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ET MANUFACTURES
« ÉCOLE CENTRALE PARIS »

THÈSE

présentée par

Ali CHEAITOU

pour l'obtention du

GRADE DE DOCTEUR

Spécialité : Génie Industriel

Laboratoire d'accueil : Laboratoire Génie Industriel

SUJET: Stochastic Models for Production/Inventory Planning: Application to Short Life-Cycle Products

soutenue le : 21/01/2008

devant un jury composé de :

Président
Michel Minoux: Professeur, LIP6, Université Paris 6

Examinateurs
Yves Dallery: Professeur, Ecole Centrale Paris
Ger Koole : Professeur, Department of Mathematics, VU University Amsterdam
Christian van Delft : Professeur associé, HEC Paris
Jean-Philippe Vial: Professeur, HEC Genève, Université de Genève

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To my family

To my Darling

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Acknowledgments

I am very pleased to begin this manuscript and to end the three years of my Ph.D. work by thanking all the persons who have contributed in the achievement of this doctoral work.

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Chapter 1

Introduction

In this chapter, we give a general introduction of the thesis. First, we provide problem statement of our work. Second, we highlight some research questions that we answer in this thesis. Third, we describe the work of the thesis and present its main contributions. Finally, we present the structure of the manuscript.

1.1 Problem statement

"God does not play dice"[1]. Even if Albert Einstein used his famous sentence in order to defend his idea of the causal determinism of the universe, he agreed with the fact that the predictive determinism is not always achievable due to the lack in our knowledge about the reality of things. Some scientists went further and assumed that the uncertainty is inherent to the nature of things and therefore to their behavior. Among these scientists we cite the members of the "Copenhagen School" that founded the "quantum physics" and formulated the famous Heisenberg's "uncertainty" principle. Other scientists disagreed with that principle and join Einstein's point of view, which supposes that the uncertainty is due to our ignorance of how things really are: "the uncertainty is not in things but in our head: uncertainty is a misunderstanding"[2].

Nevertheless, whatever is the source of uncertainty, this phenomenon exists and must be faced in many domains. During the last decades, many mathematical techniques, permitting to deal with the uncertainty, have been emerged. The use of these techniques allows one to master the uncertainty in order to reduce its negative impact, and consequently that of our misunderstanding.

In the supply chain management domain, a lot of sources of uncertainty can be encountered. Essentially, these uncertainty sources reflect our inability to predict the future behavior of a part or the whole of a system with certainty. For example, the future production capacity of a machinery park is governed by different factors including the ambient temperature, that can not be predicted with precision for a long time period. The knowledge of the exact future demand of a given internal combustion engine spare part is impossible, despite the knowledge of the past demand of that spare part.

The impact of the uncertainty on the supply chain management is considerable. For example, due to the demand uncertainty, many economic issues generate crucial (negative) impacts on the enterprise performance. We cite, as examples, the following two issues:

- Out-of-stock: (Gruen, 2007) calculates an average out-of-stock level of 8.3% worldwide, for the retail industry, based on the analysis of 52 studies that examine the out-of-stocks. This means that for every 13 items a shopper plans to purchase, one will be out-of-stock. The out-of-stock varies between Europe (8.6%) and the US (7.9%). The main cause of this out-of-stock, according to the authors, is the "ordering and forecasting" (47%), which is equivalent to the demand uncertainty. The response of the customers varies from substituting the required item with different brand (26%), not purchasing the item (9%) and buying the item at another place (31%). That induces huge revenues lost for both the retailer and the manufacturer of the out-of-stock product.

- Unsaleable products: according to the 2003 Unsaleables Benchmark Report (Lightburn, 2003)[3], unsaleable products cost the entire grocery industry more than 1% in annual sales. For example for the Supermarket Distributers, the unsalable products costed around 1.12% in 2003. This study shows that more that 40% of the Unsaleables are due to an inadequate inventory management and

[1] Albert Einstein
[2] Jacques Bernoulli
[3] performed by Industry Unsaleables Steering Committee and its industry sponsors, Food Marketing Institute (FMI) and Grocery Manufacturers of America (GMA)

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excess volume is at the heart of the issue, due to poor planning for example.

The globalization of the industrial world, increases the competition between the different manufacturers due to the difference between the costs corresponding to the different countries. This increase in the competition comes to be added to the other issues of the supply chain and therefore makes the production and procurement decision making, inside the companies, more complicated. These aspects make from the planning process a crucial task with high economic and managerial value. Indeed, the planning process takes into account the stochastic aspect of the demand, the out-of-stock issue, the unsaleable problem, the difference between the production and procurement costs and other issues. In this sense, this Ph.D. work constitutes a contribution in the production planning and inventory control domain. In this work, some of the issues and aspects that do not appear in the literature models are considered, and also some other models are improved, in order to make them more reactive and more flexible.

1.2 Research questions

The major goals of this Ph.D. dissertation is to present a quantitative analysis of the planning models of a special type of products, namely the style-goods type products and then to try to generalize the obtained results for the long life cycle type products.

More precisely, the following research problems and questions are to be answered:

- are there any planning models for the short life cycle products that are more flexible or more reactive than those existing in the literature?
- In a short life cycle products framework, how does a return or an anticipated salvage opportunity (payback option) impact the optimal production/ordering policy?
- How does a production or procurement system with dual production mode behave?
- What is the impact of the information on the optimal policy of a production system, especially on a use of the different production modes and the payback (return) option?
- How does the production capacity constraints influence the decisions, especially those related to the choice between the production modes in the different planning periods?
- Can the insights and techniques developed for the short life cycle products be applied in the case of the long life cycle products. In other words, can the results obtained for small production models be generalized to the context of big planning models?

1.3 Thesis scope

In this Ph.D. dissertation, we are especially interested in the planning models for a special category of products, the "style-goods" type products. We develop some models and some results are obtained for this type of products. We try then, in a single chapter of this work, to generalize one of the frameworks developed for the short life cycle products, to be applied on the long life cycle products.

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Therefore, in this work we provide some stochastic models to answer the questions asked in the previous section, in a style-goods type products context, which induce a short selling season. For the first question we propose many models in which there exist more flexibility due to more action opportunities and more decision variables. In order to answer the second question, in the majority of our models, we permit at the beginning of each planning period, via a decision variable, the return or the salvage of a certain quantity of the available inventory and we show the impact of this decision on the optimal policy.

In order to answer the third question, we introduce in the majority of our models, production systems with two production or procurement modes, permitting to satisfy the demand using two different modes with different production costs. The fourth question is answered via two chapters, where we model and show the impact of the information on the optimal policy: a two-period production model with demand forecasts update and a two-stage contract model with demand forecast update also. These two updates are performed using external information collected during the first stage of the model.

We introduce also another two-period production model that shows the impact of the production capacity constraints on the planning decisions, which answers the fifth question asked above.

Since it is difficult, even impossible to provide complete analytical solutions for a multi-periodic planning models even in a two-period planning setting, we give some analytical insights for two-period models and we see that these insights may be applied in a multi-periodic setting. On the other hand, we introduce a new multi-periodic planning model, in which we use the production framework with two production modes, developed for the small planning models. We define then upper and lower bounds on its optimal policy, which permit to provide an approximated solution and to answer our last research question.

1.4 Thesis structure

The remaining part of this Ph.D. dissertation contains eight chapters which are:

Chapter 2: this chapter aims at defining the main concepts used in this thesis. We begin by giving a brief historical background of the logistics, the supply chain and their military beginnings. We introduce then the supply chain and its concepts, such as the flow, the capacity, the inventory, the inventory management and the information in the supply chain. Then, we detail the notions and parameters of two important pillars of our study: the inventory management and the information in the supply chain. For the inventory management, we define the role and the functions of the inventory and the costs related to the use of inventories. For the information in the supply chain, we emphasize on the demand, especially on the demand forecasts, showing the methods used to develop demand forecasts and the impact of the stochastic (uncertain) nature of the forecasts on the supply chain. Then we provide a study of the planning in the supply chain and especially the production planning and inventory control.

Chapter 3: in this chapter, we present an inventory management model, which constitutes the simplest planning model of this thesis and the basic work for the other chapters. We define in this chapter an extension of the *newsvendor* model with initial inventory. In addition to the classical decisions and parameters of the *newsvendor* model with initial inventory, we introduce a new decision variable: a

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salvage opportunity at the beginning of the horizon, which may be very beneficial in the case of high initial inventory level. We develop the expression of the optimal policy for this extended model, for a general demand distribution. The structure of this optimal policy is particular and is characterized by two threshold levels. Some managerial insights are given via numerical examples.

Chapter 4: in this chapter we develop a two-period production planning and inventory control model with two production modes and multiple return opportunities. We provide a general modeling, which considers an initial inventory at the beginning of the planning horizon and many preliminary fixed orders in a backlog framework. The model is then solved by a dynamic programming approach. A closed-form analytical solution is developed for the second period, based on the results of Chapter 3 and a semi-analytical solution is provided for the first period using an algorithm developed in this chapter. We then provide some insights regarding this type of two-stage inventory decision process with the help of numerical examples.

Chapter 5: in this chapter we provide an extension of the model presented in Chapter 4. We use the same framework of Chapter 4 in a constrained production capacities setting. We formulate this model using a dynamic programming approach. We prove the concavity of the global objective function and we establish the closed-form expression of the second period optimal policy. Then, via a numerical solution approach, we solve the first period problem and exhibit the structure of the corresponding optimal policy. Some insights are provided, via numerical examples, that characterize the basic properties of our model and the effect of some significant parameters, especially the capacity constraints. We show how do the capacity constraints influence the optimal policy in combination with the costs differences between the different production modes.

Chapter 6: in this chapter we provide another extension of the model presented in Chapter 4. In the context of the framework shown in Chapter 4, we introduce a new and important parameter representing an external information. This market information permits the update of the second period demand distribution. The information is stochastic at the beginning of the first period and becomes deterministic at the beginning of the second period. We define the information via a joint distribution with the second period demand. We develop the optimal policy of the second period subproblem. Then, using dynamic programming, we show that the optimal policy of the first period has the same structure as the first period optimal policy of the model introduced in Chapter 4. We provide a numerical study showing the impact of the information quality on the optimal policy and on the optimal expected objective function. This information quality is modeled by the correlation coefficient between the information and the second period demand.

Chapter 7: in the same context of short life cycle products, we consider in this chapter, a two-stage supply contract model with single period planning horizon, for advanced reservation of capacity or advanced procurement supply. At the first stage, two decisions are made: a first ordered quantity and a certain amount of reserved capacity. At the second decision stage two other decisions are fixed: the use of the already reserved capacity (exercise of options) to order a supplementary quantity and the return of a certain quantity from the available inventory to the supplier. Between these two decision stages, an external information is collected that serves to update the demand forecast. The updated demand forecast

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permits to adjust the decisions of the first stage by exercising options or by returning some units to the supplier. The information is defined via a joint distribution function with the demand. At the end of the horizon, the remaining inventory, if any, is sold (or returned to the supplier) at a salvage value that is in general less than the initial unit production/procurement cost. During the selling season, any satisfied demand is charged with a unit selling price, and any unsatisfied demand is lost and a penalty shortage cost is paid. Between the two decision stages, The objective of the model is to determine the quantities to be ordered before the beginning of the selling season or the amount of capacity to be reserved, in order to satisfy optimally the demand.

Chapter 8: in this chapter, we generalize the framework of dual production mode, developed for the short life cycle products in the previous chapters, to be applied on products of long life cycle. The difference between procurement costs in a multi-periodic planning setting is modeled, by permitting to fix two orders at each period: the first order with a fast production mode, which permits an immediate delivery and the second order with a slow production mode, which has one period delivery delay. The developed model is a discounted backlog one, with proportional production, inventory holding and shortage costs. We allow all these costs to be period dependent. The demands are random variables with probability distribution functions that are independent and possibly different from one period to another. We prove that some of the analytical properties, developed for the short life cycle products problems, are valid for the multi-periodic planning problems (long life cycle products). Since the development of a complete closed-form optimal policy is difficult, even impossible, we provide upper and lower bounds for the optimal decision variables. Then, we extend a known heuristic in order to find approximations for the optimal order sizes. The provided numerical examples show the validity of our approximations.

Chapter 9: this chapter is dedicated to the general conclusions of this work and some propositions for future research.

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Chapter 2

Supply Chain Management, Production and Inventory Planning Generalities

This chapter aims at introducing and defining the different concepts used in this thesis. We begin by giving a brief historical background of the logistics and the supply chain and their military origins. Then we introduce the supply chain and its concepts, such as the capacity, the flow, the inventory and the information. We detail the notions and the parameters of the two important pillars of our study: the inventory management and the information in the supply chain. For the inventory management, we define the role and the functions of the inventory and the costs related to the use of inventories. For the information in the supply chain, we emphasize on the demand and especially on the demand forecasts, showing the methods used to develop demand forecasts and the impact of their stochastic (uncertain) nature on the supply chain. Then we provide a study of the supply chain planning and especially the production and the inventory planning.

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2.1 Brief historical background

In this section, we provide a brief historical background of the logistics and the supply chain. According to (Pimor, 2005), the roots of the logistics come from military practices. The first definition of the *logistics* in the history, has been formulated by Antoine-Henri de Jomini[1]:"the logistics is the practical art of moving and providing the armies by establishing and organizing their lines of provisioning".

Moving an army or some troops of an army can not be accomplished without providing it with ammunition and food. This supply problem has become a real problem in the modern history with the colossal increase in the armies manpower, during the few last centuries, and has got as consequences:

- either the principle of a continuous movement of the armies, from a region to another, in order to find new resources,
- either the use of an embarked logistics, to ensure the provisioning of the troops, during a long campaign, each time where it is impossible to ensure a local supply (from the region where the army is deployed), which is the case of the marine for example,
- or the use of provisioning lines between some stocks and the regions where the army is deployed.

This last mode of provisioning (provisioning lines), has profited from the revolution of the transportation means, during the 20^{th} century, to dominate the other supply modes.

Julius Caesar[2] for example, who commanded a huge army, has quoted in *The Comments*, the corn problem, that was necessary to supply his army, and that was transported by rivers and by carriage columns. He sent his legates to negotiate with the different populations in order to buy the food for the army, and he stocked the bought quantities in the regions where the legions might pass the winter.

At the end of the 16^{th} century and during the first middle of the 17^{th} century, new armies with several tens of thousands of soldiers have begun to appear, involving a lot of logistic problems. Some historians conclude that the movement of the armies, at that era, were commanded rather than constrained by logistic requirements: the armies had to remain moving in order to supply themselves.

To solve this problem, the system of stores (or stocks) has been created by Michel Le Tellier and François Michel Le Tellier de Louvois[3]. These can be considered as the founders of modern military logistics. They have set up a system composed of a network of stores with strategic reserves, permanent vehicle fleets, supply standards and intendant council with a real supply administration. During the 18^{th} century, this system has been spread out and became a complex system constituted of a network of stocks connected by a transportation lines.

In 1805, Napoléon replaced this system by a system of an extreme mobility, where for each military campaign, he built a concentration of supply on the passages of the army.

The wars in Europe between 1870 and 1914 represent two important logistics innovation constituted of the use of the railway and the huge need in supply and ammunition.

[1] 1977-1869, he was a part of Napoleon's staff and instructor of the heir of the throne of Russia

[2] 100 BC-44 BC, roman military and political leader and one of the most influential men in world history

[3] 1603-1685 and , 1641-1691 successively. Two French politicians who has contributed to the reform of the army of Louis XIV

During the second world war, real complex military supply chains have been set up. This war has induced considerable developments, like the transportation management, the development of the handling means and the use of sophisticated production and transportation planning methods. From a theoretical point of view, the war of 1939-1945 has seen the birth of the "Operational Research" domain.

From an industrial point of view, in the 1950s and 1960s most manufacturers emphasized mass production to minimize unit production cost as the primary operations strategy, with little product or process flexibility. New product development was slow and relied exclusively on in-house technology and capacity. "Bottleneck" operations were cushioned with inventory to maintain a balanced line flow, resulting in huge work in process inventory (see (Tan, 2001) and (Farmer, 1997)).

Until the 1970s, the industrial policy consisted of "pushing" the production to the market in order to flood the market and to motivate the customers to buy the products. In general, the production was higher than the demand. In most of the businesses, the responsible of each domain or department tried to minimize the costs related to its activities without worrying about the impact of his decisions on the other parts of the company. At the end of the 1970s, the increase of the number of companies in each of the different industry segments, has increased the competition between these companies. It became therefore necessary to take into account not only the production activities, but also all the industrial activities, including the supply, the distribution and the other activities that are related to the production process. The object of this change was not only to minimize the global costs, but also to increase customers' service level. As a consequent, the "modern" or the industrial supply chain was born.

This domain has been improved after the development of the "Computer Science", which has permitted the development of two branches of the logistics which are "Production Planning" and "Inventory Management" that use the techniques of "Operations Research". During the last decades, these two domains with "Operations Research" techniques have known very important improvements and new branches and practices were born. For example, the first oil crisis has pushed Toyota to create the Kanban system, connected with the "Just in Time Production" principle.

2.2 Supply chain notions

2.2.1 Definition

(Chopra and Meindl, 2007) define the supply chain as the set of parties involved, directly or indirectly, in fulfilling a customer's request. The supply chain, which is also referred to as the logistics network, consists not only of manufacturers and their suppliers, but also transporters, warehouses, distribution centers, retail outlets, as well as raw material, work-in-process inventory and finished products that flow between the facilities. A supply chain involves flows of information, materials and funds between its different stages. Each stage of the supply chain is connected with other stages through these flows.

The objective of each supply chain should be to maximize the overall value generated. The value a supply chain generates is the difference between what the final product is worth to the customer and the costs the supply chain incurs in filling the customers' requests. At the same time, other objective would be the increase (maximizing) of the customer's service level in order to satisfy in an optimal manner its

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requirements. Indeed, the two objectives could be connected via some costs like the backlogging costs (Simchi-Levi et al., 2000).

The supply chain is dynamic and evolves over time. The different parameters of the supply chain, such as the customer's demand, the cost parameters, the strength and importance of the different stages of the supply chain, fluctuate considerably across the time.

2.2.2 Supply chain management

The term "supply chain management" was originally introduced by consultants in early 1980*s* ((Chen and Paulraj, 2004), (Oliver and Webber, 1992)) and has subsequently gained tremendous attention (La Londe, 1998). (Croom et al., 2000) have provided a critical literature review of supply chain management.

Supply chain management is the process of planning, implementing and controlling the operations of the supply chain as efficiently as possible. Supply chain management spans all movement and storage of raw materials, work-in-process inventory and finished goods from point-of-origin to point-of-consumption.

Supply chain management encompasses planning and management of all activities involved in sourcing, procurement, conversion and logistics management activities. Importantly, it also includes coordination and collaboration with channel partners, which can be suppliers, intermediaries, third-party service providers and customers. In essence, supply chain management integrates supply and demand management within and across companies.

An other definition of supply chain management is given by (Berry et al., 1994): supply chain management aims at building trust, exchanging information on market needs, developing new products and reducing the supplier base to a particular original equipment manufacturer (OEM) so as to release management resources for developing meaningful, long term relationship.

2.2.3 Decision levels in supply chain

Even if the supply chain regroups different domains, the decisions inside the supply chain are split into three different decision levels. For example, choosing the location of a factory is a strategic decision, defining or adjusting the production capacities during a given set of time periods is a tactical decision and the scheduling of four different tasks on a machine inside a workshop is an operational issue.

The boundaries between the different levels are not clear. In general, three levels are used: the strategic level, the tactical level and the operational level. Note that these decision levels are generally related to the horizon during which they are applied and to their nature. In general, the decisions of the higher levels are considered as constraints for the lower levels decisions. For example, choosing the location of a factory (strategic) constraints the transportation decisions which is taken every day (operational).

(Dallery, 2000) proposes a classification of the decision levels in a supply chain constituted of four levels (see Figure 2.1).

The strategic decision level includes decisions relative to the design of the supply chain or long-term decisions. Among these decisions we find the number, location and size of the warehouses, of the distribution centers and of the facilities. We can find also the decisions about the Information Technology

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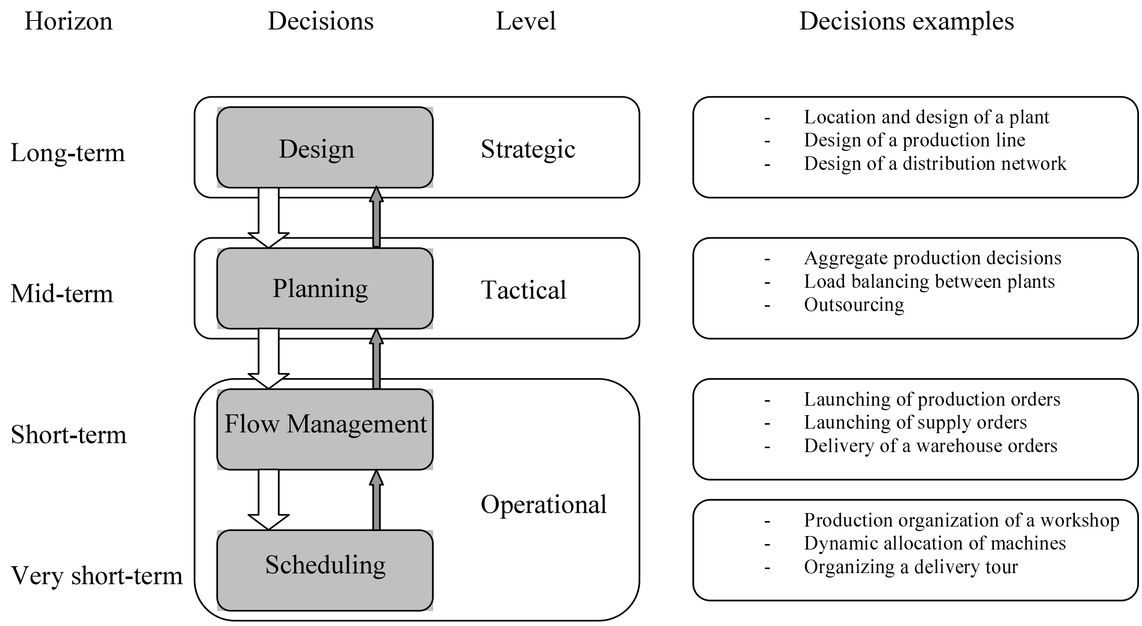

Figure 2.1: Decision levels in a supply chain

infrastructure that support supply chain operations. The strategic partnership with suppliers, distributors and customers and the creation of communication channels (cross docking, direct shipping and third-party logistics) are also strategic decisions.

The tactical (or mid-term) decisions include planning decisions aiming at finding an equilibrium between charge and capacity. These decisions include the production (contracting, locations, scheduling and planning process definition), the inventory (quantity, location and quality of inventory), the sourcing contracts and other purchasing decisions.

The operational decision level is divided into two sub-levels. The first one, called "flow management", is relative to the short time decisions; it includes the decisions of launching the production, ordering and transportation orders. The second sub-level, called "scheduling", is relative to the very short-term decisions; it includes the decisions of scheduling the different tasks inside a workshop.

In this dissertation, we present some decision making models that can be situated in the tactical decision level, which are relative to the planning of the supply chain and precisely to the production and inventory planning.

2.2.4 Supply chain concepts

(Chen and Paulraj, 2004) say that the origin of the supply chain concept has been inspired by many fields including the quality revolution (Dale et al., 1994), notions of materials management and integrated logistics ((Carter and Price, 1993) and (Forrester, 1961)), a growing interest in industrial markets and networks ((Ford, 1990) and (Jarillo, 1993)), the notion of increased focus ((Porter, 1987) and (Snow et al., 1992)) and influential industry-specific studies ((Womack et al., 1990) and (Lamming, 1993)).

In this section we define some of the supply chain concepts that we use in the following chapters of this dissertation.

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Flow (flux)

In a supply chain the flow is the movement of the raw products, components, funds and information from the supplier forward to the customer and vice-versa. The flow of material for example, is the movement of the raw material from the supplier to the manufacturer of components, to the assembly units, to the customer passing through the retailing network. We distinguish between two types of flows: internal and external (Baglin et al., 2001).

The internal flow is relative to the flows (of materials, information and funds) inside a single unit (company) of the supply chain. The external flow is the flow that circulates between two or more units. The performance of the supply chain, or of a unit of the supply chain depends on both types of flow.

If the internal and external flows of materials corresponding to a given a part of the supply chain are unsynchronized, that creates a stock (positive or negative) for that part. This phenomenon can occur in the internal flow, at the connection between two internal flows for example.

Capacity

(Baglin et al., 2001) define the resource as the set of means needed to transform the raw material into components and finished products. These resources include manpower, equipments, buildings, etc..

The capacity is the measure of the aptitude of a resource to deal with a flow. It is the number of treated (produced) units of products per unit of time. This capacity could be reduced due to breakdowns, maintenance and the setup processes for example.

Inventory(stock)

As we have seen above, the "stock" or the inventory level is defined as the accumulation of a difference between flows. We define the stock rotation as the ratio between the reference duration (a year for example) and the flow duration. It is equivalent to the number of successive fillings of the stock (Baglin et al., 2001). Let us define also the stock turnover as the ratio between the cost of goods sold divided by the average of the inventory holding level.

Inventory functions (Baglin et al., 2001) enumerate the different functions that a stock serves for. These functions are classified in different categories:

- *Service function*: the inventory permits to maintain a certain service level, in order to permit an immediate fulfillment of the customer's demands; this function is essential in the case where the delivery time is lower than the production/supply time. This function of the stock allows to avoid the shortage and therefore the lost sales or the backlogs of the customer's demands and the related cost.

- *Capacity regulation function*: the use of an inventory serves to compensate for the predictable difference between the charge (customer's demand) and the capacity. This function is very important in the case of seasonal products, where the demand varies tremendously over the time. It is useful also in the case of lack of capacity.

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- *Circulation function*: this function aims at assuring a certain continuity in the flow inside a structure (a factory for example). In fact, the inventory permits a decoupling between the different units of a structure and allows for example to a downstream machine to continue working, even if the upstream machine that feed it has failed. Therefore the inventory ensures a continuous circulation of products inside a structure, by decoupling its units.

- *Technological function*: for some types of the products, stocking is necessary for the accomplishment of the production process. For example the heat treatment of 10000 pieces in an oven at the time, the production of wine and the production of perfumes.

- *Speculation function*: this function is relative to the use of the stock in order to profit from the difference in ordering or production costs. For example, if the ordering/production costs of a certain product is not stable, then it becomes profitable to order and to stock a certain quantity of that product when the ordering costs are low and then to use these stocked units to fulfill the market demands when the ordering costs are high.

- *Other functions*: one of the other functions of the inventory can be the economy of scale. Ordering a big quantity may induce considerable reduction in the unit ordering costs, which implies the increase of the unit revenue. An important reason to establish and to use inventories may be the transportation. For some products, it is not possible to transport a small quantity, and the retailer must order a quantity sufficient to fill a truck for example each time. A last reason to use inventories could be the need to avoid the transportation (delivery) uncertainty. If the supply delay is not reliable, then to avoid shortage in satisfying customers, or in feeding the production units, one must use a stock at the entry of the system.

On the other hand the inventory presents some disadvantages. The first one that we can cite is the obsolescence of the stored products. Some products have a perishable nature, where after a certain storing period become unusable. The second one is the financial immobilization. This is due to the fact that the stored products represent a part of the company capital. If those units are stored for a certain period, the capital that they represent can not be used or invested somewhere else. The third disadvantage is the unsold articles. In fact these unsold articles represent a complete loss, unless they are sold with a salvage value, which may permit to refund a part of their value. The fourth inconvenient is the cost related to the holding inventories, which includes the handling costs, the space (buildings), etc..

Inventory costs The inventory costs can be classified in three families ((Toomey, 2000) and (Zermati and Mocellin, 2005)). The ordering costs, holding costs and shortage costs. When optimizing the decisions relative to the inventory, one must take into account all these costs.

- *Ordering/production costs*: if we are in an ordering setting, the ordering costs include the salaries of the personnel, the functioning costs (buildings, offices, etc.), reception and test costs, information systems costs and customs costs. These costs represent about 2 to 5% of the value of the ordered articles (Zermati and Mocellin, 2005).

Assume that we are in a production setting, which means that the goods are produced locally, using the proper facilities of the company. In this case the costs will include:

- raw material costs,
- labor costs,
- overheads (costs that cannot be charged directly to a specific product),
- fixed overheads (stable costs that occur regardless of whether or not goods are being produced, i.e. rent of factory, which are allocated according to the number of machine hours),
- variable overheads (changeable overhead costs that vary according to the number of goods produced, such as the energy consumption, which are allocated according to the number of labor hours).

- *Inventory holding cost*: this family of costs can be divided into two subfamilies: financial and functional costs.

  The financial costs represent the financial interest of the money invested in procuring the stocked products.

  The functional costs include the rent and maintenance of the required place, the salaries of the employees, the insurance costs, the equipments costs, the inter-depot transportation costs and the obsolescence costs.

  This family of costs represents about 12 to 25% of the value of the held products (Zermati and Mocellin, 2005). This means that $12-25\%$ of the value of the stocked products is charged per year.

- *Shortage costs*: the inventory shortage corresponds to the case where the units available at the moment when the customer's demand occurs, are not sufficient to satisfy that demand. The related costs are classified in two categories: lost sales costs and backlogging costs. In the lost sales category, if the available units are not sufficient to completely satisfy the demand, the unsatisfied demands are then completely lost and the cost in this case is the "miss to gain". In the second case, the cost will be a penalty shortage cost. This last includes the cost difference between satisfying the demand at the period at which it arrives and the period at which it is satisfied.

  In both cases, some costs can be incurred like the increase in the cost of raw material by the use of substitute materials, the cost of buying or rent of a substitute product.

  In the case where the stock is internal (located between two internal production units for example), the inventory shortage will induce the stop of production of the second (downstream) unit and therefore all the consequent costs, like technical unemployment.

Lead time

The lead time is an important parameter for the supply chain management and due to the globalization of the industry, it becomes directly coupled with the procurement costs. Indeed the lead time is defined as the laps of time between the initiation of any process of production and the completion of that process.

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An example of lead time is the time passed between ordering a new laptop on a website and receiving it at home, which could be around 2 to 3 weeks.

In the supply chain management realm, the lead time means the time from the moment the supplier receives an order to the moment he ships it in the absence of finished goods or intermediate (work in process). It is the time taken to actually manufacture the order without any inventory order other than raw materials or supply parts. As mentioned before, in the current global setting, the bigger is the lead time, the higher is the procurement costs. Indeed, in some countries, the manpower and the raw materials costs are lower than those in other countries, but the procurement from these countries and the shipment to high costs countries induces the obligation of the use of transportation means, which makes the lead time bigger than the procurement from the high costs country directly. As example, the Airbus aircraft manufacturer that has its assembling factories in Toulouse, France, has as supplier for some of the $A380$ wings components the Indonesian Aerospace company that has its factories in Indonesia. Of course, the procurement lead time in his case is higher than that of a French local supplier, but the procurement costs of the Indonesian supplier must be lower due to the difference in the manpower, the raw materials and functional costs between France and Indonesia.

Information in supply chain

In the last years, we are living in the era of the "Information" or the "Information Technology" with all the impacts and changes that have been generated on the society. One of the most important sectors of the society that has been impacted by the "Information revolution" was the management and more precisely the supply chain management. Indeed, the databases, electronic and data interchanges, Internet and Intranets and decision support systems are dominating the markets, the production activities and all the enterprizes sectors.

These technologies have permitted to the economic actors, and especially to the supply chain decision makers to get accurate information about inventory levels, demand forecasts, production operations and delivery status, which had made them more efficient and effective in their decision making processes.

Unfortunately, having more information, even if it makes the management of the supply chain more accurate and efficient, it makes the decision making process more complex.

(Simchi-Levi et al., 2000) enumerate the benefits of disposing of abundant information in the supply chain. This information plays many roles that can be detailed as follows:

- it helps to reduce the variability in the supply chain,
- it helps suppliers make better forecasts, accounting for promotions and market changes,
- it enables the coordination of the manufacturing and distribution systems and strategies,
- it enables retailers to better serve their customers by offering tools for locating desired items,
- it enables retailers to react and adapt to supply problems more rapidly,
- it enables lead time reduction.

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In the models that we present further in this Ph.D. dissertation, two information are crucial: the inventory level and the demand forecasts.

The information about the inventory level is a dynamic information and is collected at the end of each period of the planning horizon, in order to make the decisions relative to the production and ordering for the rest of the horizon periods.

The notion of demand forecasts is developed in the next section.

Demand forecast

In the so-called "push" production processes, the actions are performed in anticipation of the demand, whereas in the called "pull" production processes, they are performed in response to customer's demand. In this Ph.D. work we adopt the "push" production processes.

In a production system where the production lead time is not equal to zero, one must have information about the future demand. Otherwise, any overconsumption (non satisfied orders) and underconsumption (remaining inventory) will charge shortage and inventory holding costs. This means that the supplier must have forecasts to produce/order its goods in order to satisfy the demands of its customers and to minimize its costs.

Therefore, not using forecasts may cause loss of customers' demands, production stop and supplementary logistic costs (inventory holding).

On the other hand, the forecasts allow the decision maker to organize and to exploit his facilities (factories, production capacities, warehouses, etc.) in such a way to satisfy the demands with appropriate time delays. In fact, some production processes require long time delays in order to be achieved. Therefore, in order to organize the facilities and to respect a short delivery time for the customers' demands, the decision maker should anticipate and use estimation of the future demand at the beginning of the production process.

We conclude that the forecasts, or the estimate of the future demand is an absolute requirement to the supply chain planning and especially to production and inventory planning. An exception could be if the product is made or purchased to order. Even in this situation an estimate of the future requirements would still be needed for capacity and/or financial planning.

In a lot of industries, it is possible to improve the forecasts quality over the time. In general improving the forecasts quality means decreasing the variability of the forecasts by decreasing their standard deviation over the time. This quality improvement can be done by using two type of information: internal or external. The internal information is the information collected on the system during the previous time periods, like the realization of random variables. This information can be used in two ways: the first is to update the value of the state variables, and the second is to update the distribution of the random variables of the following periods. This update can be done due to a correlation between the realized random variables and the future periods random variables. The external information is the information collected on the external factors that may influence the system. In a supply chain context, this information can be the market information. Another example is the weather as information influencing the demand of some products.

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Forecast characteristics (Chopra and Meindl, 2007) define the characteristics of the demand forecasts as follows:

- forecasts are always wrong and should thus include both the expected value of the forecast (mean) and a measure of the error (standard deviation),
- long-term forecasts are usually less accurate than short-term forecasts,
- aggregate forecasts are usually more accurate than disaggregate forecasts as they tend to have a smaller standard deviation of error relative to the mean,
- in general, the farther up the supply chain a company is (or the farther it is from the consumer), the greater is the distortion of information it receives.

Forecast methods In general, building forecast models relies on some data like historical sales during the last periods. (Toomey, 2000) introduces a detailed study of the forecasting methods. Indeed, the choice between these methods depends on the available data and on their correlation with the anticipated future demand. If the demand data is reliable, and if there are no any external factor that impacts the anticipated future demand, then the demand history is used and the forecasting method is called "quantitative-intrinsic". If the future demand relates to external factor more than to the past product sales, then the forecasts will be a computational projection based on patterns of external data. The forecast method is therefore called "quantitative-extrinsic". In the case of new products, where there is no historical data available, the forecasting method is "qualitative" which involves intuitive or judgemental evaluation.

In many cases, forecasting future demand relies on the use of all the three methods.

Many forecasting techniques are used to determine the parameters of the future demand. The choice between them depends on the forecast requirements, the patterns of past history and the availability of the data.

For the "quantitative-intrinsic" situations, the used techniques are the "moving average", "exponential smoothing", "extrapolation", "linear prediction" and others. A "simple moving average" is the unweighted mean of the previous n data points. For example a 12-day simple moving average of closing price is the mean of the previous 12 days' closing prices.

There are various popular values of n. A moving average lags behind the latest data point, simply from the nature of its smoothing. A simple moving average can lag to an undesirable extent, and can be disproportionately influenced by old data points dropping out of the average. This is addressed by giving extra weight to more recent data points as in the weighted and exponential moving average.

Exponential smoothing is a method of forecasting based on the weighted average technique, requiring only two data (numbers): the last forecast and the actual demand for the last period (Toomey, 2000). In this method, the new forecast is equal to the old forecast added to a weighted difference between the last period and the old forecast.

For the "quantitative-extrinsic" situations, where it is possible to identify the underlying factors that might influence the variable that is being forecast. If the causes are understood, projections of

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the influencing variables can be made and used in the forecasting process. In this case there are various methods like the "regression analysis using the linear or nonlinear regression", the "autoregressive moving average" and the "autoregressive integrated moving average".

In the case of "qualitative" or "judgemental" forecasting, there exist some other techniques like the "surveys", the "scenario building" and the "technology forecasting".

In this Ph.D. dissertation we do not study these forecasting techniques, and we assume that the demand forecasts are given. Therefore we are not interested in detailing and studying the forecasting techniques.

Uncertainty in Supply chain

The demand forecasts are in general modeled using probability distribution functions. Since the forecasts are always "wrong", as we have shown in the previous section, the demand forecasts that should be taken into account in the planning process of the supply chain, constitute then the main source of uncertainty in that process.

Other uncertainty sources may be the lead time of the suppliers that is not always reliable, the produced quantities, that may vary due to a quality problem for example, the production capacities that might also vary due to the breakdowns of the machines. The economic parameters of the supply chain can also be uncertain (or stochastic), like the selling prices and the ordering or production costs.

These different uncertainties are classified by (Davis, 1993) in three distinct categories: demand uncertainty, process uncertainty and supply uncertainty. The supply uncertainty is due to the supplier processes and includes the lead time, the quantities and the quality of the products. The process uncertainty includes the production process and especially the breakdowns of the machines. The demand uncertainty is for (Davis, 1993) the most serious of all the uncertainties in the supply chain.

(Ho, 1989) regroups the uncertainties in supply chains into two groups: the environmental uncertainty and the system uncertainty. The first category includes uncertainties beyond the production process, such as demand and supply uncertainties. The second category includes the production process uncertainties such as the quality, the lead time, the breakdowns uncertainty.

These uncertainties are worsened by the complexity and the dynamic aspect and interactions between supply chain entities (Bhatnagar and Sohal, 2005). The greater the uncertainty in the supply chain, the greater is its impact on the supply chain performance.

2.3 Supply chain planning

2.3.1 Definition

"Imagine a world in which manufacturing, transportation, warehousing and even information capacity are all limitless and free. Imagine lead times of zero, allowing goods to be produced and delivered instantaneously. In this world, there would be no need to plan in anticipation of demand, because whenever a customer demands a product, the demand would be instantly satisfied. In this world supply

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chain planning plays no role" (Chopra and Meindl, 2005).

However, in the real world, the capacities are limited, the production, transportation and stocking processes are costly, the lead times are sometimes very high. Therefore, in order to fulfill the customers' demand and to minimize the cost, the companies must anticipate the demand and make decisions on the capacities, the inventory levels, the produced quantities, the contracting and subcontracting strategies and the prices, before that the demand is known.

The processes that permit to determine the decisions relevant of these problems, are called supply chain planning.

In this dissertation we are interested in the mid-term or tactical planning and especially in the production and inventory planning, or what is called in the literature the "aggregate planning" that we define in the following section.

2.3.2 Aggregate planning: production and inventory planning

Note that after this point, when we cite "planning" or "planning model", we refer to "production planning" or "production planning model" respectively, unless other indication is given.

Imagine an enterprise for which the market is quite stable, where there is no technological development, no competition with other companies and where the quantities sold per time period are constants. For this company, once the adequate production facilities are chosen, the operations management will be restricted to the supply of raw material and the launching of the production orders in a repetitive manner. The system is therefore stable over the time.

In the real world, enterprises are generally confronted with dynamic environments: the demand (charge) evolves over the time, and therefore they are unstable. These types of fluctuations are generally encountered in the case of seasonal products, or in the case of products with uncertain future demand, as we have mentioned above.

These important fluctuations of the demand over the time, induced by the market characteristics, imply corresponding fluctuations in the charge on the production facilities. Nevertheless, in a mid-time (or tactical) context, the production facilities have fixed capacities that do not change: the number of machines is fixed, the procurement contracts are in general inflexible and could allow the reception of only limited quantities per time period.

In this context, the role of the aggregate planning is to answer the following question: how to face demand fluctuations with a production system with limited flexibility?(Baglin et al., 2001)

The aggregate planning serves then to anticipate the demand evolution, in order to adapt the production facilities to their market.

(Silver and Peterson, 1985) defines the aggregate production planning as follows: given a set of (usually monthly) demand forecasts for a single product, or for some measure of output that is common across several products, the aggregate planning specifies:

- the rate of production, or the produced quantities at each time period, or the equilibrium charge/ capacity,

- the quantities of raw material needed at each period, and the related contracts with the suppliers,
- the size of the workforce, in order to adjust slightly the production capacities,
- the quantities shipped to the internal or external customers at each time period.

In our work ,we assume that the workforce, and consequently the production capacities are constants, and therefore the related decision variables do not appear in our models. We assume also that the quantity shipped are always equal to the quantity produced, and then the decision variables related to the shipped quantities do not appear also.

The problem is then to minimize the total expected incurred cost or to maximize the total expected profit. The components of the costs and of the profits that are taken into account in the planning process are:

- the selling price,
- the production or procurement cost,
- the cost of using a supplementary production capacity (overtime, hiring, training, etc.),
- the inventory holding cost,
- the cost of insufficient production (or capacity), known as the shortage penalty cost,
- the salvage value, or the return value, which is the value of the unit returned to the supplier or sold in a parallel market.

The aggregate planning is a mid-term or tactical planning that is applied not on individual products, but rather on products families. A product family is a set of similar products that have, for example, the same setup costs, the same seasonality properties, the same and approximately the same production rate (Silver et al., 1998).

The aggregate plan that results from this planning or optimization process serves as a guide for the operational decisions and establishes the parameters for the short-term production and distribution decisions.

In this work we assume that the decisions or decision variables that are taken into account in the planning process are relative to a single unit of the supply chain, namely the manufacturer (or the retailer). It would be more profitable if these decisions are taken in a coordinated manner, in such a way that the different stages of the supply chain, such as the manufacturer, the supplier, the retailer and the transporter fix their aggregated plans in a coordinated way, even if it becomes more complicated to be modeled and to be optimized.

Planning parameters

To build a mid-term planning model permitting to optimize the decision variables and to minimize the total cost, some important parameters should be chosen in a suitable manner. These parameters could be defined as follows ((Bitran and Tirupati, 1993) and (Shapiro, 1993)):

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- the aggregation level- product: the suitable aggregation level depends on the cost structure and on the production facility. The majority of the aggregated production models assume a single product which is the case studied in this Ph.D. dissertation,

- the aggregation level- facilities: most of the aggregate planning models assume that all the production facilities are grouped into a single resource. There exist some models, called "monolithic models" that treat each product and each facility alone. Note that the higher the aggregation level, the simpler is the related optimization problem. In this work, we treat only cases with a single product (or product family) and with a single production capacity.

- time unit (planning period): in general, for the mid-term planning models the planning period is the month. Sometimes, the "week" is used, and the choice depends in general on the length of the planning horizon. The "theoretical" models presented in the literature do not mention, generally, the unit of the planning period, because it does not affect the optimal solution.

- planning horizon: the planning horizon is the time interval covered by the production plan. The length of this horizon depends on the nature of the problem being modeled. For example, for the problems presenting an annual seasonality, this planning horizon should be at least of twelve months. For the short life cycle products or the style-goods type products, the planning horizon is very short and in general it is constituted of one or two one-month planning periods. The notion of the planning horizon is discussed in details in the next section.

Planning horizon

The planning horizon is the time interval on which the planning problem is defined. We can distinguish two categories of planning horizon which are differentiated by the structure of the periods that constitute the horizon:

- "frozen" horizon, for which the planning periods are defined and fixed and do not change over the time. This type of planning horizon is used in the case of a defined contract between a supplier and a retailer for example. It is used for some types of products that have a certain time-limited demand defined over a fixed duration. This is the case of style-goods type products for example.

- rolling horizon, for which the planning periods evolve in such a manner that after the end of each period, a new period is added at the end of the horizon. This type of planning horizons is used in the case of products that are produced over a long time duration. Since taking into account all the duration (that may be infinite) is costly in terms of calculation and since the forecast of the far periods is in general unreliable, one can transform the infinite planning horizon to a rolling horizon. (Garcia and Smith, 2000 a) and (Garcia and Smith, 2000 b) show that for all infinite horizon, there exists a corresponding finite horizon for which the optimal decisions of the first period are the same.

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Stochastic vs deterministic planning

As we have seen above, the uncertainty has many sources in the supply chain and then it is difficult to omit its existence. This uncertainty should be taken into account in the decision process.

However, there exist some models that do not recognize the uncertainty. These models are in general built for some problems in which the uncertainty is not very dominating and its influence could be ignored. For example if the production is made on order, then the decision maker knows a priori how much is the future demand to satisfy, and therefore it makes no sense to consider a stochastic demand in the decision process. In such a problem, the decision variables may be the production quantities, knowing the costs, the production capacities and the future demands ((Bitran et al., 1982), (Rajagopalan and Swaminathan, 2001) and (Zangwill, 1966)).

Due to the complexity generated by the stochastic aspect of some parameters, and especially of the demand, there exist in the literature some papers that provide approximated solutions to the original planning model. For example, (Bitran et al., 1982) transform the stochastic model to a deterministic one and add some constraints that guarantee a certain service level. The authors provide an upper bound on the error caused by substituting the optimal solution by the approximated one.

Others, such as (Raino and Ng, 2003) and (Haackman et al., 2002), provide models that deal with stochastic production lead time. The lead time uncertainty is due to stochastic production capacities (breakdowns), lack of supplier reliability, personnel availability, etc.. They use a rolling horizon framework, where they solve at each period the new model after gathering new information and updating the planning horizon. They also approximate the original stochastic model with a deterministic one, using some service level constraints.

(Wang and Gerchak, 1996) introduce a planning model that solves using stochastic dynamic programming in an information updating setting. They take into account, at the same time, stochastic production capacities and stochastic production rate. They show some properties of the optimal policy using a rolling horizon setting, without providing a closed-form optimal policy.

(Ciarallo et al., 1994) use the same techniques as (Wang and Gerchak, 1996) within two different frameworks: finite and infinite horizon, with information updates and with uncertainty only on demands and on production capacities.

(Mula et al., 2006) provide a literature review of the main production planning models under uncertainty. They classify the models in a chronological way, in four categories: the conceptual models (Material Requirement Planning) and supply chain planning models or MRP, the analytical models (hierarchical production planning, MRP, capacity planning, manufacturing resource planning, inventory management and supply chain planning), artificial intelligence models (aggregate planning, MRP, Manufacturing resource planning, inventory management an supply chain planning) and the simulation models (aggregate planning, MRP, capacity planning and manufacturing resource planning).

2.3.3 Supply chain planning models: classification

In this section we propose a classification of the planning problems in terms of the different parameters.

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Let us define, the following two different decision strategies:

- decision strategy with fixed decision, in which at a given period, all the decisions for the following periods are taken and fixed. This decision strategy is used in general where it is impossible to improve the information about the system, such the demand forecasts, over the time,

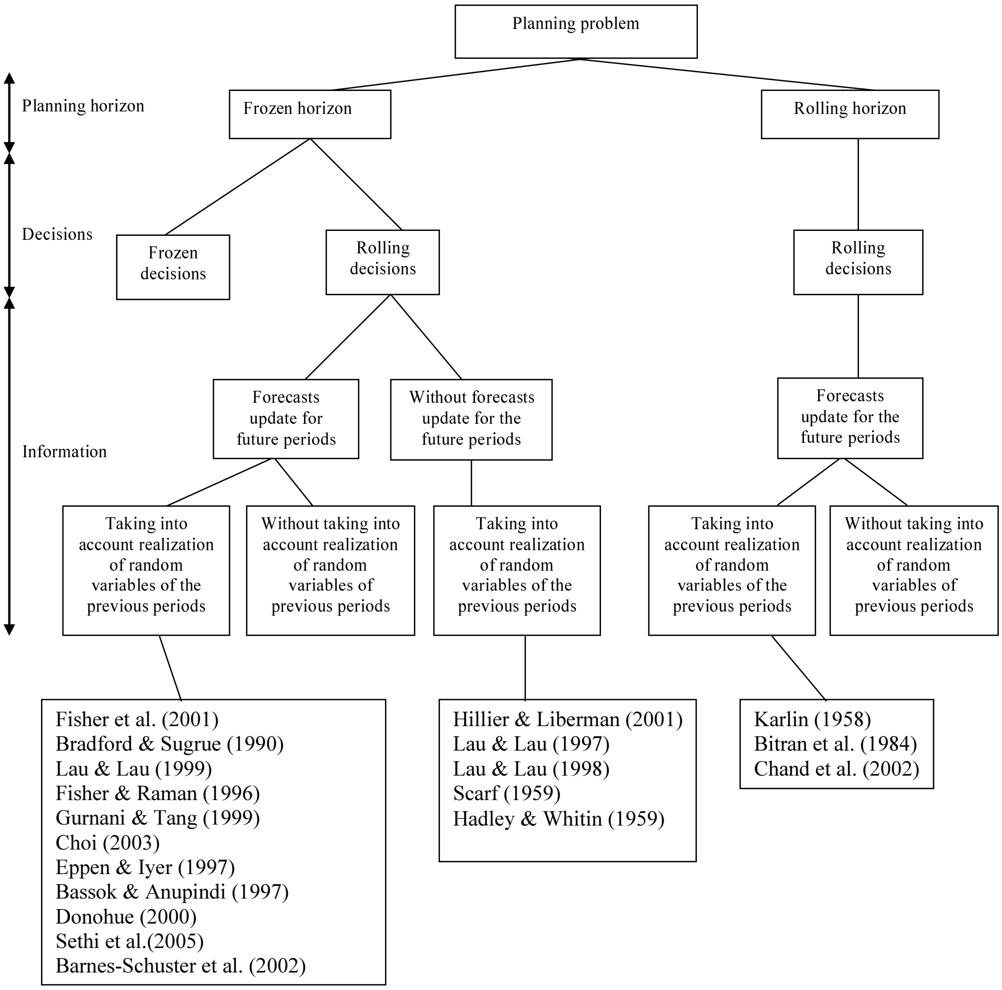

Figure 2.2: Classification of planning problems

- decision strategy with rolling decisions, where in each iteration, all the decisions relative to the planning horizon are calculated and only the decisions relative to the first period are fixed. Then, at the next period, using some new information to update the system state, the decisions of all the planning horizon are calculated again and only the first period decisions are fixed.

The choice of the type of planning horizon and the decision strategy depends on many factors such as the uncertainty of the problem parameters (demand, capacities), the reliability of the forecasts, the nature of the products (style-goods, spare parts, perishable, etc.) and the complexity of the problem, etc..

Another classification could be done in terms of the obtained solution of the proposed models. The criteria in this case is the nature of the solution, if it is analytical or numerical, and the degree of aggregation of the products or the periods.

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In Figure 2.2 we detail the different possible cases of a planning model in terms of the planning horizon, the nature of the decision strategy and the nature of the information, and we provide some examples from the literature. These possible cases are described in the following paragraphs.

The first case corresponds to the "frozen horizon" with rolling decisions, forecasts updates and with taking into account the realization of the random variables of the previous periods (Figure 2.3). In this case, the duration of the planning horizon is known from the beginning, but the decisions are fixed as the time goes on. At each time period, the last period is eliminated, and the new information about the system status is used in order to update the forecasts of the following periods random variables (demand for example), and to fix the decisions relative to the current period. This type of situation is used in the case of flexible contracts over a certain time duration, and in which gathering and using information to update the system and to reduce the variability of the random variables is possible and not costly. Examples of papers that deal with this type of problems can be (Fisher et al., 2001), (Bradford and Sugrue, 1990), (Lau and Lau, 1999), (Fisher and Raman,1996), (Gurnani and Tang, 1999), (Choi, 2003), (Eppen and Iyer, 1997), etc..

