ \langle p_{n+\frac12},q\rangle_{\mathcal{D}},\\ &\forall v\in H_{\mathcal{D}}(\Omega)^d,\forall q\in H_{\mathcal{D}}(\Omega),\ \forall n\in\mathbb{N}.\end{aligned}\tag{4.14}$$

In (4.14), we consider the approximation of $b_{\mathcal{D}}$ given by (4.4). We then define the set $H_{\mathcal{D}}(\Omega\times(0,T))$ of piecewise constant functions in each $K\times(n\delta t,(n+1)\delta t)$, $K\in\mathcal{M}$, $n\in\mathbb{N}$, and we define $(u,p)\in H_{\mathcal{D}}(\Omega\times(0,T))$ by

$$u(x,t)=u_{n+\frac12}(x),\ \text{ and } p(x,t)=p_{n+\frac12}(x),\ \text{ for a.e. } (x,t)\in\Omega\times(n\delta t,(n+1)\delta t),\ \forall n\in\mathbb{N}. \tag{4.15}$$

REMARK 4.3 (Time discretization). *It is wellknown that the Crank–Nicholson discretization is implicit. If we use the θ scheme: $u_{n+\frac12}=\theta u_{n+1}+(1-\theta)u_n$, with $\theta\in[\frac12,1]$, the convergence proof which follows applies with a few minor changes. Variable time steps may also be considered.*

Let us now prove the existence of at least one solution to scheme (4.13)-(4.15).

PROPOSITION 4.9 (Existence of a discrete solution). *Under hypotheses (1.3)-(1.7), let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of Definition 4.8. Let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$ be given. Then there exists at least one $(u,p)\in(H_{\mathcal{D}}(\Omega\times(0,T)))^d\times H_{\mathcal{D}}(\Omega\times(0,T))$, solution to (4.13)-(4.15).*

Proof. We remark that, for a given $n=0,\ldots,N-1$, taking as unknown $u_{n+\frac12}$, and noting that $u_{n+1}=2u_{n+\frac12}-u_n$, Scheme (4.14) is under the same form as scheme (4.3), with $\eta=\frac{2}{\delta t}$ and with a term in u_n included in the right hand side. Therefore the existence of at least one solution follows from proposition 4.5. □

We then have the following estimate.

PROPOSITION 4.10 (Discrete $L^2(0,T;H^1_0(\Omega))$ estimate on velocities). *Under hypotheses (1.3)-(1.7), let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of definition 4.8. Let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$. Let $(u,p)\in(H_{\mathcal{D}}(\Omega\times(0,T)))^d\times H_{\mathcal{D}}(\Omega\times(0,T))$, be a solution to (4.13)-(4.15). Then there exists $C_{28}>0$, only depending on d, Ω, ν, u_0, f, T such that the following inequalities hold*

$$\|u\|_{L^\infty(0,T;L^2(\Omega)^d)}\le C_{28}, \tag{4.16}$$

$$\|u\|_{L^2(0,T;H_{\mathcal{D}}(\Omega)^d)}\le C_{28}, \tag{4.17}$$

and

$$\lambda\ \mathrm{size}(\mathcal{D})^\alpha\sum_{n=0}^{N-1}\delta t|p_{n+\frac12}|^2_{\mathcal{D}}=\lambda\ \mathrm{size}(\mathcal{D})^\alpha\int_0^T|p(\cdot,t)|^2_{\mathcal{D}}\mathrm{d}t\le C_{28}. \tag{4.18}$$

Proof. Let $p=1,\ldots,N$. We get, setting $v=u_{n+\frac12}$ in the first equation of (4.14), summing on $K\in\mathcal{M}$ and $n=0,\ldots,p-1$ in the first equation of (4.14) and using property (4.7),

$$\frac12\sum_{n=0}^{p-1}\int_\Omega(u_{n+1}(x)^2-u_n(x)^2)\mathrm{d}x+\nu\sum_{n=0}^{p-1}\delta t[u_{n+\frac12},u_{n+\frac12}]_{\mathcal{D}}-$$
$$\sum_{n=0}^{p-1}\delta t\int_\Omega p_{n+\frac12}(x)\mathrm{div}_{\mathcal{D}}(u_{n+\frac12})(x)dx=\sum_{n=0}^{p-1}\int_{n\delta t}^{(n+1)\delta t}\int_\Omega f(x,t)\cdot u_{n+\frac12}(x)\mathrm{d}x\mathrm{d}t,$$

