# Small Viscosity Solution of Linear Scalar Hyperbolic Problems with Discontinuous Coefficients in Several Space Dimensions.

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**Abstract**

In this paper we show that, for multi-D scalar nonconservative hyperbolic problems with an expansive discontinuity of the coefficient localized on $\{x_d = 0\}$, a solution can successfully be singled out via a small viscosity approach. An interesting feature is that the so selected small viscosity solution is, in general, less regular than the data. Two stability results are also given under different assumptions on the coefficients. Finally, we give results about the small viscosity solution for discontinuous coefficients in either compressive setting or traversing setting. By doing so, we show that both the loss of regularity illustrated by $Fig.1$. and the need to make a stability assumption on the coefficients in order to get uniform Evans stability are specific to discontinuous coefficients in expansive configuration.

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# 1 Introduction

For the sake of simplicity we will restrict ourselves to Cauchy problems with piecewise constant coefficients on each side of $\{x_d = 0\}$. Let $y := (x_1, \ldots, x_{d-1})$ denote the space variable tangential to the boundary and let $t$ stand for the time parameter belonging to $(0, T)$. Let us fix $T > 0$ arbitrarily once for all and consider the following problem:

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} a_j \partial_j u = f, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\ u|_{t=0} = h. \end{cases} \tag{1.1}$$

where the source term $f$ belongs to $H^s((0, T) \times \mathbb{R}^d)$ with $s > \frac{1}{2}$ and the Cauchy data $h$ belongs to $H^s(\mathbb{R}^d)$. We underline that no corner compability conditions are assumed to hold. For the sake of simplicity, the coefficients $a_j$ are assumed to be piecewise constant and only discontinuous through $\{x_d = 0\}$. Let $a_{j,R}$ [resp $a_{j,L}$] stand for the restriction of the coefficient $a_j$ to $\{x_d > 0\}$ [resp $\{x_d < 0\}$]. Well-posedness theory for equations of the form (1.1) is well-established when the characteristics for the problem are uniquely defined, which is the case for sufficiently regular coefficients. These arguments break down when the coefficient is, for instance, discontinuous across a fixed hypersurface, which is our current framework. For conservative problems, Poupaud and Rascle, by using generalized characteristics in the sense of Filippov, extend the basic theory to a new framework including some cases in which the coefficients may be discontinuous. Their approach works very well as long as there is uniqueness of the characteristics in the sense of Filippov, but breaks down otherwise. The subject has been studied by various approaches in many works. Bouchut, James and Mancini ([3]) show for a linear scalar problem in several space dimensions existence and uniqueness of a solution, as well as a stability result. These results are proved provided a one-sided Lipschitz condition on the coefficients is satisfied. There are several works on the topic, as these kinds of problems naturally arise from mathematical modeling. Among the works on this subject, we may refer to the works of Bachmann and al. ([1],[2]), Diperna and Lions ([5]), Fornet ([6],[16]), Gallouët([8]), LeFloch and al. ([4],[11],[12],[10]).

Considering the behavior of the characteristics in a neighborhood of the area of discontinuity of the coefficients, three cases arise. This paper will mainly focus on the expansive case, for which $sign(a_d) = sign(x)$ in a neighborhood of the interface $\{x_d = 0\}$.

This case is the most troublesome as far as uniqueness is concerned. Since one of the main concerns of this paper is the interesting case where the discontinuity of the coefficient is expansive, *except for the last section of the paper devoted to the other two cases, the reader shall assume that the discontinuity of the coefficient is in expansive setting*, which writes:

**Assumption 1.1.** *For all* $(t, y) \in (0, T) \times \mathbb{R}^{d-1}$, *there holds:*

$$a_d|_{x_d=0^+}(t, y) > 0$$

*and*

$$a_d|_{x_d=0^-}(t, y) < 0.$$

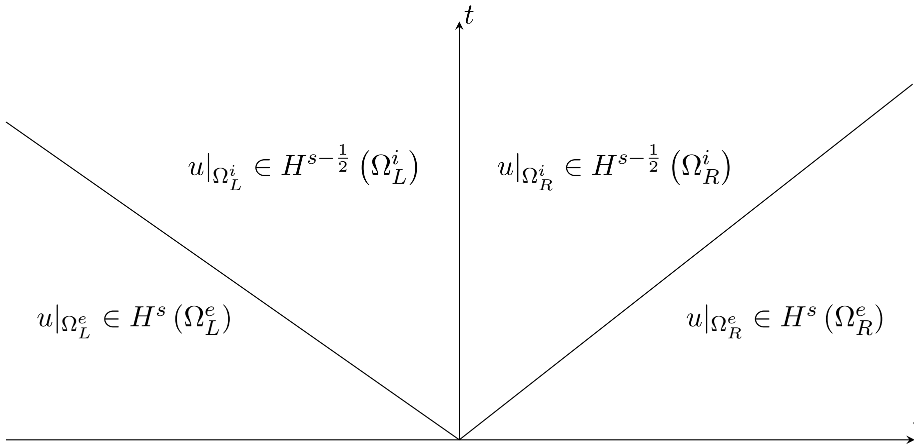
Under Assumption 1.1, there are an infinity of solutions to problem (1.1). Indeed, prescribing $u|_{x_d=0} = g$ gives an unique solution to the problem. In particular, if $g$ belongs to $H^s((0, T) \times \mathbb{R}^{d-1})$, the induced solution $u$ belongs to $H^s((0, T) \times \mathbb{R}^d)$. A crucial remark is that, as stated in the abstract, the natural solution selected by a small viscosity approach is not, generally speaking, in $H^s((0, T) \times \mathbb{R}^d)$; thus the viscous approach, viewed as a selective process to get a unique solution, does not favor smoothness in the expansive framework. The loss of regularity remains hidden for problems in one space dimension but takes place as far as several space dimensions are present.

Let us now describe our approach of the problem. We introduce the regularization $u^\varepsilon$ of $u$, which is defined as the unique solution of the following viscous problem:

$$(1.2) \qquad \begin{cases} \partial_t u^\varepsilon + \displaystyle\sum_{j=1}^{d} a_j \partial_j u^\varepsilon - \varepsilon \Delta u^\varepsilon = f, & (t, x) \in (0, T) \times \mathbb{R}^d, \\ u^\varepsilon|_{t=0} = h, \end{cases}$$

where $\Delta$ stands for the spatial Laplacian $\sum_{j=1}^{d} \partial_j^2$.

FIG. 1




This picture shows the Sobolev smoothness of the small viscosity solution $u := \lim_{\varepsilon\to 0^+} u^\varepsilon$ over the open domains $\Omega_L^e$, $\Omega_L^i$, $\Omega_R^i$ and $\Omega_R^e$.

Let us consider the following transmission problem:

$$
(1.3)\qquad \begin{cases} \partial_t u_R + \displaystyle\sum_{j=1}^{d} a_{j,R}\partial_j u_R = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*+}, \\ \partial_t u_L + \displaystyle\sum_{j=1}^{d} a_{j,L}\partial_j u_L = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*-}, \\ u_R|_{x_d=0^+} - u_L|_{x_d=0^-} = 0, \\ \partial_d u_R|_{x_d=0^+} - \partial_d u_L|_{x_d=0^-} = 0, \\ u_R|_{t=0} = h, \quad u_L|_{t=0} = h, \end{cases}
$$

where we recall that the source term $f$ belongs to $H^s((0,T)\times\mathbb{R}^d)$ and the Cauchy datum $h$ belongs to $H^s(\mathbb{R}^d)$. In what follows, we shall abbreviate $\mathbb{R}^{d-1}\times\mathbb{R}_{*+}$ [resp $\mathbb{R}^{d-1}\times\mathbb{R}_{*-}$] as $\mathbb{R}^d_+$ [resp $\mathbb{R}^d_-$].

Let us denote by $\mathcal{E}$ the set of functions whose Sobolev regularity is as described by Fig. 1. The following lemma states the well-posedness of problem (1.3), which is the limit problem satisfied by the small viscosity solution for coefficients with discontinuities in expansive setting.

**Lemma 1.2.**
*There is a unique solution* $u := u_L \mathbf{1}_{x_d<0} + u_R \mathbf{1}_{x_d>0}$ *of* (1.3) *in* $\mathcal{E}$.

**Remark 1.3.** *The function* $u$ *belongs at most to* $C^1\left((0,T)\times\mathbb{R}^d\right)$ *(when the corner compatibility conditions are checked) but not to* $C^2\left((0,T)\times\mathbb{R}^d\right)$, *hence, by Sobolev embedding,* $u$ *has an upper bound on its global Sobolev smoothness.*

This paper is mainly devoted to the proof of the following results concerning the case of one fixed line of discontinuity in expansive setting. In the first result stability estimates are obtained as a result of the uniform Evans condition holding, while in the second result estimates are proved by integration by parts. Prior to the statement of the two theorems, let us stress that problem (1.2) is proved to be always Evans-stable. However, contrary to the uniform Evans stability, Evans stability alone is not sufficient to yield (or even infirm) the $L^2$ stability of the hyperbolic-parabolic problem at hand.