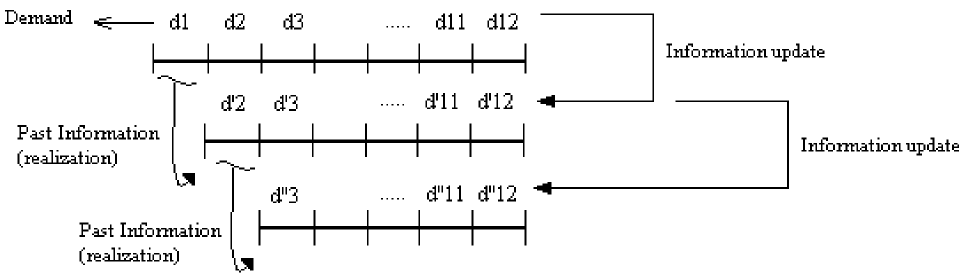

Figure 2.3: Frozen horizon with rolling decisions

The second case is similar to the first case, except the fact that in this type the realization of random variables is not taken into account. That may be due to the difficulty of collecting new information about the demand in the last periods for example, due to absence of an information system. In this case the update of the forecasts is performed with new information collected on the market for example. To our knowledge, in the literature there is any paper that presents models which can be classified in this case, but this case may be encountered in the industry.

The third case corresponds to the classical models of planning, with "frozen horizon", without information update and with taking into account the realization of the previous periods random variables. In this case, many papers can be found, such as (Lau and Lau, 1997), (Lau and Lau, 1998), (Scarf, 1959), (Hadley and Whitin, 1959) and many others.

The second family of the planning models, corresponds to the rolling horizon category, and in which there exists two types (Figure 2.4). The first type of models, corresponds to the models in which we take into account the realization of the random variables of the previous models, with forecasts update for the future periods. The information update may be of two origins: the first source of update is due to the fact that a new period is added to the end of the planning horizon, and then new demand forecast, for that period, is available. The second update source is the update of the existing demand forecasts,

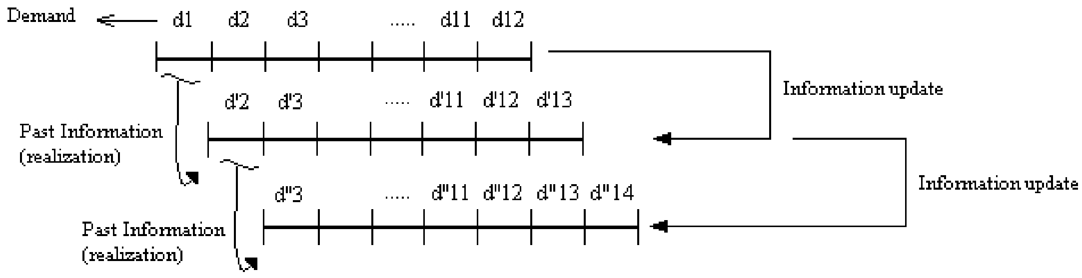

Figure 2.4: Rolling horizon with information update and with taking into account the realization of random variables

using the collected information during the last periods. A lot of papers are published for this category of models. (Chand et al., 2002) provide a literature review of the planning models with a rolling horizon framework.

The other category of models in this family corresponds to models with rolling horizon, with information update but without taking into account the realization of the random variables of the previous periods. Note that this category corresponds to cases in which it is costly or difficult to collect information about the realized values of the previous periods random variables (demand for example) due to the absence of information systems for example. To our knowledge, there is any work in the literature that deals with such models.

Note that there are no models with rolling horizon and without forecasts updates. Indeed, the fact that a new period is added at each iteration to the end of the planning horizon, implies that new information about the demand of that period is available, which can be considered as an information update.

2.4 Conclusion

This chapter constituted an introduction to the supply chain concepts and especially to the production and inventory planning notions. We have begun by giving a brief historical background of the military origins of the supply chain, then we have defined the main notions of the supply chain, such as the decision levels, the flow and the capacities. We have also given an overview of the inventory management including the role of the inventories and the related costs. Another important aspect of our work, which has been studied in this chapter, is the information in the supply chain. This information has been constituted of two elements: the inventory level and the demand forecast. Therefore, the role of the demand forecast has been detailed and an overview of the forecasting techniques has been provided. Since the demand forecasts are defined in general via random variables, we have shown the impact of the forecast uncertainty which constitutes the main source of uncertainty in the supply chain planning processes. Finally, we have provided a detailed study of the supply chain aggregate planning and in particular the production and inventory planning.

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Chapter 3

A *Newsvendor* Model with Initial Inventory and Two Salvage Opportunities

In this chapter, we develop an extension of the *newsvendor* model with initial inventory. In addition to the usual quantity ordered at the beginning of the horizon and the usual quantity salvaged at the end of the horizon, we introduce a new decision variable: a salvage opportunity at the beginning of the horizon, which might be used in the case of a high initial inventory level. We develop an expression for the optimal policy for this extended model, for a general demand distribution. The structure of this optimal policy has a particular form and is characterized by two threshold levels. Some managerial insights are given via numerical examples.

Keywords: *Newsvendor* model, initial inventory, lost sales, salvage opportunities, concave optimization, threshold levels.

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3.1 Introduction

The single period inventory model known as the *newsvendor* model is an important paradigm in operations research and operations management literature. The underlying problem consists of ordering a certain quantity of a given product, in order to optimally satisfy a future uncertain demand. The ordered quantity should optimize an objective function that includes different costs and/or profits. As there is a unique replenishment opportunity, if the ordered quantity is lower than the demand, excess demand will be lost. On the contrary, if this quantity exceeds the observed demand, the excess will be sold at a salvage value, generally lower than the wholesale price (purchasing price). The *newsvendor* model has had numerous important applications, as in style-goods products (fashion, apparel, toys, etc.) or in services management (booking on hotels, airlines, etc).

This model, which has represented the basic stochastic inventory model for several decades, has received a lot of attention in the literature. In particular, many extensions have been proposed in order to include specific additional characteristics in the model. The literature concerning the *newsvendor* model is thus very large (for extensive literature reviews, see for example (Porteus, 1990), (Silver et al., 1998) and (Khouja, 1999)). Generally speaking, a *newsvendor* model is characterized by three elements: the objective function, the demand characterization and different financial flow specifications. Most of the studies about the *newsvendor* model focus on the computation of the optimal order quantity that maximizes the expected profit (or minimizes the expected cost) (Nahmias, 1996). Nevertheless, some other works consider other criteria, such as maximizing the probability of achieving a target profit (Kabak and Schiff, 1978), (Lau, 1980) and (Khouja, 1999). The demand process can be considered exogenous (Nahmias, 1996), marketing effort dependent (Netessine and Rudi, 2000) or price-sensitive (Petruzzi and Dada, 1999). Financial flows generally introduced in the *newsvendor* problem are the wholesale price, the selling price, the salvage value and the shortage penalty cost. Many extensions exist, such as a fixed ordering cost (Silver et al., 1998), a dynamic selling price (Emmons and Gilbert, 1998), general forms of salvage value and shortage penalty cost (Lal and Staelin, 1984).

Some authors have considered other decision variables or parameters in the model. In particular, (Hillier and Lieberman, 1990) have analyzed a *newsvendor* model with an initial inventory. In this extension, the decider observes, at the beginning of the selling season, the initial inventory level and fixes his decisions as a function of this initial inventory. These authors have shown that in this case the optimal order quantity can be deduced from the classical model (without initial inventory). (Kodama, 1995) has considered a similar model in which the vendor, after observing the demand value, can carry out partial returns or additional orders in the limit of defined levels.

In the present chapter, we develop a new extension of the *initial inventory newsvendor* model in which a part of the initial inventory can be salvaged at the beginning of the selling season. As a matter of fact, when the initial inventory level is sufficiently high, it may be profitable to immediately salvage a part of this initial inventory to a parallel market, before the season. This is an extension of the classical model in which the unique salvage opportunity is placed at the end of the selling season.

In many practical situations, a potential interest exists for such a salvage opportunity before the selling season. For example, due to very long design/production/delivery lead-times a first products quantity could be ordered from the supplier a long time before the selling season. This implies that the demand distribution is not precisely known at the moment when the quantity is ordered ((Fisher and Raman, 1996) and (Fisher et al., 2001)). In this case, if the demand appears to be particularly low, it could be profitable to return a part of the received quantity to the supplier or sell it to a parallel market, with a return price which is lower than the order price.

From an intuitive point of view, we expect the corresponding optimal policy to be such that: if the initial inventory is very high, a part of this inventory is expected to be salvaged. If the initial inventory level is very low, an additional quantity should be ordered. In this chapter, we actually establish that the optimal policy corresponding to this new extended model is a threshold based policy with two different thresholds: the first corresponds to the order-up-to-level policy of the classical model with initial inventory, and the second threshold corresponds to a salvage-up-to-level policy, and is a result of the salvage opportunity at the beginning of the season. Between the two thresholds, the optimal policy consists of neither ordering, nor salvaging any quantity.

The remainder of this chapter is structured as follows. In the following section, we introduce the model, describe the decision process and define the notation used in the chapter, the objective function and the model assumptions. In section 3.3, we recall the classical *newsvendor* model with initial inventory and we show that it represents a special case of our model. In section 3.4, we solve our model and exhibit the structure of the optimal policy as a function of the initial inventory level. In section 3.5 we give some managerial insights via numerical applications. The last section is dedicated to conclusions and presentation of new avenues of research.

3.2 The model

A manager has to fill an inventory in order to face a stochastic demand. The ordering and selling processes are as depicted in Figure 3.1. Before occurrence of the demand, an initial inventory is available. Without loss of generality and because it is more coherent with the main idea of the present chapter, this inventory is assumed to be positive. Note however that a model with a negative initial inventory can also be developed, which would correspond to situations with some *firm orders* received before the beginning of the selling season. At the beginning of the season, the manager can make two decisions: first, he can sell a part of this initial inventory to a parallel market and/or second, he can order a new quantity to complete the initial inventory in order to better satisfy the future demand. After demand has occurred, the remaining inventory, if any, is salvaged or the unsatisfied orders, if any, are lost and, in this case, a shortage penalty cost is paid.

The decision and state variables corresponding to this problem (according to Figure 3.1) are denoted as follows:

I_b: the initial inventory level, available at the beginning of the selling season,

Q: the ordered quantity, which is to be received before the demand occurs,

S_b: the quantity salvaged at the beginning of the season, before the demand occurs,

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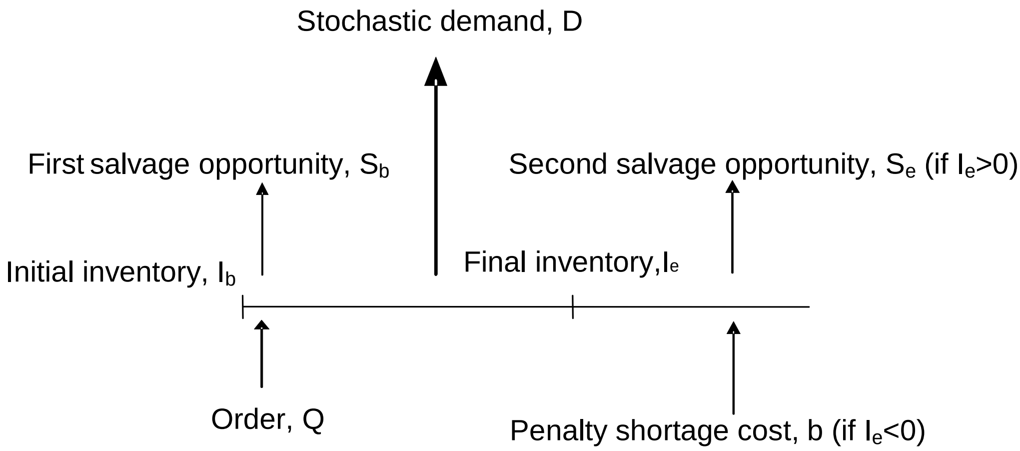

Figure 3.1: Ordering and selling process

I_e: the inventory level at the end of the selling season,

S_e: the quantity that is salvaged at the end of the selling season.

We also define the following parameters:

D: the random demand, which is characterized by a continuous probability density function $f(\cdot)$: $[0,\infty[\rightarrow \mathbb{R}^+$ and by the cumulative distribution function $F(\cdot) : [0,\infty[\rightarrow [0,1]$,

p: the unit selling price during the season,

s_b: the unit salvage value for the quantity S_b,

c: the unit order cost for the quantity Q,

s_e: the unit salvage value of the quantity S_e,

b: the unit shortage penalty cost.

As mentioned above, the objective function of the model consists of maximizing the total expected profit, denoted as $\Pi(I_b, Q, S_b, S_e)$. This expected profit, with respect to the random variable D, is explicitly given by

$$\begin{aligned}\Pi(I_b, Q, S_b, S_e) &= s_b S_b - cQ + s_e S_e + p\int_0^{I_b+Q-S_b} D f(D)\, dD \\ &\quad + p(I_b + Q - S_b)\int_{I_b+Q-S_b}^{\infty} f(D)\, dD \\ &\quad - b\int_{I_b+Q-S_b}^{\infty} (D - I_b - Q + S_b) f(D)\, dD. \qquad (3.1)\end{aligned}$$

The different terms can be interpreted as follows:

- the first term, s_bS_b, is the profit generated by salvage at the beginning of the season,
- the second term, cQ, is the order purchase cost,
- the third term, s_eS_e, is the profit generated by salvage at the end of the season,
- the fourth and fifth terms are the expected sales,
- the last term is the expected shortage penalty cost.

It is worth noting that equivalent models can be built with a cost minimization criterion ((Khouja, 1999) and (Geunes et al., 2001)). The decision variables have to satisfy the following constraints

$$0 \leq Q, \tag{3.2}$$

$$0 \leq S_b \leq I_b, \tag{3.3}$$

$$0 \leq S_e \leq I_e. \tag{3.4}$$

In the classical *newsvendor* model, some assumptions are necessary to guarantee the interest and the coherency of the model. A classical assumption is the following

$$s_e < c < p, \tag{3.5}$$

which simply states that the selling process is profitable ($c < p$) while the salvage process is not ($s_e < c$). As we introduce the new decision variable, S_b, it is necessary to adapt assumption (3.5) as follows

$$0 < s_e < s_b < c < p. \tag{3.6}$$

We assume $s_b < c$, in order to avoid those cases where it would be profitable to order a quantity at the beginning of the season and to immediately sell to the parallel market at the corresponding salvage price s_b. We furthermore assume $s_e < s_b$, which seems quite reasonable from a managerial point of view. Otherwise, namely if $s_b \leq s_e$, salvage at the beginning of the period would never be more profitable than keeping all the inventory to face demand and, eventually, to salvage the remaining inventory at the price s_e. This would eliminate the interest of the new model we consider in this chapter.

3.3 The classical *newsvendor* model with initial inventory

In the classical *newsvendor* model with initial inventory ((Silver et al., 1998) and (Hillier and Lieberman, 1990)), there is no salvage opportunity at the beginning of the single period horizon, even if the initial inventory level is very high compared to the required quantity to satisfy the stochastic demand. The

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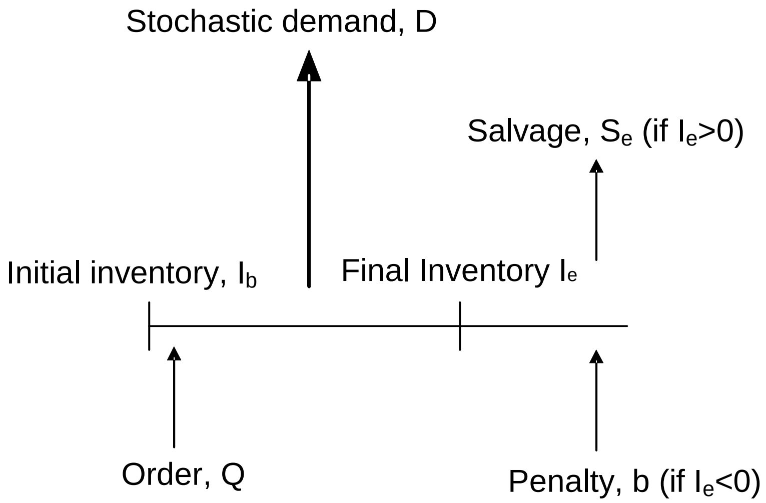

Figure 3.2: Classical *newsvendor* model with initial inventory

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decision process is as depicted in Figure 3.2, and the expected profit is given by

$$\begin{aligned}\Pi(I_b, Q) = -cQ + s_e S_e + p\int_0^{I_b+Q} Df(D)\,dD \\ + p(I_b+Q)\int_{I_b+Q}^{\infty} f(D)\,dD - b\int_{I_b+Q}^{\infty}(D - I_b - Q)f(D)\,dD.\end{aligned} \tag{3.7}$$

It is a well known result that the optimal policy at the end of the season consists of completely salvaging the remaining inventory, if any, ((Khouja, 1999), (Silver et al, 1998) and (Hillier and Lieberman, 1990)). The optimal value of S_e is thus given by

$$S_e^* = \max(0; I_e). \tag{3.8}$$

Substituting S_e by its optimal value given in (3.8), the expected objective function (3.7) becomes as follows

$$\begin{aligned}\Pi(I_b, Q) = -cQ + s_e\int_0^{I_b+Q}(I_b + Q - D)f(D)\,dD \\ + p\int_0^{I_b+Q} Df(D)\,dD + p(I_b+Q)\int_{I_b+Q}^{\infty} f(D)\,dD \\ - b\int_{I_b+Q}^{\infty}(D - I_b - Q)f(D)\,dD.\end{aligned} \tag{3.9}$$

Assuming (3.5), it is easily shown (Hillier and Lieberman, 1990) that $\Pi(I_b, Q)$ is concave in Q and that the optimal value $Q^*(I_b)$ is given by

$$Q^*(I_b) = \max\left(0 \;\;;\;\; F^{-1}\left(\frac{p+b-c}{p+b-s_e}\right) - I_b\right). \tag{3.10}$$

Clearly the sum of the lost margin $p - c$ and the shortage penalty b can be viewed as an underage cost, while the item cost c minus the salvage value s_e can represent an overage cost. As one would see, a relatively high underage cost results in a higher order quantity, whereas a relatively high overage cost leads to a lower order quantity. It is obvious that this *newsvendor* model with initial inventory, represents a special case of our model, described in paragraph 3.2, with zero initial salvage value, namely $s_b = 0$.

It is worth noticing that the optimal ordering quantity does not explicitly depend on the pair $(p\,;\,b)$ but only on the sum $p + b$. In particular, a model with a unit selling price p and a penalty cost $b > 0$ is equivalent to a model with a penalty cost $b' = 0$ and a unit selling price $p' = p + b$.

3.4 The extended model

In this section we consider the extended model described in section 3.2, with an expected objective function given by (3.1). First we show the concavity of this expected objective function with respect to the decision variables, then we explore the structure of the optimal policy.

Property 3.1 *The objective function* $\Pi(I_b, Q, S_b, S_e)$*, defined in (3.1)* is a jointly concave function with respect to Q, S_b and S_e.

Proof. The hessian of $\Pi(I_b, Q, S_b, S_e)$ with respect to Q, S_b and $S_e$is given by

$$\nabla^2\Pi(I_b, Q, S_b, S_e) = -(b + p)f(I_b + Q - S_b)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

From the model assumptions (3.6), for each vector $V = (V_1, V_2, V_3) \in \mathbb{R}^3$ we find

$$V^T\nabla^2\Pi(I_b, Q, S_b, S_e)V = -(b + p)f(I_b + Q - S_b)(V_1 - V_2)^2 \leq 0,$$

which proves that the matrix $\nabla^2\Pi(I_b, Q, S_b, S_e)$ is semi-definite negative. Consequently, the objective function $\Pi(I_b, Q, S_b, S_e)$ is jointly concave with respect to Q, S_b and S_e. □

Lemma 3.1 *The optimal value of the decision variable* S_e *is given by*

$$S_e^* = \max(0; I_e) \tag{3.12}$$

Proof. It could be easily shown that the first partial derivative of the expected objective function $\Pi(I_b, Q, S_b, S_e)$ with respect to S_e is given by

$$\frac{\partial\Pi(I_b, Q, S_b, S_e)}{\partial S_e} = s_e. \tag{3.13}$$

From assumption (3.6), one concludes that $\Pi(I_b, Q, S_b, S_e)$ is an increasing function in S_e. Thus the optimal value of S_e, considering the constraint (3.4), is $\max(0; I_e)$. □

From Lemma 3.1, one concludes that the optimal value of S_e depends on the inventory level I_e. Note that I_e is a random variable at the beginning of the selling season. Hence one could substitute S_e by

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its expected optimal value in equation (3.1), and the expected profit function for the model described in Figure 3.1 becomes

$$\begin{aligned}\Pi(I_b, Q, S_b) &= s_b S_b - cQ + s_e \int_0^{I_b+Q-S_b} (I_b + Q - S_b - D) f(D)\, dD \\ &+p \int_0^{I_b+Q-S_b} D f(D)\, dD + p(I_b + Q - S_b) \int_{I_b+Q-S_b}^{\infty} f(D)\, dD \\ &-b \int_{I_b+Q-S_b}^{\infty} (D - I_b - Q + S_b) f(D)\, dD. \end{aligned} \tag{3.14}$$

It is worth noting that this expected objective function depends only on I_b, Q and S_b. Therefore $Q^*(I_b)$ and $S_b^*(I_b)$, the optimal values of the decision variables Q and S_b, are the solution of the optimization problem

$$(Q^*(I_b), S_b^*(I_b)) = \arg \left\{ \max_{0 \le Q, 0 \le S_b \le I_b} \{\Pi(I_b, Q, S_b)\} \right\}, \tag{3.15}$$

where $\Pi(I_b, Q, S_b)$ is given in (3.14).

Let us now turn our attention to characterizing the optimal policy that defines the optimal values of Q and S_b. We have shown in Property 1 that the objective function $\Pi(I_b, Q, S_b)$ is jointly concave with respect to the decision variables Q and S_b, hence one could use the first order optimality criterion.

Consider the two partial derivatives of $\Pi(I_b, Q, S_b)$ with respect to Q and S_b, respectively given by

$$\frac{\partial \Pi(I_b, Q, S_b)}{\partial Q} = -c + b + p + (s_e - b - p) F(I_b + Q - S_b) \tag{3.16}$$

and

$$\frac{\partial \Pi(I_b, Q, S_b)}{\partial S_b} = s_b - b - p + (b + p - s_e) F(I_b + Q - S_b). \tag{3.17}$$

It can be easily seen, by the structure of (3.16) and (3.17) and by assumption (3.6), that these derivatives cannot be simultaneously equal to zero, which shows that, for a given inventory I_b, there are no optimal solutions such that $Q^*(I_b) > 0$ and $S_b^*(I_b) > 0$ simultaneously. In other words, one has the property

$$Q^*(I_b)\, S_b^*(I_b) = 0. \tag{3.18}$$

This result is fairly easy to understand. Indeed, a case for which $Q^*(I_b) >$ and $S_b^*(I_b) > 0$ would correspond to a situation where starting from an initial inventory I_b, amount of products (equal to $\min(S_b^*(I_b), Q^*(I_b))$) is sold at a unit salvage value s_b and right after purchased at a unit cost c. Since $s_b < c$ (assumption (3.6)), this would obviously be a counterproductive action.

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3.4.1 Optimality conditions for Q^*

For any given S_b value satisfying $0 \leq S_b \leq I_b$, the optimal ordering quantity $Q^*(I_b)$ is a function of $I_b - S_b$ that can be computed as the solution of the following optimization problem

$$Q^*(I_b) = \arg\left\{\max_{0\leq Q}\{\Pi(I_b, Q, S_b)\}\right\}. \tag{3.19}$$

By concavity of $\Pi(I_b, Q, S_b)$ with respect to Q, and for any given S_b value, the optimal solution $Q^*(I_b)$ is given either by

$$Q^*(I_b) = 0 \tag{3.20}$$

if $-c + b + p + (s_e - b - p)F(I_b - S_b) \leq 0$, or by

$$Q^*(I_b) = F^{-1}\left(\frac{b + p - c}{b + p - s_e}\right) - I_b + S_b \geq 0 \tag{3.21}$$

if $-c + b + p + (s_e - b - p)F(I_b - S_b) \geq 0$.

3.4.2 Optimality conditions for S_b^*

For any given Q value satisfying $0 \leq Q$, the optimal ordering quantity $S_b^*(I_b)$ is defined as the solution of the following optimization problem

$$S_b^*(I_b) = \arg\left\{\max_{0\leq S_b\leq I_b}\{\Pi(I_b, Q, S_b)\}\right\}. \tag{3.22}$$

By concavity of $\Pi(I_b, Q, S_b)$ with respect to S_b, and for any given Q value, the optimal solution $S_b^*(I_b)$ is given either by

$$S_b^*(I_b) = 0 \tag{3.23}$$

if $s_b - b - p + (b + p - s_e)F(I_b + Q) \leq 0$, or by

$$S_b^*(I_b) = F^{-1}\left(\frac{s_b - b - p}{b + p - s_e}\right) - I_b - Q \geq 0 \tag{3.24}$$

if $s_b - b - p + (b + p - s_e)F(I_b + Q) \geq 0$.

3.4.3 Critical threshold levels

From the above optimality conditions, two threshold levels appear to be of first importance in the optimal policy characterization,

$$Y_1^* = F^{-1}\left(\frac{b + p - c}{b + p - s_e}\right) \text{ and } Y_2^* = F^{-1}\left(\frac{b + p - s_b}{b + p - s_e}\right), \tag{3.25}$$

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with, from assumption (3.6), are related by:

$$Y_1^* \leq Y_2^*. \tag{3.26}$$

These threshold levels can be interpreted as values such as

$$-c + b + p + (s_e - b - p)F(Y_1^*) = 0, \tag{3.27}$$

and

$$s_b - b - p + (b + p - s_e)F(Y_2^*) = 0. \tag{3.28}$$

As the function $F(\cdot)$ is monotonously increasing, for any I_b values such that $I_b < Y_1^*$ (resp. $I_b > Y_1^*$), one finds

$$-c + b + p + (s_e - b - p)F(I_b) > 0 \quad (\text{resp.} \quad -c + b + p + (s_e - b - p)F(I_b) < 0)\,, \tag{3.29}$$

and for any I_b values such that $I_b > Y_2^*$ (resp. $I_b < Y_2^*$), one finds

$$s_b - b - p + (b + p - s_e)F(I_b) > 0 \quad (\text{resp.} \quad s_b - b - p + (b + p - s_e)F(I_b) < 0)\,. \tag{3.30}$$

3.4.4 Critical threshold levels and structure of the optimal policy

We show below that the structure of the optimal policy is in fact fully characterized by these two threshold levels as depicted in Figure 3.3.

Lemma 3.2 *For $Y_1^* \leq I_b \leq Y_2^*$, the optimal solution is given by*

$$Q^*(I_b) = S_b^*(I_b) = 0.$$

Proof. For $Y_1^* < I_b < Y_2^*$, one finds

$$\frac{\partial \Pi(I_b, 0, 0)}{\partial Q} < 0 \; and \; \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} < 0, \tag{3.31}$$

which induces, by concavity, that the solution $Q^*(I_b) = S_b^*(I_b) = 0$ is the optimum of the profit function for these I_b values. If $Y_1^* = I_b$, one finds

$$\frac{\partial \Pi(I_b, 0, 0)}{\partial Q} = 0 \text{ and } \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} < 0, \tag{3.32}$$

which leads to the same conclusion. If $Y_2^* = I_b$, one finds

$$\frac{\partial \Pi(I_b, 0, 0)}{\partial Q} < 0 \text{ and } \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} = 0, \tag{3.33}$$

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which leads to the same conclusion. □

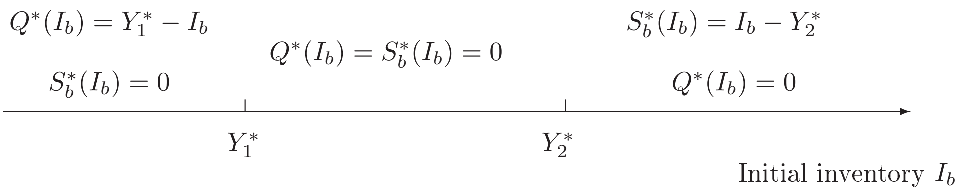

Figure 3.3: Structure of the optimal policy

Lemma 3.3 *For $I_b \leq Y_1^*$, the optimal solution is given by*

$$Q^*(I_b) = Y_1^* - I_b \quad \textit{and} \quad S_b^*(I_b) = 0. \tag{3.34}$$

Proof. For $I_b \leq Y_1^*$, one finds that

$$\frac{\partial \Pi(I_b, Y_1^* - I_b, 0)}{\partial Q} = 0 \ \text{ and } \ \frac{\partial \Pi(I_b, Y_1^* - I_b, 0)}{\partial S_b} < 0 \tag{3.35}$$

which induces, by concavity, that the solution $Q^*(I_b) = Y_1^* - I_b$ and $S_b^*(I_b) = 0$ is the optimum of the profit function for such I_b values. □

Lemma 3.4 *For $Y_2^* \leq I_b$, the optimal solution is given by*

$$Q^*(I_b) = 0 \quad \textit{and} \quad S_b^*(I_b) = I_b - Y_2^*. \tag{3.36}$$

Proof. For $Y_2^* \leq I_b$, one finds that

$$\frac{\partial \Pi(I_b, 0, I_b - Y_2^*)}{\partial Q} < 0 \text{ and } \frac{\partial \Pi(I_b, 0, I_b - Y_2^*)}{\partial S_b} = 0 \tag{3.37}$$

which induces, by concavity, that the solution $Q^*(I_b) = 0$ and $S_b^*(I_b) = I_b - Y_2^*$ is the optimum of the profit function for such I_b values. □

As in the classical *Newsvendor* model, it follows from the previous derivations that the optimal policy does not explicitly depend on the pair (p,b) but only on the sum $p+b$. In particular, a model with a unit selling price p and a penalty cost $b > 0$ is equivalent to a model with a penalty cost $b' = 0$ and a unit selling price $p' = p + b$.

3.5 Numerical examples and insights

The fundamental properties of the considered model will be illustrated by some numerical examples. In a first example, we illustrate the structure of the optimal policy as a function of the initial inventory I_b.

Then we exhibit, via a second numerical example, the impact of the demand variability on the structure of optimal policy. A third example illustrates the effect of s_b, the salvage value at the beginning of the horizon. In the last example, we compare the considered extended model with the classical *newsvendor* model with initial inventory, and we show the potential benefit associated with the initial salvage process.

For these numerical applications, we assume that the demand has a truncated-normal distribution, corresponding to a normal distributed demand, $D \sim N[\mu, \sigma]$ truncated at the zero value (we exclude negative demand values). Without loss of generality we also assume that the inventory shortage cost is zero, namely $b = 0$.

In the following figures, $Q^*(I_b)$ and $S_b^*(I_b)$ represent the optimal values of the decision variables and $E[S_e^*(I_b)]$ is the *expected* optimal value of the decision variable $S_e(I_b)$, which is given by

$$E[S_e^*(I_b)] = \int_0^{I_b+Q^*(I_b)-S_b^*(I_b)} (I_b + Q^*(I_b) - S_b^*(I_b) - D) f(D)\, dD. \qquad (3.38)$$

This is to account for the fact that the variables Q and S_b are decided before the demand is known while the variable S_e is decided after the demand is realized.

3.5.1 Optimal policy

In this first example, we depict the behavior of the optimal decision variables as a function of the initial inventory I_b. The numerical values for the parameters are the following: $\mu = 1000$, $\sigma = 400$, $p = 100$, $s_b = 30$, $c = 50$ and $s_e = 20$. The two thresholds $Y_1^* = 1127$ and $Y_2^* = 1460$ have been represented in

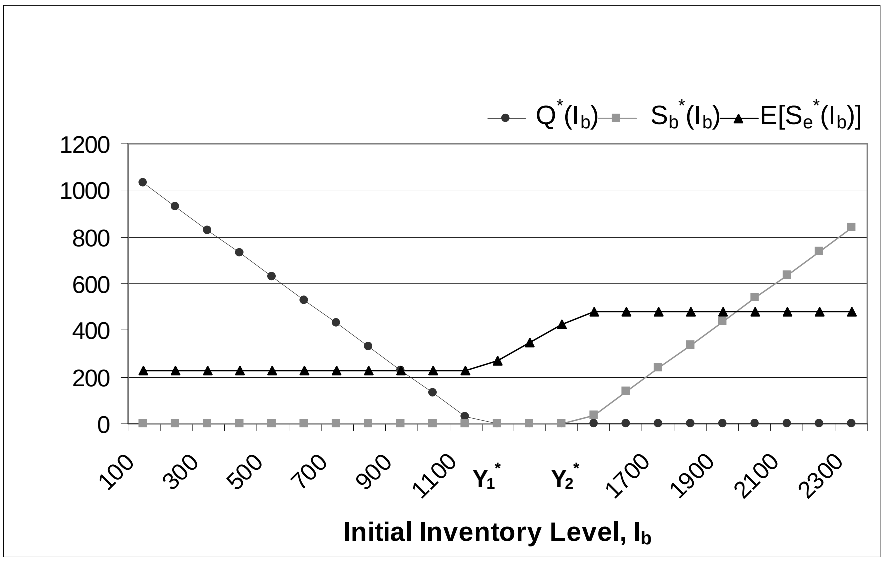

Figure 3.4: Optimal policy

Figure 3.4. For $Y_1^* \leq I_b \leq Y_2^*$, one has $Q^*(I_b) = S_b^*(I_b) = 0$, while $E[S_e^*(I_b)]$ is increasing. For $I_b < Y_1^*$, Q^* decreases linearly as a function of I_b, which corresponds to the order-up-to-level policy defined by equation (3.34). For $I_b > Y_2^*$, $S_b^*(I_b) > 0$ is a linear increasing function of I_b, which corresponds to the salvage-up-to-level policy defined by the equation (3.36).

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3.5.2 Variability effect

In this extended model, the decision variables Q and S_b are fixed before demand occurrence and are thus, in one way or another, affected by demand variability. On the other hand, the decision variable S_e is fixed once the demand is perfectly known. From an intuitive point of view, the more variable the demand the more profitable is postponement of the decisions. In order to show the consequence of the

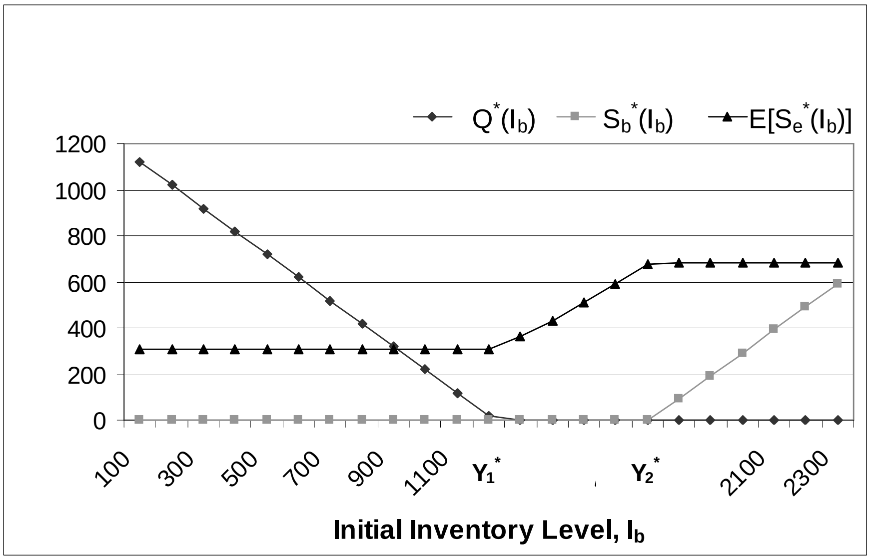

Figure 3.5: Optimal policy for high demand variability

variability, we compare the example of Figure 3.4 with two other examples with higher (Figure 3.5) and lower (Figure 3.6) variability. All the numerical parameters are the same as in the first example, except demand variability. The demand standard deviations are respectively $\sigma = 600$ for the second example in Figure 3.5 and $\sigma = 200$ for the third example in Figure 3.6. The thresholds are $Y_1^* = 1191$ and

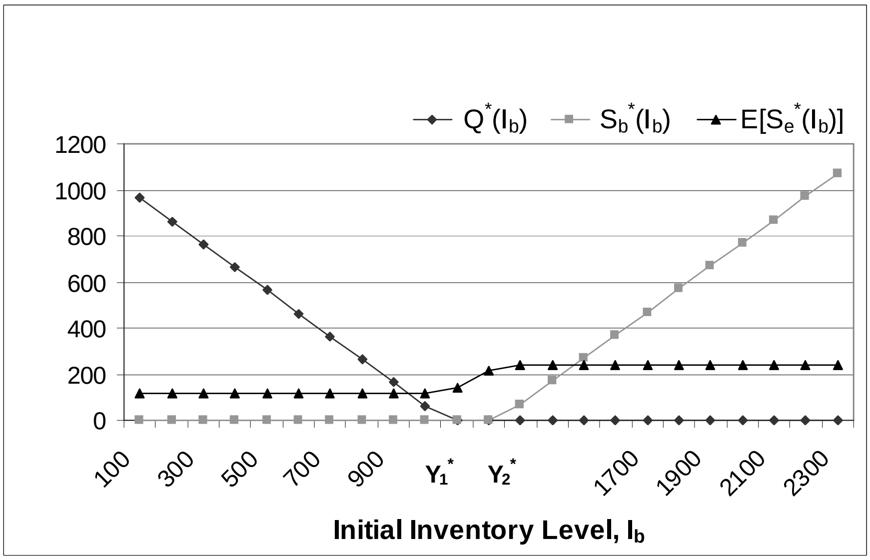

Figure 3.6: Optimal policy for low demand variability

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$Y_2^* = 1690$ for Figure 3.5 and $Y_1^* = 1064$ and $Y_2^* = 1230$ for Figure 3.6. It can be seen that the values of both thresholds increase with the standard deviation of the demand. The increase of Y_1^* is in accordance with standard *newsvendor* results with normally distributed demands, where for given costs, the order quantity increases linearly with the demand standard deviation. The same justification is valid for the increase of Y_2^*. This increase in the Y_1^* and the Y_2^* values is accompanied by an increase of the optimal Q^* or the decrease of the S_b^* for a given initial inventory value.

The increase of the optimal Q value permits the manager, for a given initial inventory, to stock a bigger quantity to face demand variability. The same is true for the decrease of the optimal S_b value.

The increase in the Y_1^* and Y_2^* values is accompanied by an increase in the difference $Y_2^* - Y_1^*$. In the interval $[Y_1^*, Y_2^*]$, the value of $E[S_e^*(I_b)]$ increases with I_b. The fact that the interval width $Y_2^* - Y_1^*$ increases with demand standard deviation leads to an increase of $E[S_e^*(I_b)]$, which may be interpreted as a postponement of the decision until the end of the season.

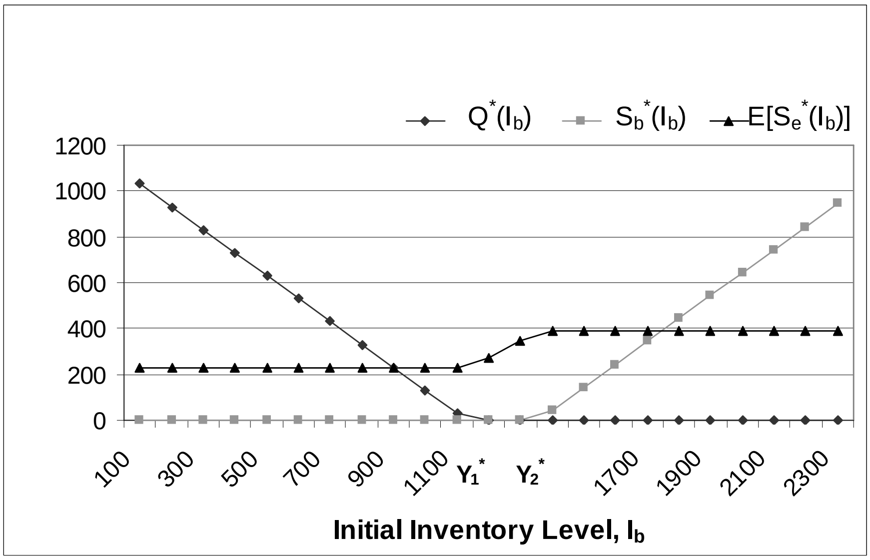

Figure 3.7: Optimal policy for high s_b value

3.5.3 Effect of initial salvage value s_b

In this section, we study the effect of the s_b value on the optimal policy. We compare the nominal example (defined in section (3.5.1), Figure 3.4), with two other examples with different s_b values. We consider for the first example a high s_b value , i.e. $s_b = 35$ (Figure 3.7), and for the second a low s_b value, i.e. $s_b = 25$ (Figure 3.8). By (3.25), it is explicit that Y_1^* does not depend on s_b. By (3.25) also, it is also explicit that Y_2^* is a decreasing function of s_b, as it is the case for the optimal policy behavior. For $s_b = 35$, we find $Y_1^* = 1127$ and $Y_2^* = 1355$, while for $s_b = 25$, these values become $Y_1^* = 1127$ and $Y_2^* = 1614$. An increase of s_b automatically induces a decrease of Y_2^*, which means that for a given value of the initial inventory I_b, the salvaged quantity $S_b^*(I_b)$ will increase. This increase will be accompanied by a decrease of the expected value of $S_e^*(I_b)$. This can be summarized as follows: the higher the salvage value of the parallel market, the higher the salvaged quantity $S_b^*(I_b)$ and the lower the expected salvaged quantity at

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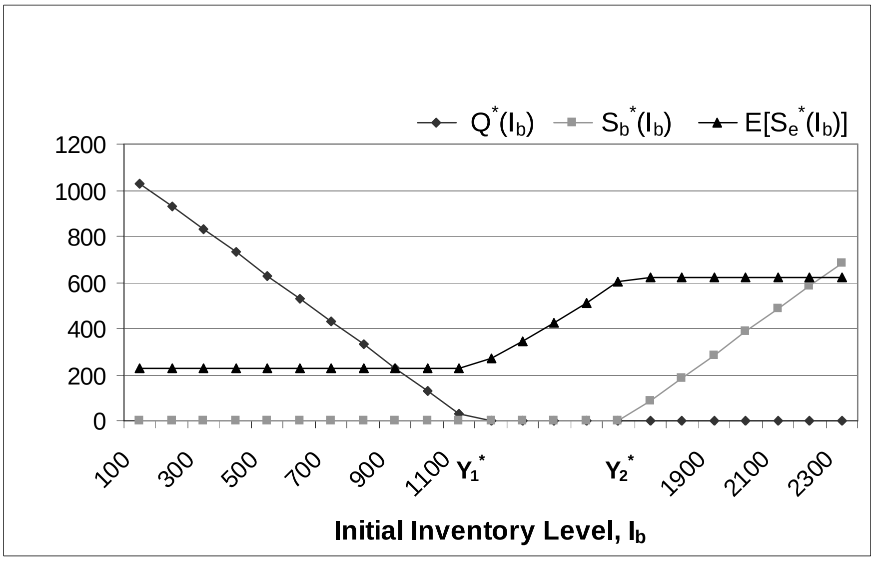

Figure 3.8: Optimal policy for low s_b value

the end of the season $E[S_e^*(I_b)]$.

3.5.4 Numerical comparison with the classical *newsvendor* model with initial inventory

Our extended model introduces the additional variable S_b, which appears to be useful in presence of high initial inventory level. In order to illustrate the magnitude of the benefits potentially associated with S_b, we compare our model with the initial inventory *newsvendor* model described in section 3.2. For the same numerical parameters values we have measured the relative difference between the expected objective functions of the two models for three values of the salvage value s_b: the nominal value, $s_b = 30$; a high value, $s_b = 35$; a low value, $s_b = 25$. The comparison is shown in Figure 3.9. Figure 3.9 shows that the benefits associated with the S_b variable can be non-negligible for high values of I_b. Clearly, it is equal to zero for the I_b values that are less than Y_2^*, where $S_b^* = 0$. Via Figure 3.9, one may conclude that:

- the difference, between the two expected optimal objective functions, is greater for high s_b values. This increase corresponds logically to the fact that the s_b term only appears in the objective function of the extended model and not in the *newsvendor* model.

- the threshold Y_2^* decreases with s_b. For high s_b value, the difference becomes positive.

This can be summarized as follows: the extended model is profitable for high s_b values and/or high I_b values.

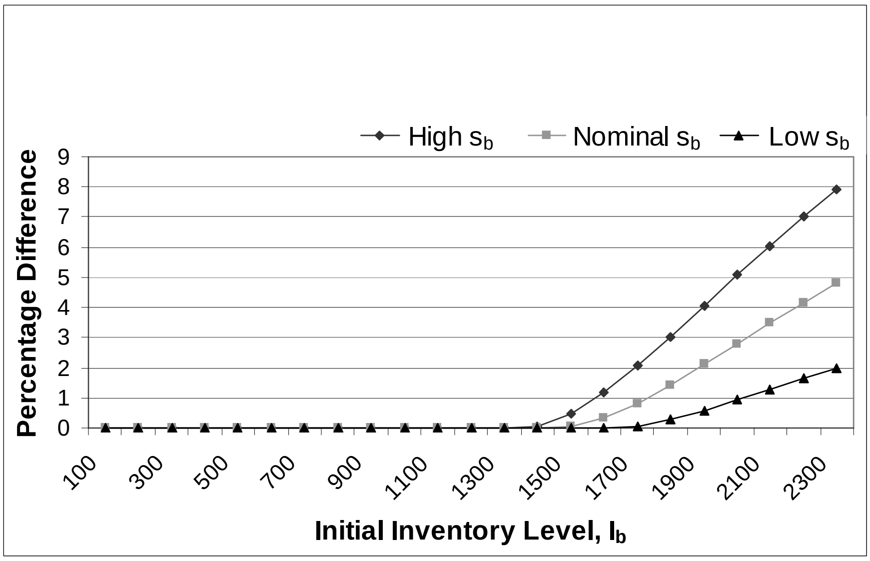

Figure 3.9: Comparison with the classical *newsvendor* model

3.6 Summary and conclusions

This chapter presents a new extension to the initial inventory *newsvendor* model in which a part of the initial inventory can be salvaged to a parallel market before demand occurrence. We have shown that in the case of a high initial inventory level, or a high initial salvage value s_b, this feature can be useful. The structure of the optimal policy is characterized by two threshold levels. Via numerical applications, we have illustrated the theoretical properties and given some managerial insight.

The extension of this model to a multi-periodic framework is an ongoing research avenue.

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Chapter 4

Two-Period Stochastic Production Planning and Inventory Control: General Modelling and Optimal Analytical Resolution

We investigate in this chapter a two-period production planning and inventory control model. Two production modes, with different production costs, can be used and multiple return opportunities are available which provides an important flexibility. We consider that, at the beginning of the decision process, an initial inventory is available and some preliminary fixed orders are to be delivered at each period. A general modelling is provided in the context of a backlog framework. The model is solved by a dynamic programming approach. A closed-form analytical solution is developed for the second period subproblem, which is similar to a modified *Newsvendor* problem and is characterized by two threshold levels. Due to the dynamic and sequential structure of the model, the first period subproblem solution is quite complex to be completely characterized by a closed-form formula. We define, therefore, an algorithm that permits to simplify that solution and to characterize it under some assumptions. We note that, in some cases, also the optimal solution of the first period subproblem is partially characterized by two threshold levels. We then provide insights regarding this type of two-stage inventory decision process with the help of numerical examples.

Keywords: Production planning, inventory control, production modes, backlog, parallel market, dynamic programming, closed-form solution.

4.1 Introduction

In this chapter, we study the same type of products that has been studied in Chapter 3, which is the *style-goods* type products. These products are characterized by a short life cycle or are produced only once and they have an uncertain future demand. Examples of such products are spare parts for a single production run of a new model of an airplane. For this kind of products, the demand is, in general, spread out only over a single period. In the literature, the model that deals with the production/inventory problems of this type of products is known as the *Newsvendor* model.

The basic *Newsvendor* model consists of producing/ordering a certain quantity of a given product, at the beginning of a single period horizon. This quantity has to optimally satisfy a future uncertain demand during the same horizon. The decision maker has no additional replenishment opportunities: as a consequence, if the ordered quantity is lower than the demand, excess demand is lost; on the other hand, if this quantity exceeds the demand, the decision maker will have to salvage, at the end of the horizon, the excess at a salvage value (generally lower than the production cost).

This single period model has been investigated and an extension to this model has been proposed in Chapter 3. In spite of a huge literature that treats this type of models, the single period model can not be really applied in many real-life applications that are fundamentally multiperiodic: several correlated decisions should be sequentially taken. It is thus quite natural to consider *two-period* models, as a fundamental building block, to analyze the structure of optimal decisions in such multi-period decision processes. Such two-period decision processes permit one to adapt the inventory levels to the demand variability. In fact, considering two periods makes the model more reactive and permits to exploit new information associated to the demand realization. In other words, in a single period model, a unique quantity is ordered at the beginning of the season, before information about the effective demand is available. On the contrary, in a two-period model, after the first order, the realized demand of the first period can be observed and a second order is made, which clearly exploits this information. This flexibility gives to the decision maker more degrees of freedom and gives to him the choice between multiple decisions, production modes, with different costs, which may be very beneficial.

Several authors have investigated such two-period production models. First, (Hillier and Liberman, 1990) analyzed a two-period model with uniformly distributed independent demands. Via a dynamic programming approach, they have analytically solved this model and they have proposed an explicit optimal order-up-to-level policy. (Lau and Lau, 1997, 1998) developed lost sales two-period models and proposed numerical solutions via dynamic programming. (Bradford and Sugrue, 1990) proposed another class of model in which the second period demand is correlated to the first period demand. A Bayesian update for the second period demand forecast can thus be used after the observation of the realized first period demand. These authors determined a conditional order-up-to-level policy for the second period and an optimal order quantity for the first period. Another important two-period model has been proposed by (Fisher and Raman, 1996); in these authors' paper, the demand of the whole horizon and the demand of the first period are characterized via a joint probability density function. Furthermore, the order size for the second period is constrained by an upper limit. (Gurnani and Tang, 1999) considered a two-period

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model with a first period demand equal to zero. In their model, the dynamic structure concerns available information for the sequential decision process: at the end of the first period, exogenous information is collected permitting to update the forecast for the second period demand. (Choi et al., 2003) proposed a quite similar two-stage newsboy model with an update of the forecast of the second-period demand via some market information. (Donohue, 2000) applies a similar approach for developing supply contracts.

In this chapter, we focalize on *style-goods* type products and on the development of a two-period production/inventory model for this type of products. The induced costs are purchasing costs, inventory holding costs and backorder costs. The demands at the first and second periods are defined by independent random variables, with known probability distributions. We assume that at the end of the second period, the remaining inventory can be sold to a specific market with a given salvage value.

In addition to these classical parameters, we suppose that some preliminary fixed orders should be delivered at each period. Also, we suppose that, at the beginning of the first period, the initial inventory level might be positive. On the other hand, we use two different production modes allowing the decision maker to produce for the second period twice and at different moments. This flexibility permits, at the same time, to exploit the lower costs of some production modes and to exploit the information about the realization of the first period demand (Cheaitou et al., a, 2006).

Note that the initial inventory can result from previous selling seasons, or from a preliminary (early) production or ordering operations. An early production is launched at a given moment, even before estimating precisely the future demand, generally in order exploit the low production costs at that moment.

An important aspect that we add to our model, and that does not exist in the literature is that the decision maker is allowed to salvage (or return) a part of his inventory at the beginning of each period to a parallel market. The salvage value of this option is more important than the salvage value at the end of the season (i.e. at the end of the second period).

There are many cases where a given quantity can be salvaged at the beginning of the selling season or at the beginning of a given period. The first case consists in selling this quantity to a parallel market, which is considered as a client that buy with a price lower than the usual market price. In the second case, this quantity can be returned to the supplier according to the terms of a supply contract. For example, if one orders the product from a supplier, and if the order is placed sufficiently in advance, the demand distribution is not well known at this moment. In this case, a specific contract with the supplier can be implemented, which permits a part of the received quantity to be returned to the supplier with a unit return value lower than the initial unit ordering cost.