This leads, setting $q=p_{n+\frac12}$ in the second equation of (4.14), to

$$\begin{aligned}&\frac12\int_\Omega(u_p(x)^2-u_0(x)^2)\mathrm{d}x+\nu\sum_{n=0}^{p-1}\delta t[u_{n+\frac12},u_{n+\frac12}]_{\mathcal{D}}+\\&\lambda\ \mathrm{size}(\mathcal{D})^\alpha\sum_{n=0}^{p-1}\delta t|p_{n+\frac12}|^2_{\mathcal{D}}=\int_0^{p\delta t}\int_\Omega f(x,t)\cdot u(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.19}$$

Setting $p = N$ in (4.19) gives (4.17) and (4.18). The discrete Poincaré inequality (2.4) and the inequality $\|u_0\|_{L^2(\Omega)^d} \le \|u_{\rm ini}\|_{L^2(\Omega)^d}$ give

$$\|u_p\|^2_{L^2(\Omega)^d} \le \frac{\mathrm{diam}(\Omega)^2}{2\nu} \|f\|^2_{L^2(\Omega\times(0,T))^d} + \|u_{\rm ini}\|^2_{L^2(\Omega)^d},\ \forall p = 1,\dots,N,$$

which proves (4.16), since $\|u_{n+\frac12}\|_{L^2(\Omega)^d} \le \frac12(\|u_n\|_{L^2(\Omega)^d} + \|u_{n+1}\|_{L^2(\Omega)^d})$ for all $n = 0,\dots,N-1$. ∎

We then have the following estimates on translations.

Proposition 4.11 (Space and time translate estimates). *Under hypotheses (1.3)-(1.7), let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of definition 4.8. Let $\lambda \in (0,+\infty)$, $\alpha\in(0,2)$ and $\theta > 0$, such that* $\mathrm{regul}(\mathcal{D}) \ge \theta$. *Let $(u,p) \in (H_{\mathcal{D}}(\Omega\times(0,T)))^d \times H_{\mathcal{D}}(\Omega\times(0,T))$, be a solution to (4.13)-(4.15). We denote by u the prolongment in $\mathbb{R}^d\times\mathbb{R}$ of u by 0 outside of $\Omega\times(0,T)$. Then there exists $C_{29} > 0$ and $C_{30} > 0$, only depending on d, Ω, ν, λ, α, u_0, f, θ and T such that the following inequalities hold:*

$$\|u(\cdot+\xi,\cdot) - u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R})} \le C_{29}\,|\xi|(|\xi| + 4\mathrm{size}(\mathcal{M})),\ \forall \xi\in\mathbb{R}^d, \tag{4.20}$$

and

$$\|u(\cdot,\cdot+\tau) - u\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))} \le C_{30}\,|\tau|^{1/2},\ \forall\tau\in\mathbb{R}. \tag{4.21}$$

Proof. In the following proof, we denote by C_i, where i is an integer, various positive real numbers which can only depend on d, Ω, ν, λ, α, u_0, f, θ and T. Inequality (4.20) is obtained from (4.17) (see [15]). Let us prove (4.21). Let $\tau\in(0,T)$ be given. We define the following norms on $(H_{\mathcal{D}}(\Omega))^d$, by:

$$\begin{aligned}&\forall\, w\in (H_{\mathcal{D}}(\Omega))^d,\\ &\|w\|^2_{\mathcal{D},\lambda} = \|w\|^2_{\mathcal{D}} +\\ &\frac{1}{\lambda\mathrm{size}(\mathcal{D})^\alpha}\left(\sup\left\{\int_\Omega \mathrm{div}_{\mathcal{D}}(w)(x) q(x)\mathrm{d}x,\ q\in H_{\mathcal{D}}(\Omega), |q|_{\mathcal{D}} = 1\right\}\right)^2\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\forall\, w\in (H_{\mathcal{D}}(\Omega))^d,\\ &\|w\|_{\star,\mathcal{D},\lambda} = \sup\left\{\int_\Omega w(x)\cdot v(x)dx, v\in (H_{\mathcal{D}}(\Omega))^d, \|v\|_{\mathcal{D},\lambda} = 1\right\}.\end{aligned} \tag{4.23}$$