**Theorem 1.4.** *Problem* (1.2) *is uniformly Evans stable if and only if there holds:*

$$a_{d,R}^{-1}a_{j,R} - a_{d,L}^{-1}a_{j,L} = 0, \quad 1 \le j \le d-1. \tag{1.4}$$

*Let* $u^\varepsilon$ *stand for the solution of problem* (1.2) *and* $u$ *be the solution of problem* (1.3)*. If the equalities* (1.4) *are checked, then there is* $C > 0$, *such that for all* $0 < \varepsilon < 1$, *we have:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\mathbb{R}^d)} \le C\varepsilon.$$

Remark that Theorem 1.4 can be extended for piecewise $C^\infty$ coefficients constant outside a compact set, which would also result in longer and more technical proofs.

In the case equalities 1.4 are not checked, assuming that $s \ge 1$, we establish the stability of the problems by integration by parts. The proof conducted here for piecewise constant coefficient seems difficult to generalize to variable coefficients.

**Theorem 1.5.** *There holds:*

$$\|u^\varepsilon - u\|_{L^\infty((0,T):L^2(\mathbb{R}^d))} \le C\varepsilon.$$

Theorem 1.5 is deduced from the $L^2$ stability of the problem which still holds even though the Evans condition is not uniformly holding. For both Theorem 1.4 and Theorem 1.5, we refer to Fig.1 for details about the small viscosity solution $u$.

## 2 Proof of Lemma 1.2

Let us consider $(u_L, u_R)$ defined as the solution of the transmission problem (1.3). We recall that the transmission conditions on the boundary $\{x_d = 0\}$ write:

$$\begin{cases} [u]_{x_d=0} := u_R|_{x_d=0^+} - u_L|_{x_d=0^-} = 0, \\ [\partial_d u]_{x_d=0} := \partial_d u_R|_{x_d=0^+} - \partial_d u_L|_{x_d=0^-} = 0. \end{cases} \tag{2.1}$$

We will focus on showing the following result, which leads to Lemma 1.2:

**Proposition 2.1.** *The Sobolev regularity of the trace of the small viscosity solution, namely $u|_{x_d=0}$, is, generally speaking, given by the worst Sobolev smoothness between the trace of the source term $f|_{x_d=0}$ and trace of the Cauchy datum $h|_{x_d=0}$.*

*Proof.* From the first transmission condition over the boundary, we get that:

$$u|_{x_d=0} := u_R|_{x_d=0^+} = u_L|_{x_d=0^-},$$

moreover, denoting $x_0 := t$, we have:

$$\partial_j u_R|_{x_d=0^+} = \partial_j u_L|_{x_d=0^-} = \partial_j u|_{x_d=0}, \quad \forall 0 \leq j \leq d-1.$$

Using the equation, we obtain that:

$$[\partial_d u]_{x_d=0} = a_{d,R}^{-1} \left( f|_{x_d=0} - \left( \partial_t + \sum_{j=1}^{d-1} a_{j,R} \partial_j \right) u|_{x_d=0} \right)$$
$$- a_{d,L}^{-1} \left( f|_{x_d=0} - \left( \partial_t + \sum_{j=1}^{d-1} a_{j,L} \partial_j \right) u|_{x_d=0} \right)$$

Hence the second transmission condition states that the trace $u_0 := u|_{x_d=0}$ is solution of the following well-posed Cauchy problem with constant coefficients:

$$\begin{cases} \partial_t u_0 + \sum_{j=1}^{d} \left( a_{d,R}^{-1} - a_{d,L}^{-1} \right)^{-1} \left( a_{d,R}^{-1} a_{j,R} - a_{d,L}^{-1} a_{j,L} \right) \partial_j u_0 = f|_{x_d=0}, \quad (t,y) \in (0,T) \times \mathbb{R}^{d-1}, \\ u_0|_{t=0} = h|_{x_d=0}. \end{cases}$$

The Sobolev regularity of $u_0$ is fixed by the Sobolev regularity of the data hence implying Proposition 2.1. As illustrated by Fig. 1, $u$ is then less regular on the zone where it results from the propagation of the trace $u_0$ along the characteristic, while it remains as regular as the Cauchy datum on the area where $u$ can be computed from the propagation of the Cauchy datum along the characteristics. □

# 3 Proof of Theorems 1.4 and 1.5

## 3.1 Construction of an approximate solution

Let us proceed with our first step of the proof. We wish to emphasize that this step is common to both the proofs of Theorem 1.4 and Theorem 1.5. We construct an approximate solution $u^\varepsilon_{app}$ of the following formulation of equation (1.2) as a transmission problem:

$$(3.1)\qquad \begin{cases} u^\varepsilon := u_L^{\varepsilon,e}\mathbf{1}_{\Omega_L^e} + u_L^{\varepsilon,i}\mathbf{1}_{\Omega_L^i} + u_R^{\varepsilon,i}\mathbf{1}_{\Omega_R^i} + u_R^{\varepsilon,e}\mathbf{1}_{\Omega_R^e}, \\ \partial_t u_R^{\varepsilon,i} + \displaystyle\sum_{j=1}^d a_{j,R}\partial_j u_R^{\varepsilon,i} - \varepsilon\Delta u_R^{\varepsilon,i} = f, \quad (t,x)\in\Omega_R^i, \\ \partial_t u_R^{\varepsilon,e} + \displaystyle\sum_{j=1}^d a_{j,R}\partial_j u_R^{\varepsilon,e} - \varepsilon\Delta u_R^{\varepsilon,e} = f, \quad (t,x)\in\Omega_R^e, \\ \partial_t u_L^{\varepsilon,i} + \displaystyle\sum_{j=1}^d a_{j,L}\partial_j u_L^{\varepsilon,i} - \varepsilon\Delta u_L^{\varepsilon,i} = f, \quad (t,x)\in\Omega_L^i, \\ \partial_t u_L^{\varepsilon,e} + \displaystyle\sum_{j=1}^d a_{j,L}\partial_j u_L^{\varepsilon,e} - \varepsilon\Delta u_L^{\varepsilon,e} = f, \quad (t,x)\in\Omega_L^e, \\ u^\varepsilon \in C^1((0,T)\times\mathbb{R}^d), \\ u^\varepsilon|_{t=0} = h. \end{cases}$$

Since for fixed positive $\varepsilon$ the exact solution $u^\varepsilon$ belongs to $C^1((0,T)\times\mathbb{R}^d)$, we will seek $u^\varepsilon_{app}$ as a function in $C^1((0,T)\times\mathbb{R}^d)$. We will construct the profiles separately on the four domains $\Omega_L^i$, $\Omega_L^e$, $\Omega_R^i$ and $\Omega_R^e$ as a first step then impose the necessary transmission conditions in order for $u^\varepsilon_{app}$ to belong to $C^1((0,T)\times\mathbb{R}^d)$.

In other words, we will construct separately pieces of the approximate solution as follows:

$$u^{\varepsilon,i}_{app,L} := u^{\varepsilon}_{app}|_{\Omega^i_L} = \sum_{j=0}^{2M} \left( \underline{\mathbf{U}}^i_{j,L}(t,y,x) + \mathbf{U}^{c,i}_{j,L}\left(t,y,\frac{x_d - a_{d,L}t}{\sqrt{\varepsilon}}\right)\right) \varepsilon^{\frac{j}{2}},$$

with $\underline{\mathbf{U}}^i_{j,L}$ belonging to $L^2(\Omega^i_L)$ and the characteristic boundary layer profiles $\mathbf{U}^{c,i}_{j,L}(t,y,\theta_L)$ belong to $e^{-\delta\theta_L}L^2((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*+})$, for some $\delta > 0$.