The model includes initial inventory and initially fixed order quantities to be delivered in the different periods. We develop then an analytical solution for the second period and a semi-analytical solution for the first period which permits to identify the optimal policy of each of the two periods. A special form, two-threshold levels optimal policy for each of the planning horizon periods has been obtained. In order to show the impact of the different model parameters on the optimal policy, we conclude the chapter by giving some numerical applications.

In summary, our model is characterized by the following contributions:

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- the periodic ordering process is quite general in the sense that at each time period orders can be made for the different subsequent periods, possibly with different costs,
- for general demand distributions, a closed-form analytical solution for the second period and a semi-analytical solution for the first period have been developed,
- the periodic selling process is quite general, in the sense that, in addition to the classical selling process, it is possible at the beginning of each period, to sell a part of the available inventory to a parallel market, at a given salvage value,
- data are dynamic which means that the selling prices, costs, salvage values and demand probability distributions are period-dependent.

The remaining part of this chapter is structured as follows: the second section describes the model (namely the complete decision process, the information structure, the costs and profits structure and the global objective function), the third section details the dynamic programming approach and the solution details. Numerical examples are solved in the fourth section. The last section is dedicated to the conclusion.

4.2 The Model

4.2.1 Model description

Note that in the model presented in this chapter, we do not consider any constraints about the time length of the first and the second periods.

In each period t, the independent random demand D_t is defined by a probability density function (PDF) $f_t(\cdot) : [0, +\infty[\to \mathbb{R}^+$ and by a cumulative distribution function (CDF) $F_t(\cdot) : [0, +\infty[\to [0, 1]$. At each period any received demand is charged at a price p_t, even if it is satisfied only at the next period.

We define the decision variables Q_{ts} (with $0 \leq t \leq s \leq 3$) as the quantities ordered at the beginning of period t to be received at the beginning of period s, with a unit order cost of c_{ts}. Q_{01} and Q_{02} have been ordered before the selling horizon and are assumed to be given. We now introduce the additional decision variables S_t (with $1 \leq t \leq 3$), which are the quantities salvaged at the beginning of period t, with unit salvage values s_t. All the decision variables, i.e. Q_{ts} and S_t, are assumed to be non-negative.

The state variables of the model are X_t, the inventory level at the beginning of period t and I_t, the inventory level at the end of period t (I_0 is given and considered as the initial inventory for the problem).

The periodic inventory holding cost is h_t, while unsatisfied orders in period t are backlogged to the next period, with a penalty shortage cost b_t. It is worth noting that the third period is used in the problem not as real period, involving a decision process to be optimized, but only as a terminal condition for the problem.

Figure 4.1 presents the structure of the decision process and the demand realization, which can be described as follows: the available inventory at the beginning of the first period, before current orders

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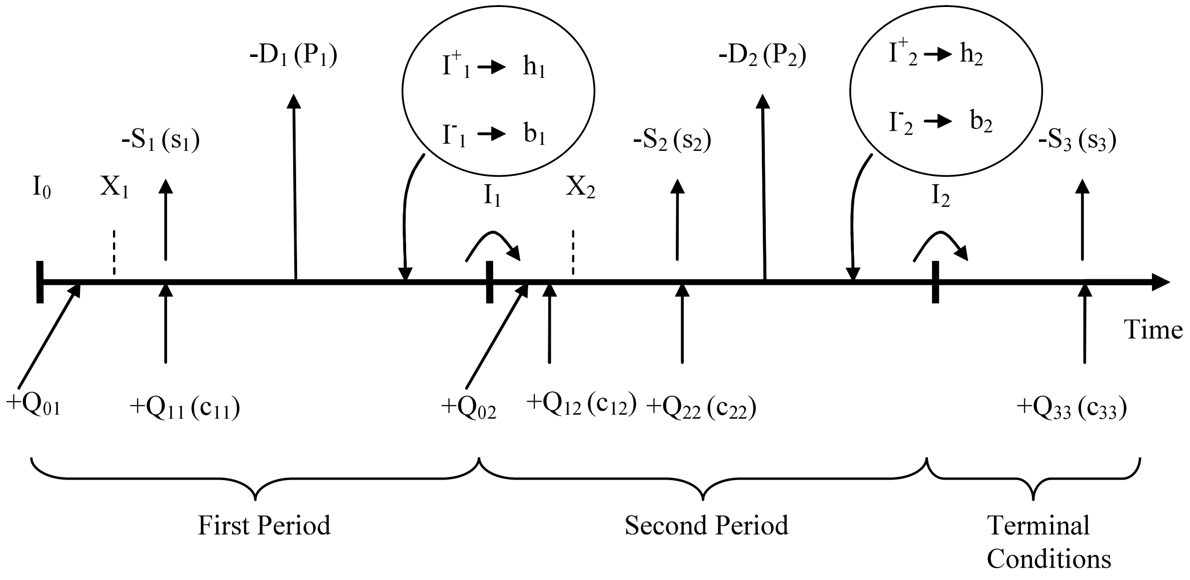

Figure 4.1: Decision process

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are chosen and demand occurs, is

$$X_1 = I_0 + Q_{01}, \tag{4.1}$$

where, I_0 and Q_{01} can be considered as given data. Then decision variables Q_{11}, Q_{12} and S_1 are fixed. Demand D_1 occurs, and the available inventory at the end of the first period is given by

$$I_1 = X_1 + Q_{11} - D_1 - S_1. \tag{4.2}$$

The available inventory at the beginning of the second period, before current orders are chosen, before demand occurs, and after Q_{12} is received, is

$$X_2 = I_1 + Q_{02} + Q_{12} = X_1 + Q_{11} - D_1 - S_1 + Q_{02} + Q_{12}, \tag{4.3}$$

where Q_{02} can be considered as data. The decision variables Q_{22} and S_2 are then fixed. Demand D_2 occurs, and the available inventory at the end of the second period is given by

$$I_2 = X_2 + Q_{22} - D_2 - S_2. \tag{4.4}$$

The ordered quantities cannot be negative and the salvaged quantities clearly cannot be higher than the available inventories. These constraints are formulated by the following inequations

$$0 \leq Q_{11}, \quad 0 \leq Q_{12}, \quad 0 \leq Q_{22}, \quad 0 \leq Q_{33}, \quad 0 \leq S_1 \leq X_1^+, \quad 0 \leq S_2 \leq X_2^+. \tag{4.5}$$

Note that the optimal values of the decision variables Q_{11}, Q_{12}, Q_{22}, S_1 and S_2 are Q_{11}^*, Q_{12}^*, Q_{22}^*, S_1^* and S_2^* successively.

Note that the third period optimal policy is defined by the following Lemma.

Lemma 4.1 *The optimal decision variables Q^*_{33} and S^*_3 are defined as follows*

$$Q^*_{33} = -I_2 \text{ if } I_2 \leq 0 \quad \text{and} \quad S^*_3 = I_2 \text{ if } I_2 > 0. \tag{4.6}$$

Proof. Let us define $\Pi_3(.)$ as the third period profit function. This function is given by

$$\Pi_3(I_2, Q_{33}, S_3) = -c_{33}Q_{33} + s_3 S_3 \tag{4.7}$$

with

$$Q_{33} \geq 0 \text{ and } 0 \leq S_3 \leq I_2. \tag{4.8}$$

An additional constraint may be the result of our decision process, where we assume that the unsatisfied demands at a period are charged with the price of that period and then satisfied at the following period. Therefore, when $I_2 < 0$, then the corresponding backlogged demand should be satisfied at the third period.

It is clear that the function $\Pi_3(I_2, Q_{33}, S_3)$ is an increasing function in S_3 and a decreasing function in Q_{33}.

Therefore, when $I_2 < 0$, then the values of Q_{33} and S_3 that maximize the objective function $\Pi_3(I_2, Q_{33}, S_3)$ and respect the constraints will be $Q^*_{33} = -I_2$ and $S^*_3 = 0$.

When $I_2 > 0$, then the values of Q_{33} and S_3 that maximize the objective function $\Pi_3(I_2, Q_{33}, S_3)$ and respect the constraints will be $Q^*_{33} = 0$ and $S^*_3 = I_2$. □

4.2.2 Model assumptions

Assumptions for the parameters

To avoid some trivial or nonrealistic cases and to guarantee the significance of the model, it is necessary to introduce some assumptions for the different parameters of the model. These assumptions are described as follows:

- **Systematic backlog assumptions**

$$c_{11} < c_{22} + b_1, \quad c_{11} < c_{12} + b_1, \quad c_{12} < c_{33} + b_2 \text{ and } c_{22} < c_{33} + b_2. \tag{4.9}$$

  These constraints aim at avoiding situations with systematic backlogs of demands of one period to the next. If the first constraint is not satisfied, i.e. if $c_{11} > c_{22} + b_1$, then it would be optimal to not order any unit with Q_{11} and therefore to backlog the first period demands to the second period to be satisfied with Q^*_{22}. If the second constraint is not satisfied, i.e. if $c_{11} > c_{12} + b_1$, then the optimal decision variable Q^*_{11} will be equal to zero. In this case, the first period demands will be backlogged to the second period, to be satisfied with Q^*_{12}. If the third constraint is not satisfied, i.e. if $c_{12} > c_{33} + b_2$, then the slow production mode will not be used ($Q^*_{12} = 0$), which implies that the

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related demands will be satisfied with Q^*_{33}. If the last constraint is not satisfied, i.e. if $c_{22} > c_{33}+b_2$, then it would be optimal to not order any unit with Q^*_{22}, and to satisfy the backlogged demands of the second period with Q^*_{33}.

- **Order/salvage assumptions (Different periods)**

$$s_2 < c_{11} + h_1, \quad s_3 < c_{12} + h_2, \quad s_3 < c_{11} + h_1 + h_2 \quad \text{and} \quad s_3 < c_{22} + h_2. \tag{4.10}$$

These constraints aim at avoiding unrealistic situations where it would be optimal to order at a given period and then to sell to the parallel market at a salvage price in another period. For example, if the first constraint is not satisfied, then the optimal policy will consist in ordering an infinite Q^*_{11} quantity in the first period and selling it at a salvage price s_2 in the second period. In the case where the second constraint is not satisfied, i.e. if $s_3 > c_{12} + h_2$, then the optimal ordered quantity with the slow production mode (Q^*_{12}) and the optimal salvaged quantity at the end of the season (S^*_3) will be equal to infinity. If the third constraint is not satisfied, i.e. if $s_3 > c_{11}+h_1+h_2$, then it will be optimal to order an infinite Q^*_{11} quantity and to sell it after the end of the second period with a salvage value of s_3. If the last constraint is not satisfied, the optimal policy will consist in ordering an infinite Q^*_{22} quantity and selling it with a salvage value of s_3.

- **Order/salvage assumptions (Same period)**

$$s_1 < c_{11}, \quad s_2 < c_{22}, \quad s_2 < c_{12} \quad \text{and} \quad s_3 < c_{33}. \tag{4.11}$$

These constraints aim at avoiding other situations where it would be profitable to order at a given period with the explicit strategy of salvaging this order during the same period, without satisfying demands with this order. For example, if the first constraint is not satisfied, i.e. if $s_1 > c_{11}$, then it would be optimal to order an infinite quantity in the first period and to salvage it in the same period. If the second constraint is not satisfied, then the optimal policy will consist in ordering an infinite Q^*_{22} quantity and in salvaging it immediately with a salvage value of s_2. If the third constraint is not satisfied, i.e. if $s_2 > c_{12}$, then the optimal decision variables Q^*_{12} and S^*_2 will be equal to infinity. If the last constraint is not satisfied, then it will be optimal to order an infinite Q^*_{33} quantity and to salvage it immediately with a salvage price of s_3.

- **Slow production mode assumption**

$$c_{12} < c_{11} + h_1. \tag{4.12}$$

This constraint aims at avoiding the situation where, *a priori*, it would never be profitable to order any quantity from the slow mode, which would render the slow mode *a priori* useless.

- **Salvage assumptions**

$$s_1 > s_2 - h_1 \text{ and } \ s_2 > s_3 - h_2. \tag{4.13}$$

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These constraints aim at avoiding the situations where it would, *a priori*, never be profitable to salvage any quantity at the beginning of a given period, because waiting for the next period and salvaging at this next period would be, *a priori*, better. If the first constraint is not satisfied, i.e. if $s_1 < s_2 - h_1$, it will be optimal not to salvage any unit with S_1, and consequently, to keep all the available units at the beginning of the first period in order to satisfy the first or second period demands, or to be salvaged at the beginning of the second period. If the second constraint is not satisfied, it will be optimal not to salvage any unit at the beginning of the second period, and then to keep these units in order to satisfy the second period demands, or to be salvaged after the end of the second period, with a salvage value of s_3.

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Demand charging assumption

Since our model is defined in a backlogging framework, and in order to simplify the solution approach, we assume in this chapter that at each period t, any received demand is charged at a price p_t, even if the demand is not satisfied immediately. Unsatisfied demands are backlogged to the next period, inducing a unit backlog penalty cost b_t.

Note that we consider in this chapter an average cost framework. Therefore, the model described above can be viewed as equivalent to a model, in which the demands that cannot be satisfied immediately are charged in the period at which they are finally satisfied, at the price of the period in which the demand originally arrived.

An alternative model would be to consider that a backlogged demand is charged when satisfied but at the price of the period at which it is satisfied. It is easy to check that such a model could equivalently be transformed into our model by using an equivalent backlog penalty $b'_t = b_t + p_{t+1} - p_t$.

Note that all these assumptions are valid for this chapter, Chapter 5 and Chapter 6.

4.2.3 Global objective function

Let us introduce $\Pi(I_0, Q_{01}, Q_{02}, Q_{11}, Q_{12}, Q_{22}, Q_{33}, S_1, S_2, S_3)$ as the expected profit, with respect to the random variables D_1 and D_2, associated to the decision variables Q_{11}, Q_{12}, Q_{22}, Q_{33}, S_1, S_2, S_3, and the initial data of the problem I_0, Q_{01}, Q_{02}. Substituting S_3 and Q_{33} by their optimal values given in

(4.6), then the expected objective function $\Pi(\cdot)$ will be

$$\begin{aligned}
&\Pi(I_0, Q_{01}, Q_{02}, Q_{11}, Q_{12}, Q_{22}, Q_{33}, S_1, S_2, S_3) = \\
&\quad s_1 S_1 - c_{11} Q_{11} - c_{12} Q_{12} + p_1 \int_0^{\infty} D_1 f_1(D_1)\, dD_1 \\
&\quad - h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1 \\
&\quad - b_1 \int_{X_1+Q_{11}-S_1}^{\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1 \\
&\quad + p_2 \int_0^{\infty} D_2 f_2(D_2)\, dD_2 + s_2 S_2 - c_{22} Q_{22} \\
&\quad - (h_2 - s_3) \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2)\, dD_2 \\
&\quad - (b_2 + c_{33}) \int_{X_2+Q_{22}-S_2}^{\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2)\, dD_2.
\end{aligned} \tag{4.14}$$

The different terms can be interpreted as follows:

- $s_1 S_1$ is the total salvage price of the first period,
- $c_{11} Q_{11}$ represents the total ordering cost of the first period using the fast mode,
- $c_{12} Q_{12}$ represents the total ordering cost of the first period using the slow mode,
- the fourth term,
$$p_1 \int_0^{\infty} D_1 f_1(D_1)\, dD_1,$$
represents the expected profit of the first period,
- the fifth term,
$$h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1,$$
is the first period expected inventory holding cost,
- the sixth term,
$$b_1 \int_{X_1+Q_{11}-S_1}^{\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1,$$
represents the first period expected penalty shortage cost,
- the seventh term,
$$p_2 \int_0^{\infty} D_2 f_2(D_2)\, dD_2,$$
is the expected second period profit,
- $s_2 S_2$ represents the total second period salvage value,
- $c_{22} Q_{22}$ represents the total second period ordering cost,
- the tenth term,
$$(h_2 - s_3) \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2)\, dD_2,$$

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is the second period expected overage cost, which is constituted of the expected inventory holding cost and the expected salvage value of the remaining inventory at the end of the second period,

- the last term,

$$(b_2 + c_{33}) \int_{X_2+Q_{22}-S_2}^{\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2)\, dD_2,$$

represents the second period expected underage costs, constituted of the second period expected shortage penalty cost and the expected ordering cost after the end of the second period (in order to satisfy the backlogged demands).

4.3 The dynamic programming approach

In this section, we will use a dynamic programming approach combined with the convex optimization to develop the optimal policies of the two periods of our planning horizon. According to the standard dynamic programming methodology, we reformulate first the two-period problem into two single-period sub-problems, by the the definition of appropriate value functions. Then we use the convex multi-variable optimization to solve each of these two sub-problems.

4.3.1 Problem decomposition

In sequential stochastic decision processes, and especially in a production planning and inventory control context, dynamic programming is often considered as a powerful solution technique (see for example (Ross, 1983), (Bertsekas, 2000), (Toomey, 2000) and (Sethi et al., 2005)). Using this technique we can decompose our problem into a couple of one-period subproblems. The first subproblem is associated with the second period. The solution of this problem permits to characterize the optimal values of the second-period decision variables, namely Q_{22}^* and S_2^*. Clearly, as the decision variables are in the second period, then they are expressed as a function of the state variable X_2. They are computed as the solution of the optimization problem

$$\max_{\xi_2(X_2)} \left\{ \Pi_2(X_2, \xi_2(X_2)) \right\}, \tag{4.15}$$

where $\Pi_2(X_2, \xi_2(X_2))$ is the expected second period profit function, and where we formally have

$$\xi_2(X_2) = (Q_{22}(X_2), S_2(X_2)). \tag{4.16}$$

Then, the subproblem associated to the first period exploits the optimal solution of the second period subproblem $\xi_2^*(X_2)$ in order to find the optimal solution for the first period, namely

$$\xi_1^*(X_1) = (Q_{11}^*(X_1), Q_{12}^*(X_1), S_1^*(X_1)), \tag{4.17}$$

which is obtained as the solution of the problem

$$\max_{\xi_1(X_1)} \left\{ \Pi_1(X_1, \xi_1(X_1)) + E_{D_1} \left\{ \Pi_2^*(X_2, \xi_2^*(X_2)) \right\} \right\}, \tag{4.18}$$

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where $\Pi_1(X_1, \xi_1(X_1))$ is the expected first period profit function, while the second term is the expectation, with respect to D_1, of the second period profit function, under the optimal policy $\xi_2^*(X_2)$.

4.3.2 Second-period subproblem

The objective function of the second period is defined by the following expression

$$\begin{aligned}\Pi_2(X_2, Q_{22}, S_2) &= p_2 \int_0^{\infty} D_2 f_2(D_2)\, dD_2 + s_2 S_2 - c_{22} Q_{22} \\ &\quad -(h_2 - s_3) \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2)\, dD_2 \\ &\quad -(b_2 + c_{33}) \int_{X_2+Q_{22}-S_2}^{\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2)\, dD_2. \end{aligned} \tag{4.19}$$

This class of models has been analyzed in Chapter 3. Nevertheless, there are some differences between the second period problem and the model presented in Chapter 3. Therefore, we prove in this section the concavity of the expected objective function defined in (4.19) with respect to the decision variables Q_{22} and S_2. Then using the concavity property, we characterize the optimal policy.

Lemma 4.2 *The objective function $\Pi_2(X_2, Q_{22}, S_2)$, defined in (4.19) is a jointly concave function with respect to Q_{22} and S_2.*

Proof. The hessian of $\Pi_2(X_2, Q_{22}, S_2)$ with respect to Q_{22} and S_2 is given by

$$\nabla^2 \Pi_2(X_2, Q_{22}, S_2) = -(b_2 + c_{33} + h_2 - s_3) f_2(Q_{22} - S_2 + X_2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{4.20}$$

From the model assumptions, for each vector $\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, where $(V_1; V_2) \in \mathbb{R}^2$, we find

$$V^T \left(\nabla^2 \Pi_2(X_2, Q_{22}, S_2)\right) V = -(b_2 + c_{33} + h_2 - s_3) f_2(Q_{22} - S_2 + X_2)(V_1 - V_2)^2 \leq 0,$$

which proves that the matrix $\nabla^2 \Pi_2(X_2, Q_{22}, S_2)$ is semi-definite negative. Consequently, the objective function $\Pi_2(X_2, Q_{22}, S_2)$ is jointly concave with respect to Q_{22} and S_2. □

Optimal policy

The second period optimal policy is given by (for the proof, see Appendix A.1)

$$\text{if } \; X_2 < Y_{12} \Rightarrow \begin{cases} Q_{22}^* = Y_{12} - X_2, \\ S_2^* = 0, \end{cases} \tag{4.21}$$

$$\text{if } \; Y_{12} \leq X_2 \leq Y_{22} \Rightarrow \begin{cases} Q_{22}^* = 0, \\ S_2^* = 0, \end{cases} \tag{4.22}$$

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and

$$\text{if } \ X_2 > Y_{22} \Rightarrow \begin{cases} Q_{22}^* &= 0, \\ S_2^* &= X_2 - Y_{22}. \end{cases} \tag{4.23}$$

These conditions amount to

$$Q_{22}^* = (Y_{12} - X_2)^+ \ \text{ and } \ S_2^* = (X_2 - Y_{22})^+, \tag{4.24}$$

with

$$Y_{12} = F_2^{-1}\left(\frac{b_2 + c_{33} - c_{22}}{b_2 + c_{33} + h_2 - s_3}\right) \ \text{ and } \ Y_{22} = F_2^{-1}\left(\frac{b_2 + c_{33} - s_2}{b_2 + c_{33} + h_2 - s_3}\right). \tag{4.25}$$

Furthermore, from the above equations, we conclude the following two properties.

Property 4.1 *The optimal values of the two decision variables Q_{22}^* and S_2^* can not be simultaneously positive.*

Property 4.2 *If $X_2 < 0$, then the optimal quantity Q_{22}^* satisfies*

$$X_2 + Q_{22}^* \geq 0.$$

The two threshold levels, given in (4.25) can be interpreted similarly to the order-up-to-level of the classical *Newsvendor* problem.

For the the first threshold level, Y_{12}, define the underage cost $C_u^1 = b_2 + c_{33} - c_{22}$ as the marginal cost of not satisfying a demand in the second period with Q_{22}, and the overage cost $C_o^1 = c_{22} - s_3 + h_2$ as the marginal cost of ordering a supplementary unit with Q_{22} over the optimal value. Therefore, in the expression of Y_{12}, the argument of the function is equal to the ratio of C_u^1 and $C_u^1 + C_o^1$.

For the the second threshold level, Y_{22}, define the underage cost $C_u^2 = b_2 + c_{33} - s_2$ as the marginal cost of salvaging a supplementary unit over the optimal at the beginning of the second period (which is equivalent to not satisfying a marginal demand in the second period), and the overage cost $C_o^2 = s_2 - s_3 + h_2$ as the marginal cost of not salvaging a supplementary unit with S_2 and therefore keeping that unit for the second period. Therefore, in the expression of Y_{22}, the argument of the function is equal to the ratio of C_u^2 and $C_u^2 + C_o^2$, which can be interpreted as a modified *Newsvendor* salvage-up-to-level.

Note that it is easily seen that under the assumptions of this chapter, one has

$$Y_{12} < Y_{22}. \tag{4.26}$$

4.3.3 First period subproblem

In this section we solve analytically the optimization problem of the first period which is an optimization problem with three decision variables, i.e. Q_{11}, Q_{12} and S_1. Indeed, using the results of the second period subproblem, we find, in this section, the optimal policy of the first period, namely $\xi_1^*(X_1)$, that permits to compute the optimal value of each of the decision variables in terms of the state variable, X_1.

Since it is not possible to provide a closed-form formula which defines that optimal policy, we introduce an algorithm that permits to define it partially in an analytical manner.

The total expected profit function $\Pi(\cdot)$, under the optimal second-period policy $\xi_2^*(X_2)$ becomes

$$\begin{aligned}\Pi(X_1, Q_{11}, Q_{12}, S_1) = & \quad \Pi_1(X_1, Q_{11}, Q_{12}, S_1) \\ & + \quad E_{D_1}\left\{\Pi_2^*(X_2, Q_{22}^*(X_2), S_2^*(X_2))\right\}, \end{aligned} \tag{4.27}$$

where $\Pi_1(X_1, Q_{11}, Q_{12}, S_1)$ is given by

$$\begin{aligned}\Pi_1(X_1, Q_{11}, Q_{12}, S_1) \quad = \quad & s_1 S_1 - c_{11}Q_{11} - c_{12}Q_{12} + p_1 \int_0^{\infty} D_1 f_1(D_1)\, dD_1 \\ & - h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1 \\ & - b_1 \int_{X_1+Q_{11}-S_1}^{\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1. \end{aligned} \tag{4.28}$$

The different terms of (4.28) have been explained in section 4.2.3.

The optimization problem to solve for this subproblem is then given by

$$\begin{aligned}\Pi^*(X_1) \quad = \quad & \max_{Q_{11}, Q_{12}, S_1} \left\{\Pi(X_1, Q_{11}, Q_{12}, S_1)\right\} \\ = \max_{Q_{11}, Q_{12}, S_1} \quad & \Bigg\{ \quad p_1 \int_0^{\infty} D_1 f_1(D_1)\, dD_1 + s_1 S_1 - c_{11}Q_{11} - c_{12}Q_{12} \\ & - \quad h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1 \\ & - \quad b_1 \int_{X_1+Q_{11}-S_1}^{\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1 \\ & + \quad E_{D_1}\left\{\Pi_2^*(X_2, Q_{22}^*(X_2), S_2^*(X_2))\right\} \Bigg\} \end{aligned} \tag{4.29}$$

under the following constraints

$$Q_{11} \geq 0, \quad Q_{12} \geq 0, \quad \text{and} \quad S_1 \geq 0. \tag{4.30}$$

Lemma 4.3 *The total expected objective function* $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ *defined in (4.27) is jointly concave with respect to* Q_{11}*,* Q_{12} *and* S_1*.*

Proof. See Appendix A.3. □

Using Lemma 4.3, the optimization problem described in equation (4.29) has a unique maximum. Thus, the first order optimality criterion can be used to develop the optimal policy that permits to find the optimal decision variables values, Q_{11}^*, Q_{12}^* and S_1^*, in terms of the different problem parameters.

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First order optimality criterion

In Lemma 4.3, we have proved the concavity of the objective function $\Pi(X_1, Q_{11}, Q_{12}, S_1)$. Thus, the optimal solution can be directly characterized by the first order optimality criterion. In the following, the optimal solution of the first period sub-problem is represented by $(Q^*_{11}, Q^*_{12}, S^*_1)$. The expressions of the first order partial derivatives of the total expected objective function $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ with respect to the decision variables Q_{11}, Q_{12} and S_1 are given in Appendix A.2. The optimality conditions for the first period decision variables (Q_{11}, Q_{12}, S_1) are then given by

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(Q^*_{11}, Q^*_{12}, S^*_1) = 0, \tag{4.31}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q^*_{11}, Q^*_{12}, S^*_1) = 0, \tag{4.32}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q^*_{11}, Q^*_{12}, S^*_1) = 0, \tag{4.33}$$

with the following constraints

$$Q^*_{11} \geq 0, \quad Q^*_{12} \geq 0, \quad \text{and} \quad S^*_1 \geq 0.$$

Lemma 4.4 *The optimal values Q^*_{11} and S^*_1 for the two first period decision variables Q_{11} and S_1 satisfy the following property*

$$Q^*_{11} S^*_1 = 0$$

Proof. See Appendix A.4.

Lemma 4.5 *In the case where $c_{12} > s_1 + h_1$, the optimal decision variables Q^*_{12} and S^*_1 satisfy the following property*

$$Q^*_{12} S^*_1 = 0$$

Proof. See Appendix A.5.

First Period: different possible cases

It is clear that the optimization problem relative to the first period subproblem, given in (4.29) and (4.30), can not be solved completely in an analytical manner. Nevertheless, under some conditions on the first order partial derivatives of the expected objective function $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ with respect to the decision variables Q_{11}, Q_{12} and S_1, we are able to provide an analytical solution. Indeed, in each partial derivative, we substitute the derivation variable by zero, and then we test the partial derivative sign in that point (zero). We combine the different cases of all the partial derivatives and we obtain eight total

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possible cases. As we have said, some of these cases can be solved completely and other cases cannot be solved. Hence, we provide an algorithm that permits to identify the optimal solution.

By concavity, there exists one and unique solution $(Q_{11}^*, Q_{12}^*, S_1^*)$, that maximize the total expected objective function (4.27). The optimization problem defined in (4.27) is constrained by the constraints given (4.30) and the optimal solution obtained by solving (4.27) must satisfy these constraints.

Using the first order optimality criteria, and taking into account the non-negativity constraints, we will consider all the possible cases of the optimal solution that may exist in terms of the different problem parameters. We distinguished these cases by the sign of the optimal value of each of the decision variables, Q_{11}, Q_{12} and S_1, obtained by the resolution of the unconstrained optimization problem defined in (4.27). The sign of the optimal value of a decision variable indicates the sign of the partial derivative of the total expected objective function with respect to that decision variable, if that decision variable is replaced, in the partial derivative, by zero. Using this rule, eight different cases can be obtained. In the following sections the conditions that should be satisfied in order to obtain each of the eight cases are given, and then an algorithm that permits to know which of these cases is the valid case is defined. This algorithm permits to provide the optimal solution.

a. case 1

This first case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) \leq 0, \tag{4.34}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) \leq 0, \tag{4.35}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) \leq 0. \tag{4.36}$$

Lemma 4.6 *If equations (4.34), (4.35) and (4.36) are satisfied, then the optimal solution of the constrained first period problem will be given by*

$$(Q_{11}^*; Q_{12}^*; S_1^*) = (0; 0; 0) \tag{4.37}$$

Proof. By concavity the unconstrained problem given by (4.29) has a unique solution.

If the first partial derivative of the expected objective function with respect to Q_{11} is negative at the point $(0; Q_{12}^*; S_1^*)$, then the optimal Q_{11} value for the unconstrained problem, Q_{11}^*, is negative. Since for the the first period optimization problem, we have $Q_{11} \geq 0$, then the optimal value of Q_{11} for that problem is $Q_{11}^* = 0$. The same reasoning is valid for Q_{12} and S_1. □

This first case corresponds to values of X_1 and of the different costs where it is not optimal to order any Q_{11} or Q_{12} quantities, and it is optimal to not return (salvage) any S_1 quantity. This could be induced by a huge value of X_1, that implies zero Q_{11}^* and Q_{12}^* with a very low s_1 value that implies a

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zero S_1^*.

b. case 2

The second case of the first period optimization problem corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) > 0, \tag{4.38}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) > 0, \tag{4.39}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) > 0. \tag{4.40}$$

Lemma 4.7 *Given the model parameters, the case 2 can not happen.*

Proof. From assumption (4.38), one could conclude, using the same reasoning as in the proof of Lemma 4.6, that $\exists\ Q_{11}^* > 0$, for which the first optimality equation (4.31) is satisfied.

From assumption (4.40), one could conclude, using the same reasoning as in the proof of Lemma 4.6, that $\exists\ S_1^* > 0$, for which the third optimality equation (4.33) is satisfied also.

Thus the optimal values of the decision variables Q_{11} and S_1 are both positive. Nevertheless, from the Lemma 4.4, one has the property

$$Q_{11}^* S_1^* = 0.$$

Consequently, the case 2 in not a feasible case. □

This case corresponds to values of the optimal decisions variables, where Q_{11}^* and S_1^* are both positive. However, from our model assumptions (section 4.2.2), it can be clearly seen that it is not profitable to order a Q_{11}^* quantity and then to completely or partially return (or salvage) it instantaneously. Therefore, this case can not be feasible.

c. case 3

The third case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) > 0, \tag{4.41}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) > 0, \tag{4.42}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) \leq 0. \tag{4.43}$$

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Lemma 4.8 *If equations (4.41), (4.42) and (4.43) are satisfied, then the optimal solution of the constrained first period optimization problem is given by*

$$(Q_{11}^*; S_1^*) = ((Y_{11} - X_1); 0)\,, \tag{4.44}$$

with

$$Y_{11} = F_1^{-1}\left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right). \tag{4.45}$$

And Q_{12}^* *is positive and verifies the following implicit equation*

$$\begin{aligned}
\Omega_1(Q_{12}^*) &= -c_{12} + c_{22} + (b_2 + c_{33} - c_{22})F_1(Q_{02} + Y_{11} + Q_{12}^* - Y_{12}) \\
&+ (-b_2 - c_{33} + s_2)F_1(Q_{02} + Y_{11} + Q_{12}^* - Y_{22}) \\
&+ (-b_2 - c_{33} + s_3 - h_2)\int_{Q_{02}+Y_{11}+Q_{12}^*-Y_{22}}^{Q_{02}+Y_{11}+Q_{12}^*-Y_{12}} f_1(x)F_2(Q_{02} + Y_{11} + Q_{12}^* - x)dx \\
&= 0
\end{aligned} \tag{4.46}$$

Proof. First, by concavity of the expected total objective function, and by taking into account the constraint of non-negativity of the decision variables, equation (4.43) implies that $S_1^* = 0$. Replacing S_1^* by its value in equations (4.31) and (4.32) gives a two-variable system.

By concavity also, equations (4.41) and (4.42) imply that $\exists\ Q_{11}^* > 0$ and $\exists\ Q_{12}^* > 0$ that satisfy the optimality equations (4.31) and (4.32) respectively.

Thus, by replacing Q_{12}^* in (4.31), by its value obtained by solving the second optimality equation (4.32), one gets

$$F_1(Q_{11}^* + X_1) = \left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right). \tag{4.47}$$

From the model assumptions (section 4.2.2), one has $c_{11} < c_{12} + b_1$ and $c_{12} < c_{11} + h_1$. These two inequations lead to

$$\frac{c_{12} - c_{11} + b_1}{h_1 + b_1} < 1,$$

and permit then to write

$$Q_{11}^* = F_1^{-1}\left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right) - X_1. \tag{4.48}$$

Then one replaces Q_{11}^* and S_1^* by their values in the second optimality equation (4.32) which leads to (4.46). □

Property 4.3 *If equations (4.41), (4.42) and (4.43) are satisfied, then the state variable* X_1 *satisfies the following property*

$$X_1 < Y_{11}.$$

Proof. Equation (4.41) implies that $\exists\ Q_{11}^* > 0$ which satisfies the first optimality equation (4.31).

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From equation (4.47), one has $Q^*_{11} + X_1 = Y_{11}$, which is equivalent to $X_1 = Y_{11} - Q^*_{11}$. As $Q^*_{11} > 0$ thus $X_1 < Y_{11}$. □

It is obvious that this case is relative to the low X_1 values and to attractive c_{12} unit ordering cost. Therefore, since X_1 is low, and since from our model assumptions (section 4.2.2) it can be easily seen that it is not profitable to backlog the first period demands to the second period, Q^*_{11} must be positive. That induces, due to our model assumptions that S^*_1 is equal to zero. Then due to relatively low ordering cost c_{12}, Q^*_{12} is positive.

d. case 4

This fourth case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q^*_{12}, S^*_1) \leq 0, \tag{4.49}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q^*_{11}, 0, S^*_1) > 0, \tag{4.50}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q^*_{11}, Q^*_{12}, 0) > 0. \tag{4.51}$$

Lemma 4.9 *If equations (4.49), (4.50) and (4.51) are satisfied, then the optimal solution of the constrained first period problem will be given by*

$$(Q^*_{11}; S^*_1) = (0; (X_1 - Y_{21}))\,, \tag{4.52}$$

with

$$Y_{21} = F_1^{-1}\left(\frac{c_{12} - s_1 + b_1}{h_1 + b_1}\right). \tag{4.53}$$

And Q^*_{12} is positive and verifies the following implicit equation

$$\begin{aligned} \Omega_2(Q^*_{12}) &= -c_{12} + c_{22} + (b_2 + c_{33} - c_{22})F_1(Q_{02} + Y_{21} + Q^*_{12} - Y_{12}) \\ &+ (-b_2 - c_{33} + s_2)F_1(Q_{02} + Y_{21} + Q^*_{12} - Y_{22}) \\ &+ (-b_2 - c_{33} + s_3 - h_2)\int_{Q_{02}+Y_{21}+Q^*_{12}-Y_{22}}^{Q_{02}+Y_{21}+Q^*_{12}-Y_{12}} f_1(x)F_2(Q_{02} + Y_{21} + Q^*_{12} - x)dx \\ &= 0 \end{aligned} \tag{4.54}$$

Proof. Using the same approach as in the preceding case, and permuting Q_{11} and S_1 the result given in Lemma 4.9 can be easily proven. □

Property 4.4 *If equations (4.49), (4.50) and (4.51) are satisfied, then the state variable X_1 satisfy the following property*

$$X_1 > Y_{21}.$$

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Proof. Equation (4.51) implies that $\exists\ S_1^* > 0$ which satisfies the third optimality equation (4.33).

On the other hand, equation (4.52) implies $S_1^* = X_1 - Y_{21}$, which is equivalent to $X_1 = Y_{21} + S_1^*$. As $S_1^* > 0$ thus $X_1 > Y_{21}$. □

Note that given the model assumptions (section 4.2.2) Y_{11} and Y_{21} satisfy

$$Y_{11} < Y_{21}. \tag{4.55}$$

Property 2 implies that case 4 corresponds to high X_1 values ($X_1 > Y_{21}$). Therefore, this available initial inventory, is sufficient to optimally satisfy the first period demands, and in addition, a part of this available inventory can be salvaged (or returned to the supplier). That implies, that the optimal Q_{11}^* should be equal to zero. On the other hand, this case corresponds to relatively low c_{12} values, which induces that the optimal Q_{12}^* is positive, and is used to satisfy the second period demands.

e. case 5

This fifth case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) \leq 0, \tag{4.56}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) > 0, \tag{4.57}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) \leq 0. \tag{4.58}$$

Lemma 4.10 *If equations (4.56), (4.57) and (4.58) are satisfied, then the optimal solution of the constrained first period subproblem is given by*

$$(Q_{11}^*; S_1^*) = (0; 0), \tag{4.59}$$

and Q_{12}^* is positive and verifies the following implicit equation

$$\begin{aligned} \Omega_3(Q_{12}^*) &= -c_{12} + c_{22} + (b_2 + c_{33} - c_{22})F_1(Q_{02} + X_1 + Q_{12}^* - Y_{12}) \\ &+ (-b_2 - c_{33} + s_2)F_1(Q_{02} + X_1 + Q_{12}^* - Y_{22}) \\ &+ (-b_2 - c_{33} + s_3 - h_2)\int_{Q_{02}+X_1+Q_{12}^*-Y_{22}}^{Q_{02}+X_1+Q_{12}^*-Y_{12}} f_1(x)F_2(Q_{02} + X_1 + Q_{12}^* - x)dx \\ &= 0 \end{aligned} \tag{4.60}$$

Proof. Using the same reasoning as in case (4), and substituting $Q_{11}^* = 0$ and $S_1^* = 0$ by their values in the second optimality equation (4.32), one can easily prove the result given in Lemma 4.10. □

Property 4.5 *If equations (4.56), (4.57) and (4.58) are satisfied, then the state variable X_1 verifies the*

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following property

$$Y_{11} \leq X_1 \leq Y_{21}.$$

Proof. Equation (4.55) gives $Y_{11} < Y_{21}$.

On the one hand, assume that $X_1 < Y_{11}$, thus property (4.3) implies that equation (4.41) is satisfied, which is contradictory with equation (4.56).

On the other hand, suppose that $X_1 > Y_{21}$, then property (4.4) implies that equation (4.51) is satisfied, which is contradictory with equation (4.58).

Therefore, we can conclude that $Y_{11} \leq X_1 \leq Y_{21}$. □

It is clear from Property 3 that the case 5 corresponds to medium initial inventory level X_1 values. For these values, it is not economically profitable to salvage any unit from this initial inventory neither to order any additional unit. Nevertheless, this initial inventory is not sufficient to satisfy the demands of the second period, and therefore, due to the attractive ordering cost c_{12}, the optimal Q_{12} is positive.

f. case 6

This case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) > 0, \tag{4.61}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) \leq 0, \tag{4.62}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) > 0. \tag{4.63}$$

Lemma 4.11 *Given the model parameters, case 6 can not happen.*

Proof. Using the same reasoning as in the proof of Lemma 4.7 one could easily prove Lemma (4.11). □

This case, according to our model assumptions (section 4.2.2), is not a feasible case, because Q_{11}^* and S_1^* can not be both positive.

g. case 7

This case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) > 0, \tag{4.64}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) \leq 0, \tag{4.65}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) \leq 0. \tag{4.66}$$

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Lemma 4.12 *If equations (4.64), (4.65) and (4.66) are satisfied, thus the optimal solution of the constrained first period problem is given by*

$$(Q_{12}^*; S_1^*) = (0; 0), \tag{4.67}$$

and Q_{11}^* *is positive and verifies the following equation*

$$\begin{aligned} \Omega_4(Q_{11}^*) &= -c_{11} + b_1 + c_{22} - (h_1 + b_1)F_1(Q_{11}^* + X_1) \\ &+ (b_2 + c_{33} - c_{22})F_1(Q_{02} + X_1 + Q_{11}^* - Y_{12}) \\ &+ (-b_2 - c_{33} + s_2)F_1(Q_{02} + X_1 + Q_{11}^* - Y_{22}) \\ &+ (-b_2 - c_{33} + s_3 - h_2)\int_{Q_{02}+X_1+Q_{11}^*-Y_{22}}^{Q_{02}+X_1+Q_{11}^*-Y_{12}} f_1(x)F_2(Q_{02} + X_1 + Q_{11}^* - x)dx \\ &= 0 \end{aligned} \tag{4.68}$$

Proof. Equations (4.65) and (4.66) and the expected objective function concavity property, with the constraints of non-negativity of the decision variables imply that $Q_{12}^* = 0$ and $S_1^* = 0$.

By substituting these values in the first optimality equation (4.31), one gets the result shown in equation (4.68). □

Case 7 should correspond to relatively low X_1 values, because the optimal quantity Q_{11}^*, that is mainly used to satisfy the first period demand, is positive, and therefore the optimal quantity S_1^* is equal to zero. In the case 7, the relative ordering cost c_{12} with respect to the ordering cost c_{22} should be relatively high, and therefore, the optimal quantity Q_{12}^* is equal to zero.

h. case 8

This last case corresponds to the following assumptions

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, Q_{12}^*, S_1^*) \leq 0, \tag{4.69}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, 0, S_1^*) \leq 0, \tag{4.70}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, 0) > 0. \tag{4.71}$$

Lemma 4.13 *If equations (4.69), (4.70) and (4.71) are satisfied, then the optimal solution of the constrained first period subproblem is given by*

$$(Q_{11}^*; Q_{12}^*) = (0; 0), \tag{4.72}$$

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and S_1^* is positive and verifies the following implicit equation

$$\begin{aligned}\Omega_5(S_1^*) &= s_1 - b_1 - c_{22} + (h_1+b_1)F_1(X_1 - S_1^*) \\ &- (b_2 + c_{33} - c_{22})F_1(Q_{02} + X_1 - S_1^* - Y_{12}) \\ &- (-b_2 - c_{33} + s_2)F_1(Q_{02} + X_1 - S_1^* - Y_{22}) \\ &- (-b_2 - c_{33} + s_3 - h_2)\int_{Q_{02}+X_1-S_1^*-Y_{22}}^{Q_{02}+X_1-S_1^*-Y_{12}} f_1(x)F_2(Q_{02}+X_1-S_1^*-x)dx \\ &= 0\end{aligned} \tag{4.73}$$

Proof. The proof is similar to that of Lemma 4.12. □

It is also clear that this last case corresponds first to relatively high ordering cost c_{12}, that makes Q_{12}^* equal to zero. In this case, also The optimal Q_{11}^* is equal to zero and the optimal S_1^* is positive, which means that the initial inventory level X_1 is relatively high.

Property 4.6 *In $\Omega_4(Q_{11}^*)$ defined in (4.68), assume that $Q_{11}+X_1 = x_1$, and in $\Omega_5(S_1^*)$ defined in (4.73) assume that $X_1 - S_1^* = x_2$.*

Define Y'_{11} as the solution of $\Omega_4(x_1) = 0$ and Y'_{21} as the solution of $\Omega_5(x_2) = 0$. We have the following property

$$Y'_{11} < Y'_{21} \tag{4.74}$$

Proof. By replacing x_2 by Y'_{11} in $\Omega_5(x_2)$ one gets

$$\Omega_5(Y'_{11}) = s_1 - c_{11}. \tag{4.75}$$

From the model assumptions (section 4.2.2), one has $s_1 - c_{11} < 0$, which implies that $\Omega_5(Y'_{11})$ is negative.

On the other hand, the first derivative of $\Omega_5(x_2)$ with respect to x_2 is given by

$$\frac{d\Omega_5(x_2)}{dx_2} = (b_1+h_1)f_1(x_2) + (b_2 + c_{33} + h_2 - s_3)\int_{Q_{02}+x_2-Y_{22}}^{Q_{02}+x_2-Y_{12}} f_1(x)f_2(Q_{02}+x_2-x)dx. \tag{4.76}$$

From the model assumptions (section 4.2.2), one has

$$c_{33} > s_3,$$

which implies

$$(b_2 + c_{33} + h_2 - s_3) > 0.$$

From equation (4.26), one has also $Y_{11} < Y_{21}$, which means that

$$\int_{Q_{02}+x_2-Y_{22}}^{Q_{02}+x_2-Y_{12}} f_1(x)f_2(Q_{02}+x_2-x)dx > 0.$$

That leads to conclude that the first derivative of $\Omega_5(x_2)$ with respect to x_2 is positive and consequently,

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$\Omega_5(x_2)$ is an increasing function in terms of x_2. By definition, Y'_{21} is the solution of $\Omega_5(x_2) = 0$, thus one has $\Omega_5(Y'_{21}) = 0$. While from (4.75) one has $\Omega_5(Y'_{11}) < 0$. Therefore, we can conclude that $Y'_{11} < Y'_{21}$. The same approach can be carried out by replacing Y'_{21} in $\Omega_4(x_1)$. □

Lemma 4.14 *If the following condition is satisfied*

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q^*_{11}, 0, S^*_1) \leq 0, \tag{4.77}$$

then the optimal policy of the first period can be given as follows:

- *for* $X_1 < Y'_{11}$, $Q^* = Y'_{11} - X_1$, $Q^*_{12} = 0$ *and* $S^*_1 = 0$,
- *for* $Y'_{11} \leq X_1 \leq Y'_{21}$, $Q^* = 0$, $Q^*_{12} = 0$ *and* $S^*_1 = 0$,
- *for* $X_1 > Y'_{21}$, $Q^* = 0$, $Q^*_{12} = 0$ *and* $S^*_1 = X_1 - Y'_{21}$.

Proof. For the three cases described above, we can get using (4.77) with the concavity property of the expected objective function and the model constraints that $Q^*_{12} = 0$.

Then replace Q^*_{12} by its value in the first order partial derivatives with respect to Q_{11} and S_1, given in (A.17) and (A.19).

Therefore, for the first case, $X_1 < Y'_{11}$, one finds

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(Y'_{11} - X_1, 0, 0) = 0 \text{ and } \frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Y'_{11} - X_1, 0, 0) < 0 \tag{4.78}$$

which induces, by concavity, that the solution $Q^*_{11} = Y'_{11} - X_1$ and $S^*_1 = 0$ is optimal.

For the second case, $Y'_{11} \leq X_1 \leq Y'_{11}$, one finds

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, 0, 0) < 0 \text{ and } \frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(0, 0, 0) < 0 \tag{4.79}$$

which induces, by concavity, that the solution $Q^*_{11} = 0$ and $S^*_1 = 0$ is optimal.

For the last case where $X_1 > Y'_{21}$ one finds

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0, 0, X_1 - Y'_{21}) < 0 \text{ and } \frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(0, 0, X_1 - Y'_{21}) = 0 \tag{4.80}$$

which induces, by concavity, that the solution $Q^*_{11} = 0$ and $S^*_1 = X_1 - Y'_{21}$ is optimal for that case. □

First period: optimal solution

Based on the results obtained in section 4.3.3, we define in this section an algorithm to provide the closed-form optimal solution of the first period, or the closed-form optimal solution with an implicit equation.

The concavity of the global expected objective function, given in (4.29), implies the existence of a single optimal solution of the first period subproblem. Therefore, for a given set of the model parameters, one and only one of the eight cases, described in section 4.3.3, is valid.

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Therefore, we propose the following algorithm, constituted of a series of five tests, that permits to know in which case one is located and consequently what is the optimal solution of the first optimization subproblem.

Test 1 If the following inequations are satisfied

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(0,0,0) \leq 0, \tag{4.81}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(0,0,0) \leq 0, \tag{4.82}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(0,0,0) \leq 0, \tag{4.83}$$

then the optimal solution is

$$(Q_{11}^*; Q_{12}^*; S_1^*) = (0; 0; 0).$$

Note that for a given X_1, the inequations $\Omega_1(0) > 0$, $\Omega_2(0) > 0$ and $\Omega_3(0) > 0$ could not be satisfied together.

Test 2 If Test (1) is not satisfied and if the inequality $\Omega_1(0) > 0$ is satisfied then the optimal solution is

$$(Q_{11}^*; S_1^*) = ((Y_{11} - X_1); 0)\,,$$

and Q_{12}^* verifies the following implicit equation

$$\Omega_1(Q_{12}^*) = 0.$$

Test 3 If Test (1) is not satisfied and if the inequality $\Omega_2(0) > 0$ is satisfied then the optimal solution is

$$(Q_{11}^*; S_1^*) = (0; (X_1 - Y_{21}))\,,$$

and Q_{12}^* verifies the following implicit equation

$$\Omega_2(Q_{12}^*) = 0.$$

Test 4 If Test (1) is not satisfied and if the inequality $\Omega_3(0) > 0$ is satisfied then the optimal solution is

$$(Q_{11}^*; S_1^*) = (0; 0),$$

and Q_{12}^* verifies the following implicit equation

$$\Omega_3(Q_{12}^*) = 0.$$

Test 5 If Tests (1) to (4) are not satisfied then the optimal solution is

$$Q^*_{12} = 0,$$

and Q^*_{11} and S^*_1 are given by the following:

- for $X_1 < Y'_{11}$, $Q^* = Y'_{11} - X_1$ and $S^*_1 = 0$,
- for $X_1 > Y'_{21}$, $Q^* = 0$ and $S^*_1 = X_1 - Y'_{21}$,

where Y'_{11} and Y'_{12} are given in Property 4.6.

First period: some analytical insights

In this section, we discuss the results obtained in the previous sections, and especially the cases 3, 4 and 5 described in section 4.3.3, where $Q^*_{12} > 0$.

In these three cases it is easily seen that the optimal values of the two decision variables Q^*_{11} and S^*_1 are completely independent, and the optimal value of the decision variable Q^*_{12} is dependent of the second period.

In the following section we treat the first period optimal decision variables Q^*_{11} and S^*_1 independence of the second period parameters. More precisely, we study the case where the optimal decision variable Q^*_{12} is positive, and we explain in this case, the independence of Q^*_{11} and S^*_1 of the ordering cost c_{22}, the unit salvage value s_1 and the second period demand D_2.

We treat this independence because its anti-intuitive nature. Indeed, normally, when the decisions related to Q_{11}, Q_{12} and S_1 are fixed, one must normally take into account the unit ordering cost c_{22} and the unit salvage value s_2 and the second period demand distribution, which is not the case here.

- **Independence of Q^*_{11} from the second period, when Q^*_{12} is positive**

  We discuss in this section the independence of the optimal decision variables Q^*_{11} from the second period parameters, namely the demand the ordering cost c_{22}, the salvage value s_2 and the demand distribution D_2.

  Note that we distinguish three different cases, based on the realization of the first period demand D_1: $X_2 < Y_{12}$, $Y_{12} \leq X_2 \leq Y_{22}$ and $Y_{22} < X_2$. In each of these cases, there exists a different optimal policy for the second period: in the first case, the optimal policy is ($Q^*_{22} > 0$;$S^*_2 = 0$). In the second case one has $Q^*_{22} = S^*_2 = 0$ and in the last case one has $Q^*_{22} = 0$ and $S^*_2 > 0$ (see 4.3.3).