We then have, for a.e. $t\in(0,T)$,

$$\|u(\cdot,t+\tau) - u(\cdot,t)\|^2_{L^2(\Omega)^d} \le \|u(\cdot,t+\tau)-u(\cdot,t)\|_{\mathcal{D},\lambda} \|u(\cdot,t+\tau)-u(\cdot,t)\|_{\star,\mathcal{D},\lambda},$$

and therefore, thanks to the Young formula,

$$\begin{aligned}\|u(\cdot,t+\tau) - u(\cdot,t)\|_{L^2(\Omega)^d} \le\ & \tfrac{\sqrt{\tau}}{2}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{\mathcal{D},\lambda}\\ &+\tfrac{1}{2\sqrt{\tau}}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{\star,\mathcal{D},\lambda}.\end{aligned} \tag{4.24}$$

We get, from (4.14), for all $q\in H_{\mathcal{D}}(\Omega)$ and for a.e. $t\in(0,T)$,

$$\int_\Omega \mathrm{div}_{\mathcal{D}}(u(\cdot,t))(x)q(x)\mathrm{d}x = -\lambda\ \mathrm{size}(\mathcal{D})^\alpha\ \langle p(\cdot,t), q\rangle_{\mathcal{D}},$$

which proves, using (4.22), that

$$\|u(\cdot,t)\|_{\mathcal{D},\lambda}^2 \le \|u(\cdot,t)\|_{\mathcal{D}}^2 + \lambda\ \mathrm{size}(\mathcal{D})^\alpha |p(\cdot,t)|_{\mathcal{D}}^2.$$

Using the Cauchy-Schwarz inequality, we have that:

$$\left(\int_0^{T-\tau} \|u(\cdot,t+\tau)-u(\cdot,t)\|_{\mathcal{D},\lambda}\mathrm{d}t\right)^2 \le 4T\int_0^T \|u(\cdot,t)\|_{\mathcal{D},\lambda}^2\mathrm{d}t,$$

and therefore, using (4.17), (4.18),

$$\int_0^{T-\tau} \|u(\cdot,t+\tau)-u(\cdot,t)\|_{\mathcal{D},\lambda}\mathrm{d}t \le C_{31}\,. \tag{4.25}$$

We now study $\|u(\cdot,t+\tau)-u(\cdot,t)\|_{\star,\mathcal{D},\lambda}$. We can write, for a.e. $t\in(0,T-\tau)$ and $x\in\Omega$,

$$u(x,t+\tau)-u(x,t) = \frac{1}{2}\sum_{n=0}^{N-1}(\chi_n(t,\tau)+\chi_{n+1}(t,\tau))(u_{n+1}(x)-u_n(x)),$$

where, for all $n\in\mathbb{N}$ and $t\in(0,T)$, $\chi_n(t,\tau)=1$ if $n\delta t\in[t,t+\tau[$, and $\chi_n(t,\tau)=0$ otherwise. This implies

$$\begin{aligned}&\|u(\cdot,t+\tau)-u(\cdot,t)\|_{\star,\mathcal{D},\lambda}\le\\ &\tfrac{1}{2}\textstyle\sum_{n=0}^{N-1}(\chi_n(t,\tau)+\chi_{n+1}(t,\tau))\|u_{n+1}-u_n\|_{\star,\mathcal{D},\lambda}.\end{aligned} \tag{4.26}$$