$$u^{\varepsilon,e}_{app,L} := u^{\varepsilon}_{app}|_{\Omega^e_L} = \sum_{j=0}^{2M} \left( \underline{\mathbf{U}}^e_{j,L}(t,y,x) + \mathbf{U}^{c,e}_{j,L}\left(t,y,\frac{x_d - a_{d,L}t}{\sqrt{\varepsilon}}\right)\right) \varepsilon^{\frac{j}{2}},$$

with $\underline{\mathbf{U}}^e_{j,L}$ belonging to $L^2(\Omega^e_L)$ and the characteristic boundary layer profiles $\mathbf{U}^{c,e}_{j,L}(t,y,\theta_L)$ belong to $e^{\delta\theta_L}L^2((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*-})$, for some $\delta > 0$.
The functions $u^{\varepsilon,i}_{app,L}$ and $u^{\varepsilon,e}_{app,L}$ also check the following transmission conditions on $\Gamma_L$ :

$$\begin{cases} \lim\limits_{x_d\to a_{d,L}t, x_d > a_{d,L}t} u^{\varepsilon,i}_{app,L} = \lim\limits_{x_d\to a_{d,L}t, x_d < a_{d,L}t} u^{\varepsilon,e}_{app,L}, \\ \lim\limits_{x_d\to a_{d,L}t, x_d > a_{d,L}t} (\partial_d - a_{d,L}\partial_t)\, u^{\varepsilon,i}_{app,L} = \lim\limits_{x_d\to a_{d,L}t, x_d < a_{d,L}t} (\partial_d - a_{d,L}\partial_t)\, u^{\varepsilon,e}_{app,L}. \end{cases}$$

In a symmetric manner, we have:

$$u^{\varepsilon,i}_{app,R} := u^{\varepsilon}_{app}|_{\Omega^i_R} = \sum_{j=0}^{2M} \left( \underline{\mathbf{U}}^i_{j,R}(t,y,x) + \mathbf{U}^{c,i}_{j,R}\left(t,y,\frac{x_d - a_{d,R}t}{\sqrt{\varepsilon}}\right)\right) \varepsilon^{\frac{j}{2}},$$

with $\underline{\mathbf{U}}^i_{j,R}$ belonging to $L^2(\Omega^i_R)$ and the characteristic boundary layer profiles $\mathbf{U}^{c,i}_{j,R}(t,y,\theta_R)$ belong to $e^{\delta\theta_R}L^2((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*-})$, for some $\delta > 0$.

$$u^{\varepsilon,e}_{app,R} := u^{\varepsilon}_{app}|_{\Omega^e_R} = \sum_{j=0}^{2M} \left( \underline{\mathbf{U}}^e_{j,R}(t,y,x) + \mathbf{U}^{c,e}_{j,R}\left(t,y,\frac{x_d - a_{d,R}t}{\sqrt{\varepsilon}}\right)\right) \varepsilon^{\frac{j}{2}},$$

with $\underline{\mathbf{U}}^e_{j,R}$ belonging to $L^2(\Omega^e_R)$ and the characteristic boundary layer profiles $\mathbf{U}^{c,e}_{j,R}(t,y,\theta_R)$ belong to $e^{-\delta\theta_R}L^2((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_{*+})$, for some $\delta > 0$.
The functions $u^{\varepsilon,i}_{app,R}$ and $u^{\varepsilon,e}_{app,R}$ also satisfy the following transmission conditions on $\Gamma_R$ :

$$\begin{cases} \lim\limits_{x_d\to a_{d,R}t, x_d < a_{d,R}t} u^{\varepsilon,i}_{app,R} - \lim\limits_{x_d\to a_{d,R}t, x_d > a_{d,R}t} u^{\varepsilon,e}_{app,R} = 0, \\ \lim\limits_{x_d\to a_{d,R}t, x_d < a_{d,R}t} (\partial_d - a_{d,R}\partial_t)\, u^{\varepsilon,i}_{app,R} - \lim\limits_{x_d\to a_{d,R}t, x_d > a_{d,R}t} (\partial_d - a_{d,R}\partial_t)\, u^{\varepsilon,e}_{app,R} = 0. \end{cases}$$

In addition $u^{\varepsilon,i}_{app,R}$ and $u^{\varepsilon,i}_{app,L}$ check the following transmission conditions on $\{x_d = 0\}$ :

$$\begin{cases} u^{\varepsilon,i}_{app,R}|_{x_d=0^+} - u^{\varepsilon,i}_{app,L}|_{x_d=0^-} = 0, \\ \partial_d u^{\varepsilon,i}_{app,R}|_{x_d=0^+} - \partial_d u^{\varepsilon,i}_{app,L}|_{x_d=0^-} = 0. \end{cases}$$

We will now show that the underlined profiles can be computed by induction as a first step. Plugging our ansatz in the equation and identifying the terms with same powers of $\varepsilon$, we get, to begin with, that $(\underline{\mathbf{U}}_{0,R}, \underline{\mathbf{U}}_{0,L})$ satisfies the following transmission problem:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{0,R} + \sum_{j=1}^{d} a_{j,R} \partial_j \underline{\mathbf{U}}_{0,R} = f, \quad (t,x) \in (0,T) \times \mathbb{R}^d_+, \\ \partial_t \underline{\mathbf{U}}_{0,L} + \sum_{j=1}^{d} a_{j,L} \partial_j \underline{\mathbf{U}}_{0,L} = f, \quad (t,x) \in (0,T) \times \mathbb{R}^d_-, \\ \underline{\mathbf{U}}_{0,R}|_{x_d=0^+} - \underline{\mathbf{U}}_{0,L}|_{x_d=0^-} = 0, \\ \partial_d \underline{\mathbf{U}}_{0,R}|_{x_d=0^+} - \partial_d \underline{\mathbf{U}}_{0,L}|_{x_d=0^-} = 0, \\ \underline{\mathbf{U}}_{0,R}|_{t=0} = h, \quad \underline{\mathbf{U}}_{0,L}|_{t=0} = h. \end{cases}$$

The profiles $\underline{\mathbf{U}}^i_{0,R}$, $\underline{\mathbf{U}}^e_{0,R}$, are then obtained as the restrictions of $\underline{\mathbf{U}}_{0,R}$ respectively to $\Omega^i_R$ and $\Omega^e_R$ and the profiles $\underline{\mathbf{U}}^i_{0,L}$ and $\underline{\mathbf{U}}^e_{0,L}$ are obtained as the restrictions of $\underline{\mathbf{U}}_{0,L}$ respectively to $\Omega^i_L$ and $\Omega^e_L$.
If $n$ is an even number greater than 1, we get that $(\underline{\mathbf{U}}_{n,R}, \underline{\mathbf{U}}_{n,L})$ is solution of the following transmission problem:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{n,R} + \sum_{j=1}^{d} a_{j,R} \partial_j \underline{\mathbf{U}}_{n,R} = \Delta \underline{\mathbf{U}}_{n-2,R}, \quad (t,x) \in (0,T) \times \mathbb{R}^d_+, \\ \partial_t \underline{\mathbf{U}}_{n,L} + \sum_{j=1}^{d} a_{j,L} \partial_j \underline{\mathbf{U}}_{n,L} = \Delta \underline{\mathbf{U}}_{n-2,L}, \quad (t,x) \in (0,T) \times \mathbb{R}^d_-, \\ \underline{\mathbf{U}}_{n,R}|_{x_d=0^+} - \underline{\mathbf{U}}_{n,L}|_{x_d=0^-} = 0, \\ \partial_d \underline{\mathbf{U}}_{n,R}|_{x_d=0^+} - \partial_d \underline{\mathbf{U}}_{n,L}|_{x_d=0^-} = 0, \\ \underline{\mathbf{U}}_{0,R}|_{t=0} = 0, \quad \underline{\mathbf{U}}_{0,L}|_{t=0} = 0. \end{cases}$$

On the other hand, if $n$ is an odd number then both $\underline{\mathbf{U}}_{n,R} = 0$ and $\underline{\mathbf{U}}_{n,L} = 0$. This shows that the right scale for observing the underlined profiles is of order $\varepsilon$, not $\sqrt{\varepsilon}$. The profiles $\underline{\mathbf{U}}^i_{n,R}$ [resp $\underline{\mathbf{U}}^e_{n,R}$] are by definition the restriction of $\underline{\mathbf{U}}_{n,R}$ to the domain $\Omega^i_R$ [resp $\Omega^e_R$].

The profiles $\underline{\mathbf{U}}^i_{n,R}$, $\underline{\mathbf{U}}^e_{n,R}$, $\underline{\mathbf{U}}^i_{n,L}$ and $\underline{\mathbf{U}}^e_{n,L}$ are deduced from $\underline{\mathbf{U}}_{n,R}$ and $\underline{\mathbf{U}}_{n,L}$ by taking the appropriate restrictions the same way as described above.