  - **Independence from the unit order cost c_{22}**

    When $Q^*_{12} > 0$, then the optimal quantity Q^*_{11} is independent of the cost c_{22}.

    When Q^*_{12} is positive, then the first period demand D_1 can be satisfied with one of the three following quantities:

    * Q^*_{11} with a unit cost of c_{11},
    * Q^*_{12} with a unit cost of $c_{12} + b_1$,

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* Q^*_{22} with a unit cost of $c_{22} + b_1$.

From the model assumptions, we have $c_{11} < c_{12} + b_1$ and $c_{11} < c_{22} + b_1$. Therefore, we can conclude that due to the cost structure, it is beneficial to satisfy D_1 first with Q^*_{11}.

Let us firstly take the case where the demand D_1 is in such a way that $X_1 < Y_{12}$.

Following our decision process described in Figure 4.1, and taking into account the above constraints, the first period demand is then satisfied, by a prioritized order: first by Q^*_{11}, with a unit cost of c_{11}, then by Q^*_{12} with a unit cost of $c_{12} + b_1$ and finally by Q^*_{22} with a unit cost of $c_{22} + b_1$. The latter conclusion is equivalent to: the demand D_1 which is satisfied first with Q^*_{11}, will be then satisfied with Q^*_{12} if Q^*_{11} is not sufficient and then with Q^*_{22} if $Q^*_{11} + Q^*_{12}$ is not sufficient.

Taking into account the above reasoning, and since the quantities Q^*_{11} and Q^*_{12} are decided at the same time, we conclude that, in the case where $X_1 < Y_{12}$, Q^*_{11} must depend on the unit ordering cost c_{12} (and not on c_{22}), whereas $(Q^*_{11} + Q^*_{12})$ must depend on the unit ordering cost of Q^*_{22}, namely c_{22}.

Secondly, we take the last two cases of the first period demand realization where $Y_{12} < X_2 < Y_{22}$ and $X_2 > Y_{22}$. In these two cases $Q^*_{22} = 0$ (see section 4.3.3), and consequently Q^*_{11} is independent of c_{22}.

We conclude that for all the possible values of the demand D_1, Q^*_{11} is independent of c_{22}, which implies that Q^*_{11} is completely independent of c_{22}, when Q^*_{12} is positive.

– **Independence from the unit salvage value s_2**

If $Q^*_{12} > 0$, then the optimal quantity Q^*_{11} is independent of the salvage value s_2.

Firstly, we consider the two possible cases of the demand D_1 values, where $X_2 < Y_{12}$ and $Y_{12} \leq X_2 \leq Y_{22}$. In these two cases, the optimal decision variables S^*_2 is equal to zero (see section 4.3.3). Consequently, it becomes obvious that, in these two cases, Q^*_{11} is independent of the unit salvage value s_2.

Secondly, consider the case where $X_2 \geq Y_{22}$, in which $S^*_2 > 0$ and $Q^*_{22} = 0$ (see section 4.3.3). If any of the salvaged units does not belong to Q^*_{11}, then Q^*_{11} is independent of s_2. Assume that at least a part of the returned quantity belongs to Q^*_{11}.

As we have seen in the previous section, it is beneficial to satisfy the demand D_1 with the fast production mode (Q^*_{11}) more than the slow production mode (Q^*_{12}).

On the other hand, from the same model assumptions, one has $s_2 < c_{11} + h_1$ and $s_2 < c_{12}$ (section 4.2.2). These assumptions means that, in the case where $S^*_2 > 0$, it is more beneficial to have as less units as possible at the beginning of the second period to minimize the number of returned units.

Taking into account all these facts, we conclude that when $S^*_2 > 0$ and $Q^*_{12} > 0$, and since it is beneficial to reduce as much as possible the returned units (S^*_2), one must begin first by reducing Q^*_{12}. We conclude that as long as one can reduce Q^*_{12}, or in other terms as long as Q^*_{12} is positive, then Q^*_{11} does not depend on s_2.

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Consequently, we note that when Q_{12}^* is positive then Q_{11}^* is independent of s_2.

– **Independence from the second period demand D_2**

If $Q_{12}^* > 0$, then the optimal quantity Q_{11}^* is independent of the second period demand, namely D_2.

The second period demand D_2 can be satisfied with one of the three following quantities:

* Q_{11}^* with a unit cost of $c_{11} + h_1$,
* Q_{12}^* with a unit cost of c_{12},
* Q_{22}^* with a unit cost of c_{22}.

From the model assumptions, one has $c_{12} < c_{11} + h_1$. Since the decisions Q_1^* and Q_{12}^* are fixed at the same moment, therefore it is more beneficial to satisfy D_2 with Q_{12}^* than with Q_{11}^*.

From the model assumptions also, we have $c_{12} < c_{22}$. Therefore it is the difference between c_{12} and c_{22}, and the first period demand variability that determine, which of Q_{12}^* or Q_{22}^* is more beneficial to satisfy D_2. Although, in all the cases, and when $Q_{12}^* > 0$, it is either Q_{12} or Q_{22} that may be used to satisfy (optimally) the second period demand. Therefore, as long as Q_{12}^* is positive, Q_{11}^* is independent of D_2.

– **Equivalent *Newsvendor* model**

In the case where $Q_{12}^* > 0$, Q_{11}^* is defined by $Q_{11}^* = (Y_{11} - X_1)^+$ with

$$Y_{11} = F_1^{-1}\left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right),$$

given in (4.44) and (4.45).

This result can be interpreted as a classical *Newsvendor* problem with the following terms:

* underage cost $C_u = c_{12} - c_{11} + b_1$,
* overage cost $C_o = c_{11} + h_1 - c_{12}$.

The underage cost C_u is interpreted as follows:

Since Q_{12}^* is positive and it arrives before ordering Q_{22}^* and after the realization of D_1(Figure 4.2), therefore, in the case where there are back-orders from the first period, they will be satisfied first with Q_{12}^* with a unit cost of $c_{12} + b_1$. Therefore, C_u represents marginal cost of backlogging a demand of the first period and satisfying it with Q_{12}^* instead of Q_{11}^*.

The overage cost C_o is interpreted as follows:

If some units ordered with Q_{11}^* are not used during the first period, they will stay until the beginning of the second period. The unit cost charged for these units is then $c_{11}+h_1$. However, if these units would be ordered with Q_{12}, that arrives at the beginning of the second period, then the unit cost charged would be c_{12}. Therefore, the marginal overage cost is the difference between $c_{11} + h_1$ and c_{12}.

• **Independence of S_1^* from the second period when Q_{12}^* is positive**

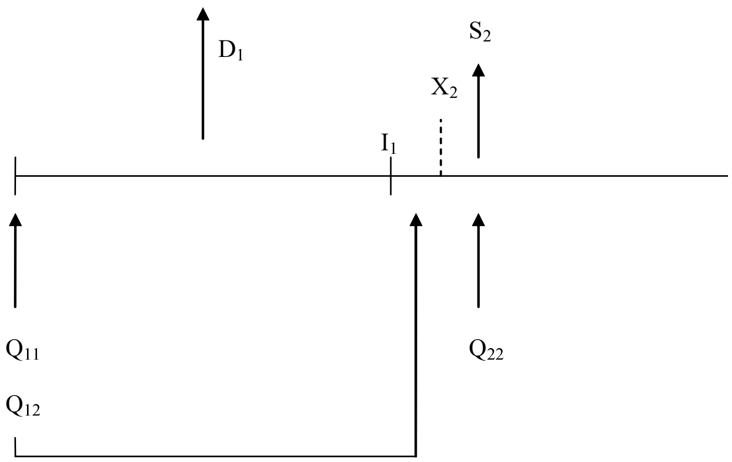

Figure 4.2: Decision process

In this paragraph we emphasize on the independence of the other first period optimal decision variable S_1^* of the second period parameters in the case where Q_{12}^* is positive.

– **Independence from the unit order cost c_{22} and of the demand D_2 distribution**

  If $Q_{12}^* > 0$, then the optimal decision variable S_1^* is independent of the cost c_{22}.

  Since the optimal decision variables S_1^* and Q_{12}^* are fixed before that the stochastic demand D_1 is realized, then in order to study the independence of S_1^* of the second period parameters, we must take separately the three possible ranges of the first period demand realization. First of all, we take the case where $X_2 < Y_{12}$, in which we have $Q_{22}^* > 0$.

  Note that when $Q_{12}^* > 0$ and $S_1^* > 0$, the Lemma 4.5 implies that $c_{12} < s_1 + h_1$. On the other hand, the second period demand D_2 can be satisfied with one (or more) of the three following quantities: I_1 (if $0 < I_1$), Q_{12}^* and Q_{22}^*. In function of the difference between c_{12} and c_{22}, one may have two cases:

  * $c_{12} < c_{22} < s_1 + h_1$,
  * $c_{12} < s_1 + h_1 < c_{22}$.

  In both of the above cases, it is clear that Q_{12}^* is more beneficial to satisfy the demand D_2, rather than the final inventory of the first period I_1 and the optimal quantity Q_{22}^*. Indeed, the marginal cost of keeping in stock a unit at the beginning of the first period is $s_1 + h_1$, and the marginal cost of ordering a unit with Q_{12}^* is c_{12}. Since the decisions Q_{12}^* and S_1^* are taken at the same moment, therefore, at that moment (the beginning of the first period) one returns as many units as possible with a salvage value s_1, and to satisfy the second period demand, one reorders units with Q_{12}^*. That implies that S_1^* depends on the ordering cost c_{12} and not on the ordering cost c_{22} neither on the demand D_2 distribution. Note that the optimal decision variable Q_{12}^* depends on the ordering cost c_{22} and the demand D_2 distribution.

  In the case where $c_{22} < c_{12} < s_1 + h_1$, the optimal decision variable Q_{12}^* is equal to zero and therefore this case does not correspond to the context studied in this section (where we have $Q_{12}^* > 0$).

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Now, we consider the cases of the demand D_1 realization, in which we have $Y_{12} < X_2 < Y_{22}$ or $X_2 > Y_{22}$. In these two cases, Q^*_{22} is equal to zero, and thus c_{22} should not be considered in the calculation of S^*_1. On the other hand, one has always $c_{12} < s_1 + h_1$ which means that Q^*_{12} has the priority to satisfy D_2, and since the decisions S^*_1 and Q^*_{12} are taken at the same moment, then only Q^*_{12} should depend on D_2 and therefore S^*_1 is independent of the demand D_2

We conclude that whatever is the realization of the first period demand D_1, and consequently whatever is the second period initial inventory level X_2, the decision variable S^*_1 is independent from the ordering cost c_{22} and from the demand D_2.

– **Independence from the unit salvage value** s_2

When $Q^*_{12} > 0$, then S^*_1 is independent from s_2.

The analysis of this independence is similar to those discussed in the previous paragraphs.

– **Equivalent *Newsvendor* model**

From section 4.3.3 and (4.53), we have that when Q^*_{12} is positive, then S^*_1 is given by $S^*_1 = (X_1 - Y_{21})^+$ with

$$Y_{21} = F_1^{-1}\left(\frac{c_{12} - s_1 + b_1}{h_1 + b_1}\right).$$

This solution can be interpreted as a modified *Newsvendor* problem solution with the following costs:

∗ underage cost $C_u = c_{12} - s_1 + b_1$,

∗ overage cost $C_o = s_1 + h_1 - c_{12}$.

The underage cost C_u represents the marginal cost of returning (salvaging) a product unit at the beginning of the second period, incurring a cost of $-s_1$, and then satisfying the related back-order, at the beginning of the next period with Q^*_{12}, incurring an additional cost of $c_{12}+b_1$

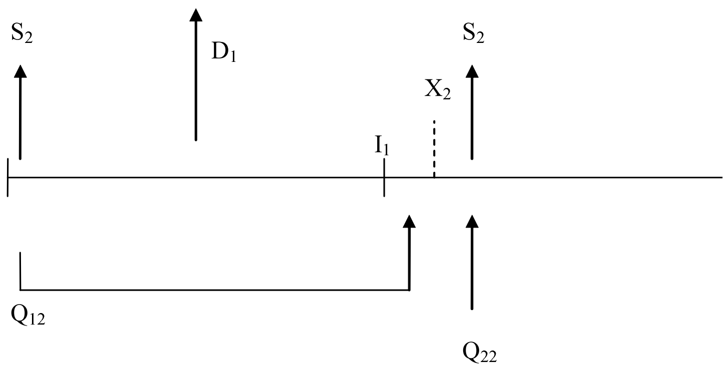

Figure 4.3: Decision process

The overage cost C_o represents the marginal cost of keeping an additional unit in stock (from the initial inventory of the first period), incurring a cost of h_1, instead of salvaging it and

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ordering a new unit with Q^*_{12} which incurs a cost of $c_{12} - s_1$. The overage cost, C_o, is then the difference between these two marginal costs.

4.4 Computational study

Based on some numerical examples, we will show, in this section, the behavior of our model as a function of the different parameters. These numerical applications give us insights which are complementary to the analytical ones, and that are compatible with the analytical solution that we have provided. The second period analytical solution is completely characterized by two threshold levels, nevertheless the first period solution is more complicated. Thus, we will emphasize on the first period decision variables and show how are they influenced by the first and second period parameters. The following sections are structured as follows:

- we begin with a nominal example that shows the first period optimal policy in terms of the first period initial inventory level X_1,
- then we show the impact of the first period demand variability on the optimal policy, and that of the second period demand variability,
- via other numerical examples, we show the effect of the ordering costs c_{12} and c_{22}.

Note that for these numerical applications, we assume that the demand has a truncated-normal distribution, corresponding to a normal distributed demand, $D \sim N[\mu; \sigma]$, for which we eliminate the negative part.

4.4.1 Nominal example

In this section, we describe via a nominal example the shape of the optimal policy of the first period in terms of its initial inventory level X_1. In this example, we also show the shape of the expected optimal policy of the second period. We represent by $E[Q^*_{22}]$ and $E[S^*_2]$ the expected optimal values of Q_{22} and S_2 in function of the demand D_1, the demand of the first period. This nominal example will be used in the following sections to provide some comparisons with other numerical examples varying different parameters.

The numerical data used in this example are the following: $D_1 \sim N[1000; 300]$, $D_2 \sim N[1000; 300]$, $h_1 = h_2 = 5$, $p_1 = p_2 = 100$, $b_1 = b_2 = 25$, $c_{11} = 50$, $c_{12} = 30$, $c_{22} = 50$, $c_{33} = 50$, $s_1 = s_2 = s_3 = 20$ and $Q_{01} = Q_{02} = 0$.

In this example we can easily identify the first period optimal policy shape, and the two thresholds structure that characterizes the optimal decision variables $Q^*_{11}(X_1)$ and $S^*_1(X_1)$.

In the numerical example defined above, one has $(c_{12} = 30) > (s_1 + h_1 = 25)$. Therefore, according to Lemma 4.5, one must have $Q^*_{12}S^*_1 = 0$, which is the case here.

One can also see that in X_1 values regions, where Q^*_{11} or S^*_1 is positive, the optimal Q^*_{12} value is constant in terms of X_1. In the region where both Q^*_{11} and S^*_1 are null, the optimal Q^*_{12} is decreasing in terms of X_1.

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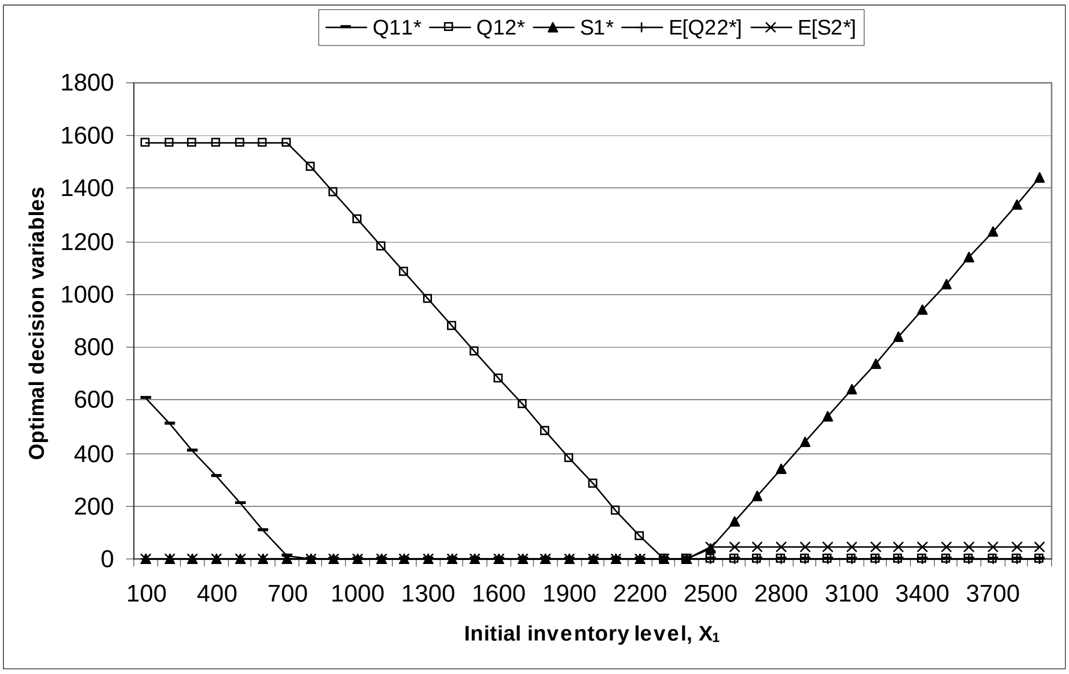

Figure 4.4: Nominal numerical example

In the region of small X_1 values, the optimal policy is an order-up to Y_{11} policy which implies that $X_1 + Q^*_{11} = Y_{11}$. Since Y_{11} is constant, then when X_1 increases, Q^*_{11} must decrease. Since the other model parameters are constant, thus Q^*_{12} does not vary. The same interpretation is valid for the high X_1 values with a salvage up-to-level policy and a positive decision variable S^*_1.

For the central region of X_1 values (between Y_{11} and Y_{21}), the two optimal decision variables Q^*_{11} and S^*_1 are equal to zero. In this region the only parameter that changes is X_1, and therefore, the optimal Q^*_{12} must decrease when X_1 increases.

4.4.2 Impact of the demand D_1 variability

In this section, we study the impact of the first period demand D_1 variability on the optimal policy. We consider the same numerical data of section 4.4.1 except the demand D_1 variability. For the first example (Figure 4.5), we consider a higher variability than that of the nominal example with $D_1 \sim N[1000; 450]$, and for the second example (Figure 4.8) we consider a lower variability with $D_1 \sim N[1000; 150]$.

As one could see from these two examples, for a given X_1 value, the optimal decision variables Q^*_{11} and S^*_1 decrease when the demand D_1 variability increases, while the optimal decision variable Q^*_{12} increases. Indeed, note that the width of the interval between Y_{11} and Y_{12} increases with the demand D_1 variability. By definition, Y_{11} and Y_{21} are the value of the inverse cumulative distribution function of D_1 at two given points, given by

$$\frac{c_{12} - c_{11} + b_1}{h_1 + b1}$$

and

$$\frac{c_{12} - s_1 + b_1}{h_1 + b1}$$

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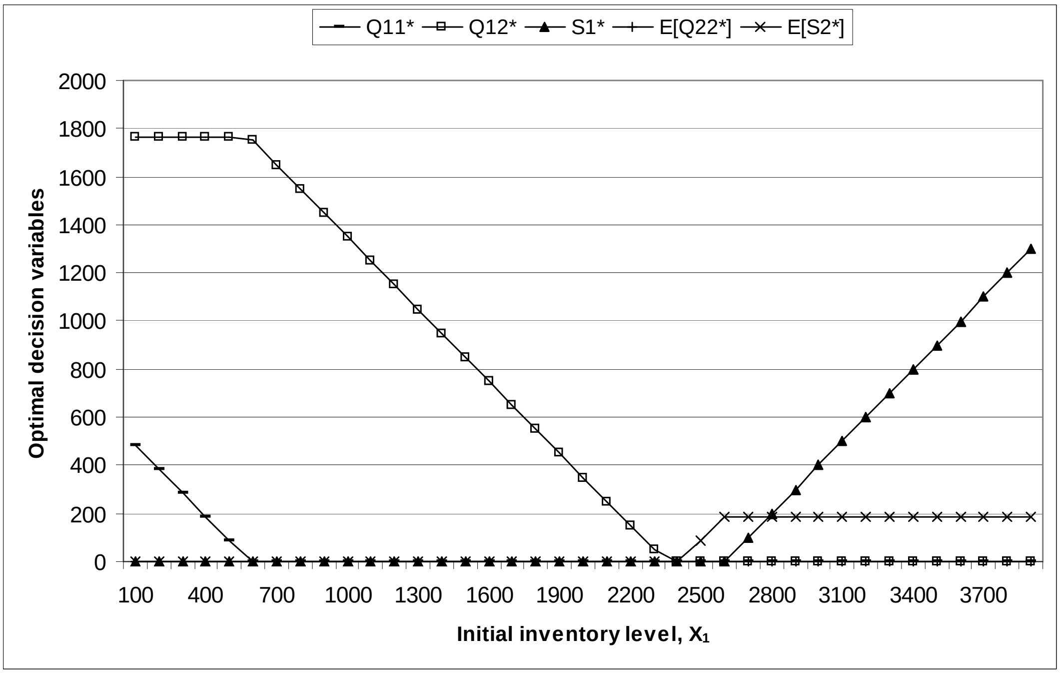

Figure 4.5: High D_1 variability

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respectively. In our numerical example, the first point, relative to Y_{11} is lower than the mean of the demand D_1 and the second point relative to Y_{21} is higher than the demand mean. Therefore, when the variability of D_1 increases, Y_{11} decreases and Y_{12} increases (see Figure 4.6 and 4.7). That implies, as a result of equations (4.44) and (4.52) that both Q^*_{11} and S^*_1 decrease. For the values of X_1 where Q^*_{11} is positive, and to compensate for the decrease in Q^*_{11}, Q^*_{12} increases. Indeed Q^*_{12} is ordered at the beginning of the first period and arrives at the beginning of the second period, and even if there is a backlog cost from the first to the second period, this quantity (Q^*_{12}) is used to satisfy backlogged orders of the first period. Therefore, when the demand D_1 variability increases, it becomes more profitable to order with the slow mode.

In the given numerical example, when $S^*_1 > 0$ one always has $Q^*_{12} = 0$ (and of course $Q^*_{11} = 0$), then when the demand D_1 variability increases, and to face this increase in variability, one find it more profitable not to sold more units with s_1 and to keep the available units at the beginning of the first period to be used in the first and eventually in the second period. In fact, as in our examples, the unit inventory holding cost h_1 is relatively low with respect to the difference between c_{12} and s_1 which makes Y_{21} higher than the mean of the demand D_1, (see (4.53)).

It is logical to get an increase in the expected optimal salvaged quantity at the beginning of the second period $E[S^*_2]$ due to the increase of Q^*_{12}.

4.4.3 Impact of the demand D_2 variability

In this section we show the impact of the second period demand D_2 variability on the optimal solution. We compare the nominal example shown in paragraph 4.4.1 with two other examples that have the same

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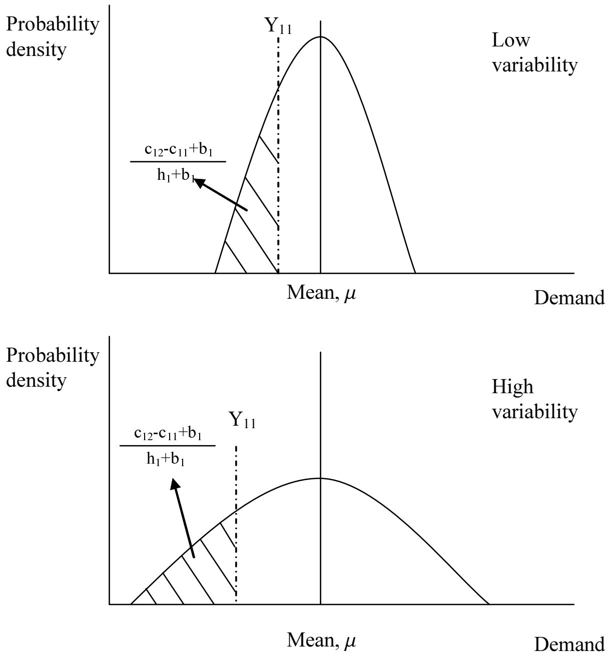

Figure 4.6: Y_{11} with two different demand D_1 variabilities

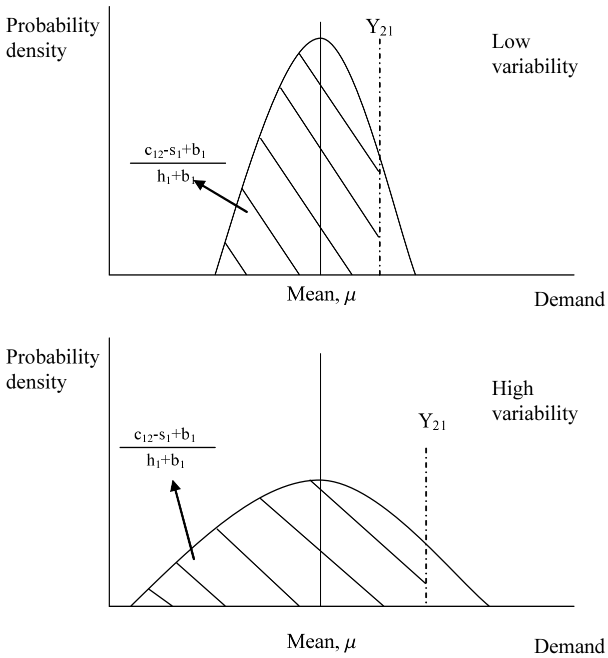

Figure 4.7: Y_{21} with two different demand D_1 variabilities

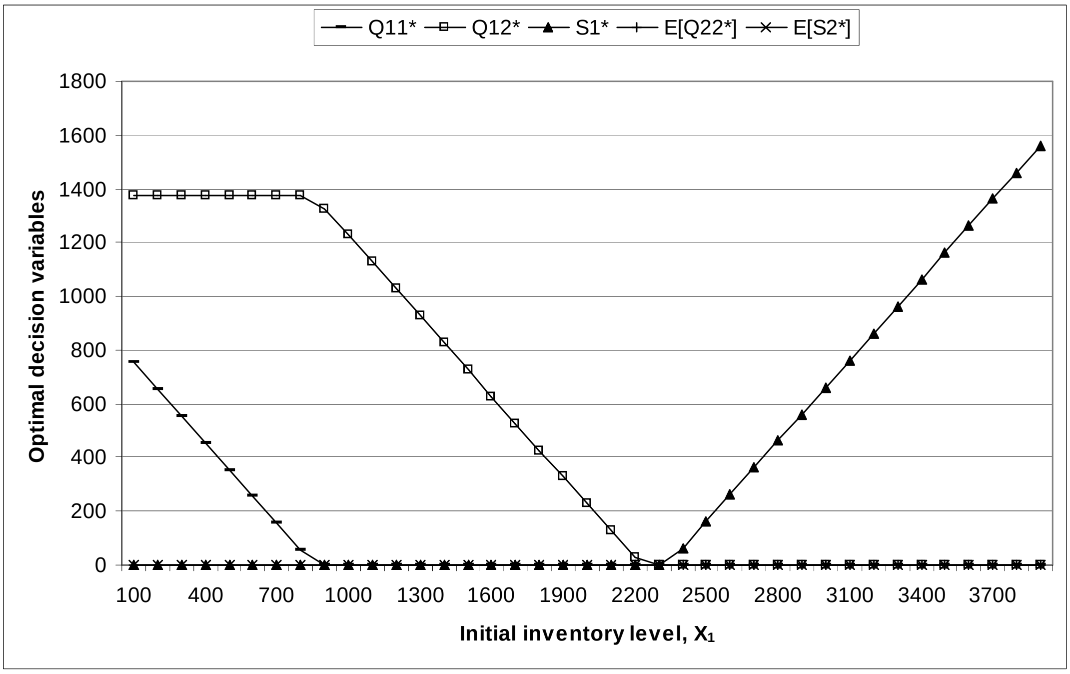

Figure 4.8: Low D_1 variability

numerical data except the standard deviation of D_2: for the first example, shown in Figure 4.9, we assume that $D_2 \sim N[1000; 450]$. For the second example shown in Figure 4.10 we suppose that $D_2 \sim N[1000; 150]$

From these two examples it can be easily see that when Q^*_{12} is positive (for low X_1 values), the optimal decision variables Q^*_{11} and S^*_1 are independent of the demand D_2 variability, whereas when $Q^*_{12} = 0$, the optimal decision variables Q^*_{11} and S^*_1 depends on the demand D_2 variability.

Take the region of the high X_1 values, where $S^*_1 > 0$. For example, for a given X_1, when the demand D_2 variability increases, the optimal decision variable S^*_1 value decreases, which permits keeping more units in stock to be eventually used in the second period, in order to face the increase in the second period demand variability.

Since in our example, Q_{12} is less costly for satisfying the demand D_2 than Q_{22} ($c_{12} < c_{22}$), therefore when the demand D_2 variability increases Q^*_{12} increases also, whereas, in these examples, Q^*_{22} is always equal to zero.

Two factors influence the expected optimal decision variable $E[S^*_2]$ when the variability of D_2 increases: firstly Y_{22} increases (see (4.25)); secondly, S^*_1 decreases (for high X_1 values).

4.4.4 Impact of the unit order cost c_{12}

In this section we consider a numerical example for which the numerical data are the same as in the nominal example given in section 4.4.1, except the initial inventory level X_1 which is equal to zero in this example ($X_1 = 0$) and the unit order cost c_{12} which is variable in this example.

Looking to this numerical example (Figure 4.11), one could immediately deduce that when c_{12} becomes higher than c_{22}, Q^*_{12} becomes automatically equal to zero. For a given c_{12} value which is less than c_{22} the

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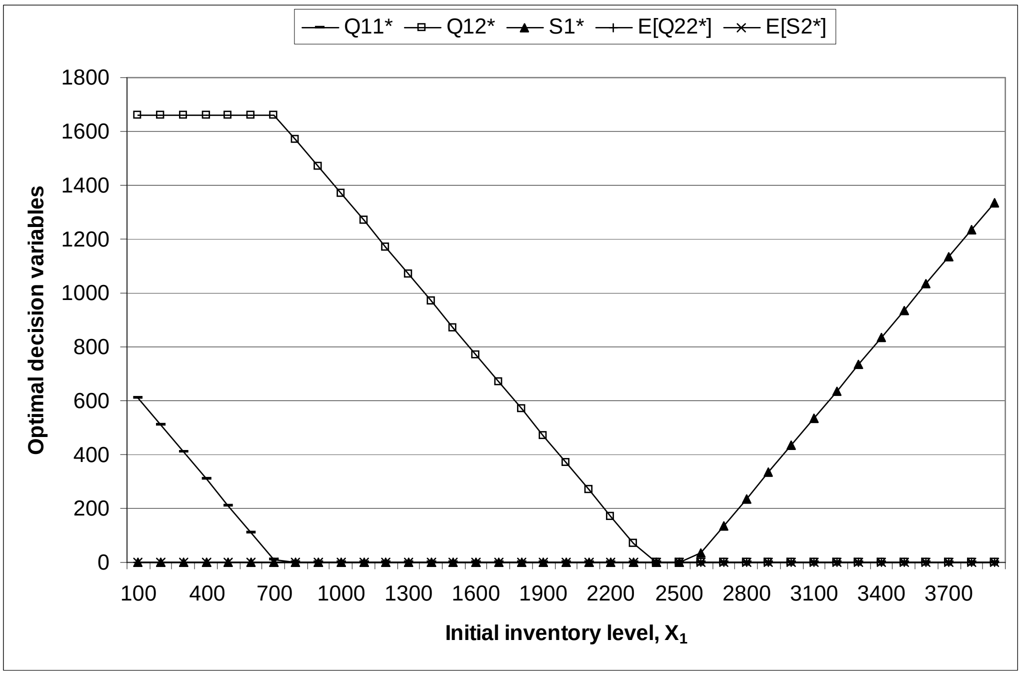

Figure 4.9: High D_2 variability

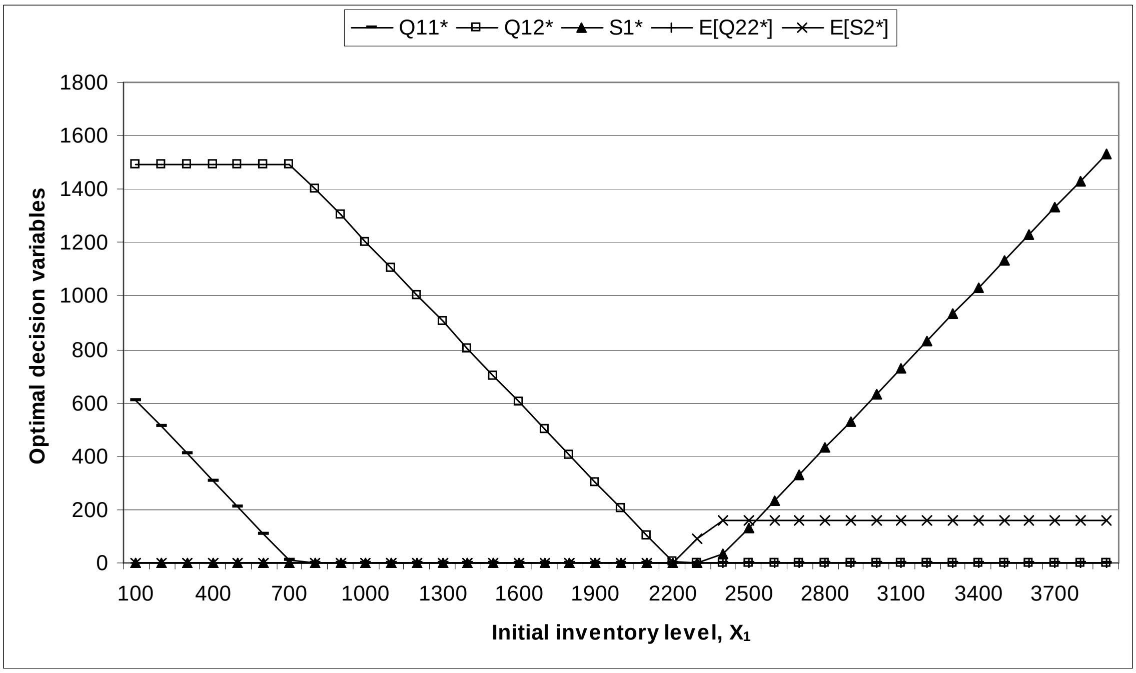

Figure 4.10: Low D_2 variability

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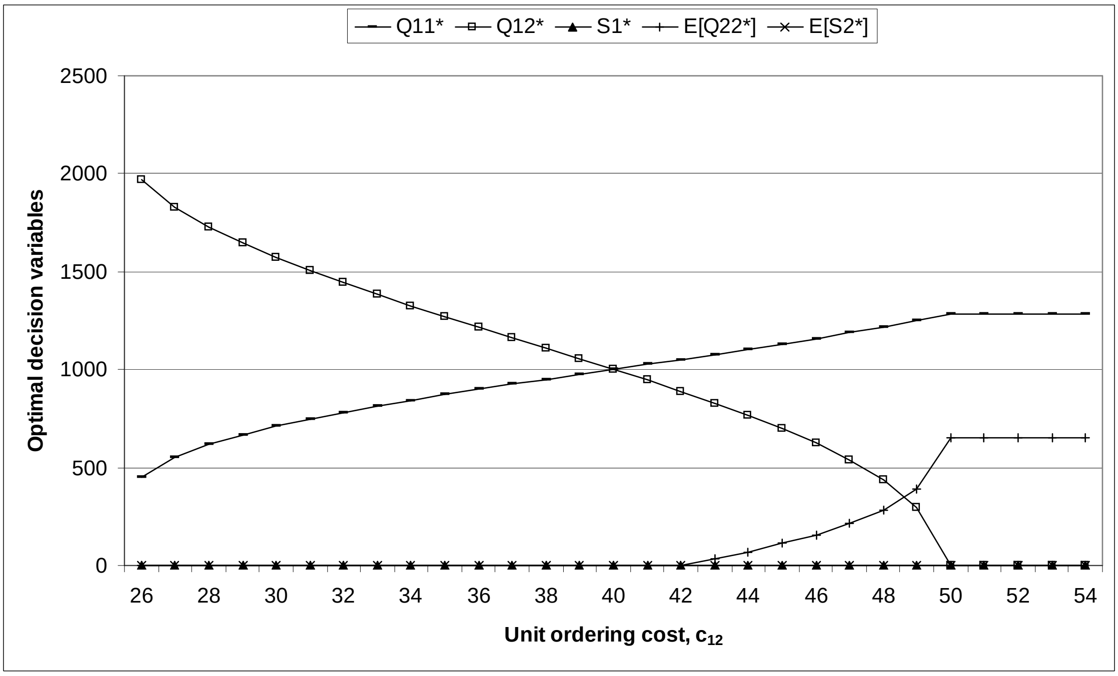

Figure 4.11: Impact of the unit order cost c_{12}

optimal expected decision variable $E[Q_{22}^*]$ becomes positive. This difference is due to the variability of the demand D_1, and to the role that Q_{12} plays in satisfying some of the backlogged demands of the first period. Indeed, when the optimal decision Q_{12}^* is taken, the demand D_1 is a random variable, while when the optimal decision Q_{22}^* is taken, the demand D_1 is known (realized). Therefore for a cost of c_{12} which is less than c_{22}, the expected Q_{22}^* becomes positive in order to compensate for this variability effect.

4.4.5 Impact of the unit order cost c_{22}

We consider in this section the nominal numerical example presented in section 4.4.1, with initial inventory level that is equal to zero ($X_1 = 0$) and with a variable unit ordering cost c_{22}.

It is clear, from Figure 4.12, that when $Q_{12}^* > 0$ the optimal decision variables Q_{11}^* and S_1^* are independent from the second period parameters and especially, for this example, from c_{22}. When c_{22} increases, the expected Q_{22}^* decreases, and becomes equal to zero for a certain value of c_{22}. This last value is higher than c_{12} (which is equal to 30 for this example). This difference is due, as we have seen in the previous section, to the fact that Q_{12}^* is decided before that D_1 is realized, while Q_{22}^* is decided after that D_1 is realized. To compensate for the decrease of expected Q_{22}^*, one increases Q_{12}^* and not Q_{11}^* to not pay the inventory holding cost at the first period.

4.4.6 Impact of the unit salvage value s_2

In this last example, we consider the same numerical data as in section 4.4.1 except the initial inventory level X_1 which is equal to zero and the unit salvage value s_2 which is variable in this example.

One can note that Q_{11}^* is independent of s_2, and this is because $Q_{12}^* > 0$. We deduce also that

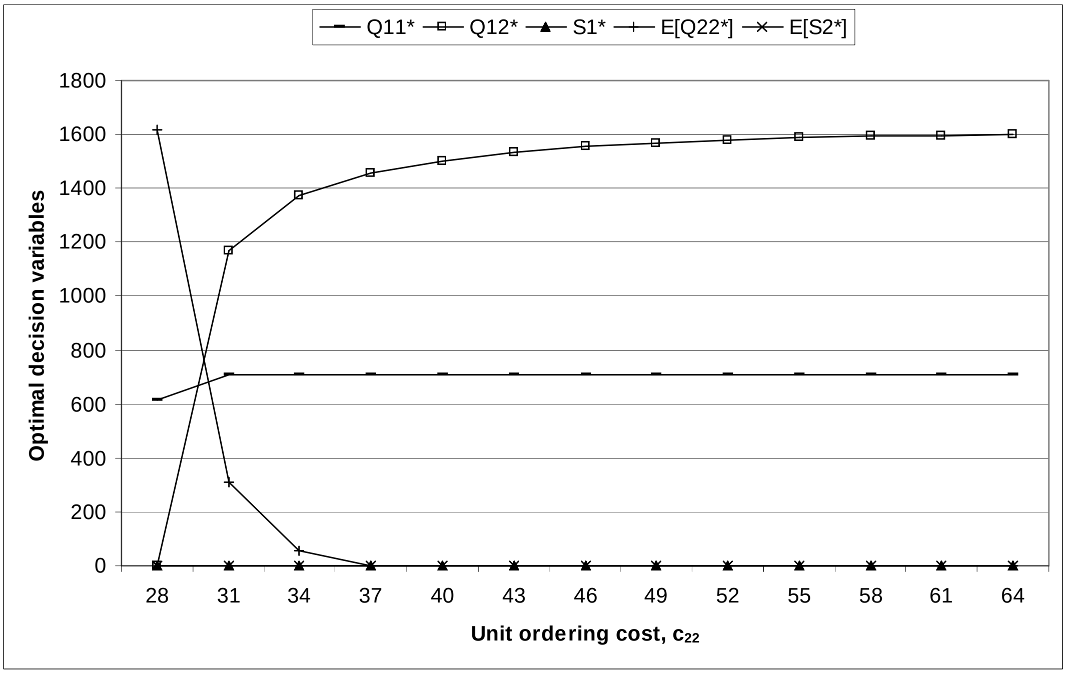

Figure 4.12: Impact of the unit order cost c_{22}

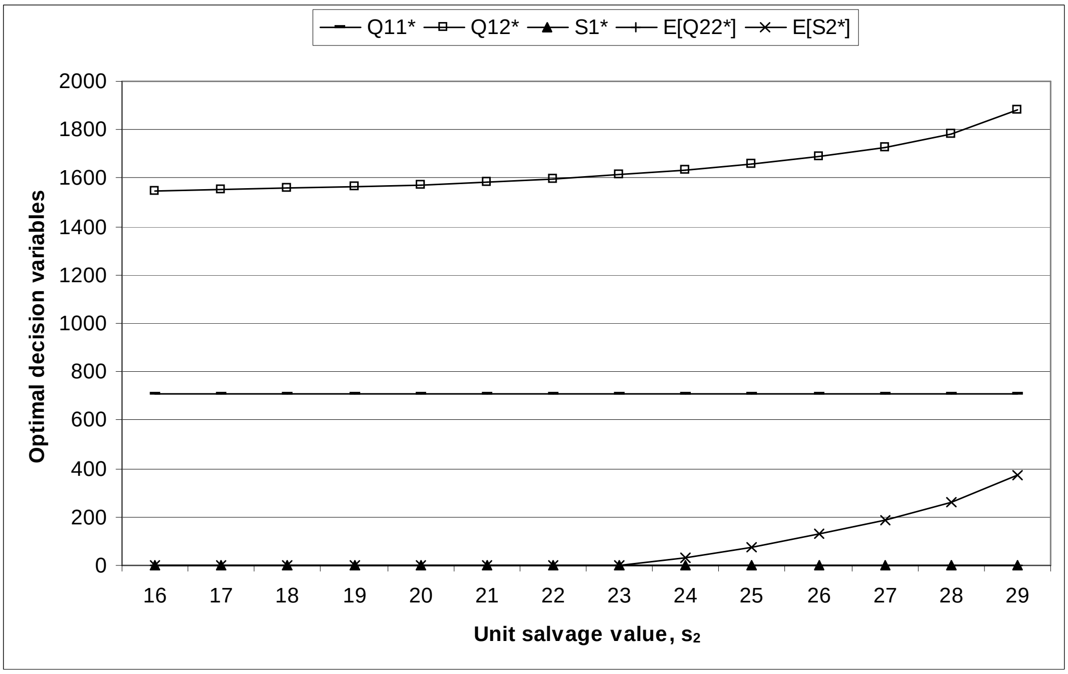

Figure 4.13: Impact of the unit salvage value, s_2, on the optimal policy

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Q^*_{12} increases with s_2 as the expected value of S^*_2. The increase of Q^*_{12} is due to the opportunity of returning a part of this ordered quantity to the supplier with an important s_2 salvage value, and is completely connected to the increase in the expected optimal salvaged quantity at the beginning of the second period.

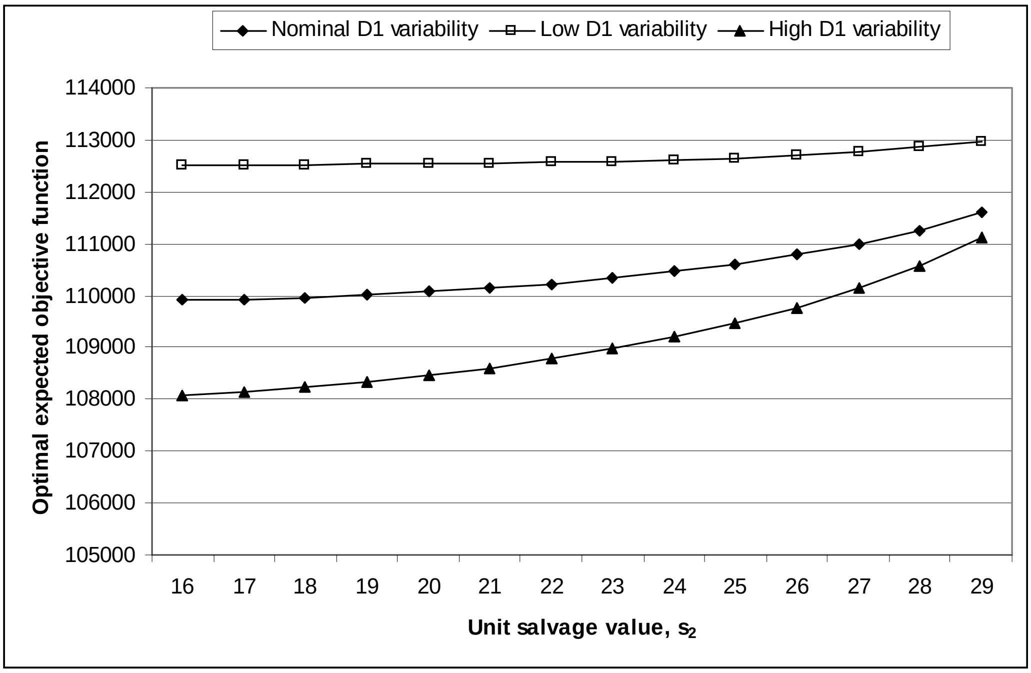

Figure 4.14: Impact of the unit salvage value, s_2, on the optimal expected objective function

In Figure 4.14 we show the impact of the unit salvage value on the optimal expected objective function. We present in this picture the expected optimal objective function of our nominal model in terms of s_2, for $X_1 = 0$, and for three different values of the demand D_1 standard deviation. The first one is the nominal value, 300. The second one is for the example with high D_1 variability, 450. The third one is for the low D_1 variability curve, 150.

Note that whatever is the variability of the demand D_1, the expected optimal objective function increases when the unit return value at the beginning of the second period, s_2, increases because the return option is very beneficial for the retailer. The increase in the objective function is more important when the variability of the demand D_1 is high. In fact, when the variability of D_1 increases, the number of potential shortages at the first period increases also, and the need to order more units at the beginning of the first period increases also: therefore when s_2 increases, that permits reduces the impact of the increase of the ordered units at the first period, because it permits a return with a better return price.

Note also that the expected optimal objective function decreases when the demand D_1 variability increases.

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4.5 Conclusion

In this chapter a new two-period production and inventory model has been presented. In this model, two production modes are allowed and return opportunities are available at the beginning of each period. A dynamic programming approach has been used in order to solve this stochastic two-period decision model. We have shown that the structure of the optimal policy of the second period is completely characterized by two threshold levels. The optimal policy of the first period has been characterized also by two threshold levels. We have shown that in the case where the optimal quantity ordered at the first period using the slow production mode is positive, the other decision variables of the first period are completely independent of the second period parameters. Since the complete analytical characterization of the first period optimal policy is not possible, an algorithm has been defined in order to characterize that optimal policy, completely in some cases, and using an implicit equation in other cases. The behavior of our model has been studied using some numerical examples. In the following two chapters (Chapter 5 and Chapter 6), we will improve the model presented in this chapter, by adding an exogenous information to update the demand forecast of the second period on the one hand, and by adding limited production capacities on the other hand.

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Chapter 5

Two-Period Production Planning and Inventory Control with Capacity Constraints

We provide in this chapter an extension of Chapter 4. We study a single product two-period production/inventory model, in which the demands at each period are independent random variables. To optimally satisfy these random demands, quantities can be produced at the beginning of each period using slow or fast production mode, under capacity constraints. In addition to the usual decision variables for such models, we consider that a certain quantity can be salvaged at the beginning of each period. Such salvage processes are useful if the initial inventory of a period is considered to be too high. The unsatisfied demands for each period are backlogged to be satisfied during the next periods. After the end of the second period, a last quantity is produced in order to satisfy remaining orders and to avoid lost sales. The remaining inventory, if any, is salvaged. We formulate this model using a dynamic programming approach. We prove the concavity of the global objective function and we establish the closed-form expression of the second period optimal policy. Then, via a numerical solution approach, we solve the first period problem and exhibit the structure of the corresponding optimal policy. We provide insights, via numerical examples, that characterize the basic properties of our model and the effect of some significant parameters such as costs, demand variabilities or capacity constraints.

Keywords: stochastic production and inventory planning, capacity constraints, salvage opportunities, dynamic programming.

5.1 Introduction

In this chapter, we extend the model developed in Chapter 4. We study, therefore, style-goods type products, characterized by a short life cycle with uncertain future demands. In the literature, the associated production/inventory management issues are modeled and analyzed, via the so-called *newsboy* model studied in Chapter 3.

However, in many cases a multi-periodic structure underlies production/inventory management problems. This is well known for long life cycle products (see (Vollmann et al., 1988)), but even for short life cycle products, as demonstrated by several recent research studies and successful applications (see (Fisher et al., 1996, 2001)). Such multi-periodic decision processes exhibit an important additional feature with respect to the classical one-period newsboy model: it permits one to be reactive and to adapt the successive orders to the successively observed demand fluctuations. In other words, in a single period model the unique order is issued once, before information about the effective demand is available. On the contrary, in a multi-period model, after each order the realized corresponding demand can be observed and future orders will clearly exploit this information.

We choose to consider here a *two-period* model. Our results can clearly be seen as building blocks permitting the analysis the structure of optimal decisions in general multi-period decision processes. In addition, we assume that the order sizes are limited by capacity constraints. These constraints could prevent the decision maker from satisfying the random future demand, inducing eventual lost-sales, or even penalties.

Several two-period models including capacity constraints have been developed in the framework of supply contracts. (Eppen and Iyer, 1997) studied a two-period lost sales model with backup agreement contracts in a forecast-update environment. In this type of contract, the quantity ordered in the second period is constrained and depends on the quantity ordered in the first period. (Bassok and Anupindi, 1997) studied supply contracts models with minimum commitment: the ordered quantity for the entire horizon has to be greater than an initially fixed commitment. Via a classical dynamic programming approach, these authors have exhibited the structure of the optimal policy for this particular multi-period inventory model with backlogs. (Donohue, 2000) applied an approach similar to that of (Choi et al., 2003) for a model with two production modes. The analysis is focused on return option in order to achieve channel coordination. (Sethi et al., 2005) analyzed, in a dynamic programming setting, a class of two-stage quantity flexibility contracts. In these contracts, one can order a first quantity before accurate forecasts are available, then after the demand forecast updates are performed, one can order a second constrained quantity and a third quantity from a spot market that is unconstrained. In (Barnes-Schuster et al., 2002), supply contracts with options are investigated. Their model is a two-period one with conditional demand distributions. The model is analyzed from the buyer and supplier points of view. Again, the theoretical analysis is mainly concerned by the channel coordination issue.

In the present chapter, we consider a two-period production/inventory model with backlogs. The induced costs are purchasing costs, inventory holding costs and backorder costs. The demands at the first and second period are described by independent random variables, with known probability distributions.

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We assume that at the end of the second period, the remaining inventory can be sold to a specific market with a given salvage value.

In addition to these classical parameters, we suppose that some preliminary fixed orders are to be delivered at each period. We suppose also that at the beginning of the first period, the initial inventory level can be given (as different from zero). This initial inventory could, for example, result from previous selling seasons, or from preliminary (early) orders.

The proposed model includes several production/ordering modes, with different delivery lead-times, in this way providing more flexibility to the decision maker.

Furthermore, in this model we consider capacity constraints: for each of the quantities ordered during the first and second periods, there is a specific bound that cannot be exceeded ((Cheaitou et al., b, 2006) and (Cheaitou et al., a, 2008)).