Let us then obtain a bound for $\|u_{n+1}-u_n\|_{\star,\mathcal{D},\lambda}$. Using the scheme (4.14), we get that, for all $v\in(H_{\mathcal{D}}(\Omega))^d$,

$$\begin{aligned}\int_\Omega (u_{n+1}(x)-u_n(x))\cdot v(x)\mathrm{d}x \quad &= \int_{n\delta t}^{(n+1)\delta t}\int_\Omega f(x,t)\cdot v(x)\mathrm{d}x\mathrm{d}t\\ &-\nu\delta t[u_{n+\frac12},v]_{\mathcal{D}} + \delta t\int_\Omega p_{n+\frac12}(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x\\ &-\frac{\delta t}{2}\int_\Omega u_{n+\frac12}^2\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x - \delta t b_{\mathcal{D}}(u_{n+\frac12},u_{n+\frac12},v).\end{aligned} \tag{4.27}$$

Using the definition of $\mathrm{div}_{\mathcal{D}}$, the fact that $\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma\mathbf{n}_{K,\sigma}=0$, and the Cauchy-Schwarz inequality, there exists C_{32} such that:

$$\int_\Omega u_{n+\frac12}^2(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x \le C_{32}\|u_{n+\frac12}^2\|_{L^2(\Omega)}\ \|v\|_{\mathcal{D}}.$$

The discrete Sobolev inequality (4.9) leads to

$$\|u_{n+\frac12}^2\|_{L^2(\Omega)} \le \sum_{i=1}^d \|(u_{n+\frac12}^{(i)})^2\|_{L^2(\Omega)} = \sum_{i=1}^d \|u_{n+\frac12}^{(i)}\|_{L^4(\Omega)}^2 \le C_{33}\|u_{n+\frac12}\|_{\mathcal{D}}^2$$

We take $\|v\|_{\mathcal{D},\lambda}=1$ and note that, from Definition (4.22), we obtain that $\|v\|_{\mathcal{D}}\le 1$, and that $\int_\Omega p_{n+\frac12}(x)\mathrm{div}_{\mathcal{D}}(v)(x)\mathrm{d}x \le (\lambda\ \mathrm{size}(\mathcal{D})^\alpha)^{1/2}\ |p_{n+\frac12}|_{\mathcal{D}}$. We then pass to the supremum in (4.27). Using the Cauchy-Schwarz inequality, the discrete Poincaré inequality, and (4.8), this yields:

$$\|u_{n+1}-u_n\|_{\star,\mathcal{D},\lambda} \le \begin{array}[t]{l} \sqrt{\delta t}\,\mathrm{diam}(\Omega)\|f\|_{L^2(\Omega\times(n\delta t,(n+1)\delta t))}\delta t \\ +\delta t\nu\|u_{n+\frac12}\|_{\mathcal{D}} + (\lambda\ \mathrm{size}(\mathcal{D})^\alpha)^{1/2}\ \ |p_{n+\frac12}|_{\mathcal{D}} \\ +\delta t(\frac{1}{2}C_{32}C_{33}+C_{19})\|u_{n+\frac12}\|^2_{\mathcal{D}}. \end{array}$$

Summing the above equation for $n=0$ to $N-1$, applying the Cauchy-Schwarz inequality to all terms of the right hand side except the last, using (4.17) and (4.18), we get that there exists C_{34} such that

$$\sum_{n=0}^{N-1}\|u_{n+1}-u_n\|_{\star,\mathcal{D},\lambda} \le C_{34}.$$

Hence, noting that for all $n=0,\ldots,N$, $\int_0^{T-\tau}\chi_n(t,\tau)\mathrm{d}t \le \tau$, we have:

$$\frac{1}{2}\int_0^{T-\tau}\sum_{n=0}^{N-1}(\chi_n(t,\tau)+\chi_{n+1}(t,\tau))\|u_{n+1}-u_n\|_{\star,\mathcal{D},\lambda}\mathrm{d}t \le C_{34}\tau,$$

which proves, using (4.26),

$$\int_0^{T-\tau}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{\star,\mathcal{D},\lambda}\mathrm{d}t \le C_{34}\tau. \tag{4.28}$$ machin

Thanks to (4.24), (4.25) and (4.28), we obtain that

$$\int_0^{T-\tau}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{L^2(\Omega)^d}\mathrm{d}t \le C_{35}\sqrt{\tau}.$$