We will now compute as second step the characteristic boundary layer profiles by induction. Since the computations of these profiles are symmetric on both half-spaces, we will focus here on describing the construction of the profiles $\mathbf{U}_L^{c,i}$ and $\mathbf{U}_L^{c,e}$. The domains $\Omega_L^i$ and $\Omega_L^e$ are separated by the characteristic curve $\Gamma_L$. The characteristic hypersurface $\Gamma_L$ is given as:

$$\Gamma_L : \left\{ (t,x) \in (0,T) \times \mathbb{R}^d_- : x_d = a_{d,L} t \right\}.$$

Let us consider a function $f$ depending of $(t,x) \in (0,T) \times \mathbb{R}^d$. The jump of $f$ through $\Gamma_L$, denoted by $[f]_{\Gamma_L}$, is defined as:

$$[f]_{\Gamma_L}(t,y) := \lim_{x_d \to a_{d,L} t, x_d > a_{d,L} t} f(t,x) - \lim_{x_d \to a_{d,L} t, x_d < a_{d,L} t} f(t,x), \quad \forall (t,y) \in (0,T) \times \mathbb{R}^{d-1}.$$

Since $u^\varepsilon_{app}$ belongs to $C^0((0,T)\times\mathbb{R}^{d-1})$, we recover the following transmission condition: $[\mathbf{U}^c_L]_{\theta_L=0} = -[\underline{\mathbf{U}}_L]_{\Gamma_L}$, where $[\mathbf{U}^c_L]_{\theta_L=0}$ is defined as

$$[\mathbf{U}^c_L]_{\theta_L=0}(t,y) := \lim_{\theta_L \to 0^+} \mathbf{U}^c_L(t,y,\theta_L) - \lim_{\theta_L \to 0^-} \mathbf{U}^c_L(t,y,\theta_L), \quad \forall (t,y) \in (0,T) \times \mathbb{R}^{d-1}.$$

In what follows, to simplify the notations, we will drop the "L" subscripts. Since $u^\varepsilon_{app}$ belongs actually to $C^1((0,T)\times\mathbb{R}^d)$, the function $\partial_d u^\varepsilon_{app} - a_d \partial_t u^\varepsilon_{app}$ is continuous through $\Gamma$. As a consequence, we get, for $j \geq 0$, the following jump condition:

$$[\partial_d \underline{\mathbf{U}}_j - a_d \partial_t \underline{\mathbf{U}}_j]_\Gamma = a_d [\partial_t \mathbf{U}^c_j]_{\theta=0} - (1+a_d^2)[\partial_\theta \mathbf{U}^c_{j+1}]_{\theta=0}.$$

Taking as a convention that the profiles indexed with a negative subscript vanishes, the above equality writes:

$$[\partial_\theta \mathbf{U}^c_j]_{\theta=0} = \frac{1}{1+a_d^2} \left( a_d [\partial_t \mathbf{U}^c_{j-1}]_{\theta=0} - [\partial_d \underline{\mathbf{U}}_{j-1} - a_d \partial_t \underline{\mathbf{U}}_{j-1}]_\Gamma \right),$$

thus leading to the following profile equation for $(\mathbf{U}^{c,+}_j, \mathbf{U}^{c,-}_j)$ :

$$\begin{cases} \left(\partial_t + \sum_{k=1}^{d-1} a_k \partial_k\right) \mathbf{U}_j^{c,+} - \partial_\theta^2 \mathbf{U}_j^{c,+} = \sum_{k=1}^{d-1} \partial_k^2 \mathbf{U}_{j-2}^{c,+}, & (t,y,\theta) \in (0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}_{*+}, \\ \left(\partial_t + \sum_{k=1}^{d-1} a_k \partial_k\right) \mathbf{U}_j^{c,-} - \partial_\theta^2 \mathbf{U}_j^{c,-} = \sum_{k=1}^{d-1} \partial_k^2 \mathbf{U}_{j-2}^{c,-}, & (t,y,\theta) \in (0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}_{*-}, \\ [\mathbf{U}_j^c]_{\theta=0} = -[\underline{\mathbf{U}}_j]_\Gamma, \\ [\partial_\theta \mathbf{U}_j^c]_{\theta=0} = \frac{1}{1+a_d^2} \left( a_d [\partial_t \mathbf{U}_{j-1}^c]_{\theta=0} - [\partial_d \underline{\mathbf{U}}_{j-1} - a_d \partial_t \underline{\mathbf{U}}_{j-1}]_\Gamma \right), \\ \mathbf{U}_j^{c,+}|_{t=0} = 0, \quad \mathbf{U}_j^{c,-}|_{t=0} = 0. \end{cases}$$

Note well that we avoid the use of the "e" and "i" superscripts since it would force us to distinguish each side of the interface. This problem reduces itself, after change of unknowns, to a parabolic Cauchy problem. Indeed let us take $\psi$ whose restriction to $(0,T) \times \mathbb{R}^d_+$ [resp $(0,T) \times \mathbb{R}^d_-$] belongs to $H^\infty((0,T) \times \mathbb{R}^d_+)$ [resp $H^\infty((0,T) \times \mathbb{R}^d_-)$] and satisfies:

$$\begin{cases} [\psi]_{\theta=0} = -[\underline{\mathbf{U}}_j]_\Gamma, \\ [\partial_\theta \psi]_{\theta=0} = \frac{1}{1+a_d^2} \left( a_d [\partial_t \mathbf{U}_{j-1}^c]_{\theta=0} - [\partial_d \underline{\mathbf{U}}_{j-1} - a_d \partial_t \underline{\mathbf{U}}_{j-1}]_\Gamma \right). \end{cases}$$

Let us denote by $\mathcal{P}$ the parabolic operator:

$$\mathcal{P} := \partial_t + \sum_{k=1}^{d-1} a_k \partial_k - \partial_\theta^2.$$

Making use of the linearity of the considered equation, the profile $\mathbf{U}_j^c$ is obtained as $\mathbf{U}_j^c = \psi + V_j^c$, with $V_j^c$ solution of the well-posed parabolic Cauchy problem:

$$\begin{cases} \mathcal{P} V_j^c = -\mathcal{P}\psi + \sum_{k=1}^{d-1} \partial_k^2 U_{j-2}^c, & (t,y,\theta) \in (0,T) \times \mathbb{R}^d, \\ V_j^c|_{t=0} = 0. \end{cases}$$

Remark that the characteristic boundary layers forming are of weak amplitude since there holds: $\mathbf{U}_0^{c,+} = 0$, $\mathbf{U}_0^{c,-} = 0$. This explains that the speed of convergence towards $u$ in $L^2$ norm occurs at a rate in $\mathcal{O}(\varepsilon)$, even though characteristic boundary layers form.

### 3.2 Stability of the problem

We will now prove stability estimates for the problem (1.2). We define the error $w^\varepsilon := u^\varepsilon_{app} - u^\varepsilon$. Let us denote by $w^{\varepsilon\pm}$ the restriction of $w^\varepsilon$ to $\{\pm x_d > 0\}$. $(w^{\varepsilon+}, w^{\varepsilon-})$ is then solution of the transmission problem:

$$\begin{cases} \partial_t w^{\varepsilon+} + \displaystyle\sum_{j=1}^d a_{R,j}\partial_j w^{\varepsilon+} - \varepsilon\Delta w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x_d > 0, (t,y) \in (0,T)\times\mathbb{R}^{d-1}, \\ \partial_t w^{\varepsilon-} + \displaystyle\sum_{j=1}^d a_{L,j}\partial_j w^{\varepsilon-} - \varepsilon\Delta w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x_d < 0, (t,y) \in (0,T)\times\mathbb{R}^{d-1}, \\ w^{\varepsilon+}|_{x_d=0^+} - w^{\varepsilon-}|_{x_d=0^-} = 0, & \\ \partial_d w^{\varepsilon+}|_{x_d=0^+} - \partial_d w^{\varepsilon-}|_{x_d=0^-} = 0, & \\ w^{\varepsilon+}|_{t=0} = 0, \quad w^{\varepsilon-}|_{t=0} = 0. & \end{cases}$$

By construction of our approximate solution, $R^\varepsilon$ belongs to $L^2((0,T)\times\mathbb{R}^d)$. We have to extend the definition of $w^\varepsilon$ to $(t,x) \in \mathbb{R}^{d+1}$. In this paper, for the sake of simplicity, we will make a slight abuse of notations and write:

$$\begin{cases} \partial_t w^{\varepsilon+} + \displaystyle\sum_{j=1}^d a_{R,j}\partial_j w^{\varepsilon+} - \varepsilon\Delta w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x_d > 0, (t,y) \in \mathbb{R}^d, \\ \partial_t w^{\varepsilon-} + \displaystyle\sum_{j=1}^d a_{L,j}\partial_j w^{\varepsilon-} - \varepsilon\Delta w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x_d < 0, (t,y) \in \mathbb{R}^d, \\ w^{\varepsilon+}|_{x_d=0^+} - w^{\varepsilon-}|_{x_d=0^-} = 0, & \\ \partial_d w^{\varepsilon+}|_{x_d=0^+} - \partial_d w^{\varepsilon-}|_{x_d=0^-} = 0, & \\ w^{\varepsilon+}|_{t<0} = 0, \quad w^{\varepsilon-}|_{t<0} = 0, & \end{cases}$$

with $R^\varepsilon$ belonging to $L^2(\mathbb{R}^{d+1})$ and vanishing in the past. We prove in [6] that we can do so.