An important feature of the proposed model is worth being highlighted : at each period, the decision maker has the opportunity of salvaging a part of current inventory. We furthermore assume that the periodic salvage values are greater than the salvage value at the end of the last period. This general salvage process corresponds to several practical cases. First, when a parallel market exists, this market can be considered as a client that buys the products at a price lower than the usual market price. A second case can occur in the framework of a buyer-supplier contract in which a fraction of the orders can be returned to the supplier if the current inventory is considered to be too high with respect to expected future demands. Clearly, in such settings the return price can be lower than the production/ordering cost.

In summary, the model studied in the present chapter has the following features :

- first, the periodic ordering process is quite general in the sense that at each time period orders can be made for the different subsequent periods, possibly with different costs and for general demand distributions,
- second, the periodic selling process is quite general, in the sense that, in addition to the classical selling process, it is possible, at the beginning of each period, to sell a part of the available inventory to a parallel market, at a given salvage value,
- third, the data are dynamic : the selling prices, costs, salvage values and demand probability distributions are period-dependent,
- fourth, the model includes initial inventory and initially fixed order quantities to be delivered in the different periods,
- fifth, the model includes many production/ordering modes and quantity-specific capacity constraints.

The remaining parts of this chapter are structured as follows: the second section describes the model (namely the complete decision process and the optimization problem). In the third section, we propose the solution approach. We first prove the objective function concavity of the second period subproblem. Then we develop an analytical solution for the second period that characterizes the optimal policy, and

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by using dynamic programming, we then provide some analytical properties of the optimal policy of the first period. Numerical examples are solved in section four, in order to give some managerial insight for this kind of production/inventory system. The last section is dedicated to the conclusion and further research ideas.

5.2 The model

5.2.1 Model description

As we have mentioned above, the model presented in this chapter is based on the model provided in Chapter 4. The main difference is the addition in this chapter of production capacity constraints.

In each period t, the random demand D_t is defined by a probability density function (PDF) $f_t(\cdot) : [0, +\infty[\to \mathbb{R}^+$ and by a cumulative distribution function (CDF) $F_t(\cdot) : [0, +\infty[\to [0, 1]$. At each period any received demand is charged at a price p_t, even if it is satisfied at the next period.

We define the decision variables Q_{ts} (with $0 \leq t \leq 3$ and $t \leq s \leq 3$) as the quantities ordered at the beginning of period t to be received at the beginning of period s, with a unit order cost of c_{ts}. Some of these orders are limited by capacity constraints, explicitly given in (5.5). Q_{01} and Q_{02} have been ordered before the selling horizon and are assumed to be given. We now introduce the additional decision variables S_t (with $1 \leq t \leq 3$), which are the quantities salvaged at the beginning of period t, with unit salvage values s_t. All the decision variables, i.e. Q_{ts} and S_t, are assumed to be non-negative.

The state variables of the model are X_t, the inventory level at the beginning of each period, and I_t, the inventory level at the end of each period (I_0 is given and considered as the initial inventory for the problem).

The periodic inventory holding cost is h_t, while unsatisfied orders in period t are backlogged to the next period, with a penalty shortage cost b_t. It is worth noting that the third period is used in the model not as real period, involving a decision process to be optimized, but only as a terminal condition for the model.

Figure 5.1 presents the structure of the decision process and demand realization, which is as follows. The available inventory at the beginning of the first period, before current orders are chosen and demand occurs, is

$$X_1 = I_0 + Q_{01}, \tag{5.1}$$

where, in fact, I_0 and Q_{01} can be considered as data. Then decision variables Q_{11}, Q_{12} and S_1 are fixed. Demand D_1 occurs in such a way that the available inventory at the end of the first period is given by

$$I_1 = X_1 + Q_{11} - D_1 - S_1. \tag{5.2}$$

The available inventory at the beginning of the second period, before current orders are chosen and demand occurs, and after Q_{12} is received, is

$$X_2 = I_1 + Q_{02} + Q_{12} = X_1 + Q_{11} - D_1 - S_1 + Q_{02} + Q_{12}, \tag{5.3}$$

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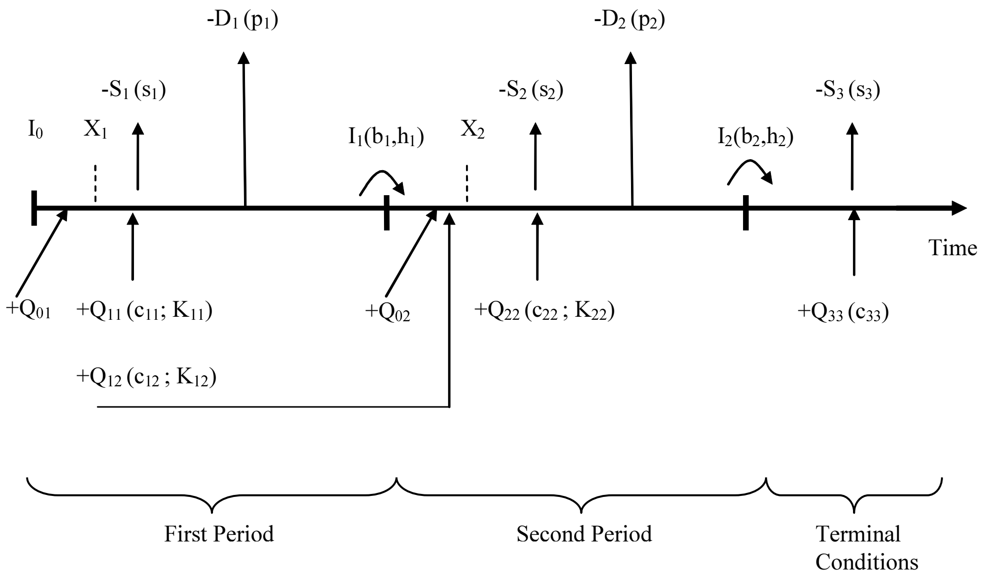

Figure 5.1: Decision process

where Q_{02} can be considered as data. Then decision variables Q_{22} and S_2 are fixed. Demand D_2 occurs in such a way that the available inventory at the end of the second period is given by

$$I_2 = X_2 + Q_{22} - D_2 - S_2. \tag{5.4}$$

The orders are constrained and the salvaged quantities clearly cannot be higher than the available inventories. These constraints are formulated by the following inequations

$$0 \leq Q_{11} \leq K_{11}, \quad 0 \leq Q_{12} \leq K_{12}, \quad 0 \leq Q_{22} \leq K_{22}, \quad 0 \leq S_1 \leq X_1^+, \quad 0 \leq S_2 \leq X_2^+. \tag{5.5}$$

The last order Q_{33} is supposed to be unconstrained. Indeed, if the quantities ordered during the first and second periods are not sufficient to satisfy demands, this last quantity, Q_{33}, is ordered from an assumed unconstrained spot market, with a unit order cost c_{33}. As in (Sethi, 2005), we assume that c_{33} is greater than the unit order costs of the preceding periods. Ordering from such a spot infinite market permits one to satisfy all the unsatisfied orders, guaranteeing a pure backlog model without lost sales.

The optimal terminal decision process in the third period is as follows (see (Khouja, 1999), (Silver et al., 1998) and (Hillier and Lieberman, 1990)): after demand D_2 has occurred, it is optimal to order Q_{33} and salvage S_3 defined as follows

$$Q_{33} = -I_2 \text{ if } \; I_2 \leq 0 \; \text{ and } \quad S_3 = I_2 \text{ if } \; I_2 > 0. \tag{5.6}$$

This result has been shown in Lemma 4.1 in Chapter 4.

Note that the assumptions on the model parameters are the same as those defined in Chapter 4 (see section 4.2.2).

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5.2.2 The optimization problem

Introduce $\Pi_1(X_1, Q_{11}, Q_{12}, S_1)$ and $\Pi_2(X_2, Q_{22}, Q_{33}, S_2, S_3)$ as the expected profit of the first and the second periods with respect to the random variables D_1 and D_2 respectively. In the following sections, the decision variables S_3 and Q_{33} will be adapted to the state of the system according to optimal feedback policy (5.6). These variables will thus be eliminated from the profit function expressions. The expected profits $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are then formulated as follows

$$\begin{aligned}\Pi_1(X_1, Q_{11}, Q_{12}, S_1) =& p_1 E[D_1] + s_1 S_1 - c_{11} Q_{11} - c_{12} Q_{12} \\ & - h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1 \\ & - b_1 \int_{X_1+Q_{11}-S_1}^{+\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\Pi_2(X_2, Q_{22}, S_2) =& p_2 E[D_2] + s_2 S_2 - c_{22} Q_{22} \\ & - h_2 \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2)\, dD_2 \\ & - b_2 \int_{X_2+Q_{22}-S_2}^{+\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2)\, dD_2 \\ & + s_3 \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2)\, dD_2 \\ & - c_{33} \int_{X_2+Q_{22}-S_2}^{+\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2)\, dD_2,\end{aligned} \tag{5.8}$$

where $E[D_1]$ and $E[D_2]$ represent the expectation of the first and second period demands respectively. Define $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}, S_1, S_2)$ as the global expected profit with respect to the random variables D_1 and D_2. This global expected profit is then

$$\Pi_1(X_1, Q_{11}, Q_{12}, S_1) + E_{D_1}\left\{\Pi_2(X_2(X_1, Q_{11}, Q_{12}, S_1), Q_{22}, S_2)\right\}, \tag{5.9}$$

where $E_{D_1}\{\cdot\}$ represents the expectation, with respect to D_1. The global optimization problem is then to maximize $\Pi(\cdot)$ with respect to the decision variables of the first and second periods under the constraints defined in (5.5), namely

$$\max_{Q_{11}, Q_{12}, Q_{22}, S_1, S_2} \Pi(X_1, Q_{11}, Q_{12}, Q_{22}, S_1, S_2), \tag{5.10}$$

subject to

$$0 \le Q_{11} \le K_{11}, \quad 0 \le Q_{12} \le K_{12}, \quad 0 \le Q_{22} \le K_{22}, \quad 0 \le S_1 \le X_1^+, \quad 0 \le S_2 \le X_2^+. \tag{5.11}$$

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5.3 The resolution approach

Via a classical dynamic programming approach (see (Bertsekas, 2005), (Barnes-Schuster et al., 2002) and (Sethi et al., 2005) and others), we decompose problem (5.10)-(5.11) into two one-period subproblems as follows. The first subproblem is associated with the second period. The optimal solution for this problem, namely the optimal policy Q^*_{22} and S^*_2, expressed as a function of the state variable X_2, is defined as the solution of the optimization problem

$$\max_{Q_{22},S_2} \{\Pi_2(X_2, Q_{22}, S_2)\}, \tag{5.12}$$

s.t.

$$0 \leq Q_{22} \leq K_{22} \quad \text{and} \quad 0 \leq S_2 \leq X_2^+. \tag{5.13}$$

Then, assuming that the optimal policy $(Q^*_{22}(X_2), S^*_2(X_2))$ will be implemented in the second period, the optimal policy for the first period, namely $(Q^*_{11}(X_1), Q^*_{12}(X_1), S^*_1(X_1))$ can be obtained as the solution of the problem

$$\begin{aligned} &\max_{Q_{11},Q_{12},S_1} \{\Pi(X_1, Q_{11}, Q_{12}, Q^*_{22}(X_2), S_1, S^*_2(X_2))\} = \\ &\max_{Q_{11},Q_{12},S_1} \{\Pi_1(X_1, Q_{11}, Q_{12}, S_1) + E_{D_1}\{\Pi^*_2(X_2, Q^*_{22}(X_2), S^*_2(X_2))\}\}, \end{aligned} \tag{5.14}$$

s.t.

$$0 \leq Q_{11} \leq K_{11}, \quad 0 \leq Q_{12} \leq K_{12} \quad \text{and} \quad 0 \leq S_1 \leq X_1^+. \tag{5.15}$$

5.3.1 Second-period subproblem

We exhibit the solution of the second period sub-problem. This kind of problems has been studied in the previous chapters. First, we prove the concavity of the expected objective function. Then, using the first order optimality condition, we provide the optimal policy.

In Chapter 4, it has been proved that the objective function $\Pi_2(X_2, Q_{22}, S_2)$, defined in (5.8) is a jointly concave function with respect to Q_{22} and S_2. Therefore, the optimal policy can be defined in the following section.

Optimal policy First, we solve the unconstrained optimization problem defined by (5.12). We get, as shown in Appendix A.1, a two-threshold optimal policy, with the threshold levels given by

$$Y_{12} = F_2^{-1}\left(\frac{b_2 + c_{33} - c_{22}}{b_2 + c_{33} + h_2 - s_3}\right) \quad \text{and} \quad Y_{22} = F_2^{-1}\left(\frac{b_2 + c_{33} - s_2}{b_2 + c_{33} + h_2 - s_3}\right). \tag{5.16}$$

By the model assumptions (see Chapter 4), it is easily seen that the two thresholds satisfy

$$Y_{12} \leq Y_{22}. \tag{5.17}$$

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Via above lemma, and using the results shown in Appendix A.1, it is clear that the optimal policy for the constrained problem (5.12)-(5.13) will be the following (see (Bassok and Anupindi, 1997)),

$$\text{if } \ X_2 \leq Y_{12} - K_{22} \Rightarrow \begin{cases} Q_{22}^* & = & K_{22}, \\ S_2^* & = & 0, \end{cases} \tag{5.18}$$

$$\text{if } \ Y_{12} - K_{22} \leq X_2 \leq Y_{12} \Rightarrow \begin{cases} Q_{22}^* & = & Y_{12} - X_2, \\ S_2^* & = & 0, \end{cases} \tag{5.19}$$

$$\text{if } \ Y_{12} \leq X_2 \leq Y_{22} \Rightarrow \begin{cases} Q_{22}^* & = & 0, \\ S_2^* & = & 0, \end{cases} \tag{5.20}$$

and

$$\text{if } \ X_2 \geq Y_{22} \Rightarrow \begin{cases} Q_{22}^* & = & 0, \\ S_2^* & = & X_2 - Y_{22}. \end{cases} \tag{5.21}$$

These conditions amount to

$$Q_{22}^*(X_2) = \big(\min(Y_{12} - X_2; K_{22})\big)^+ \ \text{ and } \ S_2^*(X_2) = (X_2 - Y_{22})^+. \tag{5.22}$$

Note that the production capacity constraint related to K_{22} is active only if K_{22} is smaller than the order-up-to-level threshold Y_{12}. Otherwise the capacity constraint of the second period will no longer be active.

5.3.2 First period subproblem

The optimization problem to solve for the first period problem is given by

$$\max_{Q_{11},Q_{12},S_1} \{\Pi_1(X_1, Q_{11}, Q_{12}, S_1) + E_{D_1}\{\Pi_2^*(X_2, Q_{22}^*(X_2), S_2^*(X_2))\}\} \tag{5.23}$$

s.t.

$$0 \leq Q_{11} \leq K_{11}, \quad 0 \leq Q_{12} \leq K_{12}. \tag{5.24}$$

Lemma 5.1 *The total objective function* $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))$*, defined in (5.14) is a jointly concave function with respect to* Q_{11}*,* Q_{12} *and* S_1*.*

Proof. See Appendix B.1.

Lemma 5.1 shows that there exists an optimal policy for the first period optimization problem and that this policy is unique. We define $Q_{11}^*(X_1)$, $Q_{12}^*(X_1)$ and $S_1^*(X_1)$ as the optimal values of the first period decision variables as a function of X_1.

Lemma 5.2 *There are no optimal solutions with $Q_{11}^*(X_1) > 0$ and $S_1^*(X_1) > 0$ simultaneously. In other words, one has the property $Q_{11}^*(X_1)\, S_1^*(X_1) = 0$.*

Proof. See Appendix B.3.

Lemma 5.3 *When the optimal value of the decision variable Q_{12} is positive, namely when $Q_{12}^*(X_1) > 0$, the optimal values of the decision variables Q_{11} and S_1 are completely characterized by two thresholds given by*

$$Y_{11} = F_1^{-1}\left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right) \quad \textit{and} \quad Y_{21} = F_1^{-1}\left(\frac{c_{12} - s_1 + b_1}{h_1 + b_1}\right). \tag{5.25}$$

In other words, one has

$$(Q_{11}^*; S_1^*) = \left(\min\left((Y_{11} - X_1)^+; K_{11}\right); (X_1 - Y_{21})^+\right) \tag{5.26}$$

Proof. See Appendix B.4.

The above lemmas partially characterize the first period optimal policy. Note that the algorithm developed in Chapter 4, in order to characterize the first period optimal policy, is still valid for the model presented in this chapter.

Unfortunately, there exists no complete closed-form solutions for this general problem (i.e. for general demand probability distributions). We thus must have recourse to numerically computed solutions for the examples analyzed in the following section.

5.4 Numerical applications and insights

We illustrate the impact of the main parameters of the model via several numerical examples. The second period analytical solution is completely determined by two threshold levels, while the first period optimal solution is determined numerically. Thus, we will emphasize especially the first period decision variables and show how these variables and policies are influenced by the model parameters. The main parameters are: the initial inventory level, the demand variability, the costs and the capacity constraints.

We first begin with a nominal example that exhibits the first period optimal policy as a function of the initial inventory level I_0. After this first example, we illustrate the effect of the first and second period demand variability. Then, via other numerical examples we show the effect of the cost parameters (we choose c_{22}) and the production capacity constraint (we choose K_{11} and K_{12}).

Note that for these numerical applications, we classically assume that the demand of period t has a truncated-normal distribution, restricted to positive values. Note also that we represent by $E[Q_{22}^*]$ and $E[S_2^*]$ the expected optimal values of Q_{22} and S_2 in function of D_1, the demand of the first period, assuming that the optimal policy is implemented in the first period.

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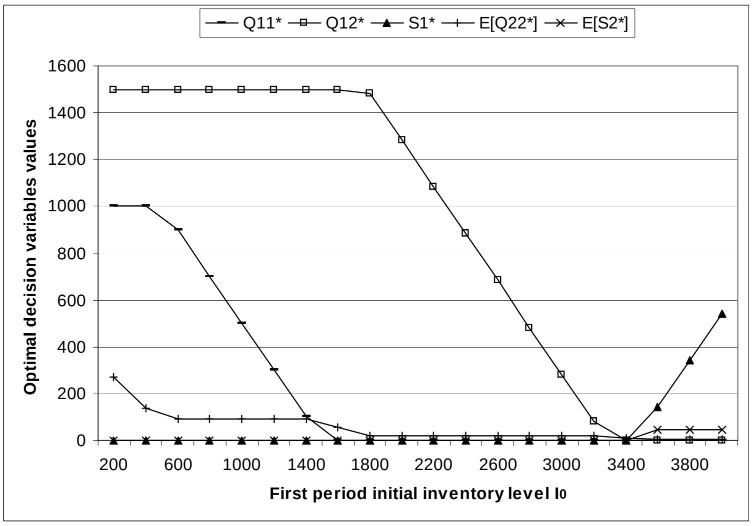

Figure 5.2: Nominal numerical example

5.4.1 Nominal example

We first provide the nominal numerical example that will be exploited in the whole numerical analysis. The numerical data for this example are the following: $D_1 \sim N[1500; 300]$, $D_2 \sim N[1500; 300]$, $h_1 = h_2 = 10$, $p_1 = p_2 = 200$, $b_1 = b_2 = 50$, $c_{11} = c_{22} = c_{33} = 100$, $c_{12} = 60$, $s_1 = s_2 = s_3 = 40$, $Q_{01} = Q_{02} = 0$, $K_{11} = K_{22} = 1000$, $K_{12} = 1500$. We first show the structure of the optimal policy of the first period in terms of I_0. In fact in this example, we also show the shape of the expected optimal policy of the second period. It is easily seen that the first period optimal policy has a shape similar to that of the second period, corresponding basically to two threshold feedback policies.

One can also see that in the regions where Q_{11}^* or S_1^* is positive, Q_{12}^* is constant in terms of I_0. In the region where both Q_{11}^* and S_1^* are zero, the optimal Q_{12}^* is decreasing in terms of I_0. Obviously, $E[Q_{22}^*]$ is also a decreasing function of I_0.

For very small I_0 values, the optimal values of both Q_{11} and Q_{12} are equal to their production capacities, respectively K_{11} and K_{12}. In this region, as the capacity constraints are active, the expected optimal value of Q_{22} is relatively high, aiming to compensate for the unproduced parts due to these constraints. In this region, Q_{11}^* decreases linearly in terms of I_0, which corresponds to an order-up-to-level policy. Once Q_{11}^* becomes equal to zero, Q_{12}^* begins to decrease to compensate the increase of I_0. For high I_0 values, the optimal value S_1^* is a linear increasing function in terms of I_0, which can be interpreted as a salvage-up-to-level policy.

5.4.2 Variability effect

We consider here the impact on the optimal policies of an increase in demand variabilities. We keep the same numerical data as in the nominal example except for the standard deviation of the periodic demands. In Figure 5.3, we have considered $\sigma_1 = 450$, for the high D_1 variability example and we have plotted the absolute difference between the optimal values of the nominal example and the high D_1

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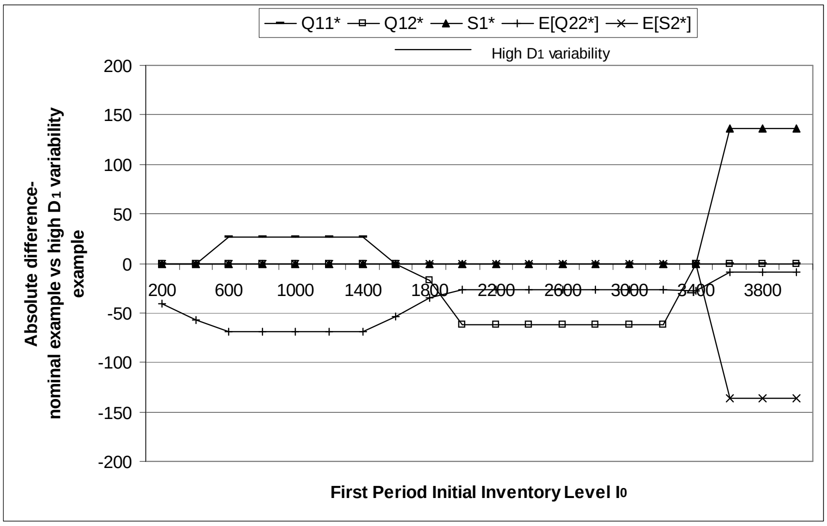

Figure 5.3: Variability effect: high D_1 variability

variability example. In Figure 5.4, we have considered $\sigma_2 = 450$ for the high D_2 variability example and we have plotted also the absolute difference between the optimal values of the nominal example and the high D_2 variability example. As appearing in these figures, for a given I_0 value in the regions where the

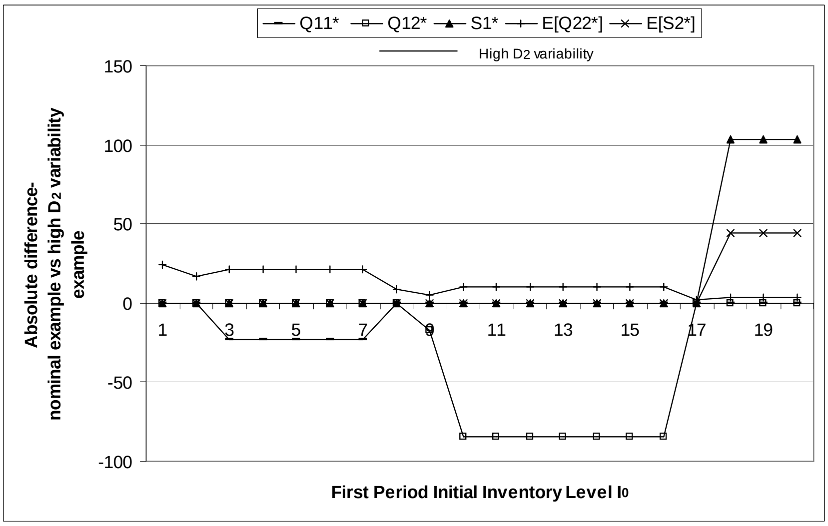

Figure 5.4: Variability effect: high D_2 variability

capacity constraints are not active, Q_{11}^* and S_1^* decrease when the demand D_1 variability increases, while the value of Q_{12}^* increases. Indeed, one could note that the size of the interval between the two threshold levels for Q_{11}^* and S_1^* increases with the demand D_1 variability. For this example, when σ_1 increases, it becomes more profitable to backlog a part of the first demand rather than to order a very high Q_{11}^* quantity, which would induce high holding costs. To satisfy the possible backlogged orders, Q_{12}^* increases as it is more profitable than Q_{22}^*. In our example, when $S_1^* > 0$, one has $Q_{12}^* = 0$ (and $Q_{11}^* = 0$). When the demand D_1 variability increases, it is not profitable to salvage units at the price s_1. On the contrary, it is better to keep the available units in inventory, in order to use them in the first and/or eventually in

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the second period. Therefore, S_1^* decreases when σ_1 increases.

On the other hand, as the optimal Q_{22} value depends on X_2, the second period initial inventory level, and on the demand D_1, it can be logically observed that $E[Q_{22}^*]$ becomes greater with an increase of the variability σ_1, while Q_{11}^* decreases simultaneously. For high I_0 values, and due to the decrease of S_1^*, the expected optimal salvaged quantity $E[S_2^*]$ increases when σ_1 increases.

Now, we compare the nominal example depicted in Figure 5.2 with the high D_2 variability case depicted in Figure 5.4. When σ_2 increases, it appears that Q_{11}^* increases and S_1^* decreases. It is worth noting that Q_{12}^* increases and not Q_{22}^*, because it is more profitable to deliver the demand D_2 via Q_{12}^* than via Q_{22}^*. In the same time, and for a given I_0 value, the expected optimal value ($E[S_2^*]$) decreases when σ_2 increases, because it is more profitable to keep the units, available at the beginning of the second period, than to salvage them at a unit price of s_2, in order to use them in the second period and to face the increase of the demand D_2 variability.

5.4.3 Cost effect on the optimal policy

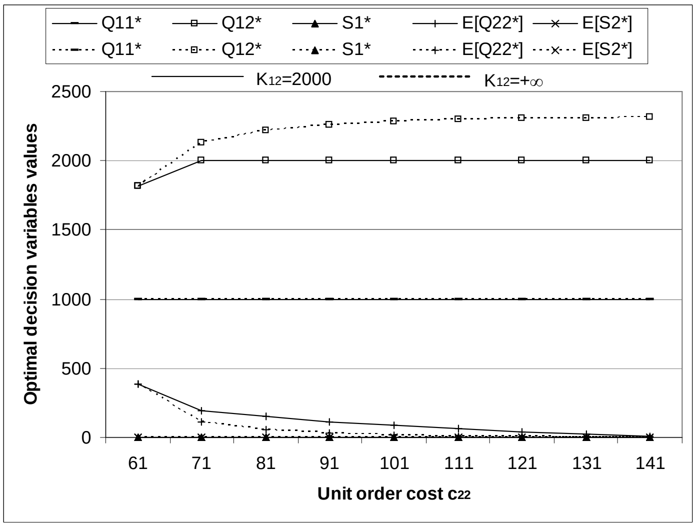

Figure 5.5: The unit order cost c_{22} and the production capacity K_{12} effect

In this example, we show the effect of the unit order cost c_{22}. We assume a zero initial inventory level I_0. We illustrate this cost effect via two models: a first model with the capacity constraint parameter $K_{12} = 2000$ and the second with $K_{12} = +\infty$. From Figure 5.5, one can conclude that it is more profitable to deliver the demand D_2 and the first period backlogged orders via Q_{12}^* than via Q_{22}^*. We thus observe $Q_{12}^* > E[Q_{22}^*]$. When the unit order cost c_{22} increases, the optimal expected Q_{22}^* decreases and is compensated by an increase of Q_{12}^* (in this case, it can be noted that Q_{11}^* is limited by the capacity constraint associated to K_{11}).

From Figure 5.5 one can see that for a given value of c_{22}, when the capacity constraint K_{12} is not active, the optimal values of Q_{12}^* and Q_{22}^* are the same in the constrained as in the unconstrained example. With the increase of c_{22} and when the capacity constraint K_{12} becomes active, the optimal Q_{12}^* is limited, and as a consequence, the expected optimal $E[Q_{22}^*]$ increases. As the unit order cost c_{22} is higher than

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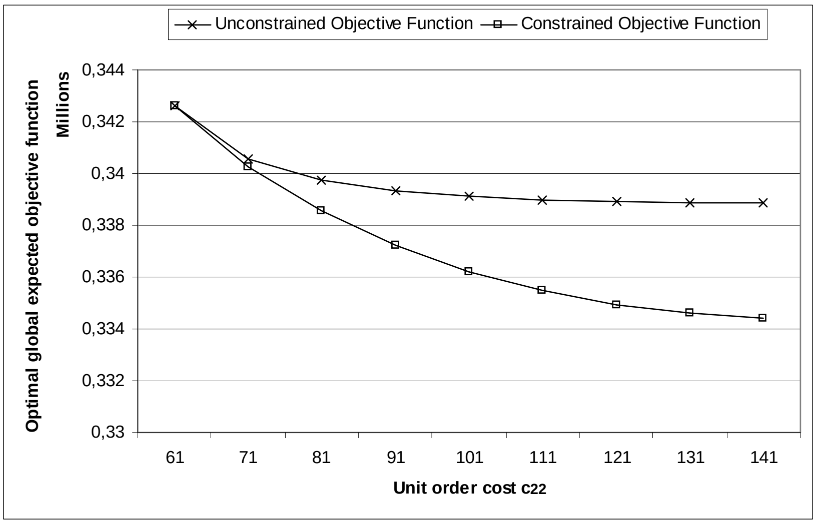

Figure 5.6: Expected Optimal objective function: unconstrained and constrained cases

c_{12}, the difference between the global expected optimal objective functions of the unconstrained model ($K_{12} = +\infty$) and the constrained one ($K_{12} = 2000$) increases when c_{22} increases, even if these two objective functions both decrease, as one can see in Figure 5.6.

5.4.4 Impact of capacity constraint

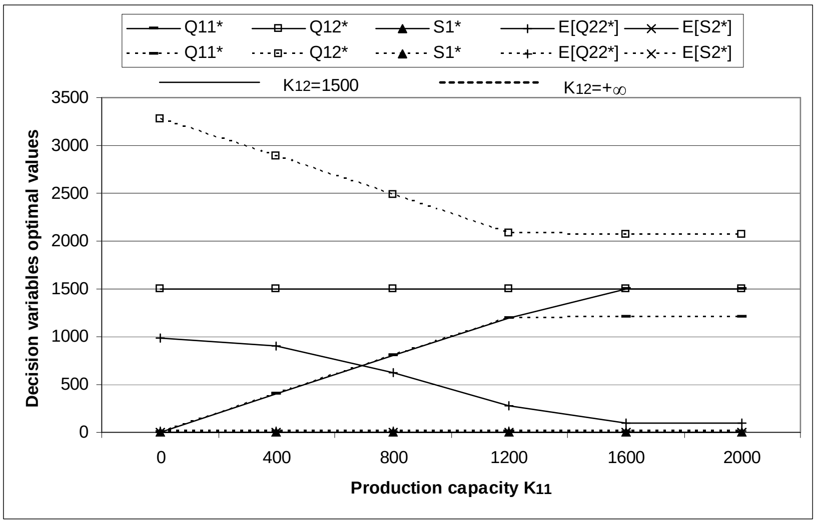

Figure 5.7: Effect of the production capacity on the optimal policy

In order to illustrate the capacity effect, we present the optimal policies and the associated optimal profit as a function of K_{11}, for two different values of K_{12}, namely $K_{12} = 1500$ and $K_{12} = +\infty$ (Figure 5.7).

For low K_{11} values, the related capacity constraint is active, and the optimal Q_{11}^* value is equal to the production capacity K_{11}. For these values of K_{11}, the optimal Q_{12}^* has to meet the first period backlogged orders. When K_{11} increases, the optimal Q_{11}^* increases and converges toward K_{11}, while simultaneously

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Q_{12}^* decreases. Once K_{11} becomes sufficiently large, the problem becomes unconstrained (with respect to K_{11}) and both Q_{11}^* and Q_{12}^* become constant. For this example, one has an infinite production capacity K_{12}, and Q_{12} is more profitable than Q_{22} (due to the cost structure), then the expected optimal Q_{22} value is very low.

In Figure 5.7, and regarding the curves related to $K_{12} = 1500$ one could see that for low K_{11} values, the capacity constraints that correspond to Q_{12} and Q_{12} are both active. In this case, the optimal values Q_{11}^* and Q_{12}^* are equal to K_{11} and K_{12} respectively. The expected optimal value of Q_{22} is relatively high (compared to the example where $K_{12} = +\infty$) to compensate for these active constraints. Note also that the value of K_{11} for which the related capacity constraint becomes inactive is higher in the case of $K_{12} = 1500$ than the case of $K_{12} = +\infty$. This is due to the fact that Q_{12}^* is limited by a capacity constraint and a part of the demand must be satisfied by Q_{11}^*. For low K_{11} values, one can have two

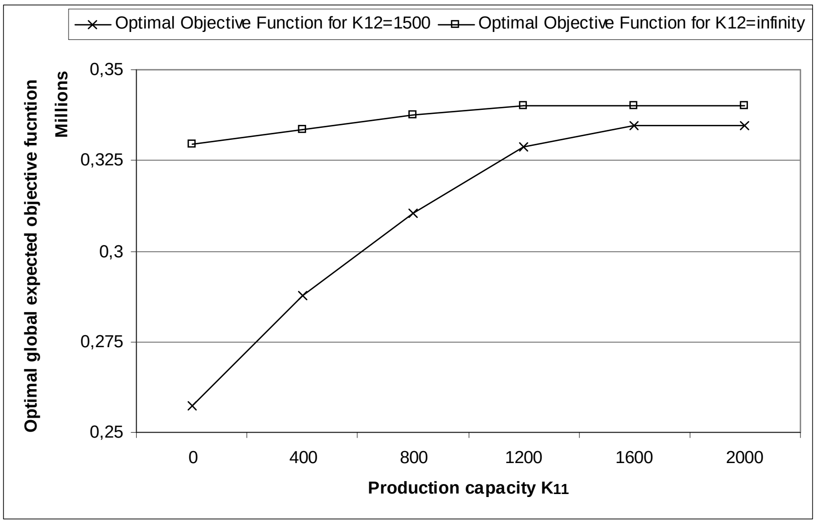

Figure 5.8: Global optimal expected objective function comparison

situations: in the first, if $K_{12} = +\infty$, the optimal Q_{12}^* is high to compensate for the unproduced part of Q_{11}^*, and $E[Q_{22}^*]$ is low. In the second situation, if $K_{12} < +\infty$, Q_{12}^* is low (and equal to K_{12}) and therefore $E[Q_{22}^*]$ is high. In our example, c_{12} is lower than c_{22}, which explains the difference between the two expected objective functions. As K_{11} increases, the need to replace Q_{11} by Q_{22} decreases, as does the difference.

5.5 Conclusion

We have proposed in this chapter a new two-period production/inventory model in which many salvage opportunities are possible and many production modes are used. Production capacities are taken into account in the calculation of the optimal decision variables. The problem has been solved using a dynamic programming approach. First, we have proved the concavity of the expected objective function of the second period, and using this property we have provided the special-form two-threshold optimal policy of that period. This result has served to show the concavity of the global expected objective function,

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and then to prove the existence of a single optimal solution. Some analytical properties that help in characterizing the optimal policy of the first period have been provided. Via numerical examples, we have shown that the first period optimal policy has a two-threshold shape similar to the second period. Then, we have provided some insights, related to the effect of the different parameters of the model, namely the demand variability, costs and capacity constraints. The introduction in the model of an information update process between successive decisions constitutes a future research.

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Chapter 6

Impact of the Information Updating on the Optimal Policy of a Two-Period Stochastic Production Planning Model

In this chapter we develop, based on the model studied in Chapter 4, a two-period production/inventory management model. In addition to the model parameters defined in Chapter 4, we introduce a new parameter that represents an external information permitting the update of the second period demand distribution. This new market information is stochastic at the beginning of the first period and becomes deterministic at the beginning of the second period. It is defined via a joint distribution with the second period demand. We develop the optimal policy of the second period subproblem, then, using dynamic programming, we show that the structure of the optimal policy of the first period is the same as that of the model in Chapter 4. Then a numerical study shows the impact of the information quality, modelled by a correlation coefficient between the information and the second period demand, on the optimal policy and on the optimal expected objective function.

Keywords: production planning, inventory control, production modes, market information, forecasts updating, signal quality.

6.1 Introduction

The two-period model constitutes a base-model that permitting to have insights about the multi-periodic inventory models. As we have explained in Chapter 4 and Chapter 5, the two-period production/inventory models apply well in the case of short-life type products. For this type of products, the demand of a single (or multi) product occurs during a defined selling season that is constituted in general of two periods. The two-period models are, in general, extensions to the very well known *Newsvendor* model, studied in Chapter 3, which deals with this type of products. The literature of the extensions of the *Newsvendor* model is huge and is composed of different families. In this chapter we are interested in two types of extensions: the first one is the two-period models, which is the case shown in Chapter 4, and the second extension is about the state of the information on the demand.

For a huge variety of products, the future demand cannot always be defined in a deterministic manner. In order to optimally satisfy this future demand, the manufacturer (or the retailer) should deal with that demand using forecasts, by defining it as a random variable with given parameters. For some products, an opportunity to improve these forecasts during the decision process is possible. In this case, if the model deals with *style-goods* type products, then the single-point decision process could be transformed into two-point or multiple-point decision process and the single period demand horizon can be changed into two or multiple periods. Between these decision stages, some information is collected and the forecast of the demand for the remaining part of the planning horizon is updated. Then new decisions are made, taking the advantage of the new available information.

In this context, this chapter constitutes an extension to Chapter 4, where we use the two-period framework developed in Chapter 4. A single product demand is defined by two independent random variables over a two-period selling season. An external information is collected during the first period and is used to update the demand of the second period (Cheaitou et al., a, 2007). This information and the second period demand are jointly distributed at the beginning of the first period. Then the information becomes deterministic at the beginning of the second period. At the beginning of the first period, one can order two quantities using two different supply modes: a fast mode with zero delivery lead time and a slow mode with one period delivery lead time. Since the model considers an initial inventory, there is the opportunity, at the beginning of the first period, to return a part of the available inventory to the supplier or to sell it in a parallel market. At the end of the first period, any unsatisfied demand is backlogged to be satisfied in the next period. At the beginning of the second period, and after the update of the demand forecast, one can order an additional quantity using a fast production mode, and/or return another quantity to the supplier or sell it in a parallel market. At the end of the planning horizon, any remaining units are salvaged at a fixed salvage value, and any unsatisfied demand is satisfied by using an emergency production mode.

The first advantage of this model is the fact that the decision process is divided into two points, with two production modes. Indeed, it permits to the decision maker to observe the realized demand in the first demand period, and then uses this information to adjust his decisions in order to better satisfy the demand, which means more reactivity or more flexibility. Many papers have been published, and that

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provide two-period extensions to the *Newsvendor* model as we have shown in Chapter 4 and Chapter 5. [See (Hillier and Liberman, 1990), (Lau and Lau, 1997, 1998) and (Choi et al., 2003)]. In these models, no demand forecast updating is permitted, and a single production mode by decision period is allowed.

The second advantage can be summarized in the forecast updating process. Since the understocking and the overstocking decisions are costing, and since it is impossible to have a perfect information or distribution for the product demand in advance, one should use the forecasts of the future demand in order to minimize the total costs. From this point of view, any information, that can be gathered during the decision process and can improve the forecasts quality, contributes in the reduction of the total costs. This information may be of two types: internal which represents the realized demand in the previous periods, or external whic represents the sales of a preseasonal product for example. To be useful, this information must be collected during the decision process, and before that the last decision is taken. In the literature many models have been proposed using the information for production/inventory problems. (Donohue, 2000) develops a contract model aiming at to determine the efficient decisions, in a two decision points framework, with a single demand period and exogenous information updating process, in terms of wholesale price and return policy, which ensures coordination between the manufacturer and the distributor. (Gurnani and Tang, 1999) developed a model similar to that of (Donohue, 2000) but with a more general situation and that is different in two points. First, they consider the case in which the cost at the second decision period is uncertain and could be higher or lower than the cost at the first decision period. In contrast, (Donohue, 2000) considers the case where the cost at the second decision period is known and is higher than that at the first instant. Second, they consider the case where the value of the information observed between the first and second instants varies from worthless to perfect.

The third advantage of the model presented in this chapter is the multi-supply modes framework. This feature permits the use of two different production modes for the second period: a fast production mode with immediate delivery and a slow and less expensive production mode with one period delivery delay. Between the instants where the decisions relative to these production modes are fixed, two information are collected: the realized demand of the first period and the realized external information. Therefore, the decisions relative to each of the two production modes are fixed with different levels of information. Indeed, the decision relative to the slow production mode is fixed before collecting the information about the demand of the first period and the external information while the decision relative to the fast mode is fixed after the collection of these information. This feature is coupled with the other feature that permits to exploit these information, and which is relative to the quantity that can be returned to the supplier at the beginning of the second period. In the literature, many papers deal with the two-mode supply problem exclusively from the buyer's perspective. Here the key question is about the quantities to be produced with each mode. A certain number of the early published works in this area assumes a multi-period periodic review setting. For example, (Daniel, 1963) examines this question when emergency shipments are bounded above and lead times are 1 and 0 periods, respectively, for the two modes. (Fukuda, 1964) extends this analysis to handle the unbounded emergency shipments and lead times of k and $k+l$ periods. (Whittemore and Saunders, 1977) consider a more general model with arbitrary lead times and identify the cases where it is optimal to use only one supply mode. (Moinzadeh and Nahmias, 1988) are the first

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who examined the basic dual supply problem in a continuous review setting. More recent work includes (Zhang, 1995), who studies a period review, infinite horizon system with up to three supply modes, and (Lawson, 1995) who considers an interesting form of lead-time flexibility which is formally modeled as a series of expedite and de-expedite opportunities. All of these models assume demand is stationary and independent between periods and there is no model that allow a return to the supplier after the forecast update.

The remaining of this chapter is structured as follows: in the following section, we present the model and its parameters. In the third section, we define the optimization method used to solve the optimization problem of our model, and then the two periods optimization problems. In the fourth section, we provide a numerical study showing the properties of the main model parameters and their impact on the optimal policy and in the last section we give conclusions.

6.2 The Model

6.2.1 Model description

In this chapter, the presented model is based on the model defined in Chapter 4. The main difference can be resumed by the introduction of a forecast updating process which uses an external information, defined by a random variable i, correlated with the second period demand.

The first period demand D_1, is defined by a probability density function (PDF) $f_1(D_1) : [0, \infty[\rightarrow \mathbb{R}^+$ and by a cumulative distribution function $F_1(D_1) : [0, \infty[\rightarrow [0, 1]$. The second period demand D_2 and the external stochastic information i, are defined by a joint probability distribution function $j(i, D_2) : [0, \infty[\times[0, \infty[\rightarrow \mathbb{R}^+$ and a joint cumulative distribution function $J(i, D_2) : [0, \infty[\times[0, \infty[\rightarrow [0, 1]$. We define also the marginal probability density function of the information i, $g(i)$ and the marginal cumulative distribution function of the information $G(i)$. For any given value i of the information, we define the conditional probability density function of the demand D_2, $f_2(D_2|i)$ and the conditional cumulative distribution function of D_2, $F_2(D_2|i)$.

Like in the original model, at each period t, any received demand is charged at a price p_t, even if it is not immediately delivered.

Let Q_{ts} be the quantity ordered at the beginning of period t to be received at the beginning of period s (with $t \leq s$ and $t, s = 1, 2, 3$). Then we introduce, for each period, the variable S_t, the quantity which is salvaged (to the parallel market) at the beginning of period t (with $t = 1, 2, 3$). These two decision variables, Q_{ts} and S_t are assumed to be non-negative.

The state variables of the model represent the inventory level at the beginning of each period, X_1 and X_2 and the inventory level at the end of each period, I_1 and I_2 (I_0 being the given initial inventory for the problem).

The unit order cost of Q_{ts} is c_{ts}. In the case of a positive inventory at the end of a period, an inventory holding cost h_t is paid. Unsatisfied orders in period t are backlogged to the next period, with a penalty shortage cost b_t. The unit salvage value at the beginning of period t is given by s_t. We note that the

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decisions that are fixed at period 3, concern the immediate moment after the end of the second period. This means that the third period is not a real period but is used only as a terminal condition.

The structure of the decision process and demand realization, is shown in Figure 6.1.

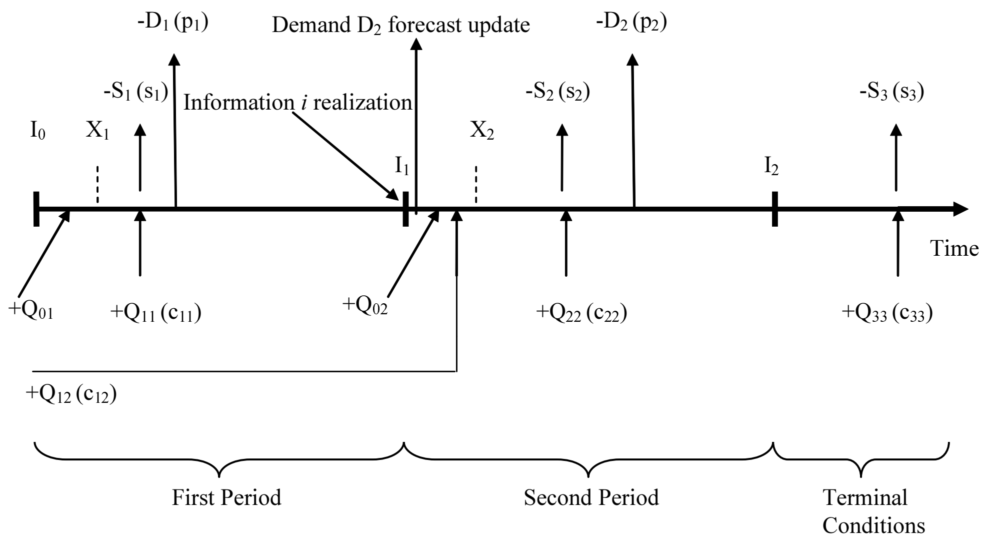

Figure 6.1: Decision process

Note that we consider the same assumptions on the model parameters (costs, prices and salvage values) that have been defined in section 4.2.2 of Chapter 4.

Note also that the optimal values of the decision variables Q_{33} and S_3 are given by (see Lemma 4.1 of Chapter 4)

$$\text{if } \ I_2 \leq 0 \quad \Rightarrow \quad Q_{33}^* = -I_2 \ \text{ and } \ S_3^* = 0, \tag{6.1}$$

$$\text{if } \ I_2 \geq 0 \quad \Rightarrow \quad Q_{33}^* = 0 \ \text{ and } \ S_3^* = I_2. \tag{6.2}$$

In order to simplify modeling and resolution of the new model with information update, we will not take into account the decision variables Q_{01} and Q_{02} defined in Chapter 4. Note that this omission does not affect the obtained results.

6.2.2 The optimization problem

In this paragraph, we introduce $\Pi_1(X_1, Q_{11}, Q_{12}, S_1)$ as the expected profit of the first period with respect to the random demand D_1 and $\Pi_2(X_2, Q_{22}, S_2|i)$ as the expected profit of the second period with respect to the random demand D_2 conditionally to the information i. These expected profits are then given by

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the following equations

$$\begin{aligned}\Pi_1(X_1, Q_{11}, Q_{12}, S_1) &= p_1 E[D_1] + s_1 S_1 - c_{11} Q_{11} - c_{12} Q_{12} \\ &- h_1 \int_0^{X_1+Q_{11}-S_1} (X_1 + Q_{11} - S_1 - D_1) f_1(D_1)\, dD_1 \\ &- b_1 \int_{X_1+Q_{11}-S_1}^{\infty} (D_1 - X_1 - Q_{11} + S_1) f_1(D_1)\, dD_1,\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\Pi_2(X_2, Q_{22}, S_2|i) &= p_2 E[D_2|i] + s_2 S_2 - c_{22} Q_{22} \\ &- (h_2 - s_3) \int_0^{X_2+Q_{22}-S_2} (X_2 + Q_{22} - S_2 - D_2) f_2(D_2|i)\, dD_2 \\ &- (b_2 + c_{33}) \int_{X_2+Q_{22}-S_2}^{\infty} (D_2 - X_2 - Q_{22} + S_2) f_2(D_2|i)\, dD_2,\end{aligned} \tag{6.4}$$

where $E[D_1]$ represents the expectation of the first period demand and $E[D_2|i]$ the expectation of the second period demand conditionally to the information i. Note that the initial inventory level of the second period, X_2, is a function of the first period parameters, decision variables and demand, where we have $X_2 = X_1 + Q_{11} + Q_{12} - S_1 + Q_{02} - D_1$.

Define $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}, S_1, S_2)$ as the global expected profit with respect to the random variables D_1, D_2 and i. This global expected profit is then

$$\begin{aligned}&\Pi(X_1, Q_{11}, Q_{12}, Q_{22}, S_1, S_2) = \\ &\quad \Pi_1(X_1, Q_{11}, Q_{12}, S_1) + E_i \left\{ E_{D_1} \left\{ \Pi_2(X_2(X_1, Q_{11}, Q_{12}, S_1, D_1), Q_{22}, S_2|i) \right\} \right\},\end{aligned} \tag{6.5}$$

where $E_i \{\cdot\}$ and $E_{D_1} \{\cdot\}$ represent the expectation with respect to i and D_1 respectively. The global optimization problem is then to maximize $\Pi(\cdot)$ with respect to the decision variables of the first and second periods. It is given by

$$\max_{Q_{11}, Q_{12}, Q_{22}, S_1, S_2} \Pi(X_1, Q_{11}, Q_{12}, Q_{22}, S_1, S_2). \tag{6.6}$$

6.3 The dynamic programming approach

As we have done for the model defined without information updates in Chapter 4, we will use the dynamic programming in order to characterize the optimal policies of the two periods of our planning horizon. Using this optimization technique, we transform the optimization problem defined in (6.6) into two optimization subproblems corresponding to the two periods of our planning horizon. The first subproblem is associated with the second period, and is defined as follows

$$\max_{Q_{22} \geq 0, S_2 \geq 0} \left\{ \Pi_2(X_2, Q_{22}, S_2|i) \right\}. \tag{6.7}$$

Solving this optimization subproblem provides us with expressions of the optimal second period decision variables, namely $(Q^*_{22}(X_2,i), S^*_2(X_2,i))$, and the optimal expected objective function $\Pi^*_2(X_2, Q^*_{22}(X_2,i), S^*_2(X_2,i))$, as functions of X_2 and i. Then using this optimal policy, we can define the second optimization subproblem associated with the first period

$$\max_{Q_{11}\geq 0, Q_{12}\geq 0, S_1\geq 0} \{\Pi(X_1, Q_{11}, Q_{12}, S_1)\} = \max_{Q_{11}\geq 0, Q_{12}\geq 0, S_1\geq 0} \{\Pi_1(X_1, Q_{11}, Q_{12}, S_1) + E_i\{E_{D_1}\{\Pi^*_2(X_2, Q^*_{22}(X_2,i), S^*_2(X_2,i))\}\}\}. \tag{6.8}$$

6.3.1 Second-period subproblem

We begin by solving the second period subproblem. The optimization problem of the second period, defined by equations (6.4) and (6.7) has been studied in Chapter 4. The only difference is the probability density and distribution function, which is a simple distribution in Chapter 4 and a conditional distribution in this chapter. This difference does not change the structure of the optimal policy provided in Chapter 4. The optimal policy of the second period is then defined by the following

$$Q^*_{22}(X_2,i) = (Y_{12}(i) - X_2)^+ \quad \text{and} \quad S^*_2(X_2,i) = (X_2 - Y_{22}(i))^+ \tag{6.9}$$

with

$$Y_{12}(i) = F_2^{-1}\left(\frac{b_2 + c_{33} - c_{22}}{b_2 + c_{33} + h_2 - s_3}|i\right) \quad \text{and} \quad Y_{22}(i) = F_2^{-1}\left(\frac{b_2 + c_{33} - s_2}{b_2 + c_{33} + h_2 - s_3}|i\right). \tag{6.10}$$

Similarly to the interpretation provided in Chapter 4 for the second period characteristic threshold levels, we interpret in this section the economic meaning of the threshold levels given in (6.10). These threshold levels are interpreted as a solution of a modified *Newsvendor* problem with and underage cost of C_u and an overage cost of C_o.