Using (4.16), we have

$$\int_{T-\tau}^{T}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{L^2(\Omega)^d}\mathrm{d}t = \int_{T-\tau}^{T}\|-u(\cdot,t)\|_{L^2(\Omega)^d}\mathrm{d}t \le C_{28}\,\tau \le \sqrt{\tau}\sqrt{T}C_{28}\,,$$

and a similar inequality holds for $\int_{-\tau}^{0}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{L^2(\Omega)^d}\mathrm{d}t$. This thus gives (4.21), for any $\tau\in(0,T)$. The case $\tau\ge T$ is obtained again using (4.16), and the case $\tau\le 0$ is obtained from $\tau\ge 0$ by the change of variable $s=t+\tau$. This completes the proof of (4.21). □

cvgcetime THEOREM 4.12 (Convergence of the scheme). *Under hypotheses (1.3)-(1.7), let $\theta>0$ be given and let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of $\Omega\times(0,T)$ in the sense of definition 4.8, such that* $\mathrm{regul}(\mathcal{D}^{(m)})\ge\theta$ *and* $\mathrm{size}(\mathcal{D}^{(m)})$ *tends to 0 as $m\to\infty$. Let $\lambda\in(0,+\infty)$ and $\alpha\in(0,2)$ be given. Let, for all $m\in\mathbb{N}$, $(u^{(m)},p^{(m)})\in(H_{\mathcal{D}^{(m)}}(\Omega\times(0,T)))^d\times H_{\mathcal{D}^{(m)}}(\Omega\times(0,T))$, be a solution to (4.13)-(4.15) with $\mathcal{D}=\mathcal{D}^{(m)}$. Then there exists a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, such that the corresponding subsequence of solutions $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^2(\Omega\times(0,T))$ to a weak solution $\bar{u}$ of (1.1)-(1.2) in the sense of definition 1.1.*

Proof. Let us assume the hypotheses of the theorem. Using translate estimates (4.20) and (4.21) in the space $L^1(\mathbb{R}^d\times\mathbb{R})$, we can apply Kolmogorov's theorem. We get that there exists $\bar{u}\in L^1(\Omega\times(0,T))$ and a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, again denoted

$(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, such that the corresponding subsequence of solutions $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^1(\Omega\times(0,T))$ to $\bar u$ as $m\to\infty$. Using (4.17), we get $\|u^{(m)}\|_{L^2(0,T;H_{\mathcal{D}_m}(\Omega))}\le C_{28}$, for all $m\in\mathbb{N}$, which gives, using the discrete Sobolev inequalities, $\|u^{(m)}\|_{L^1(0,T;L^4(\Omega))}\le C_{36}$, for all $m\in\mathbb{N}$. Using a classical result on spaces $L^p(0,T;L^q(\Omega))$, we get that $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^1(0,T;L^2(\Omega))$ to $\bar u$ as $m\to\infty$. Thanks to (4.16), we have $\|u^{(m)}\|_{L^\infty(0,T;L^2(\Omega)^d)}\le C_{28}$, for all $m\in\mathbb{N}$. The same result on spaces $L^p(0,T;L^q(\Omega))$ implies that $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^2(0,T;L^2(\Omega))$ to $\bar u$ as $m\to\infty$. We can therefore pass to the limit in (4.20). The resulting inequality implies $\bar u\in L^2(0,T;H^1_0(\Omega)^d)$ (see [15]). Passing to the limit in (4.16) leads to $\bar u\in L^\infty(0,T;L^2(\Omega)^d)$.

Let us now prove that $\bar u$ is a weak solution of (1.1)-(1.2) in the sense of definition 1.1.

Let $\varphi\in C_c^\infty(\Omega\times(-\infty,T))^d$ be given, with $\mathrm{div}\varphi(x,t)=0$ for all $(x,t)\in\Omega\times(-\infty,T)$. Let $\mathcal{D}^{(m)}$ be a given admissible discretization extracted from the considered subsequence. Omitting some of the indices m for the simplicity of notation, we then set $v=P_{\mathcal{D}}\varphi(\cdot,n\delta t)$ in (4.14), and we sum for $n=0,\dots,N-1$. We thus get

$$T_{16}^{(m)}+T_{17}^{(m)}+T_{18}^{(m)}+T_{19}^{(m)}+T_{20}^{(m)}=T_{21}^{(m)}, \tag{4.29}$$