We will now reformulate this problem into an equivalent problem, posed on one side of the boundary. Defining $\tilde{w}^\varepsilon := \begin{pmatrix} w^{\varepsilon+}(t,x) \\ w^{\varepsilon-}(t,-x) \end{pmatrix}$, the error equation rewrites as the doubled problem on one side of the boundary:

$$\begin{cases} \tilde{\mathcal{H}}^\varepsilon \tilde{w}^\varepsilon = \varepsilon^M \tilde{R}^\varepsilon, \quad \{x_d > 0\}, \\ \Gamma \tilde{w}^\varepsilon|_{x_d=0} = 0, \\ \tilde{w}^\varepsilon|_{t<0} = 0. \end{cases}$$

where $\mathcal{H}^\varepsilon = \partial_t + \sum_{j=1}^{d} \tilde{A}_j \partial_j - \varepsilon \Delta$,

$$\tilde{A}_d = \begin{bmatrix} a_{d,R} & 0 \\ 0 & -a_{d,L} \end{bmatrix},$$

$$\tilde{A}_j = \begin{bmatrix} a_{j,R} & 0 \\ 0 & a_{j,L} \end{bmatrix}, \quad \forall j = 1, \cdots, d-1 \ and \ \Gamma = \begin{bmatrix} 1 & -1 \\ \partial_x & \partial_x \end{bmatrix}.$$

As established for instance in [16], if our linear mixed parabolic problem satisfies a Uniform Evans Condition, the following stability estimate holds:

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2((0,T)\times\mathbb{R})} = \mathcal{O}\left(\varepsilon^{\frac{M-1}{2}}\right),$$

taking $M$ large enough achieves then the proof of Theorem 1.4. Uniform Evans stability of the problem will be investigated in section 4 while stability through integration by parts will be established in section 5 under no assumption at all on the coefficients of the tangential derivatives, leading then to Theorem 1.5.

# 4 Evans Stability analysis of the problem

Let us introduce:

$$\mathbb{A}_R(\zeta) = \begin{pmatrix} 0 & 1 \\ i\left(\tau + \sum_{j=1}^{d-1} \eta_j a_{j,R}\right) + \gamma & a_{d,R} \end{pmatrix}$$

$$\mathbb{A}_L(\zeta) = \begin{pmatrix} 0 & 1 \\ i\left(\tau + \sum_{j=1}^{d-1} \eta_j a_{j,L}\right) + \gamma & a_{d,L} \end{pmatrix}$$

In what follows let $\kappa_R$ be

$$\kappa_R(\tau,\eta) := \tau + \sum_{j=1}^{d-1} a_{j,R}\eta_j$$

and $\kappa_L$ be

$$\kappa_L(\tau,\eta) := \tau + \sum_{j=1}^{d-1} a_{j,L}\eta_j.$$

As shown in the 1-D framework in [16], the uniform Evans condition is checked if and only if for all $\zeta := (\gamma,\tau,\eta) \in \mathbb{R}_{*+} \times \mathbb{R}^d$, there holds:

$$|det\left(\mathbb{E}_-\left(\mathbb{A}_R(\zeta)\right), \mathbb{E}_+\left(\mathbb{A}_L(\zeta)\right)\right)| \geq C > 0,$$

where, in the case $M \in \mathcal{M}_N(\mathbb{C})$, $\mathbb{E}_-(M)$ [resp $\mathbb{E}_+(M)$]denotes the space spanned by the generalized eigenvectors associated to the eigenvalues of $M$ with negative [resp positive] real part. In addition, if $\mathbb{E}$ and $\mathbb{F}$ are two linear subspaces of $E$ such that $\dim \mathbb{E} + \dim \mathbb{F} = \dim E$, then the notation $det(\mathbb{E}, \mathbb{F})$ stands for the determinant obtained by taking orthonormal bases for both $\mathbb{E}$ and $\mathbb{F}$. In our case, for fixed $\zeta$, $\mathbb{E}_-\left(\mathbb{A}_R(\zeta)\right)$ and $\mathbb{E}_+\left(\mathbb{A}_L(\zeta)\right)$ are two linear subspaces of dimension one of $\mathbb{C}^2$.

### 4.1 Computation of the Evans function for medium frequencies

There holds:

$$\mathbb{E}_-(\mathbb{A}_R(\zeta)) = Span \left\{ \begin{pmatrix} 1 \\ \mu_R^-(\zeta) \end{pmatrix} \right\}$$

where $\mu_R^-$ denotes the eigenvalue of $\mathbb{A}_R$ with negative real part and is given by:

$$\mu_R^-(\zeta) = \frac{1}{2} a_{d,R} - \frac{1}{4} \left((a_{d,R}^2 + 4\gamma)^2 + 16\kappa_R^2\right)^{\frac{1}{4}} \left( \left( 1 + \frac{16\kappa_R^2}{\left(a_{d,R}^2 + 4\gamma\right)^2} \right)^{-\frac{1}{2}} + 1 \right)$$

$$-i\ sign(\kappa_R) \frac{1}{4} \left((a_{d,R}^2 + 4\gamma)^2 + 16\kappa_R^2\right)^{\frac{1}{4}} \left( 1 - \left( 1 + \frac{16\kappa_R^2}{\left(a_{d,R}^2 + 4\gamma\right)^2} \right)^{-\frac{1}{2}} \right)$$

Moreover, we have:

$$\mathbb{E}_+(\mathbb{A}_L(\zeta)) = Span \left\{ \begin{pmatrix} 1 \\ \mu_L^+(\zeta) \end{pmatrix} \right\}$$

where $\mu_L^+$ denotes the eigenvalue of $\mathbb{A}_L$ with positive real part and is given by:

$$\mu_L^+(\zeta) = \frac{1}{2} a_{d,L} + \frac{1}{4} \left((a_{d,L}^2 + 4\gamma)^2 + 16\kappa_L^2\right)^{\frac{1}{4}} \left( \left( 1 + \frac{16\kappa_L^2}{\left(a_{d,L}^2 + 4\gamma\right)^2} \right)^{-\frac{1}{2}} + 1 \right)$$

$$+i\ sign(\kappa_L)\frac{1}{4}\left((a_{d,L}^2+4\gamma)^2+16\kappa_L^2\right)^{\frac{1}{4}}\left(1-\left(1+\frac{16\kappa_L^2}{\left(a_{d,L}^2+4\gamma\right)^2}\right)^{-\frac{1}{2}}\right)$$

If we consider $\zeta$ such that $0 < c \leq |\zeta| \leq C < \infty$, an Evans function is the modulus of the following determinant:

$$\begin{vmatrix} 1 & 1 \\ \mu_R^-(\zeta) & \mu_L^+(\zeta) \end{vmatrix}$$

that is to say: $|\mu_L^+(\zeta) - \mu_R^-(\zeta)|$, since $\mu_L^+$ keeps a positive real part and $\mu_R^-$ keeps a negative real part, for all $\zeta$ such that $0 < c \leq |\zeta| \leq C < \infty$, there holds:

$$\left|\mu_L^+(\zeta) - \mu_R^-(\zeta)\right| > 0.$$

Hence the Evans Condition is checked for medium frequencies.

### 4.2 Computation of the asymptotic Evans function when $|\zeta| \to \infty$.

As in [13], to deal properly with high frequencies, we introduce the weight $\Lambda$ defined by:

$$\Lambda(\zeta) = \left(1+\tau^2+\gamma^2+|\eta|^2\right)^{\frac{1}{2}}$$

We recall that the scaled eigenspaces for high frequencies write then:

$$\mathbb{E}_-(\mathbb{A}_R(\zeta)) = Span\left\{\begin{pmatrix} 1 \\ \Lambda^{-1}\mu_R^-(\zeta) \end{pmatrix}\right\}$$

$$\mathbb{E}_+(\mathbb{A}_L(\zeta)) = Span\left\{\begin{pmatrix} 1 \\ \Lambda^{-1}\mu_L^+(\zeta) \end{pmatrix}\right\}$$

An asymptotic Evans function for high frequencies writes:

$$\lim_{|\zeta|\to\infty}\left|\frac{\mu_L^+(\zeta) - \mu_R^-(\zeta)}{\Lambda(\zeta)}\right|.$$

Since there is $C > 0$ such that, for all $\rho \geq C > 0$, $\Re e\frac{\mu_L^+(\zeta)}{\Lambda(\zeta)} \geq C$ and $\Re e\frac{\mu_R^-(\zeta)}{\Lambda(\zeta)} \leq -C$, making $|\zeta| \to \infty$, we have:

$$\left|\frac{\mu_L^+(\zeta) - \mu_R^-(\zeta)}{\Lambda(\zeta)}\right| \geq C' > 0.$$

Therefore, the Evans Condition is checked for high frequencies.