For the first threshold level $Y_{12}(i)$, define the underage cost $C^1_u = b_2 + c_{33} - c_{22}$ as the marginal cost of not satisfying a demand in the second period with Q_{22}, and the overage cost $C^1_o = c_{22} - s_3 + h_2$ as the marginal cost of ordering a supplementary unit with Q_{22} over the optimal value. Therefore, in the expression of $Y_{12}(i)$, the argument of the function is equal to the ratio of C^1_u and $C^1_u + C^1_o$.

For the second threshold level $Y_{22}(i)$, define the underage cost $C^2_u = b_2 + c_{33} - s_2$ as the marginal cost of salvaging a supplementary unit at the beginning of the second period (which implies not satisfying a marginal demand in the second period), and the overage cost $C^2_o = s_2 - s_3 + h_2$ as the marginal cost of not salvaging a supplementary unit at the beginning of the second period (which means keeping that unit to be used in the second period). Therefore, in the expression of $Y_{22}(i)$, the argument of the function $F_2^{-1}(\cdot)$ is equal to the ratio of C^2_u and $C^2_u + C^2_o$. That permits to interpret $Y_{22}(i)$ as a modified *Newsvendor* salvage-up-to level.

It can be easily seen that for a given value of the information i, and under the cost assumptions of

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this chapter, one has

$$Y_{12}(i) < Y_{22}(i). \tag{6.11}$$

6.3.2 First period subproblem

In this section we solve the optimization problem of the first period defined in (6.4) and (6.8).

It should be noticed that even if we have added a new and important element to the model presented in Chapter 4, the structure of the optimal policy does not change. Note that the first order partial derivatives of the expected objective function (6.8), given in Appendix C.1, are similar to the partial derivatives of the first period expected objective function of the model shown in Chapter 4 given in Appendix A.2. The only difference between the two models partial derivatives is the fact that in the partial derivatives of the model shown in this chapter, some terms are defined as an expected value with respect to the external information i.

Consequently, we will not enumerate the different possible cases of the optimal policy, which are similar to those of Chapter 4, and we will only show a single case of that optimal policy. For the other possible cases, one can refer to Chapter 4. Instead, we will provide a numerical study to show the impact of the external information on the optimal policy, the importance of the correlation (between the information and the second period demand) and its impact on the optimal policy.

Before providing the numerical study, we prove the concavity of the expected first period objective function, then we provide two Lemmas that partially characterize the first period optimal policy.

Lemma 6.1 *The total expected objective function $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ defined in (6.4) is jointly concave with respect to the decision variables Q_{11}, Q_{12} and S_1.*

Proof. The proof of this Lemma is similar to that of the Lemma 4.3 presented in Chapter 4, except that the function $G(X_1, Q_{11}, Q_{12}, S_1)$ defined in the proof of the Lemma 4.3 should be replaced by the function $W(X_1, Q_{11}, Q_{12}, S_1)$. This function can be defined as follows

$$\begin{aligned} &W(X_1, Q_{11}, Q_{12}, S_1) = \\ &\int_0^\infty \left[\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}(i)}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}(i)} f_1(x) f_2(Q_{02}+Q_{11}+Q_{12}-S_1+X_1-x|i)dx \right] g(i)\, di, \end{aligned} \tag{6.12}$$

which completes the proof. □

Using Lemma 6.1, we can conclude that the optimization problem described in (6.4) and (6.8) has a unique maximum. Thus, one could use the first order optimality criterion to develop the optimal policy.

Lemma 6.2 *There are no optimal solutions with $Q^*_{11}(X_1) > 0$ and $S^*_1(X_1) > 0$ simultaneously. In other words, one has the property*

$$Q^*_{11}(X_1)\, S^*_1(X_1) = 0.$$

Proof. See the Proof of Lemma 4.4 in Chapter 4.

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Lemma 6.3 *When the optimal value of the decision variable Q_{12} is positive, namely when $Q_{12}^*(X_1) > 0$, then the optimal values of the decision variables Q_{11} and S_1 are completely characterized by two thresholds given by*

$$Y_{11} = F_1^{-1}\left(\frac{c_{12} - c_{11} + b_1}{h_1 + b_1}\right) \quad \textit{and} \quad Y_{21} = F_1^{-1}\left(\frac{c_{12} - s_1 + b_1}{h_1 + b_1}\right) \tag{6.13}$$

In other words, one has

$$(Q_{11}^*; S_1^*) = \left((Y_{11} - X_1)^+; (X_1 - Y_{21})^+\right) \tag{6.14}$$

Proof. See the Proof of Lemma 4.8 and Lemma 4.9 in Chapter 4.

6.4 Numerical examples

We remind that some analytical results concerning the optimal policy of the first period, can be obtained from Chapter 4. However, providing a complete closed-form analytical solution is not possible due to the sequential nature of the decision process. Therefore, in this section we give some numerical examples in order to show the behavior of our model and the impact of the different model parameters on the optimal policy and the optimal expected objective function.

Note that for these numerical applications, we assume that the first period demand has a truncated-normal distribution, corresponding to a normal distributed demand, $D_1 \sim N[\mu_1; \sigma_1]$, for which we eliminate the negative part. In fact the numerical mean and standard deviation given in the following examples are those of the initial normal distribution. In the calculations we use the truncated-normal that corresponds to the initial normal distribution.

We suppose that the joint probability density function of the information i and the second period demand D_2, namely $j_2(I, D_2)$, is a bivariate normal distribution, with means θ and μ_2, standard deviations δ and σ_2 and a correlation coefficient ρ. Hence, the bivariate normal distribution allows us to solve numerically the first period subproblem, and then to capture how the information enables the decision maker to obtain more accurate demand forecast. It is then well known that the conditional second period demand $(D_2|i)$ is also normally distributed with mean μ_2' and standard deviation σ_2' (Bickel and Doksum, 1977), where

$$\mu_2' = \mu_2 + \rho\frac{i - \theta}{\delta} \quad \text{and} \quad \sigma_2' = \sigma_2\sqrt{1 - \rho^2} \tag{6.15}$$

Note that the correlation coefficient ρ always satisfies the inequality $\rho \leq 1$ which implies that $\sigma_2' \leq \sigma_2$. The last inequality means that the conditional distribution of the second period demand is more accurate than the demand distribution which does not take into account the realized information.

Let $f_2(\cdot)$ and $F_2(\cdot)$ be, consecutively, the PDF and CDF of this normal distribution (with mean μ_2' and standard deviation σ_2').

We can, using the definitions given above, rewrite the two threshold levels defined in (6.10) as follows

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$$Y_{12}(i) = \mu_2 + \rho(i-\theta)\frac{\sigma_2}{\delta_2} + (\sigma_2\sqrt{1-\rho^2})\Phi^{-1}\left(\frac{b_2+c_{33}-c_{22}}{b_2+c_{33}+h_2-s_3}\right), \quad (6.16)$$

and

$$Y_{22}(i) = \mu_2 + \rho(i-\theta)\frac{\sigma_2}{\delta} + (\sigma_2\sqrt{1-\rho^2})\Phi^{-1}\left(\frac{b_2+c_{33}-s_2}{b_2+c_{33}+h_2-s_3}\right). \quad (6.17)$$

This new definition of the threshold levels, permits to formulate the optimization problem of the first period more practically and to perform some numerical examples that are shown in the following sections.

Note that the particular case where $\rho = 0$ is equivalent to the model presented in Chapter 4, where the second period demand is independent of any information.

6.4.1 Nominal example

In this section we define the basic numerical values on which we build our numerical examples. Firstly, note that we will depict in these examples the optimal values of the decision variables of the first period, namely Q_{11}^*, Q_{12}^* and S_1^*, and the expected optimal values of the second period decision variables, namely $E[Q_{22}^*]$ and $E[S_2^*]$. The expectation of these optimal decision variables is with respect to the random information i and the random second period demand D_2.

The numerical data used in this example are the following: $D_1 \sim N[1000; 300]$, $\mu_2 = 1000$, $\theta = 1000$, $\sigma_2 = 300$, $\delta = 300$, $h_1 = 5$, $h_2 = 5$, $p_1 = 100$, $p_2 = 100$, $b_1 = 25$, $b_2 = 25$, $c_{11} = 50$, $c_{12} = 30$, $c_{22} = 50$, $c_{33} = 50$, $s_1 = 20$, $s_2 = 20$, $s_3 = 20$.

The correlation coefficient ρ takes two values: the first value is equal to 0.1 and corresponds to a low correlation between the information and the second period demand, or a low signal quality, while the second value is equal to 0.9 and corresponds to a high signal quality, or a strong correlation.

In Figure 6.2 we can see the structure of the two-period optimal policy in terms of the initial inventory level I_0. Note that the optimal policy of the first period is characterized, like the optimal policy of the second period, by two threshold levels. These two threshold levels divide the values of the initial inventory level I_0 into three intervals. The first interval corresponds to the low I_0 values, in which Q_{11}^* is decreasing in I_0, Q_{12}^* is positive and constant and S_1^* is equal to zero. The second interval corresponds to the medium I_0 values, in which both Q_{11}^* and S_1^* are equal to zero, and Q_{12}^* is decreasing in I_0. The last interval corresponds to the high I_0 values, in which only S_1^* is positive.

6.4.2 Information quality

In this section we compare the nominal example presented in Figure 6.2, where the correlation coefficient is $\rho = 0.9$ with another example depicted in Figure 6.3, where the information does not have a great importance due to a low correlation coefficient with the demand of the second period ($\rho = 0.1$).

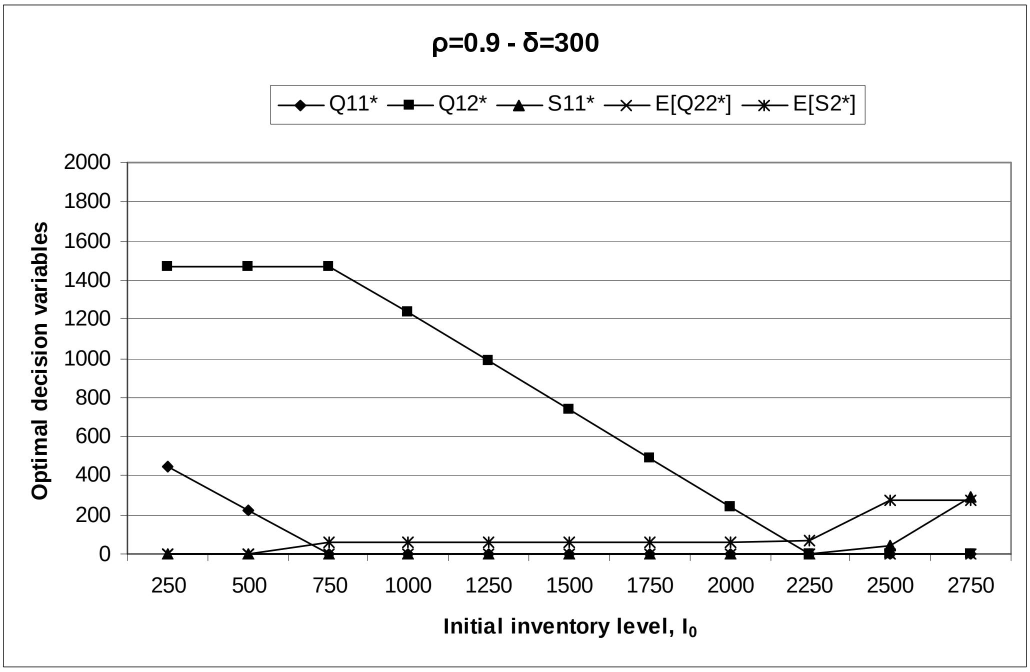

Figure 6.2: Nominal example: structure of the optimal policy, with high information quality ($\rho = 0.9$)

The other numerical parameters of the example depicted in Figure 6.3 are the same as those of the nominal example, given in the previous section.

From the comparison between these numerical example, we can deduce that when the quality of the information increases (ρ increases):

- the optimal quantity ordered with the slow mode, Q_{12}^*, decreases,
- the expected optimal quantity $E[S_2^*]$ increases,
- the optimal quantity Q_{11}^*, the optimal quantity S_1^* and the expected optimal quantity $E[Q_{22}^*]$ do not change.

From these observations, we can conclude that in the interval of the I_0 values where Q_{12}^* is positive, the optimal Q_{11}^* is independent of the information and of the second period demand distribution. The second conclusion is that when the information quality increases, the use of the slow production mode decreases. This means that instead of ordering a quantity Q_{12}^* to be used in the second period, one would prefer to wait and to profit from the collected information during the first period, and then fix the decisions related to the second period. Since in each of the two numerical examples, the second period fast production mode (Q_{22}^*) is not used, then in both of these examples $E[Q_{22}^*]$ does not change and is always equal to zero.

In the interval of medium I_0 values, and due the the decrease of the demand D_2 variability, the expected $E[S_2^*]$ increases. This increase is related to the decrease in the expected threshold level $E_i[Y_{22}(i)]$. Indeed, from (6.17) one can easily see that when ρ increases $Y_{22}(i)$ and $E_i[Y_{22}(i)]$ decrease. In fact, in this region, when the demand D_2 variability decreases, then the need to additional units to satisfy the

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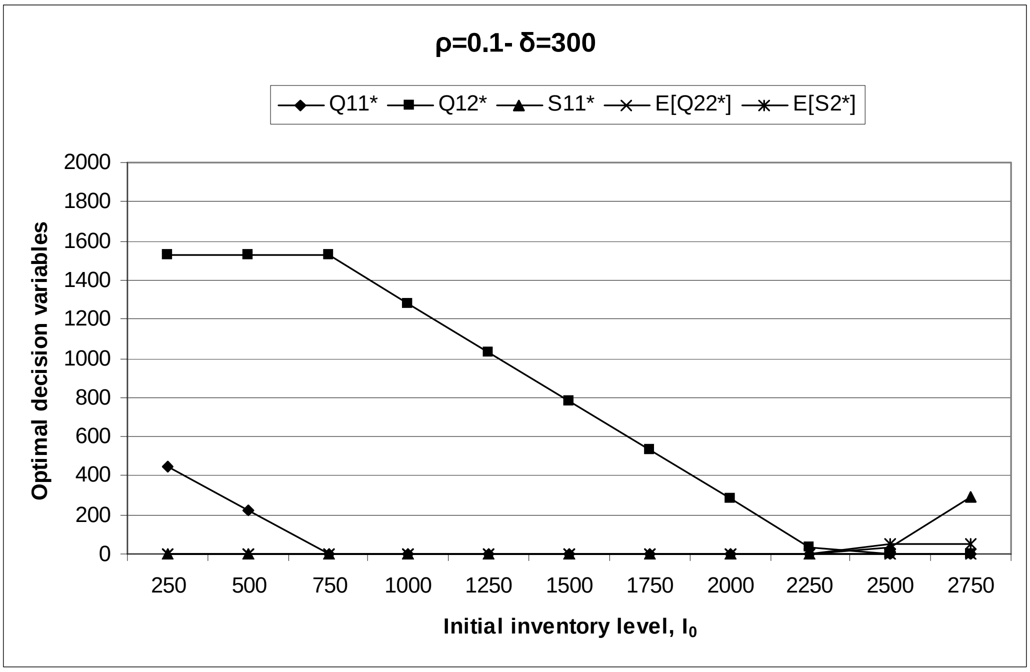

Figure 6.3: Structure of the optimal policy, with low information quality ($\rho = 0.1$)

demands of the second period decreases, and as we are in the interval in which $E[S_2^*]$ is positive, then $E[S_2^*]$ increases. In the high I_0 values interval, where Q_{12}^* is equal to zero, S_1^* and $E[S_2^*]$ increase in order to compensate for the decrease of the demand variability.

6.4.3 Impact of the information variability

In this section we study the effect of the information quality, showing the impact of the information variability δ on the optimal policy and on the expected optimal objective function.

In Figure 6.4, the optimal policy is depicted in terms of the initial inventory level I_0. The numerical data of this example are the same as the nominal data given in the nominal example except the information standard deviation $\delta = 100$ and the correlation coefficient $\rho = 0.9$.

By comparing Figure 6.2 and Figure 6.4, one can see that, in the interval of positive Q_{12}^* values, when the information variability decreases then the optimal Q_{11}^* does not change. This is due to the fact that in this region, Q_{11}^* is independent of the second period demand distribution and of the information distribution. Note also that when the variability of the information i decreases, the optimal Q_{12}^* slightly decreases. This slight decrease is due to the fact that Q_{12}^* is decided before the knowledge of the information i.

As we have mentioned before, it is optimal to satisfy the demand of the second period roughly with Q_{12}^* (due to the structure of our costs) and then the optimal $E[Q_{22}^*]$ is equal to zero in the numerical examples presented in this section. In this case, the fast mode of the second period (Q_{22}) serves as an *emergency* mode which permits to satisfy the backlogged demands from the first period. Therefore, due to the strong correlation between the information and the second period demand ($\rho = 0.9$) and to the fact

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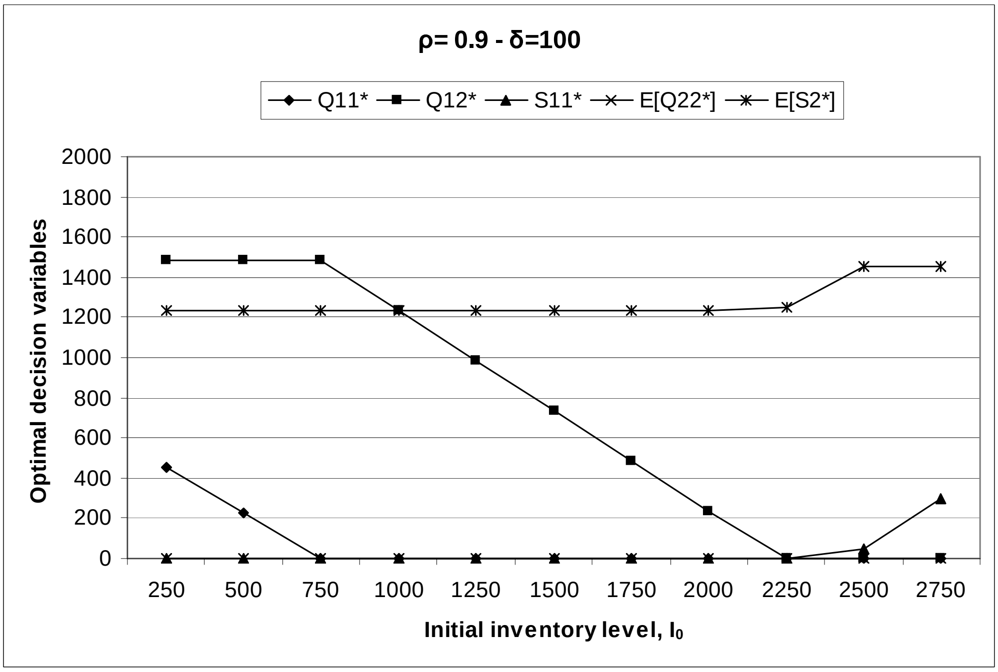

Figure 6.4: Structure of the optimal policy, with low information variability ($\delta = 100$) and high information quality ($\rho = 0.9$)

that Q_{12}^* is ordered before the realization of the information i, when the variability of i decreases then the optimal decision variable Q_{12}^* decreases slightly and therefore $E[S_2^*]$ increases tremendously. This big increase is the consequence of the decrease of $E_i[Y_{22}(i)]$. Indeed, from the (6.17) it can be easily seen that for a high value of ρ, when δ increases then $Y_{22}(i)$ and consequently $E_i[Y_{22}(i)]$ decrease. This decrease is proportional to the value of ρ: the higher the value of ρ, the higher the decrease of $E_i[Y_{22}(i)]$ for the same decrease of δ.

In Figure 6.5 we depict the relative difference between the expected optimal objective function in the case where $\rho = 0.9$ and that in the case where $\rho = 0.1$. We plot this relative difference for two values of the information variability, $\delta = 100$ and $\delta = 500$.

The first remark that can be captured is the fact that the higher the correlation coefficient, the higher the expected optimal objective function. This means that when the signal quality increases, the expected optimal profit increases. This conclusion can be seen from Figure 6.5 where the difference between the expected optimal objective function with $\rho = 0.9$ and that with $\rho = 0.1$ is always positive.

On the other hand, from Figure 6.5 it can be seen that when the variability of the information increases, the impact of the correlation between the information and the second period demand D_2 on the optimal expected profit increases also. Therefore, when the variability of the information increases, the difference between expected optimal profit in the case where the correlation is high ($\rho = 0.9$) and the expected optimal profit in the case where the correlation is low ($\rho = 0.1$) increases. That means, to get a better performance (higher expected profit), it is better to have a high correlation in the case where the information variability is high. In the case where the information variability is low, it does not really matter if the correlation coefficient is high or low.

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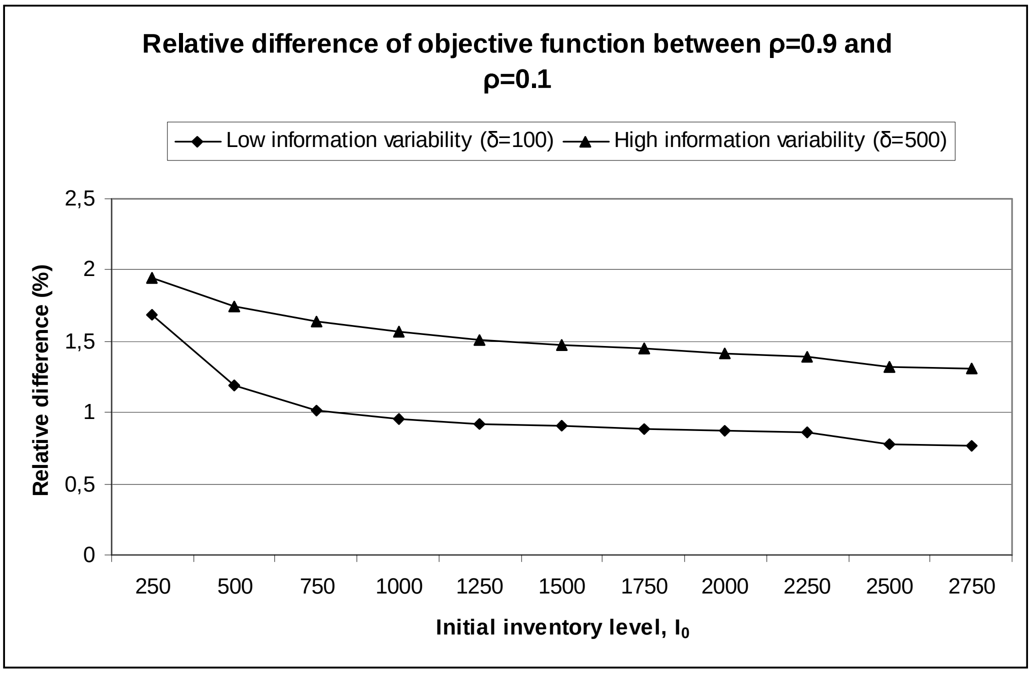

Figure 6.5: Impact of the information variability and quality on the expected optimal objective function

6.4.4 Impact of the difference between ordering costs

In this section we plot the optimal policy of our model in terms of the unit order cost of the slow production mode c_{12}. In both of the numerical examples shown in Figure 6.6 and Figure 6.7, we use the nominal numerical data given above except the unit order cost c_{12} that we vary and the correlation coefficient ρ that is equal to 0.1 in the first figure and to 0.9 in the second one. Note that in the two figures, when the unit order cost of the slow production mode c_{12} increases, the optimal quantity ordered with this production mode, namely Q_{12}^*, decreases. On the other hand, the optimal quantities produced with the fast production mode, Q_{11}^* and Q_{22}^* increases which is an intuitive result. For the low c_{12} values, the optimal expected decision variable $E[Q_{22}^*]$ is equal to zero, which means that for low c_{12} values, the slow production mode is the mode that is roughly used to satisfy the second period demand. The fast production mode of the second period, is used as a backup option to satisfy the backlogged demands from the first period.

By comparing the two figures, one can see that, for the values where Q_{12}^* is positive, when the information quality increases then the optimal values of the decision variables relative to the first period do not change, which can be deduced from (6.13). This result confirm the fact that when the optimal quantity ordered with the slow mode is positive, then the optimal policy of the first period is completely independent of the demand of the second period and of the information.

As we have mentioned in section 6.4.2, when the information quality increases, then the use of the slow production mode decreases and it becomes more profitable to wait until the end of the first period to exploit the collected information, in order to update the second period demand and to fix the ordering decisions relative to the second period. Therefore, when ρ increases, the expected optimal decision

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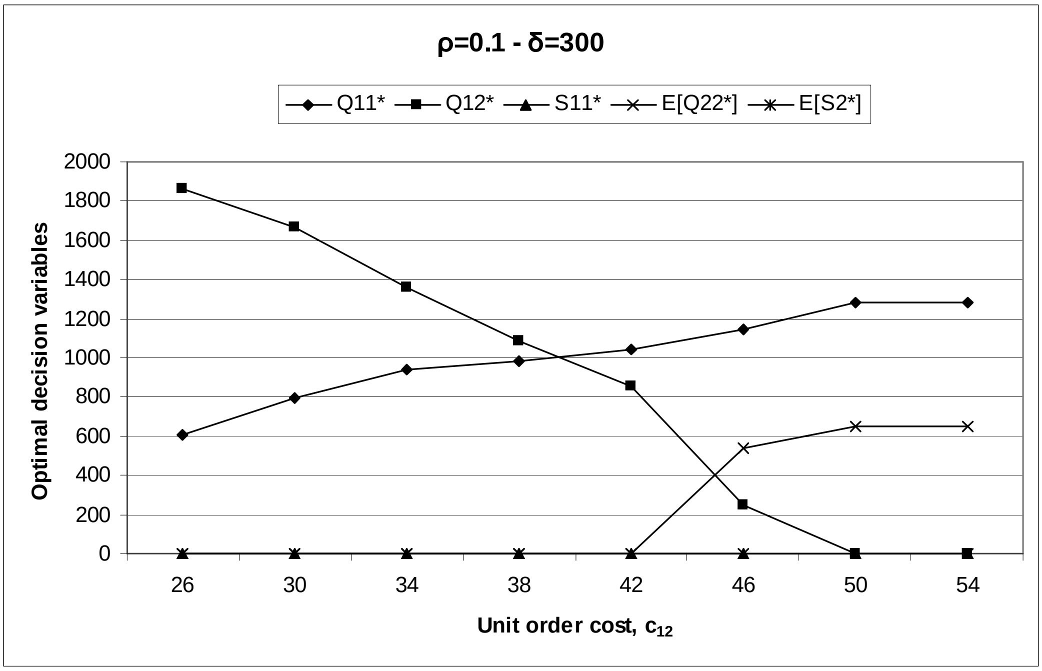

Figure 6.6: Impact of the difference between the ordering costs, with low information quality

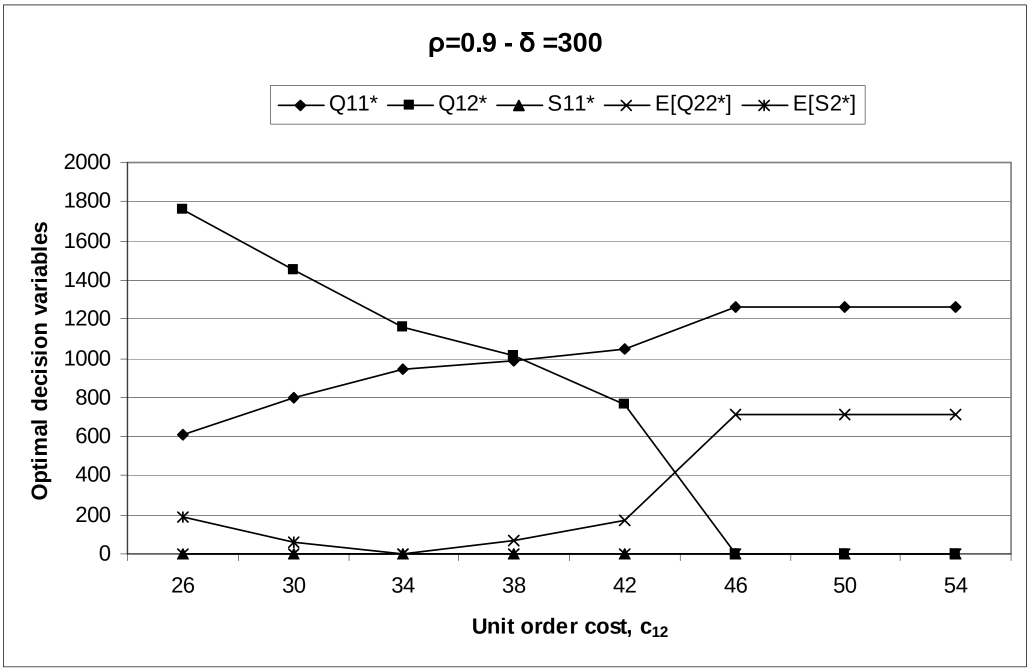

Figure 6.7: Impact of the difference between the ordering costs, with high information quality

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variable $E[Q^*_{22}]$ increases also. The expected optimal decision variables $E[S^*_2]$ increases also due to the decrease of $E_i[Y_{22}(i)]$ as we have shown above.

Note that for a given value of the unit ordering cost c_{12}, the optimal decision variable Q^*_{12} becomes equal to zero, and the optimal policy becomes insensitive to the variation of c_{12}. At this point, c_{12} is approximately equal to c_{22} when the information is worthless ($\rho = 0.1$). In this case it is roughly the economic difference (costs) between the slow and fast production modes that determine which production mode to use. When the information quality is high ($\rho = 0.9$), the value of c_{12} at this same point become lower than c_{22}. These two examples show that when one use an updating of the demand distribution, with two production modes, then it is not only the costs difference between the two production modes that determine which mode to use, but also the value of the information used to update the demand forecast.

6.5 Conclusion

In this chapter we have generalized the model presented in Chapter 4. We have used the same framework of Chapter 4, and we have introduced a new and important element that was an updating process of the second period demand forecast. The information used to update the demand distribution is an external information that can be the sales of a similar product whose demand is correlated to the concerned product. First of all, we have assumed a general correlation between the information and the demand, and then to provide some insights via the numerical applications, we have assumed that the information and the second period demand are defined by a bivariate normal distribution. We have shown that the introduction of this new information updating process does not change the the structure of the optimal policy. Then, using some numerical applications we have shown the structure of the optimal policy, and the impact of the information quality and its variability on that policy. We have also provided an analysis of the impact of the information quality on the use of the different available production modes.

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Chapter 7

Two-Stage Flexible Supply Contract with Payback, Information Update and Stochastic Costs

In this chapter, we consider a two-stage supply contract model for advanced reservation of capacity or advanced procurement supply, with payback option at the beginning of the selling horizon. Between the two decision stages, an external information is collected that permits to update the demand forecast. The updated demand forecast serves then to adjust the decisions of the first stage by exercising options or by returning some units to the supplier. This type of contracts can be applied in the case of products with short life cycle, or, in other words, with the *style-goods* type products. For this type of products, the demand occurs during a single selling period (season). At the end of this period, the remaining inventory, if any, is sold (or returned to the supplier) at a salvage value that is usually less than the initial unit production/procurement cost. During the selling season, any satisfied demand is charged with a unit selling price and any unsatisfied demand is lost and a penalty shortage cost is paid. The demand is characterized by a probability density function with parameters that are known at the beginning of the period. The objective of the model is to determine the quantities to be ordered before the beginning of the selling season which can be interpreted as the amount of capacity to be reserved, in order to satisfy optimally the demand.

Keywords: supply contract, information update, payback, *newsvendor*, dynamic programming, particular cases, capacity reservation.

7.1 Introduction

The *style-goods* type products are characterized by a short product-life. Demand uncertainty management has long been regarded as crucial. Since it is impossible to have perfect knowledge about the demand before the selling season and both understocking and overstocking are undesirable, policies that advocate the use of market information to improve stocking decisions have been proposed (Choi, 2007). Therefore, many firms recognize opportunities to collect information permitting the improvement of the demand forecast. This information may have different sources: advanced custom forecasts, early season demand, advanced bookings or sales of a pre-seasonal product. For example a firm can collect information from few key customers and use them in order to develop a total demand forecasts. In fashion industry (Fisher and Raman, 1996) and in the computer industry (Padmanabhan, 1999), the early season information collected by firms constitutes a strong indicator for the total season demand. Many studies on the use of information for inventory problems have been proposed in the literature (Scarf, 1959), (Murray and Silver, 1966), (Azoury and Miller, 1984), (Azoury, 1985) and (Lovejoy, 1990).

To be useful, the collected information must be available before the last decision stage, or before taking the last decisions. An example may be a firm that should take some capacity reservation decisions, and after that it should take decisions concerning the use of these production capacities. In this case, the information must be collected at least between the first and the second decision stages.

An important factor that the firm must understand in order to use in a better manner the external information, is the quality of that information. Take the example of a firm collecting some information about the demand of a key customer before the selling season. If this information is not representative of the whole set of customers then it can not be used to improve tremendously the quality of the demand forecasts of all the customers. Therefore, in this case the demand quality is low. In other cases, the collected information from the key customer may be very representative of the other customers' demand, and therefore, the collected information constitutes a strong indicator on the demand forecast of the whole set of customers. In this case the quality of the collected information is high. This quality of the collected information could be modeled by the correlation between the demand and the information. In the case where the collected information and the demand are random variables, the quality of the information is then the correlation coefficient between these two random variables.

The benefit of a flexible contract, versus an inflexible one, is directly related to the quality of the collected information. If the collected information is not very correlated to the demand, then the flexible contract offers a little profit gain (Brown and Lee, 1998b).

The literature of the supply contracts is very rich. This literature can be divided into two main categories (see (Lariviere, 1999) and (Tsay and al., 1999)): in the first category, a particular contract is studied in order to examine the optimal action of the firm given that the contract terms are fixed. (Anupindi and Bassok, 1999) refer to this category of research as contract analysis. In the second category, a simple two-party model is studied in order to show whether or not contract terms which improve or coordinate the supply chain can be found. (Anupindi and Bassok, 1999) refer to this category as contract design. Our chapter fits into the first category of work.

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Many factors distinguish the models of supply contracts in the literature. The most important are the structure (e.g. options-futures, quantity flexibility or backup agreement), the number of period in the planning horizon (one period, two periods or multiple periods), the correlation between demand and information (is the demand independent from period to period, or is there an external signal that permits the improvement of future demand forecasts).

The first factor which distinguishes the contract models is the structure of the contract. The first structure can be the options-futures contract ((Brown and Lee, 1998a), (Barnes-Schuster et al., 1998) and (Cachon and Lariviere, 1997)). In this type of contracts, the firm has two decision stages. At the first stage two decisions are fixed: the number of futures (a non-refundable and unchangeable commitment) and the number of options (a flexible commitment). At the second decision stage the firm can exercise a number or the totality of the prescribed options by paying an exercise cost. The second contract type is the backup contract (Eppen and Iyer, 1997). In this case, the firm makes an initial order decision and at the final decision point, a part of the initial order may be cancelled, up to a certain predefined percentage. The third contract type is the quantity flexibility contract ((Bassok and Anupindi, 1995) and (Tsay, 1999)). In this case, the firm makes an initial order decision and can later revise this order decision within a certain upside and downside percentages.

The second factor which differentiates contracts is the number of periods. (Bassok and Anupindi, 1995) analyse a rolling horizon flexibility contract without information updates. (Tsay and Lovejoy, 1999) consider the contract of (Bassok and Anupindi, 1995) and allow for forecast updates. The resulted multiple period models are difficult to be solved and no analytical insights can be obtained.

In the literature, most papers treating the supply contracts study two-period models, which may permit some analytical insights. (Milner and Rosenblatt, 1997) consider a two-period model in which demands are assumed to be independent like (Bassok and Anupindi, 1995). Most papers allow the demand updates using an external information (see for example (Eppen and Iyer, 1997), (Donohue, 1996, 2000), (Tsay, 1999), (Brown and Lee, 1998a) and (Brown, 1999)).

We assume in this chapter, that the demand is characterized by a probability density function that is updated, using an external market information, before the beginning of the selling season. We assume also that the demand and the information are jointly distributed using a bivariate normal distribution. The external information could be for example, the sales of a pre-seasonal product where the demand is closely related to our product demand. The information could be also a completely external information about an external condition (like the weather forecasts). We assume also that the ordering cost and the return value of the second decision stage are stochastic.

In the model presented in this chapter, we develop a new type of contract that is more flexible than those existing in the literature (Cheaitou et al., b, 2008). In addition to the classical contract process, we assume that at the beginning of the selling period, a certain quantity can be returned to the supplier. In general, the manufacturer (supplier) has many retailers or many retail channels. Therefore, if any of these retailers decides to return a certain quantity to this supplier (in application of the contract), then he could resell it to another retailer before the beginning of the selling horizon, or he can resell it via another retail channel. This important option permits the retailer to profit from the external information that is collected between the two decision stages, and from the realization of the stochastic parameters (costs and return value). Therefore, the retailer can adjust his previous decisions taken at the first decision stage. More precisely, if the updated demand distribution is relatively low, then a part of the already ordered quantity will be returned to the supplier, and if the updated demand distribution is relatively high, then a part of the already reserved capacity will be used. Based on our knowledge, the return option does not exist in any of the existing contracts papers.

In this chapter, the decision process is divided into two stages: in the first stage a first group of decisions about the production and the capacity reservation (options) is made. In the second stage, an exogenous information which is correlated with the demand distribution is collected, which permits the update of the demand forecast. Another information related to the costs in the second stage is known. After updating the demand forecast and the costs information, another decision group is made. These new decisions are relative to the production of new quantities and to the return of a certain quantity, from the quantity which was already received.

The remaining part of the presented chapter is structured as follows: in the following section we define the model, the parameters and the assumptions, the objective function and the optimal policy for the two decision stages. In section 7.3, we study a particular case that represents the worthless information case. In section 7.4, we develop another particular case, in which the correlation between the information and the demand is perfect. In section 7.5, we provide a numerical study and in section 7.6 we give conclusions and perspectives.

7.2 The Model

7.2.1 Model parameters

The stochastic demand of a mono-product is defined with an exogenous information by bivariate normal probability density function. The random exogenous information becomes deterministic and completely known between two different stages of the decision process. To satisfy the demand, the retailer can order a first time with a low cost production mode and he can reserve a certain amount of his supplier capacity (options), but the available information about the demand is not accurate and the procurement costs and return values in the second stage are stochastic. Once the exogenous information is known the demand forecast is updated and at the same time the procurement cost and the return value become deterministic. In the case of a sufficiently high correlation between the collected information and the demand distribution, the variability of the demand decreases, and the retailer has an accurate demand distribution that permits him to adjust his previous decisions. The retailer could either exercise a certain number of the options bought from his supplier, or choose to return a part of the order that he had previously bought from his supplier. After the end of the single period horizon, the unsatisfied demands are lost and the remaining inventory is returned to the supplier or sold in a parallel market.

The model parameters are defined as follows:

- D : the stochastic demand,

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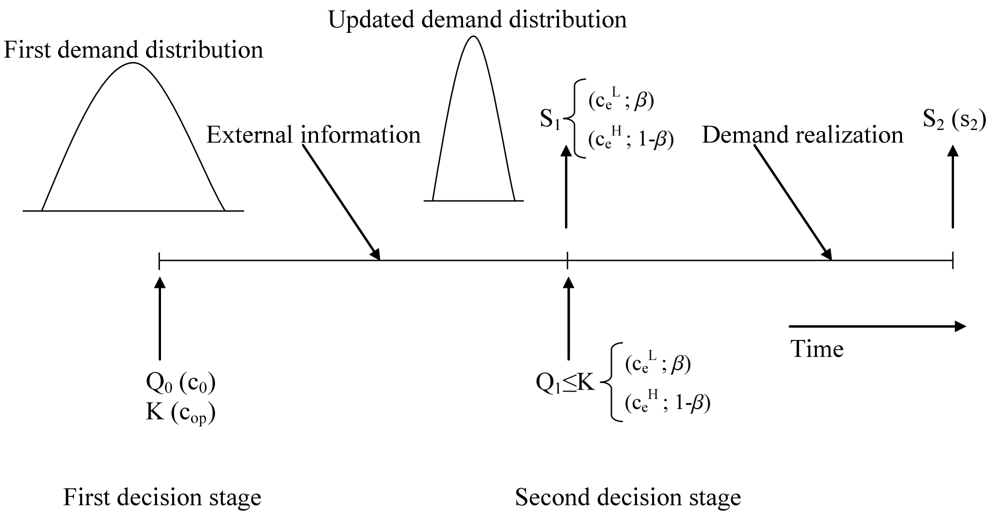

Figure 7.1: Decision process

- i: the stochastic exogenous information collected between the two stages of the decision process,
- $f(i, D)$: the bivariate normal probability density function of the demand D and the information i,
- μ_0 and θ_0: the means of the bivariate normal distribution with respect to D and i, respectively,
- σ_0 and δ_0: the standard deviations of the bivariate normal distribution with respect to D and i respectively,
- ρ: the correlation coefficient between D and i,
- $g(i)$ and $G(i)$: the marginal probability density and cumulative distribution functions of the information i, respectively,
- $h(D|i)$ and $H(D|i)$: the conditional probability density and cumulative distribution functions of the demand D conditionally to the given the information i, respectively,
- $\Phi(\cdot)$: the standard normal cumulative distribution function,
- Q_0: the quantity ordered at the first decision stage,
- K: the amount of capacity reserved (or number of options bought) at the first decision stage,
- Q_1: the quantity ordered at the second decision stage (number of exercised options). $(Q_1 \leq K)$,
- Q_T: the maximal number of units that might be ordered until the second decision stage ($Q_T = Q_0 + K$),
- S_1: the quantity returned to the supplier or salvaged in a parallel market at the beginning of the selling horizon (at the second decision stage),
- c_0: the unit order cost for the quantity Q_0,

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- c_{op}: the unit cost for the capacity reservation,

- c_e: the unit exercise cost for the options (for the quantity Q_1). At the first decision stage, this parameter is stochastic. At the second decision stage it becomes deterministic and may take one of the following values: a low value, c_e^L, that occurs with a probability β, and a high value, c_e^H, that occurs with a probability $1-\beta$,

- s_1: the unit price (salvage value) of the returned quantity at the beginning of the selling horizon, S_1. Like c_e, s_1 is stochastic at the first decision stage and becomes deterministic at the second decision stage. It may take one of the following values: a low value s_1^L that occurs with a probability β, and a high value s_1^H that occurs with a probability $1-\beta$,

- p: the unit selling price during the selling horizon,

- b: the unit shortage penalty cost at the end of the horizon,

- s_2: the unit salvage value at the end of the horizon.

Note that the reserved capacity (K) at the first decision stage, plays two essential roles. On the one hand, the capacity reservation made by the retailer allows the supplier to prepare his raw materials and production facilities in order to probably satisfy the demand corresponding to this reservation in the second decision stage. On the other hand, the capacity reservation allows the retailer to exploit the demand forecast updates by ordering an additional quantity at the second stage in order to optimally satisfy the demand.

The random variables c_e and s_1 are realized (their values are known) at the same time with the same level (low or high). That means that if the realized value of c_e is low, then the realized value of s_1 is also low, and vice-versa.

Since $f(D,i)$ is a bivariate normal distribution, then it is well known that, for a given value i of the information, the conditional demand $(D|i)$ is normally distributed (Bickel and Doksum, 1977), with mean μ_1 and standard deviation σ_1, which are given by

$$\mu_1 = \mu_0 + \rho\frac{(i-\theta_0)\sigma_0}{\delta_0} \quad \text{and} \quad \sigma_1 = \sigma_0\sqrt{1-\rho^2}. \tag{7.1}$$

Note that since ρ always satisfies $\rho \leq 1$, then one always has $\sigma_1 \leq \sigma_0$. The last inequality implies that the variability of the conditional demand distribution given the information i is lower than the variability of the marginal demand distribution which does not take into account the realization of the stochastic information.

To develop the optimal policy, we will begin by determining the optimal policy of the second stage of the decision process, then using the dynamic programming, we will use the optimal policy of the second stage in order to develop the optimal policy of the first decision stage.

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7.2.2 Model parameters assumptions

Some assumptions on the model parameters are made in order to avoid classes of specific cases for which the optimal solution is trivial and does not permit to have any interesting insight.

The first one is relative to the ordering costs. We assume that $c_{op}+c_e^L < c_0 < c_{op}+c_e^H$ in order to have a general situation that makes the assumption of stochastic costs more interesting. If this assumption is not satisfied, then one will have only a single ordering mode and the other one will not be used. For example if one has $c_{op}+c_e^L < c_{op}+c_e^H < c_0$, the optimal Q_0^* will be automatically equal to zero.

An other intuitive assumption is taken into account and is relative to the selling price: $c_{op}+c_e^H < p$.

A third assumption defines the ranges of the salvage and the unit payback values and that gives sense to the payback option at the beginning of the selling horizon: $s_2 < s_1^L < s_1^H$. If for example we have $s_2 > s_1^H > s_1^L$, then the optimal returned quantity at the beginning of the selling horizon, S_1^* will be equal to zero.

We assume also that $s_1^L < s_1^H < c_e$. This assumption aims at avoiding situations in which it will be profitable to exercise options at the beginning of the second decision stage, in order to receive articles from the supplier and then to return these articles immediately to the same supplier, which is not logical.

The last two assumptions are relative to the relation between the ordering costs and the return values: $s_1^L < c_{op}+c_e^L < c_0$ and $s_1^H < c_0 < c_{op}+c_e^H$. If these assumptions are not respected, for example if one has $s_1^L > c_{op}+c_e^L$, then at the beginning of the second decision stage, if the realized values of the exercise cost and salvaged value are $(c_e^L; s_1^L)$, then one will use all the reserved capacity K to order $Q_1^* = K$, and then to return these units immediately after receiving them with a unit payback value of s_1^L.

7.2.3 Second decision stage subproblem

Objective function

Let $E_X(\cdot)$ be the expectation taken over random variable X, $x^+ = \max(0,x)$, $x \wedge y = \min(x,y)$.

Let us define the objective function of the second stage by the following equation

$$\begin{aligned}&\Pi_1(Q_1, S_1|i, Q_0, Q_T) = \\ &\quad p\left((Q_0+Q_1-S_1)\wedge D\right) - b(D-Q_0-Q_1+S_1)^+ \\ &\quad +s_2(Q_0+Q_1-S_1-D)^+ - c_eQ_1 + s_1S_1. \end{aligned} \tag{7.2}$$

The expected value of the objective function with respect to the demand will be

$$\begin{aligned}&E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right] = \\ &\quad p\int_0^{Q_0+Q_1-S_1} Dh(D|i)dD + p\int_{Q_0+Q_1-S_1}^{\infty} (Q_0+Q_1-S_1)h(D|i)dD \\ &\quad -b\int_{Q_0+Q_1-S_1}^{\infty} (D-Q_0-Q_1+S_1)h(D|i)dD \\ &\quad +s_2\int_0^{Q_0+Q_1-S_1} (Q_0+Q_1-S_1-D)h(D|i)dD - c_eQ_1 + s_1S_1 \end{aligned} \tag{7.3}$$

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Lemma 7.1 *The expected objective function $E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right]$ is jointly concave with respect to the decision variables Q_1 and S_1.*

Proof. The hessian of $E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right]$ with respect to q_1 and S_1 is given by

$$\nabla^2 E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right] = -(p+b-s_2)h(Q_0+Q_1-S_1|i)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{7.4}$$

From the model assumptions (section 7.2.2), for each vector $\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, where $(V_1; V_2) \in \mathbb{R}^2$, we find

$$V^T\left(\nabla^2 E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right]\right)V = -(p+b-s_2)h(Q_0+Q_1-S_1|i)(V_1-V_2)^2 \leq 0,$$

which proves that the matrix $\nabla^2 E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right]$ is semi-definite negative. Consequently, the expected objective function $E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right]$ is jointly concave with respect to Q_1 and S_1. □

Optimization problem

By proving the concavity of the expected objective function of the second decision stage subproblem, we can now write the optimization problem of that stage as follows

$$\max_{0\leq Q_1\leq K,\quad 0\leq S_1\leq Q_0} E_{D|i}\left[\Pi_1(Q_1, S_1|i, Q_0, Q_T)\right] \tag{7.5}$$

The unconstrained type of this optimization problem (with an infinite capacity K) has been studied in details in Chapter 4 and Chapter 6. Therefore, the optimal policy of the unconstrained problem can be completely characterized by two threshold levels, that are given by

$$Y_1(i) = H^{-1}\left(\frac{p+b-c_e}{p+b-s_2}|i\right) \quad \text{and } Y_2(i) = H^{-1}\left(\frac{p+b-s_1}{p+b-s_2}|i\right), \tag{7.6}$$

with, from the model assumptions (section 7.2.2), are related by

$$Y_1(i) \leq Y_2(i). \tag{7.7}$$

These two threshold levels could be expressed differently as follows

$$Y_1(i) = \mu_0 + \rho(i-\theta_0)\frac{\sigma_0}{\delta_0} + (\sigma_0\sqrt{1-\rho^2})\Phi^{-1}\left(\frac{p+b-c_e}{p+b-s_2}\right), \tag{7.8}$$

and

$$Y_2(i) = \mu_0 + \rho(i-\theta_0)\frac{\sigma_0}{\delta_0} + (\sigma_0\sqrt{1-\rho^2})\Phi^{-1}\left(\frac{p+b-s_1}{p+b-s_2}\right). \tag{7.9}$$

Note that these threshold levels depend on the value of two stochastic parameters, namely c_e and s_1. At the second decision stage, these two stochastic parameters become deterministic.

In section 7.2.1, we have assumed that c_e and s_1 take, at the second decision stage, either their high or their low value together. This means that at the second decision stage we will have one of the two following cases:

- with a probability β the following threshold levels

$$Y_1^L(i) = H^{-1}\left(\frac{p+b-c_e^L}{p+b-s_2}|i\right) \text{ and } Y_2^L(i) = H^{-1}\left(\frac{p+b-s_1^L}{p+b-s_2}|i\right), \tag{7.10}$$

- with a probability $1-\beta$ the following threshold levels

$$Y_1^H(i) = H^{-1}\left(\frac{p+b-c_e^H}{p+b-s_2}|i\right) \text{ and } Y_2^H(i) = H^{-1}\left(\frac{p+b-s_1^H}{p+b-s_2}|i\right), \tag{7.11}$$

with

$$Y_1^H(i) \leq Y_1^L(i) \text{ and } Y_2^H(i) \leq Y_2^L(i). \tag{7.12}$$

Optimal policy of the constrained problem

In the remaining parts of this chapter, the decision variables of the first decision stage will be Q_0 and Q_T. The decision variable K could be easily deduced from these two decision variables.

Using the concavity of the expected objective function of the second decision stage, with respect to the decision variables Q_1 and S_1, the optimal policy of the second decision stage can be defined as a function of the threshold levels $Y_1(i)$ and $Y_2(i)$, as follows:

- if $Y_2(i) < Q_0$ then $Q_1^* = 0$ and $S_1^* = Q_0 - Y_2(i)$,
- if $Y_1(i) \leq Q_0 \leq Y_2(i)$ then $Q_1^* = S_1^* = 0$,
- if $Q_0 < Y_1(i) < Q_T$ then $Q_1^* = Y_1(i) - Q_0$ and $S_1^* = 0$,
- if $Q_T < Y_1(i)$ then $Q_1^* = Q_T - Q_0$ and $S_1^* = 0$.

Note that if $(c_e; s_1) = (c_e^L; s_1^L)$ then $(Y_1(i); Y_2(i)) = (Y_1^L(i); Y_2^L(i))$ and if $(c_e; s_1) = (c_e^H; s_1^H)$ then $(Y_1(i); Y_2(i)) = (Y_1^H(i); Y_2^H(i))$.