with

$$T_{16}^{(m)}=\sum_{n=0}^{N-1}\int_\Omega (u_{n+1}(x)-u_n(x))\cdot P_{\mathcal{D}}\varphi(x,n\delta t)\mathrm{d}x,$$

$$T_{17}^{(m)}=\sum_{n=0}^{N-1}\delta t[u_{n+\frac12},P_{\mathcal{D}}\varphi(\cdot,n\delta t)]_{\mathcal{D}},$$

$$T_{18}^{(m)}=-\sum_{n=0}^{N-1}\delta t\int_\Omega p_{n+\frac12}(x)\mathrm{div}_{\mathcal{D}}(P_{\mathcal{D}}\varphi(\cdot,n\delta t))(x)\mathrm{d}x,$$

$$T_{19}^{(m)}=\frac12\sum_{n=0}^{N-1}\delta t\int_\Omega u_{n+\frac12}(x)^2\mathrm{div}_{\mathcal{D}}(P_{\mathcal{D}}\varphi(\cdot,n\delta t))(x)\mathrm{d}x,$$

$$T_{20}^{(m)}=\sum_{n=0}^{N-1}\delta t\, b_{\mathcal{D}}(u_{n+\frac12},u_{n+\frac12},P_{\mathcal{D}}\varphi(\cdot,n\delta t)),$$

and

$$T_{21}^{(m)}=\sum_{n=0}^{N-1}\int_{n\delta t}^{(n+1)\delta t}\int_\Omega f(x,t)\cdot P_{\mathcal{D}}\varphi(x,n\delta t)\mathrm{d}x\mathrm{d}t.$$

In the following, we denote by C_i various positive reals which can only depend on d, Ω, T, $u_{\rm ini}$, f, ν, θ and λ. We first start with the study of T_{17}. We classically have (see [15])

$$\lim_{m\to\infty}T_{17}^{(m)}=\int_0^T\int_\Omega\nabla\bar u(x,t):\nabla\varphi(x,t)\mathrm{d}x\mathrm{d}t. \tag{4.30}$$

The proof that

$$\lim_{m\to\infty} T_{18}^{(m)} = 0 \tag{4.31}$$

is a consequence of (4.18) and of a direct adaptation of Proposition 2.3 to time-dependent functions. Let us now prove that

$$\lim_{m\to\infty} T_{19}^{(m)} = 0. \tag{4.32}$$

Since $(u^{(m)})^2$ tend to $\bar u^2$ as $m \to \infty$ in $L^1(\Omega \times (0,T))$, the same argument as in the steady state case (see proof of theorem 4.7) provides (4.32).

We now turn to the study of T_{20}. Following the proof of proposition 4.7, the proof that

$$\lim_{m\to\infty} T_{20}^{(m)} = \int_0^T b(\bar u(\cdot,t), \bar u(\cdot,t), \varphi(\cdot,t))\mathrm{d}t. \tag{4.33}$$

is a direct consequence of the convergence of u to $\bar u$ in $L^2(\Omega\times(0,T))$ and proposition 2.3. The study of T_{21} is classical, and we have

$$\lim_{m\to\infty} T_{21}^{(m)} = \int_0^T \int_\Omega f(x,t)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t. \tag{4.34}$$

Let us now prove that

$$\lim_{m\to\infty} T_{16}^{(m)} = -\int_0^T \int_\Omega \bar u(x,t)\partial_t\varphi(x,t)\mathrm{d}x\mathrm{d}t - \int_\Omega u_{\mathrm{ini}}(x)\varphi(x,0)\mathrm{d}x. \tag{4.35}$$

Indeed, we have

$$T_{16}^{(m)} = -\int_\Omega u_0(x)\cdot P_{\mathcal D}\varphi(x,0)\mathrm{d}x - T_{22}^{(m)} - \frac{1}{2}T_{23}^{(m)},$$

with

$$T_{22}^{(m)} = \sum_{n=0}^{N-1}\int_\Omega u_{n+\frac12}(x)\cdot(P_{\mathcal D}\varphi(x,(n+1)\delta t) - P_{\mathcal D}\varphi(x,n\delta t))\mathrm{d}x.$$

and

$$T_{23}^{(m)} = \sum_{n=0}^{N-1}\int_\Omega (u_{n+1}(x)-u_n(x))\cdot(P_{\mathcal D}\varphi(x,(n+1)\delta t) - P_{\mathcal D}\varphi(x,n\delta t))\mathrm{d}x$$