### 4.3 Low frequency analysis of the Evans condition in the 1-D framework.

Due to the expansive setting of the discontinuity the two eigenvalues $\mu_L^+$ and $\mu_R^-$ are hyperbolic, which means that:

$$\mu_L^+|_{\zeta=0} = 0,$$

$$\mu_R^-|_{\zeta=0} = 0.$$

As a result, both linear subspaces $\mathbb{E}_-(\mathbb{A}_R(\zeta))$ and $\mathbb{E}_+(\mathbb{A}_L(\zeta))$ cease to be well-defined. Since $\mathbb{A}_L$ and $\mathbb{A}_R$ have similar definitions, let us focus on proving the continuous extension of the linear subspace $\mathbb{E}_-(\mathbb{A}_R)$ to low frequencies.

$\mathbb{A}_R(\zeta)$ appears in an ODE of the form:

$$\partial_z \begin{pmatrix} w_R \\ \partial_z w_R \end{pmatrix} = \mathbb{A}_R(\zeta) \begin{pmatrix} w_R \\ \partial_z w_R \end{pmatrix} + F_R,$$

This time let $\rho$ be $\rho := (\tau^2 + \gamma^2)^{1/2}$. We have then:

$$\partial_z \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix} := \begin{pmatrix} 0 & \rho Id \\ \rho^{-1}(i\tau + \gamma) Id & a_{d,R} \end{pmatrix} \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix} := \rho \check{\mathbb{A}}_R(\check{\zeta}, \rho) \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix},$$

where

$$\check{\mathbb{A}}_R(\check{\zeta}, \rho) := \begin{pmatrix} 0 & 1 \\ \rho^{-1}(i\check{\tau} + \check{\gamma}) & \rho^{-1} a_{d,R} \end{pmatrix}$$

with $\check{\tau} := \frac{\tau}{\rho}$ and $\check{\gamma} := \frac{\gamma}{\rho}$.

A continuous extension of the positive and negative spaces of $\mathbb{A}_L$ and $\mathbb{A}_R$ has to be performed if we want to study the Evans function for low frequencies. These extended spaces will be denoted by $\mathbb{E}_-^{lim}(\mathbb{A}_R)$ and $\mathbb{E}_+^{lim}(\mathbb{A}_L)$, and are computed as follows:

$$\mathbb{E}_-^{lim}(\mathbb{A}_R) = \mathbb{E}_-(\check{\mathbb{A}}_R)|_{\check{\tau}=1, \check{\gamma}=0, \rho=0},$$

and

$$\mathbb{E}_+^{lim}(\mathbb{A}_L) = \mathbb{E}_+(\check{\mathbb{A}}_L)|_{\check{\tau}=1, \check{\gamma}=0, \rho=0}.$$

The low frequency asymptotic Evans condition writes then:

$$\mathbb{E}_-^{lim}(\mathbb{A}_R) \bigcap \mathbb{E}_+^{lim}(\mathbb{A}_L) = \{0\}.$$

Let us look at the negative eigenvalue of $\check{\mathbb{A}}_R(\check{\zeta},\rho)$ that we will note $\check{\lambda}_R(\check{\zeta},\rho)$ and compute its associated eigenvector:

$$\check{\mathbb{A}}_R \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \check{\lambda}_R \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

We get:

$$v_2 = \check{\lambda} v_1,$$

and multiplying by $\rho > 0$ the second coordinate of our vector gives:

$$(i\check{\tau} + \check{\gamma})v_1 + a_{d,R} v_2 = \rho \check{\lambda} v_2$$

Making $\rho \to 0^+$, we obtain that:

$$\check{\lambda}_R(\check{\zeta},\rho) = -\frac{i\check{\tau} + \check{\gamma}}{a_{d,R}}$$

As a result

$$\lim_{\rho\to 0^+} \mathbb{E}_- \left(\check{\mathbb{A}}_R(\check{\zeta},\rho)\right) = Span \left\{ \begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a_{d,R}} \end{pmatrix} \right\}$$

The same way, we have:

$$\lim_{\rho\to 0^+} \mathbb{E}_+ \left(\check{\mathbb{A}}_L(\check{\zeta},\rho)\right) = Span \left\{ \begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a_{d,L}} \end{pmatrix} \right\}$$

Since, by assumption, $a_{d,L} < 0$ and $a_{d,R} > 0$, taking $\check{\gamma} = 0$, we get that the asymptotic Evans condition for low frequencies always holds.

## 4.4 Study of the uniform Evans stability in several space dimensions.

Let $\rho$ be $\rho := (\tau^2 + |\eta|^2 + \gamma^2)^{1/2}$, using similar notations as the one introduced in 1-D, we have:

$$\partial_z \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix} := \begin{pmatrix} 0 & \rho Id \\ \rho^{-1}(i\kappa_R + \gamma) Id & a_{d,R} \end{pmatrix} \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix} := \rho \check{\mathbb{A}}_R(\check{\zeta},\rho) \begin{pmatrix} w_R \\ \rho^{-1}\partial_z w_R \end{pmatrix},$$

where

$$\check{\mathbb{A}}_R(\check{\zeta},\rho) := \begin{pmatrix} 0 & 1 \\ \rho^{-1}(i\check{\kappa}_R + \check{\gamma}) & \rho^{-1} a_{d,R} \end{pmatrix}$$

with $\check{\kappa}_R := \frac{\kappa_R}{\rho}$ and $\check{\gamma} := \frac{\gamma}{\rho}$.
Proceeding like in 1-D, we get:

$$\mathbb{E}_-^{lim}(\mathbb{A}_R) = \mathbb{E}_-(\check{\mathbb{A}}_R)|_{\check{\tau}^2+|\check{\eta}|^2=1,\check{\gamma}=0,\rho=0},$$

and

$$\mathbb{E}_+^{lim}(\mathbb{A}_L) = \mathbb{E}_+(\check{\mathbb{A}}_L)|_{\check{\tau}^2+|\check{\eta}|^2=1,\check{\gamma}=0,\rho=0}.$$

The low frequency Evans condition is satisfied if and only if for all $(\check{\eta},\check{\tau})$ such that $|\check{\eta}|^2+\check{\tau}^2=1$, there holds:

$$a_{d,L}^{-1}\left(\check{\tau}+\sum_{j=1}^{d-1}a_{j,L}\check{\eta}_j\right) \neq a_{d,R}^{-1}\left(\check{\tau}+\sum_{j=1}^{d-1}a_{j,R}\check{\eta}_j\right)$$

which means that for all $0\leq|\check{\eta}|\leq 1$ :

$$\left(a_{d,L}^{-1}-a_{d,R}^{-1}\right)\sqrt{1-|\check{\eta}|^2}+\sum_{j=1}^{d-1}\check{\eta}_j(a_{d,L}^{-1}a_{j,L}-a_{d,R}^{-1}a_{j,R})\neq 0$$

As a first step, we assume that $d=2$. In this case, our geometric stability criterion for low frequencies consists in checking whether for all $-1\leq\check{\eta}\leq 1$, there holds:

$$\left(a_{2,L}^{-1}-a_{2,R}^{-1}\right)\sqrt{1-\check{\eta}^2}+\check{\eta}(a_{2,L}^{-1}a_{1,L}-a_{2,R}^{-1}a_{1,R})\neq 0$$

This property has no chance to hold true if we have not $a_{d,L}^{-1}a_{1,L}=a_{d,R}^{-1}a_{1,R}$. Indeed, taking $\check{\eta}=\pm1$, the studied expression would change sign otherwise, which would mean vanishing of the low frequency asymptotic Evans function for some $\check{\eta}$.

In several space dimensions taking $\check{\eta}_j=0,\ \forall j\neq k,\ \check{\tau}=0$ and $\check{\eta}_k=\pm1$ the same reasoning leads to the following conditions:

$$a_{d,R}^{-1}a_{j,R}-a_{d,L}^{-1}a_{j,L}=0 \quad ,\forall 1\leq j\leq d-1 \tag{4.1}$$

If these conditions are checked, the low frequency Evans function does not depend of $\eta$. As a consequence, the problem is then uniformly Evans stable as the stability analysis becomes identical to the one-dimensional one performed section 4.3.