This solution is simply a modified newsvendor solution using the updated demand distribution $H(D|i)$, constrained by the initial decisions Q_0 and Q_T of the first decision stage.

7.2.4 First decision stage subproblem

Note that we make the assumption that increasing values of realization of the information i denote increasing forecasts of demand. In other words, the demand D is positively correlated with the information i. For example, if the information is an aggregation of key customer forecasts, then a larger key customer forecast indicates larger overall demand. Mathematically, we can express this using the concept of stochastically larger (Ross, 1983). Consider two realization of i: i_1 and i_2. Then, $i_1 > i_2$ implies

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that $D|i_1$ is stochastically larger than $D|i_2$, expressed $D|i_1 \geq_{st} D|i_2$. This stochastic relationship implies that $H(D|i_1) \leq H(D|i_2)$ for all D (Brown and Lee, 1998b).

Since D is stochastically increasing in i, $H(D|i)$ is decreasing in i for all D. Thus,

$$H^{-1}\left(\frac{p+b-c_e}{p+b-s_2}|i\right)$$

and

$$H^{-1}\left(\frac{p+b-s_1}{p+b-s_2}|i\right)$$

are increasing in i, so $Y_1(i)$ and $Y_2(i)$ are increasing in the information i. Because of this monotonic behavior, we can express the optimal policy of the second stage as a function of the observed external information as follows:

- if $i < U_2(Q_0)$ then $Q_1^*(i, Q_0) = 0$ and $S_1^*(i, Q_0) = Q_0 - Y_2(i)$,
- if $U_2(Q_0) \leq i \leq U_1(Q_0)$ then $Q_1^*(i, Q_0) = S_1^*(i, Q_0) = 0$,
- if $U_1(Q_0) < i < V_1(Q_T)$ then $Q_1^*(i, Q_0) = Y_1(i) - Q_0$ and $S_1^*(i, Q_0) = 0$,
- if $V_1(Q_T) < i$ then $Q_1^*(i, Q_0) = Q_T - Q_0$,

with $U_2(Q_0)$ is the value of i so that $Y_2(i) = Q_0$. $U_2(Q_0)$ is given by the following equation

$$U_2(Q_0) = \frac{\delta_0}{\rho\sigma_0}\left[Q_0 - \sigma_0\sqrt{1-\rho^2}\Phi^{-1}\left(\frac{p+b-s_1}{p+b-s_2}\right) - \mu_0\right] + \theta_0. \tag{7.13}$$

$U_1(Q_0)$ and $V_1(Q_T)$ are also the values of i so that $Y_1(i) = Q_0$ and $Y_1(i) = K$ respectively. $U_1(Q_0)$ and $V_1(K)$ are given by the following equations

$$U_1(Q_0) = \frac{\delta_0}{\rho\sigma_0}\left[Q_0 - \sigma_0\sqrt{1-\rho^2}\Phi^{-1}\left(\frac{p+b-c_e}{p+b-s_2}\right) - \mu_0\right] + \theta_0, \tag{7.14}$$

and

$$V_1(Q_T) = \frac{\delta_0}{\rho\sigma_0}\left[Q_T - \sigma_0\sqrt{1-\rho^2}\Phi^{-1}\left(\frac{p+b-c_e}{p+b-s_2}\right) - \mu_0\right] + \theta_0. \tag{7.15}$$

Note that the value of $U_2(Q_0)$, $U_1(Q_0)$ and $V_1(K)$ depends on the realized value of the couple $(c_e; s_1)$.

Objective function

Using the dynamic programming principle, and using the obtained results in the previous sections, we can write the expected objective function of the first decision stage as follows

$$\Pi_0(Q_0, Q_T) = -c_{op}(Q_T - Q_0) - c_0 Q_0 + E_{(c_e;s_1)}\left[E_i\left[E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]\right]\right]. \tag{7.16}$$

Let $E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ be the optimal expected objective function of the second period (with respect to $D|i$). $E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ is then defined as follows

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$$E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] =$$

$$\begin{cases} E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] & if\ i < U_2(Q_0) \\ E_{D|i}\left[\Pi_{12}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] & if\ U_2(Q_0) < i < U_1(Q_0) \\ E_{D|i}\left[\Pi_{13}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] & if\ U_1(Q_0) < i < V_1(Q_T) \\ E_{D|i}\left[\Pi_{14}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] & if\ V_1(Q_T) < i \end{cases} \tag{7.17}$$

with

$$\begin{aligned} &E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = \\ &\quad p\int_0^{Y_2(i)} Dh(D|i)dD + p\int_{Y_2(i)}^{\infty} Y_2(i)h(D|i)dD \\ &\quad -b\int_{Y_2(i)}^{\infty} (D - Y_2(i))h(D|i)dD \\ &\quad +s_2\int_0^{Y_2(i)} (Y_2(i) - D)h(D|i)dD + s_1(Q_0 - Y_2(i)), \end{aligned} \tag{7.18}$$

$$\begin{aligned} &E_{D|i}\left[\Pi_{12}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = \\ &\quad p\int_0^{Q_0} Dh(D|i)dD + p\int_{Q_0}^{\infty} Q_0h(D|i)dD \\ &\quad -b\int_{Q_0}^{\infty} (D - Q_0)h(D|i)dD + s_2\int_0^{Q_0} (Q_0 - D)h(D|i)dD, \end{aligned} \tag{7.19}$$

$$\begin{aligned} &E_{D|i}\left[\Pi_{13}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = \\ &\quad p\int_0^{Y_1(i)} Dh(D|i)dD + p\int_{Y_1(i)}^{\infty} Y_1(i)h(D|i)dD \\ &\quad -b\int_{Y_1(i)}^{\infty} (D - Y_1(i))h(D|i)dD \\ &\quad +s_2\int_0^{Y_1(i)} (Y_1(i) - D)h(D|i)dD - c_e(Y_1(i) - Q_0), \end{aligned} \tag{7.20}$$

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and

$$E_{D|i}\left[\Pi_{14}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = \tag{7.21}$$
$$p\int_0^{Q_T} Dh(D|i)dD + p\int_{Q_T}^{\infty} Q_T h(D|i)dD$$
$$-b\int_{Q_T}^{\infty}(D-Q_T)h(D|i)dD + s_2\int_0^{Q_T}(Q_T-D)h(D|i)dD$$
$$-c_e(Q_T-Q_0). \tag{7.22}$$

Then the expected (with respect to $D|i$, i and $(c_e; s_1)$) optimal objective function of the second period is given in Appendix D.1.

Lemma 7.2 *The expected objective function $\Pi_0(Q_0, Q_T)$ is jointly concave with respect to the decision variables Q_0 and Q_T.*

Proof. See Appendix D.2.

Lemma 7.3 *The optimal values of the decision variables Q_0 and K are characterized by a system of two independent equations given by*

$$\frac{\partial \Pi_0(Q_0, Q_T)}{\partial Q_T} = 0 \ \textit{and}\ \frac{\partial \Pi_0(Q_0, Q_T)}{\partial Q_0} = 0 \tag{7.23}$$

where the two partial derivatives are given in Appendix D.3 (equations (D.4) and (D.5)).

These equations characterize the optimal policy of the first decision stage, but they are very complicated to be handled in order to get simple formulas that determine this optimal policy. For this reason, we will develop in the following sections some particular cases in which we will provide closed-form formulas of the optimal policy.

7.3 Worthless information particular case

In this section we will solve the particular case in which the information i is worthless, so that the correlation coefficient ρ is equal to zero. In this case the optimal threshold levels of the second decision stage will be given by the following equations

$$Y_1(i) = \mu_0 + \sigma_0\Phi^{-1}\left(\frac{p+b-c_e}{p+b-s_2}\right) \quad \text{and} \quad Y_2(i) = \mu_0 + \sigma_0\Phi^{-1}\left(\frac{p+b-s_1}{p+b-s_2}\right). \tag{7.24}$$

Since the threshold levels defined above are independent of the information i, therefore, for any value of the information i, one will have deterministic problem at the beginning of the second decision stage. This problem depends on these two threshold levels and on the two optimal decision variables Q_0^* and K^*.

One can identify two different cases, in terms of the values of the parameters s_1^L, s_1^H, c_e^L and c_e^H: case A and case B.

The case A corresponds to the following assumptions

$$s_1^L < s_1^H < c_e^L < c_e^H, \tag{7.25}$$

while the case B corresponds to the following assumption

$$s_1^L < c_e^L < s_1^H < c_e^H. \tag{7.26}$$

7.3.1 Case A

This case corresponds to the following inequalities on the threshold levels of the second decision stage

$$Y_1^H(i) < Y_1^L(i) < Y_2^H(i) < Y_2^L(i). \tag{7.27}$$

Second decision stage optimal policy

Depending on the realized value of the couple $(c_e; s_1)$, one might have one of the two possible cases in the second decision stage.

The optimal policy of this case and the possible combinations of situations, depending on the values of the decision variables and the costs parameters, may be summarized in the following equations:

- case A.1: if $Q_0 \leq Q_T \leq Y_1^H$, then $Q_1^* = K$ and $S_1^* = 0$,
- case A.2: if $Q_0 \leq Y_1^H \leq Q_T$, then $S_1^* = 0$ and
  - if $(c_e; s_1) = (c_e^L; s_1^L)$, then $Q_1 = K$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = Y_1^H - Q_0$,
- case A.3: if $Y_1^H \leq Q_0 \leq Q_T \leq Y_2^L$ then $S_1^* = 0$ and
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1 = K$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1 = 0$.

First decision stage optimal policy

Using the optimal policy developed in the previous section, we will provide in this section the optimal policy of the first stage of this particular case.

In order to develop the first period optimal policy, we begin by defining the following expressions

$$\bar{s}_1 = \beta s_1^L + (1-\beta)s_1^H, \quad \bar{c}_e = \beta c_e^L + (1-\beta)c_e^H \quad \text{and} \quad c_{Moy} = \bar{c}_e + c_{op}. \tag{7.28}$$

Since we have proved the concavity of the expected objective function of the first decision stage, then we will use the first order optimality criterion in developing the optimal policy.

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Whatever are the values of the costs of our model, we will have only three possible cases:

- the first case corresponds to $c_{op} > \beta(c_e^H - c_e^L)$, which corresponds to the case A.1 of the second decision stage. In this case, the optimal value of K is

$$K^* = H^{-1}\left[\frac{p+b-c_{Moy}}{p+b-s_2}\right] - Q_0^*.$$

  For this case, if $c_{Moy} < 0$ then the optimal value of Q_0, $Q_0^* = 0$ and if $c_{Moy} > 0$ then

$$Q_0^* = H^{-1}\left[\frac{p+b-c_{Moy}}{p+b-s_2}\right],$$

- the second case applies if $\beta(c_0 - c_e^L) < c_{op} < \beta(c_e^H - c_e^L)$ which corresponds to the case A.2 of the second decision stage. The optimal value of K is

$$K^* = H^{-1}\left[\frac{p+b-c_e^L - c_{op}/\beta}{p+b-s_2}\right] - Q_0^*.$$

  For this case if $c_{Moy} < 0$ then the optimal value of Q_0, $Q_0^* = 0$ and if $c_{Moy} > 0$ then

$$Q_0^* = H^{-1}\left[\frac{p+b-c_e^H}{p+b-s_2}\right],$$

- the third case applies if $c_{op} < \beta(c_0 - c_e^L)$ and $c_0 < c_{Moy}$ which corresponds to the case A.3 of the second decision stage. In this case, the optimal values of the decision variables are

$$K^* = H^{-1}\left[\frac{p+b-c_e^L - c_{op}/\beta}{p+b-s_2}\right] - Q_0^*$$

  and

$$Q_0^* = H^{-1}\left[\frac{p+b+\frac{c_{op}+\beta c_e^L - c_0}{1-\beta}}{p+b-s_2}\right].$$

7.3.2 Case B

This case corresponds to the following inequalities on the threshold levels of the second decision stage

$$Y_1^H(i) < Y_2^H(i) < Y_1^L(i) < Y_2^L(i). \tag{7.29}$$

Depending on the values of the cost parameters of our model, and taking into account the cost assumptions defined in section 7.2.2, one will have in the Case B one of six possible cases.

Second decision stage optimal policy

The optimal policy of the second decision stage depends on the values of the decision variables of the first decision stage and on the realization of the cost parameters of the second decision stage. These different situations may be summarized in the following equations:

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- case B.1: if $Q_0 < Q_T < Y_1^H$ then $Q_1^* = K$ and $S_1^* = 0$,
- case B.2: if $Q_0 < Y_1^H < Q_T < Y_2^H$ then
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1^* = K$ and $S_1^* = 0$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = Y_1^H - Q_0$ and $S_1^* = 0$,
- case B.3: if $Q_0 < Y_1^H < Y_2^H < Q_T < Y_1^L$ then
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1^* = K$ and $S_1^* = 0$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = Y_1^H - Q_0$ and $S_1 = 0$,
- case B.4: if $Y_1^H < Q_0 < Q_T < Y_2^H$ then
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1^* = K$ and $S_1^* = 0$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = 0$ and $S_1^* = 0$,
- case B.5: if $Y_1^H < Q_0 < Y_2^H < Q_T < Y_1^L$ then
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1^* = K$ and $S_1^* = 0$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = 0$ and $S_1^* = 0$,
- case B.6: if $Y_1^H < Y_2^H < Q_0 < Q_T < Y_1^L$ then
  - if $(c_e; s_1) = (c_e^L; s_1^L)$ then $Q_1^* = K$ and $S_1^* = 0$,
  - if $(c_e; s_1) = (c_e^H; s_1^H)$ then $Q_1^* = 0$ and $S_1^* = Q_0 - Y_2^H$.

First decision stage optimal policy

The optimal policy of the first decision stage of this particular case could be developed using the optimal policy of the second decision stage provided in the previous section. Whatever the model parameters are, one can resume the solution in six different cases that correspond to those of the second decision stage. These cases are:

- case B.1: this case applies if $c_{op} > \beta(c_e^H - c_e^L)$. The optimal value of K, is then

$$K^* = H^{-1}\left[\frac{p+b-c_{Moy}}{p+b-s_2}\right] - Q_0^*.$$

  If $c_0 > c_{Moy}$ then $Q_0^* = 0$, else $Q_0^* = Q_T$,

- case B.2: this case applies if $\beta(s_1^H - c_e^L) < c_{op} < \beta(c_e^H - c_e^L)$. In this case

$$K^* = H^{-1}\left[\frac{p+b-c_e^L - \frac{c_{op}}{\beta}}{p+b-s_2}\right].$$

  If $c_0 > c_{Moy}$ then $Q_0^* = 0$, else $Q_0^* = Y_1^H$,

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- case B.3: this case applies if $0 < c_{op} < \beta(s_1^H - c_e^L)$. In this case

$$K^* = H^{-1}\left[\frac{p+b-c_e^L-\frac{c_{op}}{\beta}}{p+b-s_2}\right].$$

If $c_0 > c_{Moy}$ then $Q_0^* = 0$, else $Q_0^* = Y_1^H$,

- case B.4: this case applies if $\beta(s_1^H - c_e^L) < c_{op} < \beta(c_e^H - c_e^L)$ and $c_{op} + (1-\beta)s_1^H + \beta c_e^L < c_0 < c_{op} + (1-\beta)c_e^H + \beta c_e^L$. In this case

$$K^* = H^{-1}\left[\frac{p+b-c_e^L-\frac{c_{op}}{\beta}}{p+b-s_2}\right]$$

and

$$Q_0^* = H^{-1}\left[\frac{p+b+\frac{c_{op}-c_0+\beta c_e^L}{1-\beta}}{p+b-s_2}\right],$$

- case B.5: this case applies if $0 < c_{op} < \beta(s_1^H - c_e^L)$ and $c_{op} + \beta c_e^L + (1-\beta)s_1^H < c_0 < c_{Moy}$. In this case

$$K^* = H^{-1}\left[\frac{p+b-c_e^L-\frac{c_{op}}{\beta}}{p+b-s_2}\right]$$

and

$$Q_0^* = H^{-1}\left[\frac{p+b+\frac{c_{op}-c_0+\beta c_e^L}{1-\beta}}{p+b-s_2}\right],$$

- case B.6: this case applies if $0 < c_{op} < \beta(s_1^H - c_e^L)$ and $c_0 < c_{op} + \beta c_e^L + (1-\beta)s_1^H$. In this case one gets

$$K^* = Q_0^* = H^{-1}\left[\frac{p+b-c_e^L-\frac{c_{op}}{\beta}}{p+b-s_2}\right].$$

7.4 Perfect information particular case

In this section we will develop the optimal policy of the entire problem in the case where the correlation between the external information and the demand is perfect, or in other words when $\rho = 1$. In this case, the conditional demand distribution will be a normal distribution with a mean equals to $\mu_0+\rho(i-\theta_0)\sigma_0/\delta_0$ and a standard deviation equals to zero, which is equivalent to a deterministic demand that is equal to $D|i = \mu_0 + \rho(i-\theta_0)\sigma_0/\delta_0$.

We will use the same methodology as for the first particular case, studied in the previous section, in order to develop the optimal policy of this particular case of our model.

We will begin by determining the optimal policy of the second decision stage, then using a dynamic programming approach, we determine that of the first decision stage.

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7.4.1 Second decision stage optimal policy

The problem of the second decision stage is deterministic. Once the information i is known, the demand D becomes deterministic and gets the value of $D|i = \mu_0 + \rho(i - \theta_0)\sigma_0/\delta_0$. In this case whatever the value of the couple $(c_e; s_1)$ is, the optimal values of the decision variables Q_1 and S_1 will not be influenced. Therefore depending on the value of the realized information i, we can have one of the following three cases:

- if $Q_0 \leq Q_T \leq D|i$ then $Q_1^{*L} = Q_1^{*H} = Q_T - Q_0$ and $S_1^{*L} = S_1^{*H} = 0$ and the optimal expected objective function (with respect to $(c_e; s_1)$) is

$$\Pi_1^*(Q_1^*, S_1^*|i) = pQ_T - b(D|i - Q_T) - \bar{c}_e(Q_T - Q_0),$$

- if $Q_0 \leq D|i \leq Q_T$ then $Q_1^{*L} = Q_1^{*H} = D|i - Q_0$ and $S_1^{*L} = S_1^{*H} = 0$ with the following expected optimal objective function

$$\Pi_1^*(Q_1^*, S_1^*|i) = pD|i - \bar{c}_e(D|i - Q_0),$$

- if $D|i \leq Q_0 \leq Q_T$ then $Q_1^{*L} = Q_1^{*H} = 0$ and $S_1^{*L} = S_1^{*H} = Q_0 - D|i$ with

$$\Pi_1^*(Q_1^*, S_1^*|i) = pD|i + \bar{s}_1(Q_0 - D|i).$$

7.4.2 First decision stage optimal policy

Using dynamic programming we can write the expected objective function of the first decision stage as follows

$$\begin{aligned}
\Pi_0(Q_0, Q_T) \quad &= -c_{op}(Q_T - Q_0) - c_0 Q_0 + E_i\left[E_{(c_e;s_1)}[\Pi_1^*(Q_1^*, S_1^*|i)]\right] \\
&= -c_{op}(Q_T - Q_0) - c_0 Q_0 \\
&+ \int_{\delta_0/\sigma_0(Q_T-\mu_0)+\theta_0}^{\infty} \left[pQ_T - \bar{c}_e(Q_T - Q_0) - b(D|i - Q_T)\right] g(i)di \\
&+ \int_{\delta_0/\sigma_0(Q_0-\mu_0)+\theta_0}^{\delta_0/\sigma_0(Q_T-\mu_0)+\theta_0} \left[pD|i - \bar{c}_e(D|i - Q_0)\right] g(i)di \\
&+ \int_{0}^{\delta_0/\sigma_0(Q_0-\mu_0)+\theta_0} \left[pD|i + \bar{s}_1(Q_0 - D|i)\right] g(i)di
\end{aligned} \tag{7.30}$$

To derive the optimal values of the decision variables Q_0 and Q_T, we will use the first order optimality criterion.

The optimal values of Q_0 and Q_T will be the solutions of the following equations

$$\frac{\partial \Pi_0(Q_0, Q_T)}{\partial Q_0} = -c_0 + c_{op} + \bar{c}_e - (\bar{c}_e - \bar{s}_1)G\left[\delta_0/\sigma_0(Q_0 - \mu_0) + \theta_0\right] = 0, \tag{7.31}$$

and

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$$\frac{\partial \Pi_0(Q_0, Q_T)}{\partial Q_T} = p + b - \bar{c}_e - c_{op} - (p + b - \bar{c}_e) G\left[\delta_0/\sigma_0(Q_T - \mu_0) + \theta_0\right] = 0. \quad (7.32)$$

These two equations gives the following optimal values of the decision variables

$$Q_0^* = \frac{\sigma_0}{\delta_0}\left[G^{-1}\left(\frac{\bar{c}_e + c_{op} - c_0}{\bar{c}_e - \bar{s}_1}\right) - \theta_0\right] + \mu_0, \quad (7.33)$$

and

$$Q_T^* = \frac{\sigma_0}{\delta_0}\left[G^{-1}\left(\frac{p + b - c_{Moy}}{p + b - \bar{c}_e}\right) - \theta_0\right] + \mu_0. \quad (7.34)$$

Proposition If $c_0 < \bar{c}_e + c_{op}$ then the optimal Q_0 is positive and is equal to the value defined in (7.33). In the other cases $Q_0^* = 0$.

7.5 Numerical analysis

In this section we provide some numerical applications to show the impact of each of our model parameters on the structure of the optimal policy.

Firstly, we define a nominal numerical set of parameters. Then, based on that set of parameters, we define other examples by varying one or more of the nominal numerical parameters. We make then comparisons between the different provided examples, in order to show the effect of some parameters of our model on the optimal policy.

The nominal numerical values of our model parameters are defined as follows: $\mu_0 = 1000$, $\theta_0 = 1000$, $\sigma_0 = 300$, $\delta_0 = 300$, $\rho = 0.5$, $\beta = 0.5$, $p = 100$, $b = 30$, $s_2 = 15$, $c_0 = 50$, $c_{op} = 5$, $c_e^L = 40$, $c_e^H = 50$, $s_1^L = 30$, $s_1^H = 40$.

Note that in all the following numerical examples, we plot the optimal values of the decision variables of the first decision stage Q_0 and Q_T, namely Q_0^* and Q_T^*, and the expected optimal values of the decision values of the second decision stage, namely $E[Q_1^*]$ and $E[S_1^*]$. Indeed, the expectation of the optimal values of the decision variables Q_1 and S_1 is with respect to the stochastic couple $(c_e; s_1)$ and with respect to the stochastic information i.

7.5.1 Impact of the unit order cost c_0

In this section we show the effect of the unit order cost c_0 on the optimal policy of the first decision stage. We use the nominal parameters defined above, and we vary the unit order cost of the first decision stage c_0 to show its impact on the optimal policy.

Note that for this example one has $\bar{c}_e = 45$, $c_{Moy} = 50$ and $\bar{s}_1 = 35$.

From Figure 7.2, one can notice the existence of three different intervals of values of c_0. The first one belongs to the lower values of c_0, where the expected optimal value of the decision variable S_1 is positive.

In this region, the optimal value of the decision variable Q_0 is high, due to the attractive ordering cost c_0, and consequently, the expected optimal decision variable $E[Q_1^*]$ is equal to zero. Therefore, in this region the optimal decision variable Q_T^* is equal to the optimal decision variable Q_0^*. Note that in this region, even if the expected salvage value $\bar{s_1}$ is lower than the ordering cost c_0, it is optimal to order some units and to salvage them later using the payback option at the beginning of the selling period. This is due to the existence of the correlation between the information i and the demand D ($\rho = 0.5$) and to the attractive ordering cost c_0 that is lower (in this region) than the expected cost $c_{Moy} = 50$.

The second region belongs to the medium values of c_0. In this region the optimal value of the decision variable Q_0 is still positive but is decreasing rapidly when c_0 increases. The expected optimal value of the decision variable S_1 is equal to zero. In fact it is not profitable to order some units at the first decision stage and then to salvage them at the second decision stage. On the other hand, when c_0 increases, the optimal decision variable Q_T^* is still constant, and as the optimal decision variable Q_0^* decreases, then the optimal K^* increases, to permit an increase in the number of units ordered at the second decision stage, namely Q_1^*.

In the third region, which corresponds to the high values of c_0, the optimal value of the decision variable Q_0 is equal to zero. In this region, it is normal that $S_1^* = 0$, because there are no units that are able to be returned to the supplier. As $Q_0^* = 0$, then in this region the optimal policy is not affected by the increase of c_0.

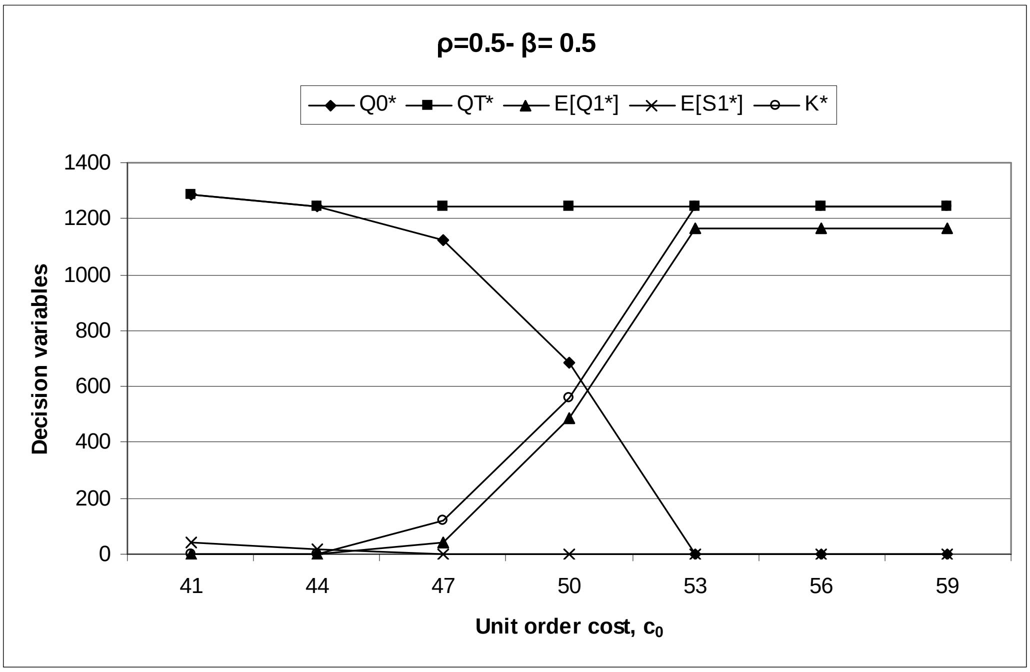

Figure 7.2: Effect of the unit order cost, c_0

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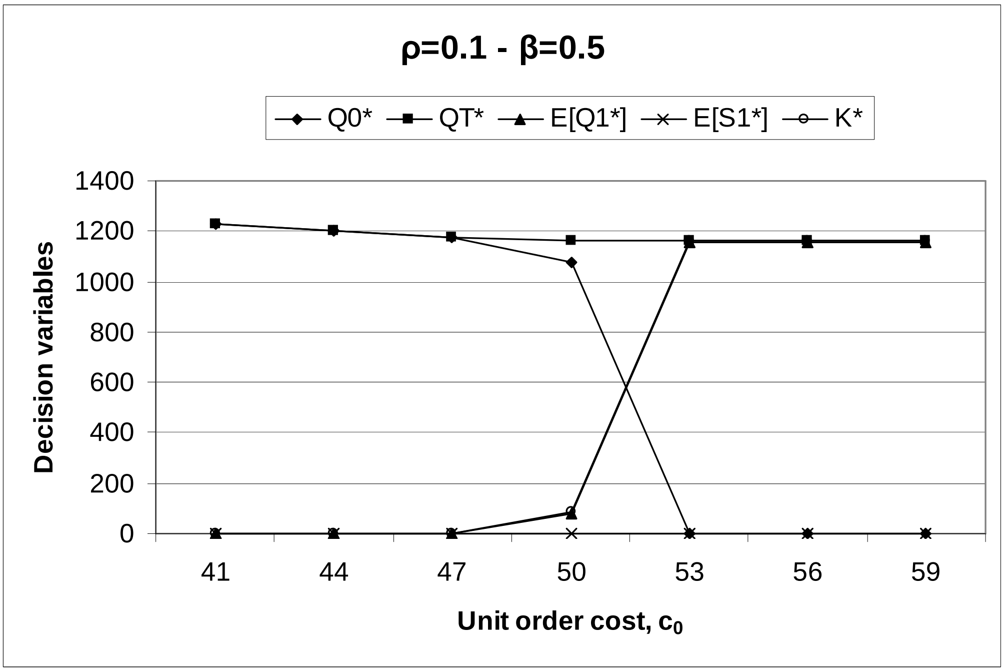

Figure 7.3: Effect of information quality, $\rho = 0.1$

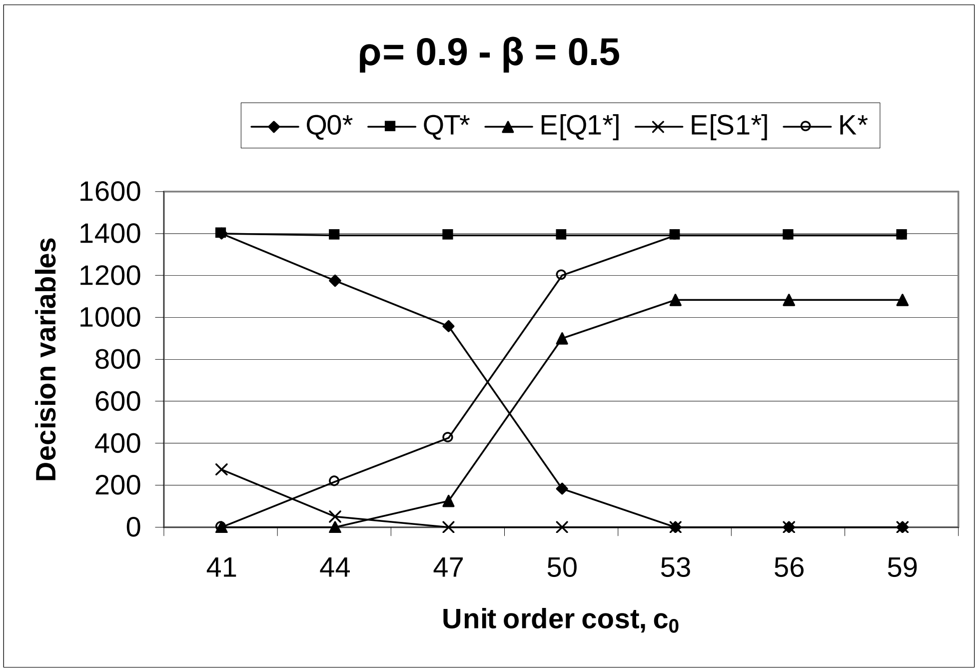

Figure 7.4: Effect of information quality, $\rho = 0.9$

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7.5.2 Impact of the information quality

In this section, we compare the example presented in the previous section, in Figure 7.2, (with correlation coefficient $\rho = 0.5$) with two other examples, that have the same parameters except the correlation coefficient: the first one is with less important information, with $\rho = 0.1$ and the second one is with more important information with $\rho = 0.9$.

The first remark to be taken into account is that when the correlation between the information and the demand decreases, the optimal policy becomes insensitive to the difference between the unit order cost c_0 and the expected cost c_{Moy}, in the regions where this difference is considerable, and is very sensitive in the zone where this difference is small. This implies that the width of the c_0 values interval, where both Q_0^* and $E[Q_1^*]$ are positive (the interval of medium c_0 values) increases when ρ increases. In this region the optimal policy is to order with Q_0 and Q_1 together.

Note also that when ρ decreases, the expected returned quantity $E[S_1^*]$ decreases. Indeed, in the case where ρ is low, the quality of the information captured between the two decision stages is very bad, and it does not permit a reduction in the demand variability. As the payback value is less than the ordering value, and one knows a priori that the collected information will not change tremendously the demand distribution, then it will not be profitable to return units ordered with Q_0 with a payback value s_1, and consequently $E[S_1^*]$ decreases.

Notice that when ρ increases then the optimal reserved capacity amount increases. That gives a bigger chance to profit from the information i and the variability reduction of the demand. This can be done by adjusting the first ordered quantity Q_0 using a part or the totality of the reserved capacity which implies an important ***reactivity***.

From Figures 7.2, 7.3 and 7.4, we can note that the effect of the information quality coupled with the unit order cost c_0 on the optimal policy can be divided into three cases.

The first case corresponds to the very low c_0 values. In this region, the difference between c_{Moy} and c_0 is very big so that the correlation coefficient ρ has no importance. Hence, the optimal decision variable K^* and consequently the optimal expected decision variable $E[Q_1^*]$ are equal to zero. In this region, when the information quality increases, the optimal decision variable Q_0^* increases due to two reasons. Firstly, if the value of the realized information i is high, then the demand will be high (due to the strong correlation). Therefore, a high value of the optimal decision variable Q_0^* satisfies well the demand. Secondly, if the value of the realized information i is low, then due to the low difference between c_0 and s_1 a part of Q_0^* could be returned to the supplier.

The second case corresponds to the medium c_0 values. In this region, the difference between c_{Moy} and c_0 is low. Then it is the quality of the information that will determine which of the procurement modes Q_0 or Q_1 will be more profitable. Indeed, when ρ increases, K^* increases, it becomes more profitable to postpone the ordering decision to the second decision stage, in order to use better the information i. Then Q_0^* decreases and to compensate $E[Q_1^*]$ increases .

The third case is relative to the high c_0 values. In this region it is logical to have $Q_0^* = 0$, for all ρ values, due to the high difference between c_{Moy} and c_0. When ρ increases, the optimal K^* increases also which gives a better reactivity in the case of high i realization at the second decision stage. On the other hand $E[Q_1^*]$ decreases when ρ increases as it is decided after that i is known. In fact, first of all, when ρ increases, and as $Q_0^* = 0$ for all ρ values, there is no need to postpone any orders from the first to the second decision stage or to compensate for any unordered units (with Q_0^*). In addition, for high ρ values, the collected information during the first period will be very useful to reduce the demand variability. Then after observing the information i, the variability of the demand decreases, and consequently the expected optimal quantity $E[Q_1^*]$ ordered to face this variability decreases also. Note that this induces an increase in the difference between K^* and $E[Q_1^*]$.

7.5.3 Impact of the probability β

In this section we compare two different examples that have both the same nominal numerical parameters defined above except the probability β. For the first example shown in Figure 7.5 the probability that the couple $(c_e; s_1)$ takes its high value $(c_e^H; s_1^H)$ is equal to $\beta = 0.9$. In the second example shown in Figure 7.6 we have $\beta = 0.1$.

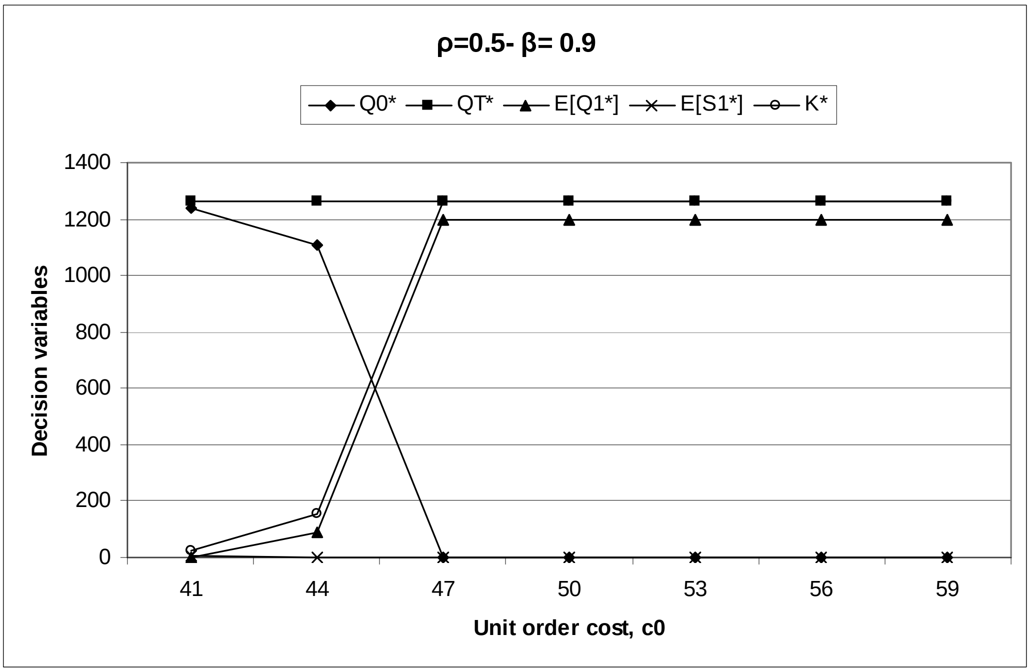

Figure 7.5: Effect of probability β, $\beta = 0.9$

Note that when the probability β decreases, the expected unit cost c_{Moy} and the expected unit payback value $\bar{s}_1$ increase.

From Figures 7.5 and 7.6 one can easily see that when the probability β decreases, it becomes more profitable to order more at the first decision stage with Q_0 for two reasons: because the expected cost of the use of the second decision stage options, c_{Moy}, increases and at the same time the unit payback value $\bar{s}_1$ increases also. This directly implies the increase in the expected S_1^* and the decrease of the optimal capacity amount reserved K^* and the optimal expected exercised options at the second decision stage $E[Q_1^*]$.

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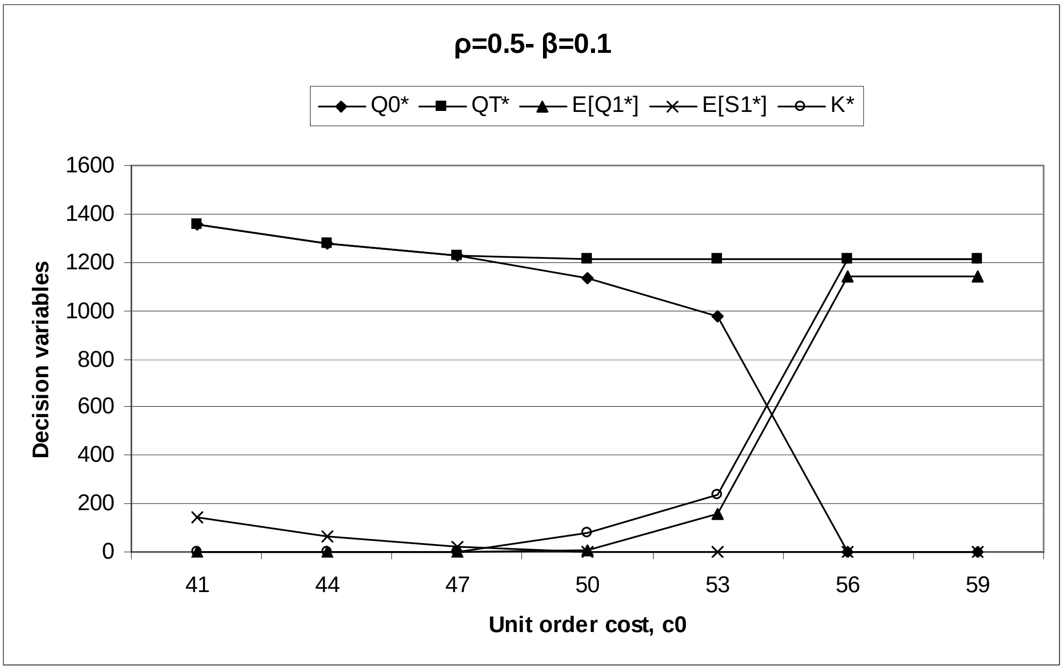

Figure 7.6: Effect of probability β, $\beta = 0.1$

7.5.4 Impact of the unit payback value s_1^H

In this section we study the effect of the high value of the unit payback value s_1^H on the optimal policy of the two decision stages and on the expected optimal objective function. Firstly, we plot two numerical examples, showing the optimal policy, based on the nominal numerical data defined above except the probability β and the value of s_1^H. In the first example shown in Figure 7.7 we vary s_1^H and we assume that $\beta = 0.5$. In Figure 7.8 we vary s_1^H and we take $\beta = 0.1$. Secondly, we plot an other example (Figure 7.9) in which we show the effect of s_1^H and of the correlation coefficient ρ on the expected optimal objective function of our model. This example (Figure 7.9) is based also on the nominal numerical values defined above, except the probability $\beta = 0.1$, the correlation coefficient ρ that we give three different values (0.1, 0.5 and 0.9) and the unit salvage value s_1^H that we vary in the admissible interval of values.

For the first two examples, Figures 7.7 and 7.8, note that when the higher value of s_1, namely s_1^H increases the expected optimal value of the returned quantity, $E[S_1^*]$ increases. This increase is accompanied with an increase in the optimal ordered quantity Q_0^*, from which some units are returned at the second decision stage. Note also that the optimal amount of reserved capacity K^* and the expected optimal ordered quantity at the second decision stage $E[Q_1^*]$ increase. This means that when s_1^H increases, the optimal policy tends to a policy where one orders more at the first decision stage and less at the second decision stage. In addition, a part of the ordered quantity at the first decision stage could be returned to the supplier.

Note also that when the probability value β decreases, the impact of the increase of s_1^H on the optimal policy may be seen more clearly, and especially for the low and medium s_1^H values.

In these two examples, we can find the same characteristics of the impact of the increase of the

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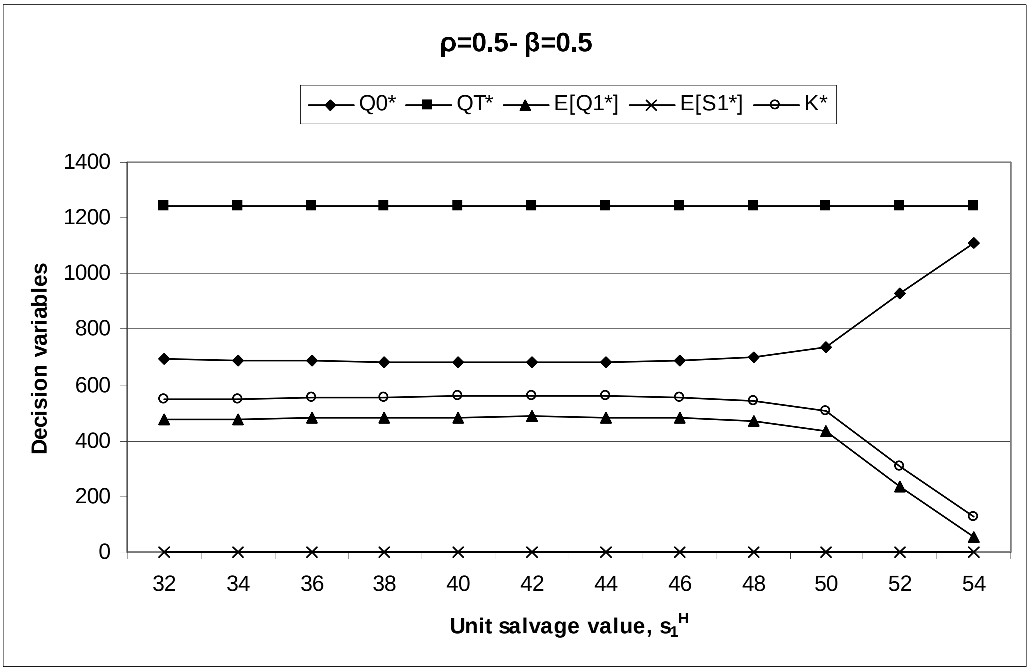

Figure 7.7: Effect of the high unit payback value s_1^H, with $\beta = 0.5$

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probability β on the oprimal policy and that have been described in section 7.5.3.

From Figure 7.9, the first important and rather intuitive issue that one can notice is the effect of the quality of the collected information on the expected optimal objective function. It is clear that the higher the quality of the information, the higher the expected optimal objective function. Note that in this example we have assumed that the probability $\beta = 0.1$, in order to emphasize on the effect of the higher value of the unit payback value, s_1^H. The second important issue is that the impact of the quality of the information on the expected optimal objective function increases when the unit payback value s_1^H increases. This is due to the fact that, for the high ρ values, when s_1^H increases, the increase in $E[S_1^*]$ is higher than that in the case of low ρ values.

7.6 Conclusion

In this chapter we have presented a new production/iventory model, for short life cycle products, or *newsboy* type products. During a single period selling horizon, a single product for which the stochastic demand is characterized by a probability density function, which is distributed jointly with an exogenous market information. The decision process is divided into two stages. In the first stage, two decisions are fixed: the first concerning a first ordered quantity, and a second decision concerning a capacity reservation. In the second stage two decisions are also fixed: the first one is relative to the use of the totality or of a part of the reserved capacity (purchased options) at the first decision stage. The second one represents the quantity returned (payback) to the supplier at the beginning of the selling horizon, from the quantity already ordered at the first stage. During the first stage, the exogenous information is stochastic. The exercise costs (of the options) and the unit value of the returned quantity at the second

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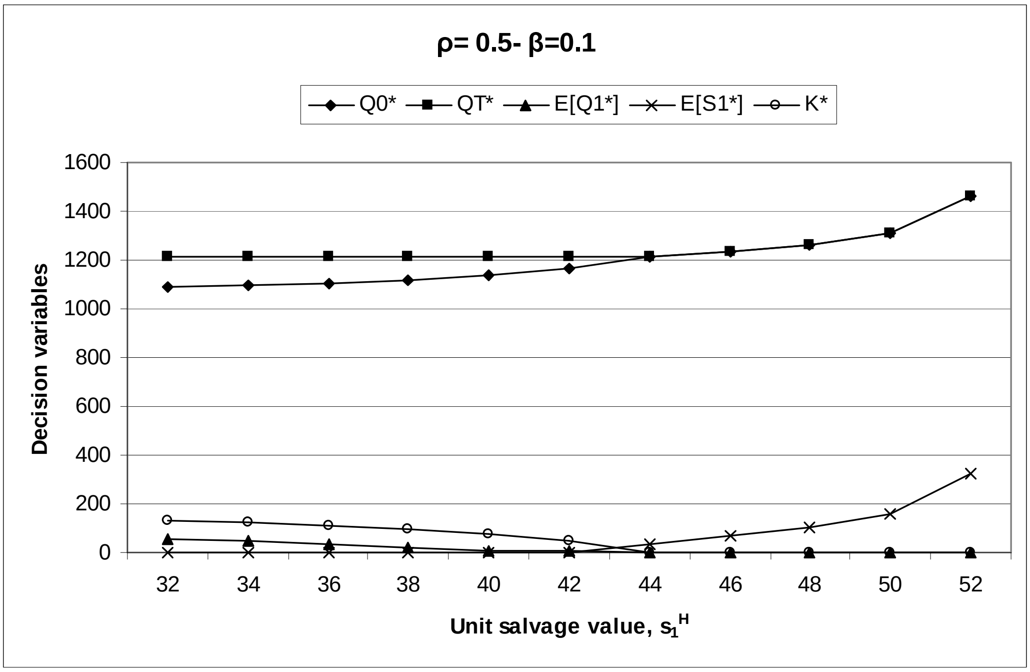

Figure 7.8: Effect of the high unit payback value s_1^H, with $\beta = 0.1$

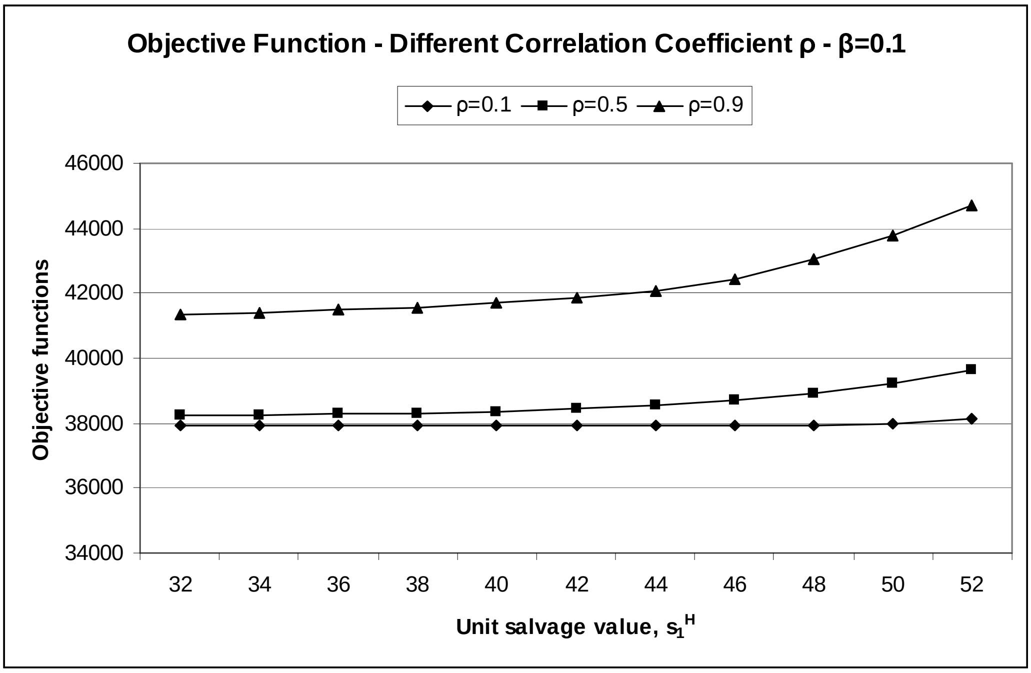

Figure 7.9: Effect of s_1^H on the optimal expected objective function, with different ρ

decision stage (payback) are also uncertain and each of them has two possible values. Between the two decision stages, the market information is collected and the parameters of the conditional distribution of the demand are known. The values of the exercise cost and of the unit return value are also known. At the end of the selling horizon, each remaining unit, if any, is salvaged and any unsatisfied order is lost. We have provided the optimal policy of the second decision stage and the optimality equations of the first decision stage. Since there is no closed-form analytical expression for the optimal policy, we have solved numerically the first decision stage. Then, via numerous numerical examples we have provided some insights on the optimal policy and the effect of the main model parameters on this policy. We have also solved analytically two particular cases: the first one is a model with worthless information and the second one is with perfect information.

Note that in this chapter we have analyzed a new type of contract from the point of view of the retailer. We have shown the increase of the expected optimal objective function of the retailer and of the total ordered quantity Q_T with the increase of the unit payback value s_1^H. In the perspectives of this chapter we can find the study of the impact of the increase of s_1^H on the optimal policy and the optimal expected objective function of the supplier, and as Q_T^* increases with s_1^H, we can assume that there exists a certain value s_1^{H*} that coordinates the channel and maximizes both the supplier and retailer expected objective functions.

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Chapter 8

Finite Horizon Dynamic Nonstationary Stochastic Inventory Problem with Two Production Modes: Near-Myopic Bounds

We model in this chapter the difference between procurement costs in a multiperiodic planning setting. We propose a model which is characterized by the possibility of fixing two orders at each period: the first order with a fast production mode, which permits an immediate delivery and the second order with a slow production mode, which has one period delivery delay. Clearly, the slow production mode is less expensive, and thus more attractive from this point of view. We develop a discounted backlog model, with proportional production, inventory holding and shortage costs. We allow all these costs to be period dependent. The demands are random variables with probability distribution functions that are independent and possibly different from one period to another. Since there is no closed-form optimal solution, then the main contribution of this chapter, which is the development of upper and lower bounds on the optimal decision variables, permits to find an approximation on the optimal solution. The approximated solutions are developed by exploiting the upper and lower bounds and by using an extension of a known heuristic. We also provide some numerical examples in order to test the efficiency of our approximations.

Keywords:production planning, inventory control, slow and fast production modes, myopic policies, upper and lower bounds.

8.1 Introduction

Nowadays, in the production planning and inventory control context, the competition between suppliers becomes rougher due to the globalization. Therefore, the gap between the correspondent procurement costs becomes more and more important essentially due to raw materials and workforce costs difference between the different countries. However, the procurement cost is connected, in general, to the delivery time. Indeed, in general, the higher the delivery time, the lower the unit ordering cost.