We classically have

$$\lim_{m\to\infty}\int_\Omega u_0(x)\cdot P_{\mathcal D}\varphi(x,0)\mathrm{d}x = \int_\Omega u_{\mathrm{ini}}(x)\varphi(x,0)\mathrm{d}x.$$

We also easily have, thanks to the convergence properties of $u^{(m)}$, that

$$\lim_{m\to\infty} T_{22}^{(m)} = \int_0^T\int_\Omega \bar u(x,t)\partial_t\varphi(x,t)\mathrm{d}x\mathrm{d}t.$$

Let us prove that the term $T_{23}^{(m)}$ tends to 0 as $m \to \infty$. We have $T_{23}^{(m)} = T_{24}^{(m)} - T_{16}^{(m)}$, with

$$T_{24}^{(m)} = \sum_{n=0}^{N-1} \int_\Omega (u_{n+1}(x) - u_n(x)) \cdot P_\mathcal{D}\varphi(x,(n+1)\delta t)\mathrm{d}x.$$

Thanks to the limits given by (4.30), (4.31), (4.32), (4.33) and (4.34), and thanks to (4.29), we obtain that $\lim_{m\to\infty} T_{16}^{(m)} = T_{25}$, with

$$T_{25} = \quad -\nu\sum_{i=1}^{d}\int_0^T\int_\Omega \nabla u^{(i)}(x,t)\cdot\nabla\varphi^{(i)}(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T b(u(\cdot,t),u(\cdot,t),\varphi(\cdot,t))\mathrm{d}t + \int_0^T\int_\Omega f(x)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t.$$

Since (4.30), (4.31), (4.32), (4.33) and (4.34) are available as well, replacing $P_\mathcal{D}\varphi(\cdot,n\delta t)$ by $P_\mathcal{D}\varphi(\cdot,(n+1)\delta t)$ in T_{17}, T_{18}, T_{19}, T_{20} and T_{21}, we also get using (4.14) with $v = P_\mathcal{D}\varphi(\cdot,(n+1)\delta t)$ that $\lim_{m\to\infty} T_{24}^{(m)} = T_{25}$. Thus we get that $\lim_{m\to\infty} T_{23}^{(m)} = 0$, which concludes the proof of (4.35). Thanks to (4.29), (4.35), (4.30), (4.31), (4.32), (4.33) and (4.34), we thus obtain (1.9), provided that we can prove

$$\mathrm{div}\bar{u}(x,t) = 0, \quad \text{for a.e. } (x,t) \in \Omega\times(0,T).$$

This last relation can be shown, following the proof of (3.20). This completes the proof of the above theorem. □

REMARK 4.4. *Using the above proof of convergence, we get the energy inequality for $d=2$ or 3 from inequality (4.19), since we have the property*

$$\int_0^T\int_\Omega (\nabla u^{(i)}(x,t))^2\mathrm{d}x\mathrm{d}t \le \liminf_{m\to\infty}\sum_{n=0}^{N^{(m)}-1}\delta t[u_{n+\frac12}^{(m,i)},u_{n+\frac12}^{(m,i)}]_{\mathcal{D}^{(m)}}$$

5. Numerical results.

An industrial implementation of a colocated finite volume scheme may be found in [1] for instance, where complex applications are considered. Focusing in this paper on properties of convergence and error estimates, some simple numerical experiments are described here to observe the convergence rate of Schemes (3.7) and (4.13)-(4.14) with respect to the space and time discretizations. To that purpose, we use a prototype code where the nonlinear equations are solved by an underrelaxed Newton method, and the linear systems by a direct band Gaussian elimination solver. This code handles Stokes or Navier-Stokes problems with various boundary conditions, using non uniform rectangular or triangular meshes on general 2D polygonal domains.