**Remark 4.1.** *1. Basically equalities* (4.1) *being checked means that the uniform Evans stability of our multi-D problem behaves in a similar manner as the stability of a 1-D problem. Indeed, the operator* $\partial_t+\sum a_j\partial_j$ *also writes:* $\partial_t+a_d\mathbb{X}$ *where* $\mathbb{X}:=\partial_x+\sum_{j=1}^{d-1}(a_d)^{-1}a_j\partial_j$. *For* $d=1$, $\mathbb{X}:=\partial_x$, *our geometric condition of stability states that, for* $d\geq 2$ $\mathbb{X}$ *remains a differential operator with continuous coefficients.*

*2. Be it in one or several space dimensions no such geometric stability condition appears in the compressive case ($a_{d,R} < 0$ and $a_{d,L} > 0$) or the traversing case ($sign(a_{d,R} = sign(a_{d,L}))$). Our stability condition is specific to the multi-D expansive case. Let us try to explain the specificity of the setting inducing a drastic change in behavior from the 1-D feature to the multi-D feature. Actually, our transmission conditions satisfied for the viscous problem couples two hyperbolic modes by the uniform Evans condition. This coupling bodes well in one space dimension but induces instabilities in a multi-D framework. For compressive or traversing discontinuities of the coefficient, at least one parabolic mode is present in the coupling, which results in uniform Evans stability be it in one or several space dimensions.*

We underline that the uniformity of the Evans condition is a crucial matter as far as we are interested in the stability of the problem (see [13]). It is actually a sufficient condition to obtain stability estimates although it is not the case of the (not uniform) Evans condition.

# 5 Stability by integration by parts

Our goal here is to show the stability of the problem holds under no assumption on the coefficients of tangential derivatives, thus establishing Theorem 1.5.

Let us consider $w^\varepsilon := w^{\varepsilon+}\mathbf{1}_{x_d>0} + w^{\varepsilon-}\mathbf{1}_{x_d<0}$ the solution of the error equation. In order to get bounds on $w^\varepsilon$ a preliminary step is to control the normal derivative $W^\varepsilon := \partial_d w^\varepsilon$. We remark that $W^\varepsilon = W^{\varepsilon+}\mathbf{1}_{x_d>0} + W^{\varepsilon-}\mathbf{1}_{x_d<0}$ is solution of the transmission problem:

(5.1)

$$\begin{cases} \partial_t W^{\varepsilon+} + \sum_{j=1}^{d} \partial_j(a_{R,j}W^{\varepsilon+}) - \varepsilon\Delta W^{\varepsilon+} = \varepsilon^M \partial_d R^{\varepsilon+}, & x_d > 0, (t,y) \in (0,T)\times\mathbb{R}^{d-1}, \\ \partial_t W^{\varepsilon-} + \sum_{j=1}^{d} \partial_j(a_{L,j}W^{\varepsilon-}) - \varepsilon\Delta W^{\varepsilon-} = \varepsilon^M \partial_d R^{\varepsilon-}, & x_d < 0, (t,y) \in (0,T)\times\mathbb{R}^{d-1}, \\ W^{\varepsilon+}|_{x_d=0^+} - W^{\varepsilon-}|_{x_d=0^-} = 0, \\ a_{d,R}\partial_d W^{\varepsilon+}|_{x_d=0^+} - \varepsilon\partial_d W^{\varepsilon+}|_{x_d=0^+} = a_{d,L}\partial_d W^{\varepsilon-}|_{x_d=0^-} - \varepsilon\partial_d W^{\varepsilon-}|_{x_d=0^-}, \\ W^{\varepsilon+}|_{t=0} = 0, \quad W^{\varepsilon-}|_{t=0} = 0. \end{cases}$$

Constructing the approximate solution up to order $M = 2$ is sufficient.

Assuming that the data $f$ and $h$ belong to $H^1$ of their respective definition domains, we get that $\partial_d R^{\varepsilon+}$ belongs to $L^2((0,T) \times \mathbb{R}^d_+)$ and that $\partial_d R^{\varepsilon-}$ belongs to $L^2((0,T) \times \mathbb{R}^d_-)$. Note that $W^\varepsilon$ is also the solution of the following Cauchy problem:

$$\begin{cases} \partial_t W^\varepsilon + \sum_{j=1}^{d} \partial_j (a_j W^\varepsilon) - \varepsilon \Delta W^\varepsilon = \varepsilon^M \partial_d R^\varepsilon & , (t,x) \in (0,T) \times \mathbb{R}^d, \\ W^\varepsilon|_{t=0} = 0. \end{cases}$$

$W^\varepsilon$ is the error obtained when replacing the exact viscous solution

$$U^\varepsilon := \partial_d u^\varepsilon$$

by the approximate solution

$$U^\varepsilon_{app} := \partial_d u^\varepsilon_{app}.$$

We emphasize that, for all fixed positive $\varepsilon$, $U^\varepsilon$ satisfies the following two transmission conditions through the hypersurface $\{x_d = 0\}$ :

$$U^\varepsilon|_{x_d=0^+} - U^\varepsilon|_{x_d=0^-} = 0$$

$$(a_{d,R} - \varepsilon \partial_d) U^\varepsilon|_{x_d=0^+} - (a_{d,L} - \varepsilon \partial_d) U^\varepsilon|_{x_d=0^-} = 0$$

For the sake of simplicity, we introduce the following notations: $\|.\|_{L^2_+} = \|.\|_{L^2(\mathbb{R}^d_+)}$ and $\|.\|_{L^2_-} = \|.\|_{L^2(\mathbb{R}^d_-)}$. Let us consider the conservative error equation (5.1) whose unknown is $\partial_d w^\varepsilon$. We proceed with the estimations in two steps. In the first step, we multiply each equation of (5.1) by respectively $W^{\varepsilon+}$ and $W^{\varepsilon-}$ then, for fixed time $t$, integrate the obtained formulae separately on both half-space. We sum the obtained estimates and make use of the transmission conditions over the boundary to get global estimates. Secondly, we consider the following nonconservative error equation:

$$\begin{cases} \partial_t w^\varepsilon + \sum_{j=1}^{d} a_j \partial_j w^\varepsilon - \varepsilon \Delta w^\varepsilon = \varepsilon^M R^\varepsilon, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ w^\varepsilon|_{t=0} = 0. \end{cases}$$

In a second step, we multiply the equation by $w^\varepsilon$ then integrate on the whole space. The obtained estimate for $W^\varepsilon := \partial_d w^\varepsilon$ yields then the desired stability estimates.

Multiplying by the solution $W^{\varepsilon+}$ and integrating by parts on the half-space $\{x_d > 0\}$, we get:

$$\frac{1}{2}\frac{d}{dt}\|W^{\varepsilon+}\|^2_{L^2_+} + \varepsilon\sum_{j=1}^{d}\|\partial_j W^{\varepsilon+}\|^2_{L^2_+}$$

$$+\int_{\mathbb{R}^{d-1}}\left(-\frac{a_{d,R}|_{x_d=0}}{2}\left(W^{\varepsilon+}|_{x_d=0}\right)^2+\varepsilon\left(W^{\varepsilon+}\partial_d W^{\varepsilon+}|_{x_d=0}+\sum_{j=1}^{d-1}W^{\varepsilon+}\partial_j W^{\varepsilon+}|_{x_d=0}\right)\right)\,dy$$

$$=\varepsilon^M\int_{\mathbb{R}^d_+}\partial_d R^{\varepsilon+}W^{\varepsilon+}\,dx.$$

Remark that:

$$-\frac{a_{d,R}|_{x_d=0}}{2}\left(W^{\varepsilon+}|_{x_d=0}\right)^2+\varepsilon W^{\varepsilon+}\partial_d W^{\varepsilon+}|_{x_d=0}$$

$$=\frac{a_{d,R}|_{x_d=0}}{2}\left(W^{\varepsilon+}|_{x_d=0}\right)^2-W^{\varepsilon+}|_{x_d=0}\left(a_{d,R}W^{\varepsilon+}|_{x_d=0}-\varepsilon\partial_d W^{\varepsilon+}|_{x_d=0}\right).$$

We multiply the equation on $\{x_d < 0\}$ by $W^{\varepsilon-}$ and integrate by parts on this half-space, which gives:

$$\frac{1}{2}\frac{d}{dt}\|W^{\varepsilon-}\|^2_{L^2_-} + \varepsilon\sum_{j=1}^{d}\|\partial_j W^{\varepsilon-}\|^2_{L^2_-}$$

$$+\int_{\mathbb{R}^{d-1}}\left(\frac{a_L|_{x_d=0}}{2}\left(W^{\varepsilon-}|_{x_d=0}\right)^2-\varepsilon\left(W^{\varepsilon-}\partial_d W^{\varepsilon-}|_{x_d=0}+\sum_{j=1}^{d-1}W^{\varepsilon-}\partial_j W^{\varepsilon-}|_{x_d=0}\right)\right)\,dy$$

$$=\varepsilon^M\int_{\mathbb{R}^d_-}\partial_d R^{\varepsilon-}W^{\varepsilon-}\,dx.$$

Let us underline that:

$$\frac{a_L|_{x_d=0}}{2}\left(W^{\varepsilon-}|_{x_d=0}\right)^2-\varepsilon W^{\varepsilon-}\partial_d W^{\varepsilon-}|_{x_d=0}$$

$$=-\frac{a_L|_{x_d=0}}{2}\left(W^{\varepsilon-}|_{x_d=0}\right)^2+W^{\varepsilon-}|_{x_d=0}\left(a_L W^{\varepsilon-}|_{x_d=0}-\varepsilon\partial_d W^{\varepsilon-}|_{x_d=0}\right).$$

Thanks to our boundary condition, there holds:

$$\begin{cases} W^{\varepsilon+}|_{x_d=0}\left(a_R W^{\varepsilon+}|_{x_d=0}-\varepsilon\partial_d W^{\varepsilon+}|_{x_d=0}\right)=W^{\varepsilon-}|_{x_d=0}\left(a_L W^{\varepsilon-}|_{x_d=0}-\varepsilon\partial_d W^{\varepsilon-}|_{x_d=0}\right)\\ W^{\varepsilon+}|_{x_d=0}\partial_j W^{\varepsilon+}|_{x_d=0}=W^{\varepsilon-}|_{x_d=0}\partial_j W^{\varepsilon-}|_{x_d=0},\quad \forall j=1\cdots d-1.\end{cases}$$

Thus, by adding our estimates, we obtain that:

$$\frac{1}{2}\frac{d}{dt}\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\varepsilon\sum_{j=1}^d\|\partial_j W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\frac{a_R|_{x_d=0}-a_L|_{x_d=0}}{2}\|W^\varepsilon|_{x_d=0}\|^2_{L^2(\mathbb{R}^{d-1})}$$

$$=\varepsilon^M\int_{\mathbb{R}^d}(\partial_d R^\varepsilon)\,W^\varepsilon\;dx.$$

$$\left|\int_{\mathbb{R}^d}(\partial_d R^\varepsilon)\,W^\varepsilon\;dx\right|\leq\frac{1}{2}\|\partial_d R^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\frac{1}{2}\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}.$$

Since $a_R|_{x_d=0}>0$ and $a_L|_{x_d=0}<0$, by Gronwall Lemma, there is a constant $C>0$ such that:

$$\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}(t)\leq C\varepsilon^M\int_0^T e^{C(t-s)}\left(\|\partial_d R^{\varepsilon+}\|^2_{L^2(\mathbb{R}^d_+)}(s)+\|\partial_d R^{\varepsilon-}\|^2_{L^2(\mathbb{R}^d_-)}(s)\right)\;ds.$$

Constructing the profiles up to order $M=1$, we have then:

$$\|\partial_d w^\varepsilon\|_{L^\infty\left((0,T):L^2(\mathbb{R}^d)\right)}=\mathcal{O}(\varepsilon)$$

and, since $a_R|_{x_d=0}-a_L|_{x_d=0}>0$, we also have:

$$\|\left(\partial_d w^\varepsilon\right)|_{x_d=0}\|_{L^\infty\left((0,T):L^2(\mathbb{R}^{d-1})\right)}=\mathcal{O}(\varepsilon).$$

Now going back to our initial error equation with unknown $w^\varepsilon$, we multiply it by $w^\varepsilon$ then integrate on the whole domain. With $\|.\|_{L^2}$ standing for $\|.\|_{L^2(\mathbb{R}^d)}$, we get then:

$$\frac{1}{2}\left(\frac{d}{dt}\|w^\varepsilon\|^2_{L^2}+(a_L-a_R)\|\left(\partial_d w^\varepsilon\right)|_{x_d=0}\|^2_{L^2(\mathbb{R}^{d-1})}\right)+\varepsilon\sum_{j=1}^d\|\partial_j w^\varepsilon\|^2_{L^2}=\varepsilon^2\int_{\mathbb{R}^d}R^\varepsilon w^\varepsilon\;dx.$$

Remark that $a_L-a_R$ is negative, that is why we had to control $\|\left(\partial_d w^\varepsilon\right)|_{x_d=0}\|^2_{L^2(\mathbb{R}^{d-1})}$ as a first step. Since $\|\left(\partial_d w^\varepsilon\right)|_{x_d=0}\|_{L^\infty\left((0,T):L^2(\mathbb{R}^{d-1})\right)}=\mathcal{O}(\varepsilon)$, proceeding the same way as before, we obtain that:

$$\|w^\varepsilon\|_{L^\infty\left((0,T):L^2(\mathbb{R}^d)\right)}=\mathcal{O}(\varepsilon),$$

which achieves the proof of Theorem 1.5.

## 6 Small viscosity solution when the setting of the discontinuity is either traversing or compressive

We add this section for the sake of completeness. Its aim is to recall results, which can be obtained as corollaries of the main Theorem proved in [16]. Corollary 6.1 concerns the compressive case while the object of Corollary 6.2 is the small viscosity solution in the traversing case. In order to be able to use the results of [16] as they stand, we assume moreover that the data vanishes in the past as well as the solution, which ensures that the compatibility conditions hold for the considered hyperbolic problem. We aim to emphasize two main points, which are:

1. Other than in the expansive case, no additional stability assumption on the coefficients need to be added to ensure uniform Evans stability of the problem in several space dimensions.

2. Other than in the expansive case, the restriction of the obtained solutions to the half-spaces $\{x_d > 0\}$ and $\{x_d < 0\}$ have the same Sobolev regularity as the data, even when several space dimensions are involved.

**Corollary 6.1.** *Let us consider the case where* $a_{d,R} < 0$ *and* $a_{d,L} > 0$*. Let* $u^\varepsilon$ *denote the solution of* (1.2)*, where we recall that* $f$ *belongs to* $H^s((0,T)\times\mathbb{R}^d)$*. Let us define* $\mathbf{u} := \mathbf{u}_R \mathbf{1}_{x_d>0} + \mathbf{u}_L \mathbf{1}_{x_d<0}$ *as the solution of the well-posed problem:*

$$\begin{cases} \partial_t \mathbf{u}_R + \displaystyle\sum_{j=1}^{d} a_{j,R}\partial_j \mathbf{u}_R = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_+^*, \\ \partial_t \mathbf{u}_L + \displaystyle\sum_{j=1}^{d} a_{j,L}\partial_j \mathbf{u}_L = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_-^*, \\ \mathbf{u}_R|_{t<0} = 0, \quad \mathbf{u}_L|_{t<0} = 0. \end{cases}$$

*Problem* (1.2) *is uniformly Evans stable and there holds:*

$$\|u^\varepsilon - \mathbf{u}\|_{L^2((0,T)\times\mathbb{R}^d)} = \mathcal{O}(\varepsilon).$$

*Moreover* $\mathbf{u}_R$ *belongs to* $H^s((0,T)\times\mathbb{R}^d_+)$ *and* $\mathbf{u}_L$ *belongs to* $H^s((0,T)\times\mathbb{R}^d_-)$*.*

Let us see now the Corollary giving the small viscosity solution in the case of a discontinuity in traversing setting.

**Corollary 6.2.** *Let us consider the case where* $sign(a_{d,R}) = sign(a_{d,L})$*. Let* $u^\varepsilon$ *denote the solution of* (1.2)*. Let us define* $\underline{\mathbf{u}} := \underline{\mathbf{u}}_R \mathbf{1}_{x_d>0} + \underline{\mathbf{u}}_L \mathbf{1}_{x_d<0}$ *as the solution of the following well-posed transmission problem:*

$$\begin{cases} \partial_t \underline{\mathbf{u}}_R + \displaystyle\sum_{j=1}^{d} a_{j,R}\partial_j \underline{\mathbf{u}}_R = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_+^*, \\ \partial_t \underline{\mathbf{u}}_L + \displaystyle\sum_{j=1}^{d} a_{j,L}\partial_j \underline{\mathbf{u}}_L = f, \quad (t,y,x_d) \in (0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_-^*, \\ \underline{\mathbf{u}}_R|_{x_d=0} - \underline{\mathbf{u}}_L|_{x_d=0} = 0, \\ \underline{\mathbf{u}}_R|_{t<0} = 0, \quad \underline{\mathbf{u}}_L|_{t<0} = 0. \end{cases}$$

*Problem* (1.2) *is uniformly Evans stable and there holds:*

$$\|u^\varepsilon - \underline{\mathbf{u}}\|_{L^2((0,T)\times\mathbb{R}^d)} = \mathcal{O}(\varepsilon).$$

*Moreover* $\underline{\mathbf{u}}_R$ *belongs to* $H^s((0,T)\times\mathbb{R}^d_+)$ *and* $\underline{\mathbf{u}}_L$ *belongs to* $H^s((0,T)\times\mathbb{R}^d_-)$*.*

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