The linear Stokes equations are first considered in the case $d=2$, $\Omega = (0,1)\times(0,1)$, $\nu = 1$, and f is taken to satisfy (3.1) with a solution equal to

$$\begin{aligned}
\bar{u}^{(1)}(x^{(1)},x^{(2)}) &= -\partial^{(2)}\Psi(x^{(1)},x^{(2)})\\
\bar{u}^{(2)}(x^{(1)},x^{(2)}) &= \partial^{(1)}\Psi(x^{(1)},x^{(2)})\\
\bar{p}(x^{(1)},x^{(2)}) &= 100\ \left((x^{(1)})^2 + (x^{(2)})^2\right),
\end{aligned}$$

denoting by $\Psi(x^{(1)},x^{(2)}) = 1000\ [x^{(1)}(1-x^{(1)})x^{(2)}(1-x^{(2)})]^2$. The approximate solution (u,p) is computed with the scheme (3.7). The observed numerical order of convergence, considering the norms $\|u - P_{\mathcal{D}}\bar{u}\|_{L^2(\Omega)}$ and $\|p - P_{\mathcal{D}}\bar{p}\|_{L^2(\Omega)}$, is equal to 2 for the velocity components, and to 1 for the pressure in the cases of non uniform rectangular and square meshes (from 400 to 6400 grid blocks). Note that in these cases, there is apparently no need for a significant positive value of the stabilization coefficient λ. The observed numerical order of convergence is similar in the case of triangular meshes (from 1400 to 5600 grid blocks), but values such as $\lambda = 10^{-4}$, $\alpha = 1$ have to be used in order to avoid oscillations in the pressure field. This confirms that in the case of triangles, the approximate pressure space is too large to avoid stabilization. In fact, other tests were performed (e.g. the classical backward step) which show that stabilization is also needed in the case of rectangles when more severe problems are considered. Note that in industrial implementations, stabilization may be performed with other means, see [28], [1], (see also [6] in the triangular case).

We then proceed to a similar comparison in the case of transient nonlinear problems. Considering a transient adaptation of the above steady-state analytical solution, the continuous problem is then defined by zero initial and boundary conditions, $T = 0.1$, and the function f is taken to satisfy (1.1) with a solution equal to

$$\begin{array}{ll} \bar{u}^{(1)}(x^{(1)}, x^{(2)}, t) & = -t\ \partial^{(2)}\Psi(x^{(1)}, x^{(2)}) \\ \bar{u}^{(2)}(x^{(1)}, x^{(2)}, t) & = t\ \partial^{(1)}\Psi(x^{(1)}, x^{(2)}) \\ \bar{p}(x^{(1)}, x^{(2)}, t) & = 100\ t\ \left((x^{(1)})^2 + (x^{(2)})^2\right), \end{array}$$

with the same function Ψ as above. We again observe an order 2 of convergence of the approximate solution at times $t = .05$ and $t = .1$, when the space and the time discretizations are simultaneously modified with the same ratio (from $\delta t = 0.01$ to $\delta t = 0.0025$ as the size of the mesh is divided by 4). Similar observations are still valid for the classical Green-Taylor example.

6. Conclusions. The above numerical results show that the theoretical error estimate which is proved in Section 3 for the linear Stokes equations is non optimal; a sharper estimate is currently being written [20] under more regularity assumptions on the mesh.

The proof of convergence of the full space-time discrete approximation of (1.1) given by (4.14) uses estimates on the time translates, which were introduced in the $L^2(\Omega \times (0,T))$ framework for the proof of convergence of the finite volume method for degenerate parabolic equations [19, 15] and used for several other cases, see e.g.[21]. A major difficulty which arises here is the handling on the nonlinear convective term, as in the continuous case, which leads us to establish an estimate on the time translates in $L^1(0,T;L^2(\Omega))$. This new technique may be used for parabolic problems with other type of nonlinearities.

We remarked that industrial codes use other types of stabilizations than the one used here. Further works will be devoted to the mathematical study of such stabilizations, for which, to our knowledge, no proof of convergence is known up to now.

Finally, let us also mention undergoing work on a generalization of the scheme studied here to the full transient Navier-Stokes equations including the energy balance, under the Boussinesq approximation.

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