In the previous chapters we have modeled this difference in the procurement costs, in a two-period setting. In this chapter, we model this difference in a multi-periodic framework by allowing to order at each period of the planning horizon twice: the first time with a fast production mode, with an immediate delivery and the second time with a slow production mode, that has one period delivery delay. It is obvious that the slow production mode is less expensive and then more attractive. We develop a discounted backlog model, with proportional production, inventory holding and shortage costs which are period dependent. The demand distributions are independent and non-stationary.

(Whittemore and Saunders, 1977) characterized the optimal ordering policies under stochastic demand when two supply options (slow and fast) are available, with different costs and different delivery times. Using some assumptions on the holding-penalty cost functions, ordering costs and backlogging, they provided theoretical conditions under which the optimal solution consists in using exclusively the fast mode (respectively the slow mode). They provided explicit formulas defining the optimal slow mode ordering quantity.

(Sethi et al., 2001, 2005) developed a periodic review inventory model with fast and slow delivery modes and demand forecast updates. At the beginning of each period, on-hand inventory and demand information are updated. At the same time, decisions on how much to order using fast and slow delivery modes are made. Fast and slow orders are delivered, respectively, at the end of the current and the next periods. They proved existence of an optimal Markovian policy, corresponding to a modified base-stock policy. No analytical expression are available for the optimal solutions.

(Bensoussan et al., 1983) analyzed an inventory model with two supply modes, one instantaneous and the other with a one period lead time. They considered fixed and variable costs associated with ordering decisions. They obtained an optimal policy, which represents a generalization of the well-known (s, S) policy.

(Zhang, 1996) proposed a model of an inventory system with three different supply modes. This author exhibited, under some assumptions, the structure of the optimal ordering policy. He derived explicit formulas, valid in some particular cases, for the optimal order-up-to levels. For the general case no closed-form formulas are available, but some properties of the optimal solutions are discussed.

(Morton and Pentico, 1995) developed a methodology to cope with the problem of ordering once at each period of a finite horizon non-stationary inventory problem. They provided upper and lower bounds on the optimal orders at each period.

In the present chapter, we address a multi-periodic non-stationary production/ inventory model with random demands and fast and slow production modes (Cheaitou et al., b, 2007), for which it is well

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known that no analytical solutions are available. We thus focus on bounds and approximations for the optimal solution, in a similar way as in (Morton and Pentico, 1995).

Using the methodology provided in (Morton and Pentico, 1995), we first perform a formal cost transformation (exclusively for the fast production mode), which permits to replace the decision variable relative to the ordered quantity by a new decision variable relative to the order-up-to level. This transformation, which is well known in the literature (Veinott, 1965), permits to develop the bounds more simply, and independently of the initial inventory level of each planning period. Via this formal transformation, we exhibit an upper bound on the optimal required inventory level at the beginning of each period. In this process, we heuristically assume that, at each period, the demand of the next period will be satisfied exclusively with units ordered at the current period with the slow mode. This last assumption is numerically justified. Under this assumption, the fast mode is exclusively used in order to deliver the backlogged orders from the previous period. In other words, the fast production mode is used, as an *emergency* mode, in order to adjust the decisions on the slow production mode, after observing the inventory level at the beginning of the period. It is thus direct, using the same methodology as for the fast mode, to provide bounds for the optimal slow mode orders. The Near-Myopic heuristic given in (Morton and Pentico, 1995), that interpolates linearly between the stockout probabilities induced by the upper and lower bounds, is then used in our setting. A numerical analysis is provided in order to show some insights and to compare the solution given by the bounds, provided in this chapter, to the optimal solution given by stochastic programming.

We note that the main contribution of this chapter is the development of the upper and lower bounds on the optimal decision variables. The difference between this chapter and the model presented in (Morton and Pentico, 1995) is the introduction of two delivery modes, with different costs and different time delays, and the fact that we model the problem in a *backlog* framework instead of a *lost sales* one. In general, the production cost is lower for the slow mode. It is worth noting that we consider a non-stationary model, with time dependent parameters and backlogs.

The main difference between the work presented in this chapter and the work of (Whittemore and Saunders, 1977), is that we provide upper and lower bound on the optimal policy whatever are the model parameters. In some cases, the approximated values that we provide are optimal.

The remainder of this chapter is structured as follows. In section 8.2 we introduce the model and the parameters. In section 8.3 we develop upper and lower bounds for the decisions variables (namely the order-up-to-level for the fast mode and the order quantity for the slow mode). In section 8.4 we provide the heuristic used to find approximated values for the decision variables via the upper and lower bounds. In section 8.5, we provide a numerical analysis that shows the efficiency of our approximations. In section 8.6 we conclude and we give some perspectives.

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8.2 The Model

8.2.1 Model parameters

Consider a nonstationary production/inventory problem in which there are N time periods. At each period, a mono-product stochastic demand is defined by a probability density function with known parameters. The demands of the different periods are independent between each other, but are time-dependent. To satisfy the demand, one can order, at the beginning of each period, twice: a first quantity produced (delivered) with a fast production (delivery) mode, that is delivered immediately, and a second quantity, produced (delivered) with a slow mode, that is delivered at the beginning of the next period. The slow production mode is assumed to be less expensive than the fast way of ordering. At the end of each period, the unsatisfied demands are backlogged to be satisfied in next period, and a proportional penalty shortage cost is charged. The remaining inventory, if any, is kept to be used in the next period, and a proportional holding cost is charged. After the end of the last period the remaining inventory is salvaged. Let us define the following model parameters:

- D_t: the demand at period t, $t = 1, ..., N$,
- $F_t(D_t)$: the cumulative distribution function of the demand at period t, $t = 1, ..., N$,
- Q_t: the quantity ordered at the beginning of period t and received immediately, $t = 1, ..., N$,
- $Q_{t,t+1}$: the quantity ordered at the beginning of period t received at the beginning of period $t + 1$, $t = 1, ..., N - 1$,
- x_t: the initial inventory level at period t (after reception of $Q_{t-1,t}$ and before ordering Q_t), $t = 1, ..., N$,
- $y_t(\geq x_t)$: the "physical" inventory level, at the beginning of period t, after ordering and reception of the quantity Q_t, $t = 1, ..., N$, namely we have

$$y_t = x_t + Q_t,$$

- $\tilde{c}_t$: the proportional unit order cost of the fast mode (Q_t), $t = 1, ..., N$,
- $\tilde{c}_{t,t+1}$: the proportional unit order cost of the slow mode ($Q_{t,t+1}$), $t = 1, ..., N - 1$,
- $\tilde{b}_t$: the unit penalty shortage cost at period t, $t = 1, ..., N$,
- $\tilde{h}_t$: the unit inventory holding cost at period t, $t = 1, ..., N$,
- $\tilde{s}_{N+1}$: the unit salvage value at the end of the planning horizon (after the end of period N),
- α: the discount factor at each period.

Note that our model is a dynamic one. Indeed, the cost parameters are period dependent.

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The decision process of the model is then described as follows: the initial inventory level x_t at the beginning of period t, is a function of the previous period parameters as follows

$$x_t = y_{t-1} - D_{t-1} + Q_{t-1,t}. \tag{8.1}$$

The decision variables Q_t and $Q_{t,t+1}$ are then fixed. The ordered quantity Q_t is then received and the new inventory level y_t is as follows

$$y_t = x_t + Q_t. \tag{8.2}$$

The demand D_t occurs and the inventory level at the end of period t is then equal to

$$y_t - D_t.$$

For a given on-hand inventory x, define the expected inventory holding and penalty shortage cost in period t, for $t = 1, ..., N$, by the following expression

$$\tilde{L}_t(x) = \tilde{h}_t E_t\left[(x - D_t)^+\right] + \tilde{b}_t E_t\left[(D_t - x)^+\right], \quad t \leq N, \tag{8.3}$$

where the first term in (8.3) is the expected inventory holding cost and the second term is penalty shortage cost, with $E_t[\cdot]$ is the expectation with respect to the random demand D_t, and $(a)^+ = \max(a, 0)$.

Since we assume that after the end of the planning horizon, the remaining inventory, if any, is salvaged, then the expected cost function of period $N+1$ (a fictitious period that represents the time that follows the end of the planning horizon) is given by

$$\tilde{L}_{N+1}(x) = -\tilde{s}_{N+1}x, \tag{8.4}$$

For each x, the derivative of the expected inventory holding and penalty shortage cost function, $\tilde{L}_t(x)$, with respect to x is given by

$$\tilde{L}'_t(x) = \tilde{h}_t F_t(x) - \tilde{b}_t\left[1 - F_t(x)\right]. \tag{8.5}$$

Define $\tilde{\Pi}_t(x_t)$ as the expected cost function from period t to the end of the horizon when entering period t with initial inventory level of x_t, and when pursuing an optimal policy from period $t+1$ to the end of the horizon. Define $\tilde{\Pi}_t^{\,*}(x_t)$ as the value of the expected cost function $\tilde{\Pi}_t(x_t)$ when pursuing an optimal policy in period t. Then

$$\begin{aligned}\tilde{\Pi}_t^*(x_t) = \quad & \min_{Q_t, Q_{t,t+1} \geq 0} \Big[\tilde{c}_t Q_t + \tilde{c}_{t,t+1} Q_{t,t+1} + \tilde{L}_t(x_t + Q_t) \\ & + \alpha \int_0^\infty \tilde{\Pi}_{t+1}^*(x_t + Q_t + Q_{t,t+1} - D_t) dF_t(D_t)\Big], t \leq N-1, \end{aligned} \tag{8.6}$$

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where $\tilde{c}_t Q_t$ represents the total ordering cost with the fast production mode in period t, $\tilde{c}_{t,t+1} Q_{t,t+1}$ is the total ordering cost with the slow production mode in period t, $\tilde{L}_t(x_t + Q_t)$ is the expected inventory holding and penalty shortage costs defined in (8.3), and

$$\alpha \int_0^\infty \tilde{\Pi}^*_{t+1}(x_t + Q_t + Q_{t,t+1} - D_t) dF_t(D_t)$$

represents the the discounted expected optimal objective function, from period $t+1$ until the end of the planning horizon, entering period $t+1$ with an initial inventory level of $x_t + Q_t + Q_{t,t+1} - D_t$.

The optimal cost function for period N, is then given by the following equation

$$\tilde{\Pi}^*_N(x_N) = \min_{Q_N \geq 0} \left[\tilde{c}_N Q_N + \tilde{L}_N(x_N + Q_N) + \alpha \int_0^\infty \tilde{\Pi}^*_{N+1}(x_N + Q_N - D_N) dF_N(D_N) \right], \tag{8.7}$$

with

$$\tilde{\Pi}^*_{N+1}(x_{N+1}) = \tilde{L}_{N+1}(x_{N+1}) = -\tilde{s}_{N+1} x^+_{N+1}. \tag{8.8}$$

8.2.2 Model costs transformation

In order to develop the upper and the lower bounds on the optimal values of our model decision variables, we will proceed a cost transformation that has been introduced in (Veinott, 1965). The aim of this transformation is to replace the decision variable Q_t, the quantity ordered with the fast production mode, by y_t, the inventory level at the beginning of period t, in such a way that y_t becomes the new decision variable instead of Q_t. As it can be easily seen from (8.2), one has $y_t = x_t + Q_t$, which means that for a given initial inventory level x_t, it is equivalent to decide of the ordered quantity Q_t or of the order-up-to level y_t. Since the optimal value of the decision variable Q_t depends on the value of the initial inventory, x_t, which implies the dependency on the previous period parameters, then the change in the decision variables decouples the problem. Indeed, the order-up-to level does not depend normally on the previous periods parameters. This decoupling makes the near myopic nature of the problem more apparent.

The cost is not performed for the decision variable relative to the slow production mode $Q_{t,t+1}$. This is due to the fact that the decision variables y_t and $Q_{t,t+1}$ are fixed at the same moment, which means that once the decision variable y_t is decided, then the decision variable $Q_{t,t+1}$ can be fixed independently of the previous periods parameters.

We rewrite equations (8.6), (8.7) and (8.8) in another way. These equations are then respectively equivalent to the following equations

$$\begin{aligned} \tilde{\Pi}^*_t(x_t) = & \min_{y_t \geq x_t, Q_{t,t+1} \geq 0} \Big[\tilde{c}_t(y_t - x_t) + \tilde{c}_{t,t+1} Q_{t,t+1} + \tilde{L}_t(y_t) \\ & + \alpha \int_0^\infty \tilde{\Pi}^*_{t+1}(y_t + Q_{t,t+1} - D_t) dF_t(D_t) \Big], t \leq N-1, \end{aligned} \tag{8.9}$$

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$$\tilde{\Pi}_N^*(x_N) = \min_{y_N \geq 0} \left[\tilde{c}_N(y_N - x_N) + \tilde{L}_N(y_N) + \alpha \int_0^\infty \tilde{\Pi}_{N+1}^*(y_N - D_N) dF_N(D_N) \right], \tag{8.10}$$

and

$$\tilde{\Pi}_{N+1}^*(x_{N+1}) = \tilde{L}_{N+1}(y_N - D_N)^+ = -\tilde{s}_{N+1}(y_N - D_N)^+. \tag{8.11}$$

The cost transformation is made first by assuming that the new minimum cost function is equal to $\Pi_t^*(x_t) = \tilde{c}_t x_t + \tilde{\Pi}_t^*(x_t)$. This assumption managerially means that the initial inventory x_t in period t, has a value of $\tilde{c}_t x_t$, which should be (and now is) charged to period t (and is credited back to period $t-1$, after discounting, as a transfer). Thus, the available inventory in a period, is judged no differently whether it comes from previous periods or it is ordered at the present period.

Add then $\tilde{c}_t x_t$ to both sides of (8.9) and (8.10), and replace $\tilde{\Pi}_t^*(x_t) + \tilde{c}_t x_t$ by $\Pi_t^*(x_t)$ and $\tilde{\Pi}_{t+1}^*(y_t + Q_{t,t+1} - D_t)$ by $-\tilde{c}_{t+1}(y_t + Q_{t,t+1} - D_t) + \Pi_{t+1}^*(y_t + Q_{t,t+1} - D_t)$. The transformation is completed by noting the following identity

$$E_t[(x - D_t)^+] - E_t[(D_t - x)^+] + E_t[D_t] = x, \tag{8.12}$$

that is used to decompose y_t. After arranging the terms one gets

$$\begin{aligned} \Pi_t^*(x_t) = & \min_{y_t \geq x_t, Q_{t,t+1} \geq 0} \Big[L_t(y_t) + (\tilde{c}_{t,t+1} - \alpha \tilde{c}_{t+1}) Q_{t,t+1} \\ & + \alpha \int_0^\infty \Pi_{t+1}^*(y_t + Q_{t,t+1} - D_t) dF_t(D_t) \Big], \quad t \leq N-1. \end{aligned} \tag{8.13}$$

The new optimal cost function, relative to period N, becomes as follows

$$\Pi_N^*(x_N) = \min_{y_N \geq x_N} \left[L_N(y_N) + \alpha \int_0^\infty \Pi_{N+1}^*(y_N - D_N) dF_N(D_N) \right], \tag{8.14}$$

with

$$\Pi_{N+1}^*(x_{N+1}) = L_{N+1}(x_{N+1}) = \nu x_{N+1}^+ = \nu (y_N - D_N)^+, \tag{8.15}$$

and

$$L_t(x) = h_t E_t\left[(x - D_t)^+\right] + b_t E_t\left[(D_t - x)^+\right] + \tilde{c}_t E_t[D_t], \quad t = 1, ..., N, \tag{8.16}$$

with the following new costs

$$b_t = \tilde{b}_t - \tilde{c}_t + \alpha \tilde{c}_{t+1}, \quad h_t = \tilde{h}_t + \tilde{c}_t - \alpha \tilde{c}_{t+1}, \quad \text{with } t = 1, ..., N-1, \tag{8.17}$$

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and

$$\nu = -\tilde{s}_{N+1}, \quad b_N = \tilde{b}_N - \tilde{c}_N, \quad h_N = \tilde{h}_N + \tilde{c}_N. \tag{8.18}$$

8.3 Near-Myopic bounds

In this section, we use the new cost functions defined in the previous section in order to develop upper and lower bounds on the optimal new decision variables, namely y_t and $Q_{t,t+1}$. We begin by providing an upper bound on the decision variable y_t, which is then used in the development of the upper bound on the decision variable $Q_{t,t+1}$. Then, in order to develop the lower bounds on the optimal decision variables, and since we do not have the values of these latter, we replace them (the optimal decision variables) by their corresponding upper bounds.

The new expected cost functions that correspond to the new cost parameters, defined in (8.17) and (8.2.2), are then the following

$$\Pi_t(x_t, y_t, Q_{t,t+1}) = \left[L_t(y_t) + (\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1})Q_{t,t+1} + \alpha \int_0^\infty \Pi^*_{t+1}(y_t + Q_{t,t+1} - D_t)dF_t(D_t)\right], \tag{8.19}$$

where $t = 1, ..., N-1$, and

$$\Pi_N(x_N, y_N) = \left[L_N(y_N) + \alpha \int_0^\infty \Pi^*_{N+1}(y_N - D_N)dF_N(D_N)\right]. \tag{8.20}$$

As we have mentioned above, the expected objective function $\Pi_t(x_t, y_t, Q_{t,t+1})$ represents the expected cost from period t to the end of the horizon, while pursuing an optimal policy from period $t+1$ to the end of the horizon.

Using the property that L_t and Π^*_t are known to be convex function (Morton and Pentico, 1995), we can deduce that $\Pi_t(\cdot)$ is a convex function also (Heyman and Sobel, 1984). Then to optimize $\Pi_t(\cdot)$, one can use the first order optimality criterion.

The first order partial derivatives of $\Pi_t(x_t, y_t, Q_{t,t+1})$ with respect to y_t and $Q_{t,t+1}$ are given by the following equations

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial y_t} = L'_t(y_t) + \alpha \int_0^\infty \frac{\partial \Pi^*_{t+1}(Q_{t,t+1} + y_t - D_t)}{\partial y_t} dF_t(D_t), t = 1, ..., N-1, \tag{8.21}$$

and

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial Q_{t,t+1}} = \tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha \int_0^\infty \frac{\partial \Pi^*_{t+1}(Q_{t,t+1} + y_t - D_t)}{\partial Q_{t,t+1}} dF_t(D_t), \tag{8.22}$$

where $t = 1, ..., N-1$.

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The partial derivative of $\Pi_N(x_N, y_N)$ with respect to y_N is defined as follows

$$\frac{\partial \Pi_N(x_N, y_N)}{\partial y_t} = L'_N(y_N) + \alpha \int_0^\infty \frac{\partial \Pi^*_{N+1}(y_N - D_N)}{\partial y_N} dF_t(D_N). \tag{8.23}$$

Then, using the convexity property of $g(\cdot)$, the optimality equations that the decision variables y_t and $Q_{t,t+1}$ must satisfy are given in the following equations

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial y_t}(y_t^*, Q^*_{t,t+1}) = 0, \quad t = 1, ..., N-1, \tag{8.24}$$

and

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial Q_{t,t+1}}(y_t^*, Q^*_{t,t+1}) = 0 \quad t = 1, ..., N-1, \tag{8.25}$$

where y_t^* and $Q^*_{t,t+1}$ are the optimal values of the decision variables y_t and $Q_{t,t+1}$ respectively.

Theorem 8.1 *When the optimal decision variable $Q^*_{t,t+1}$ is positive, then the optimal decision variable y_t^* is completely characterized by the threshold level y_t^{Max}, given by*

$$y_t^{Max} = F_t^{-1}\left(\frac{c_{t,t+1} - \alpha c_t + b_t}{b_t + h_t}\right), t = 1, ..., N-1. \tag{8.26}$$

where one has

$$y_t^* = \max\left(x_t; y_t^{Max}\right). \tag{8.27}$$

Proof. See Appendix E.1. □

Note that (8.26) constitutes a modified *Newsvendor* formula. In the expression of y_t^{Max}, the underage cost is equal to $c_{t,t+1} - \alpha c_t + b_t$, which is equivalent to marginal cost of backlogging a demand from period t to be satisfied in period $t+1$ with a unit ordered with $Q_{t,t+1}$. The overage cost is equal to $\alpha c_t - c_{t,t+1} + h_t$ which is equal to the marginal cost of ordering a unit with the fast mode, and keeping it in stock for period $t+1$, instead of ordering it with the slow mode.

On the one hand, in the case where $Q^*_{t,t+1} = 0$, it is impossible to characterize with a closed-form expression the optimal decision variable y_t^*. On the other hand, in the case where $Q^*_{t,t+1} > 0$, and even if we have shown that y_t^* is completely characterized, the optimal decision variable $Q^*_{t,t+1}$ can not be characterized via a closed-form formula. For these reasons, it is obvious that one needs to develop approximated solutions that allow to provide good approximated values of the optimal decision variables under all conditions. In the following sections we provide closed-form formulas that define upper and lower bound on y_t^* and $Q^*_{t,t+1}$.

8.3.1 Upper bound on y_t^*

In this section we develop the upper bound on y_t using the same approach provided in (Morton and Pentico, 1995).

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Theorem 8.2 *In an N-period problem, with regularity condition $b_i > \alpha b_{i+1} - h_i$, the following term constitutes an upper bound on the optimal decision variable y_t*

$$y^{*m}_{t,N} = F^{-1}_{t,N}\left[\frac{b_t}{b_t + \sum_{k=t}^{N} h_k \alpha^{k-t} + \alpha^{N-t+1}\nu}\right]. \tag{8.28}$$

Proof. See Appendix E.2. □

The idea here can be explained as follows. If all demand distributions are convoluted and moved to the first period, then a percentile of the convoluted demand distribution constitutes an upper bound on the optimal order-up-to-level of that period. The order-up-to-levels for remaining periods go to zero. This means that, to satisfy the demands of all the remaining periods, from period t to period N, we order a single quantity at period t. On the other hand, after the end of period t, each remaining unit that is still in stock, will not be used until the end of the planning horizon and therefore a corresponding inventory holding cost will be paid during each of the remaining planning periods.

The formula shown in (8.28) may be interpreted as a modified News-vendor formula, with an underage cost of b_t and and overage cost of

$$\sum_{k=t}^{N} h_k \alpha^{k-t} + \alpha^{N-t+1}\nu. \tag{8.29}$$

Note that b_t includes the unit shortage penalty cost (backlog cost $\tilde{b}_t$) and the cost difference between satisfying the demand with units ordered in period t, with the fast mode ($\tilde{c}_t$) and satisfying the demand with units ordered in period $t+1$ with the fast mode ($\alpha\tilde{c}_{t+1}$) (see (8.17)).

The overage cost, is equal to the marginal discounted inventory holding cost of a unit ordered at period t, and kept in stock until the end of the planning horizon.

Proposition 1 Let us define

$$y^{*m}_{t,j} = F^{-1}_{t,j}\left[\frac{b_j}{b_j + \sum_{k=t}^{j} h_k \alpha^{k-t}}\right], \qquad t \leq j < N. \tag{8.30}$$

Then $y^{*m}_{t,j}$, $t \leq j < N$, is also an upper bound on the optimal decision variable y^*_t.

Proof. $y^{*m}_{t,j}$ can be calculated by considering a $j+1-t$ period problem with zero salvage value at the end of period j and by using the same approach that has been used to develop the upper bound given in (E.12). □

Proposition 2 Define the k-myopic policy as the tightest of the bounds given in (8.30)

$$y^{*km}_t = min_{t \leq j < N}(y^{*m}_{t,j}). \tag{8.31}$$

Then the k-myopic policy is an upper bound on the ordering policy for the fast mode.

Define y^U_t as the tightest upper bound on the optimal decision variable y^*_t. This tightest upper bound

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is then the minimum between $y^{*m}_{t,N}$ and y^{*km}_t given by

$$y^U_t = \min\left(y^{*m}_{t,N}, y^{*km}_t\right). \tag{8.32}$$

8.3.2 Upper bounds on $Q^*_{t,t+1}$

After having determined the upper bound on y_t, we will provide in this section the upper bound on the quantity ordered with the slow production mode $Q_{t,t+1}$.

Rewrite the first order optimality equation with respect to the decision variable $Q_{t,t+1}$, given in equation (8.25), in the following way

$$\begin{aligned}
&\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial Q_{t,t+1}}(x_t, y^*_t, Q^*_{t,t+1}) \\
&= \tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha \int_0^\infty \frac{\partial \Pi^*_{t+1}(Q^*_{t,t+1} + y^*_t - D_t)}{\partial Q_{t,t+1}} dF_t(D_t) \\
&= \tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha \int_0^\infty L'_{t+1}(y^*_t + Q^*_{t,t+1} - D_t) dF_t(D_t) \\
&+\alpha \int_0^\infty \left[\alpha \int_0^\infty \frac{\partial \Pi^*_{t+2}}{\partial Q_{t,t+1}}\left(Q^*_{t,t+1} + y^*_t + Q^*_{t+1,t+2} - D_t - D_{t+1}\right) dF_{t+1}\right] dF_t = 0.
\end{aligned} \tag{8.33}$$

Assumption: We assume that $c_{t,t+1} < \alpha c_{t+1}$. Therefore, ordering with the slow mode ($Q_{t,t+1}$) in order to satisfy the demand D_{t+1} becomes more attractive than ordering with the fast mode (Q_t). If this assumption is not satisfied, then the optimal decision variable $Q^*_{t,t+1}$ will be equal to zero, and therefore the slow production mode will not be used. Thus, in order to develop an upper bound on $Q_{t,t+1}$ we will assume that all the demand of period $t+1$ is satisfied with x_{t+1} ($Q^*_{t,t+1}$), and then $Q^*_{t+1} = 0$.

At the moment when the decision variable $Q_{t,t+1}$ is fixed, the realized value of the demand of period t, namely D_t, is unknown (D_t is still a random variable). Due to that fact, no closed formula may be obtained for the upper bound on the optimal value of the decision variable $Q_{t,t+1}$. Therefore, in order to simplify the development of the upper bound of $Q^*_{t,t+1}$ we will replace in (8.33), the demand D_t of period t by its mean, namely μ_t. Note that, we have performed many numerical examples to show the impact of replacing the stochastic demand D_t by its mean. In these examples, we have calculated the upper bound on $Q^*_{t,t+1}$ by two ways: in the first way, we have taken into account the stochastic demand D_t as a random variable. In the second way, we have replaced the stochastic demand D_t by the mean value μ_t. Then we have compared the two upper bounds obtained from the two different methods. The results showed that the values of the upper bound, in both cases, are always the same. This result can be interpreted by the fact that $Q_{t,t+1}$ is mainly ordered to satisfy the demand of period $t+1$, and not the backlogged demands of period t.

Theorem 8.3 *An upper bound on the optimal value of the decision variable $Q_{t,t+1}$ is defined as follows*

$$\tilde{Q}_{t,t+1} = \left(F^{-1}_{t+1,N}\left[\frac{\alpha\tilde{c}_{t+1} - \tilde{c}_{t,t+1} + \alpha b_{t+1}}{\left(\sum_{j=t+1}^{N} h_j \alpha^{j-t} + \alpha^{N-t+1}\nu\right) + \alpha b_{t+1}}\right] - y^*_t + \mu_t\right)^+. \tag{8.34}$$

Proof. See Appendix E.3. □

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To calculate the upper bound on $Q_{t,t+1}$, namely $\tilde{Q}_{t,t+1}$, and since we can not determine the optimal y_t^*, we use the upper bound (or a combination of the upper bound and the lower bound provided in the following sections) instead of y_t^* in (E.21).

Proposition 3 Let us define

$$\tilde{Q}^k_{t,t+1} = \left(F^{-1}_{t+1,k} \left[\frac{\alpha \tilde{c}_{t+1} - \tilde{c}_{t,t+1} + \alpha b_{t+1}}{\left(\sum_{j=t+1}^{k} h_j \alpha^{j-t}\right) + \alpha b_{t+1}} \right] - y_t^* + \mu_t \right)^+, \quad t+1 \leq k < N. \tag{8.35}$$

Like the k-myopic bound provided on the optimal order-up-to-level at the beginning of each period, equation (8.35) provides also an upper bound on the optimal quantity ordered with the slow mode. It could be easily seen that $\tilde{Q}^k_{t,t+1}, \quad t \leq k < N$, is also an upper bound on the optimal decision variable $Q^*_{t,t+1}$. This upper bound can be calculated by considering a $k-t$ period problem with zero salvage penalty at the end of period k.

Define the k-myopic policy for the slow production mode, as the tightest of the bounds defined in (8.35)

$$Q^{*km}_{t,t+1} = min_{t+1 \leq j < N} (\tilde{Q}^j_{t,t+1}). \tag{8.36}$$

Then $Q^{*km}_{t,t+1}$ constitutes an upper bound on the ordering policy for the slow production mode.

To determine the tightest upper bound on $Q^*_{t,t+1}$, $Q^{*U}_{t,t+1}$, we take the minimum between $\tilde{Q}_{t,t+1}$ and $Q^{*km}_{t,t+1}$

$$Q^{*U}_{t,t+1} = \min\left(\tilde{Q}_{t,t+1}, Q^{*km}_{t,t+1}\right). \tag{8.37}$$

8.3.3 Lower bound on y_t^*

In order to confine the optimal decision variable y_t^* by using the upper bound developed above, we need a lower bound on that optimal decision variable, that we will provide in this section.

As we have done for the upper bounds, in order to develop the lower bound we will write a near-myopic equation that considers only the cases where units ordered at period t are used at periods $t+1$, $t+2$,..., N. This consideration will help us to develop lower bounds on the decision variables $y_t, \quad t = 1, ..., N$.

Theorem 8.4 *The following term constitutes a lower bound on the optimal decision variable* y_t^*

$$y_t^{*L} = F_t^{-1} \left[\frac{b_t - T_t^N - \nu P_t^N}{h_t + b_t} \right], \tag{8.38}$$

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where

$$P_t^N = \alpha^{N-t+1} P_{t,N} \quad \textit{and} \quad T_t^N = \sum_{j=t}^{N-1} \alpha^{j-t+1} P_{t,j} h_{j+1}, \tag{8.39}$$

with

$$\begin{aligned} P_{t,j} &= P_{t,j}(y_t^*, y_{t+1}^*, ..., y_{j+1}^*) \\ &= \left(\prod_{k=t+1}^{j+1} \int_0^{(y_t^* - D_{t,k-2} - y_{j+1}^*)^+} \right) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t), \end{aligned} \tag{8.40}$$

Proof. See Appendix E.4. □

In Theorem 8.4, $P_{t,j}$ is the probability that any unit will be ordered from period $t+1$ to period $j+1$. P_t^N represents the discounted probability that there is no order until the end of the horizon. T_t^N is the discounted partial expectation of the time from period $t+1$ until the first order in a period ($\geq t+1$), times the unit holding cost. This is equivalent to the discounted partial expectation of holding cost of a unit ordered at period t until the period ($\geq t+1$) where a first order is passed.

8.3.4 Lower bound on $Q_{t,t+1}^*$

In this section we develop the lower bound on the optimal value of the decision variable relative to the slow production mode, $Q_{t,t+1}$.

As we have done in the development of the upper bound on $Q_{t,t+1}$ we will assume that it is worth satisfying the demand D_{t+1} with $Q_{t,t+1}$ than with Q_{t+1}. This assumption is justified by the fact that the ordering cost of the slow mode is lower than that of the fast mode, and by many numerical examples that we have performed and that have shown the optimality of that assumption in the quasi totality of cases. Therefore, in order to develop a lower bound on the optimal decision variable $Q_{t,t+1}^*$, we assume that all the demands of period $t+1$ are satisfied with units ordered with the slow production mode ($Q_{t,t+1}$), and consequently we have $Q_{t+1}^* = 0$.

Theorem 8.5 *$Q_{t,t+1}^{*L}$, shown in (8.41), defines a lower bound on the optimal decision variable $Q_{t,t+1}^*$*

$$Q_{t,t+1}^{*L} = \left(F_{t+1}^{-1} \left[\frac{\alpha \tilde{c}_{t+1} - \tilde{c}_{t,t+1} - T_{t+1}^{'N} - P_t^{'N} \nu + \alpha b_{t+1}}{\alpha (h_{t+1} + b_{t+1})} \right] + \mu_t - y_t^* \right)^+, \tag{8.41}$$

where

$$T_{t+1}^{'N} = \sum_{j=t+1}^{N-1} \alpha^{j+1-t} P'_{t,j} h_{j+1} \quad \textit{and} \quad P_t^{'N} = \alpha^{N+1-t} P'_{t,N}, \tag{8.42}$$

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with

$$\begin{aligned} P'_{t,j} &= P'_{t,j}(Q^*_{t,t+1}, y^*_t, y^*_{t+1}, ..., y^*_{j+1}) \\ &= \left(\prod_{k=t+1}^{j} \int_0^{(Q^*_{t,t+1}+y^*_t-\mu_t-D_{t+1,k-1}-y^*_{j+1})^+} \right) dF_j(D_j)...dF_{t+1}(D_{t+1}), \end{aligned} \quad (8.43)$$

Proof. See Appendix E.5. □

In Theorem 8.5, $P'_{t,j}$ represents the probability that no order is placed in periods $t+1$ to $j+1$, or in other term, $P'_{t,j}$ represents the probability that the expected available inventory at the beginning of period $t+1$ is sufficient to satisfy all the demands from period $t+1$ until period $j+1$. P'^N_t is then the discounted probability that there is no order until the end of the horizon and, therefore, that there will be a certain inventory left to be salvaged in period $N+1$. T'^N_{t+1} represents the marginal discounted expectation of the inventory holding cost of a unit available at the beginning of period $t+1$ and left until the end of the horizon.

8.4 Heuristic to provide the approximated value of the optimal decision variables

In this section, we provide a heuristic that allows us to calculate an approximated solution for our problem. This heuristic is based on the heuristic developed by (Morton and Pentico, 1995) and constitutes a possible way to exploit the obtained upper and lower bounds.

We begin by using propositions 1 , 2 and 3 in order to find a tight upper bound on the optimal value of each of the two decision variables, y_t and $Q_{t,t+1}$. We calculate then the lower bounds on the optimal values of the decision variables, using (8.38) and (8.41). We interpolate then, for each optimal decision variable, between the upper and lower bound, in order to minimize the shortage probabilities. The approximated value of each of the optimal decision variables, interpolates between the shortage probabilities implied by its relative upper and lower bounds.

The approximated value of the optimal order-up-to-level y_t, is given as follows

$$y^{*A}_t = F^{-1}_t \left[AF_t(y^{*L}_t) + (1-A)F^{-1}_t(y^U_t) \right]. \quad (8.44)$$

The approximated value of the optimal decision variable $Q^*_{t,t+1}$ is given by the following equation

$$Q^{*B}_{t,t+1} = F^{-1}_{t+1} \left[BF_{t+1}(Q^{*L}_{t,t+1}) + (1-B)F^{-1}_{t+1}(Q^{*U}_{t,t+1}) \right], \quad (8.45)$$

with

$$0 \leq A \leq 1 \text{ and } 0 \leq B \leq 1. \quad (8.46)$$

Note that we have used the value found by (Morton and Pentico, 1995) for the interpolation coefficient

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A that was 0.25. The authors have performed a pilot study permitting them to identify this value for the interpolation parameter. For the interpolation coefficient B, of the slow production mode, we have performed a small pilot study, which shows that the best value is 0.1. Therefore, we have adopted this value for our numerical examples.

Note that performing a pilot study can be a good extension of our work and can complete it, in order to find the best value of the interpolation coefficient B. That permits to exploit the developed upper and lower bounds in the best way, permitting to find the best approximated value of the decision variables.

8.5 Numerical examples

We illustrate the effectiveness of the proposed approximations via some numerical examples. We have solved optimally these examples using a stochastic programming approach, then, using our approach, we have obtained the approximated solutions. Therefore we have compared the optimal solution with the approximated solution.

Note that for these numerical applications, we classically assume that the demand of period t has a truncated-normal distribution $N[\mu_t, \sigma_t]$, with a mean of μ_t and a standard deviation of σ_t which is restricted to only positive values. For the optimal solutions, obtained with the stochastic optimization approach, the demand distributions have been transformed into discrete distributions.

8.5.1 Nominal numerical data

We first provide the nominal numerical data that constitutes a base for all the numerical analysis section. Those numerical data are the following: $D_1 \sim N[1000; 300]$, $D_2 \sim N[1000; 300]$, $D_3 \sim N[1000; 300]$, $\tilde{h}_1 = \tilde{h}_2 = \tilde{h}_3 = 5$, $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 60$, $\tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 = 50$, $\tilde{c}_{12} = \tilde{c}_{23} = 30$, $s_4 = 15$, $\alpha = 1$.

Note that the exponential increase in the size of the stochastic program in terms of the number of periods, has limited our study to three planning periods.

This example serves as a reference to show the effect of the parameters of the model on the quality of the approximation.

8.5.2 Results

In this section we use the numerical data defined above in order to provide comparisons between the optimal solution and the approximated solution given by our approach, for the first period of the three-period problem.

It is worth noting that, since the stochastic optimization is a sequential approach, then using this approach does not permit to get a single optimal value for the decision variables of the second and third periods. Indeed, the optimal values of the second stage (period) decision variables depend on the value of the first period realized demand. The optimal values of the third period decision variables, depend on the first and second period realized demands. For this reason, we compare only the optimal and approximated values of the decision variables of the first period. In our point of view, we see that this comparison is sufficient, since one can apply this approach in a rolling horizon framework.

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The new value of the changed parameter(s)	Relative error on y_1^*, %	Relative error on Q_{12}^*, %
Nominal numerical data	0	-3.55
$\tilde{c}_2$=60	0	-1.57
$\tilde{c}_2$=70	0	-1.08
$\tilde{c}_1$=60	0	5.49
$\tilde{c}_1=\tilde{c}_2=\tilde{c}_3$=70	0	6.03
$\tilde{s}_4$=0	0	1.83
σ_3=600	0	-2.85
$\tilde{c}_1$=70	0	3.44
$\tilde{c}_3$=60	0	-2.92
$\tilde{c}_1$=60, $\tilde{c}_3$=70	0	2.55
$\tilde{b}_1=\tilde{b}_2=\tilde{b}_3$=40	0	0.27
c_{12}=90	0.9	0
c_{12}=90, $\sigma_1 = 600$	0	0

Table 8.1: Relative Error on the Decision Variables of the First Period

Table 1 shows some numerical examples based on the nominal numerical data defined in the previous section, where for each example (each line in the table) we have changed one or more of the nominal numerical data. The changed parameter(s) of each line is given in the first column of that line. In the second column, we give the relative difference between the optimal value of the decision variable y_1 obtained via stochastic optimization and the approximated value obtained by our approximation. In the last column we show the same relative difference for the decision variable Q_{12}.

From Table (1), we can easily see that the relative difference between the optimal values of the decision variables and their approximated values, is in the most of cases, very low. The relative difference between the optimal value of the order-up-to-level y_1 and the approximated value of y_1, is always very low. This low difference is due to the fact that in the cases where the optimal value of Q_{12} is positive, the solution for y_t, provided by the approximation, is optimal. In the case where the optimal value of the decision variable Q_{12} is equal to zero (the case where $c_{12} = 90$), we can see that the approximated value of y_t is still quite close to the optimal value.

What we should improve is surely the choice of the interpolation coefficient B, relative to the slow production mode, given in (8.46). This improvement, could participate in the improvement of the quality of the approximated solution of the slow production mode.

8.6 Conclusion

In this chapter we have developed a production and inventory model in which two production modes are possible. The first is a fast mode, that permits an immediate delivery with a higher cost, and the second is a slow mode with one period delivery delay and lower cost. In each period the unsatisfied demands are backlogged to be satisfied in the following period. At the end of each period, an underage and an overage costs are charged in terms of the state variable that represents the inventory level. At the end of the planning horizon, any remaining quantity is salvaged at a certain salvage value. We have provided closed-form expressions for the upper and lower bounds on the optimal values of the decision

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variables, and then using a heuristic, we have exploited these bounds in order to develop an approximated value of each of the optimal decision variables. The numerical results were satisfying. It is obvious that the first improvement that should be brought to the model is the procedure with which we choose the interpolation coefficient (between the upper and lower bounds) especially for the slow production mode. The other improvement should be the fact that we could take into account a certain information updates that permit to improve the knowledge of the demand distribution.

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Chapter 9

Conclusions and Perspectives

In this chapter, we give general concluding remarks and we present directions for future research. For further details, we refer the reader to the concluding sections of the previous chapters.

9.1 Conclusion

The production/inventory planning process is a crucial activity in the industrial world with a considerable economic value. This activity permits to the different actors of the Supply Chain to satisfy the demands of their respective customers in the best delays and with minimum costs. The demand is the most important uncertainty source in the Supply Chain. Therefore, the production/inventory planning permits to minimize its negative impact.

In fact, the production planning and the inventory management activities have many roles in the optimization of the enterprise performance. The first role may be the service function, in the sense of maintaining a certain service level and permitting the immediate fulfillment of the customers demands. The second role is the capacity regulation function, which allows to compensate for the predictable difference between the charge and the capacity. The third role is the circulation role; it permits to ensure a certain continuity in the flow inside a structure and therefore to decouple its different entities. The last important role may be the speculation role which takes its importance from the differences between the ordering/ production costs of the different suppliers and between the different planning periods.

In this Ph.D. dissertation we presented production planning models essentially for short life cycle type products. This type of products is generally characterized by a short selling season. The main goal of the presented models were to help the decision maker in optimizing his production/ ordering, return and capacity reservation decisions in order to optimally satisfy the demands of his customers and to minimize his total costs (or maximize his total profit). In the presented models, we tried to give to the decision maker more action opportunities than those existing in the literature models. These actions makes our models more flexible and more reactive.

The main contributions of this thesis are detailed in six different chapters. In addition to the conclusions given in each chapter, we provide in the following paragraphs some general concluding remarks.

In Chapter 2 we presented a general overview of the Supply Chain and of the production and inventory planning and we defined the different related aspects and issues.

In Chapter 3 we presented an extension to the very well known *Newsvendor* model. This model which is a paradigm in the literature of the operations research and management science is a single period planning model in which a single ordering opportunity, at the beginning of the planning period, is available and a single salvage (or return) opportunity at the end of the planning period is allowed. We modeled in this Chapter, a *Newsvendor* model with initial inventory and two salvage opportunity: at the beginning and the end of the planning period. We showed that the additional salvage opportunity is very beneficial in the case of a high initial inventory level.

Next, we presented a general two-period planning framework, with initial inventory, two production modes and multiples salvage opportunities, in Chapter 4. This model allows the retailer to return a part of his inventory to the supplier or to sell it in a parallel market, even before the beginning of the planning period. The different production modes with different production costs give the decision maker more degrees of freedom and allow him to profit from the difference between the production costs. In Chapter 5, we generalized this framework by adding capacity constraints to the model, which makes it more

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general permitting to consider many real cases where the production capacities are finite. In Chapter 6, we modeled an important aspect of the Supply Chain, which is the information update. Based on the model shown in Chapter 4, we have considered a demand forecast updating process for the demand of the second period. We studied the impact of the information updating process on the structure of the optimal policy and showed the impact of the information quality on that optimal policy.

In addition, we presented in Chapter 7 an advanced capacity reservation contract model. This model is constituted of a single planning period with two decision stages. We modeled in this chapter the demand forecast updating process also. We studied then two particular cases, where the information was perfect in the first case and worthless in the second case. The optimization process in this contract model was considered from the point of view of the retailer.

Finally, we tried to generalize the framework comprising two production modes to the long life cycle type products. We formulated in Chapter 8 a multi-periodic production planning problem with two production modes. Since there is no closed-form solution for such problems, we provided upper and lower bounds on the optimal decision variables. These bounds allowed us, using a known heuristic, to calculate numerically approximated solutions.

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9.2 Future Research

Worrying about practical and efficient models and results, much is left to be done. Several interesting areas of future research arise. In the following section we detail some of these research directions.

In the whole presented work, we have assumed single product models, in which the production and procurement decisions of only one product are modeled. It is interesting to see the case of multi-product models, in which the different produced articles share the same limited resources.

We note that the fact that the production capacities are constrained makes the development of an optimal solution more complicated than the development of the solution in the unconstrained problem, as we have seen in Chapter 5. Nevertheless, a first extension to our different models would be the generalization of the obtained results to multiple products in a framework with capacity constraints.

Most of our models were constituted of a single period or of two periods and had as objective to solve planning problems for products with short life cycle. These models, and especially those constituted of two periods, represent an important base to study and to understand the behavior of the multi-periodic models corresponding to long life cycle products. An interesting study would be the extension of the obtained results in Chapter 4, Chapter 5 and Chapter 6 to multi-periodic cases. We see that at least some of the obtained results in the two-period framework would be valid for the multi-periodic framework.

In this Ph.D. we have proposed two different planning models with demand forecast update, in which the information that permitted to update the demand forecast was an external market information. Two extensions might be interesting in this context: the first extension would be the generalization of the updating process to the other models presented in this work, especially to the models presented in Chapter 5 and Chapter 8. The second extension would be the use of another updating process, in which the demand distributions in the different planning period are correlated. In this case, the update of the demand forecast in a given planning period will use the information about the realization of the demand in the previous periods.

In Chapter 8 we have proposed a multi-periodic production planning model with two production modes. Since the complete analytical solution for this model is very difficult even impossible to be obtained, we have provided upper and lower bounds on the optimal values of the decision variables. These bounds allowed us, using a given heuristic, to calculate numerically an approximated solution. An important extension of our work would be the improvement of the heuristic that calculated the approximated solution and especially the choice of the interpolation coefficient. This coefficient permitted to interpolate between the upper and the lower bounds in order to find the approximated solution.

Note that in each model provided in this work, the objective function of the optimization problem has been defined as an expected function with respect to the random variables of the model. Therefore, the optimization problems of this work have been either maximization or minimization problems of an expected function. In this type of optimization problems, one does not take into account the variability of the objective function. It would be therefore interesting to use other optimization methods which include in the development of the optimal policy the variability of the objective function.

From the studies done in this Ph.D. thesis we have concluded that modeling and solving production and inventory problems in a multi-periodic setting is very difficult and the characterization of a closed-form solution in some cases is impossible. Therefore, the used of some approximations or heuristics would be more efficient from this point of view.

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Appendix A

Appendix of Chapter 4

This appendix deals with the analysis of Chapter 4. In Appendix A.1, we provide the development and the analysis of the second period optimal policy. In Appendix A.2 we give the partial derivatives of the total expected objective function with respect to the three decision variables of the first period. Finally, in Appendix A.3, Appendix A.4 and Appendix A.5 we develop the proofs of Lemma 4.3, Lemma 4.4 and Lemma 4.5 respectively.

A.1 Second period optimal policy

In this section we characterize the optimal policy of the unconstrained second period problem defined in equations (4.15) and (4.19). Consider the two partial derivatives of $\Pi_2(X_2, Q_{22}, S_2)$ with respect to Q_{22} and S_2, respectively given by

$$\frac{\partial \Pi_2(X_2, Q_{22}, S_2)}{\partial Q_{22}} = -c_{22} + b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3)F_2(X_2 + Q_{22} - S_2) \tag{A.1}$$

and

$$\frac{\partial \Pi_2(X_2, Q_{22}, S_2)}{\partial S_2} = s_2 - b_2 - c_{33} + (b_2 + c_{33} + h_2 - s_3)F_2(X_2 + Q_{22} - S_2). \tag{A.2}$$

Optimality conditions for Q_{22}^*

For any given S_2 value satisfying $0 \leq S_2 \leq X_2$, the optimal ordering quantity $Q_{22}^*(X_2)$ is a function of $X_2 - S_2$ that can be computed as the solution of the following optimization problem

$$Q_{22}^*(X_2) = \arg\left\{\max_{0 \leq Q_{22}} \{\Pi_2(X_2, Q_{22}, S_2)\}\right\}. \tag{A.3}$$

By concavity of $\Pi_2(X_2, Q, S_b)$ with respect to Q_{22}, and for any given S_2 value, the optimal solution $Q_{22}^*(X_2)$ is given either by

$$Q_{22}^*(X_2) = 0, \tag{A.4}$$

if $-c_{22} + b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3)F_2(X_2 - S_2) \leq 0$, or by

$$Q_{22}^*(X_2) = F_2^{-1}\left(\frac{b_2 + c_{33} - c_{22}}{b_2 + c_{33} + h_2 - s_3}\right) - X_2 + S_2 \geq 0, \tag{A.5}$$

if $-c_{22} + b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3)F_2(X_2 - S_2) \geq 0$.

Optimality conditions for S_2^*

For any given Q_{22} value satisfying $0 \leq Q_{22}$, the optimal ordering quantity $S_2^*(X_2)$ is defined as the solution of the following optimization problem

$$S_2^*(X_2) = \arg\left\{\max_{0 \leq S_2 \leq X_2} \{\Pi_2(X_2, Q_{22}, S_2)\}\right\}. \tag{A.6}$$

By concavity of $\Pi_2(X_2, Q_{22}, S_2)$ with respect to S_2, and for any given Q_{22} value, the optimal solution $S_2^*(X_2)$ is given either by

$$S_2^*(X_2) = 0, \tag{A.7}$$

if $s_2 - b_2 - c_{33} + (b_2 + c_{33} + h_2 - s_3)F_2(X_2 + Q_{22}) \leq 0$, or by

$$S_2^*(X_2) = X_2 + Q_{22} - F_2^{-1}\left(\frac{b_2 + c_{33} - s_2}{b_2 + c_{33} + h_2 - s_3}\right) \geq 0, \tag{A.8}$$

if $s_2 - b_2 - c_{33} + (b_2 + c_{33} + h_2 - s_3)F_2(X_2 + Q_{22}) \geq 0$.

Critical threshold levels

From the above optimality conditions, two threshold levels appear to be of great importance in the second period optimal policy characterization

$$Y_{12} = F_2^{-1}\left(\frac{b_2 + c_{33} - c_{22}}{b_2 + c_{33} + h_2 - s_3}\right) \quad \text{and} \quad Y_{22} = F_2^{-1}\left(\frac{b_2 + c_{33} - s_2}{b_2 + c_{33} + h_2 - s_3}\right). \tag{A.9}$$

Critical threshold levels and structure of the optimal policy

We show below that the structure of the optimal policy of the second period problem is, in fact, fully characterized by the two threshold levels given in equation (A.9) as depicted in Figure 15.

Lemma A.1 *For $Y_{12} \leq X_2 \leq Y_{22}$, the optimal solution is given by*

$$Q_{22}^*(X_2) = S_2^*(X_2) = 0.$$

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Proof. For $Y_{12} \leq X_2 \leq Y_{22}$, one finds

$$\frac{\partial \Pi_2(X_2,0,0)}{\partial Q_{22}} < 0 \; and \; \frac{\partial \Pi_2(X_2,0,0)}{\partial S_2} < 0, \tag{A.10}$$

which induces, by concavity, that the solution $Q^*_{22}(X_2) = S^*_2(X_2) = 0$ is the optimum of the profit function for these X_2 values. If $X_2 = Y_{12}$, one finds

$$\frac{\partial \Pi_2(X_2,0,0)}{\partial Q_{22}} = 0 \text{ and } \frac{\partial \Pi_2(X_2,0,0)}{\partial S_2} < 0, \tag{A.11}$$

which leads to the same conclusion. If $X_2 = Y_{22}$, one finds

$$\frac{\partial \Pi_2(X_2,0,0)}{\partial Q_{22}} < 0 \text{ and } \frac{\partial \Pi_2(X_2,0,0)}{\partial S_2} = 0, \tag{A.12}$$

which leads to the same conclusion. □

Lemma A.2 *For $X_2 \leq Y_{12}$, the optimal solution is given by*

$$Q^*_{22}(X_2) = Y_{12} - X_2 \;\; and \;\; S^*_2(X_2) = 0. \tag{A.13}$$

Proof. For $X_2 \leq Y_{12}$, one finds that

$$\frac{\partial \Pi_2(X_2, Y_{12}-X_2, 0)}{\partial Q_{22}} = 0 \;\text{ and }\; \frac{\partial \Pi_2(X_2, Y_{12}-X_2, 0)}{\partial S_2} < 0 \tag{A.14}$$

which induces, by concavity, that the solution $Q^*_{22}(X_2) = Y_{12} - X_2$ and $S^*_2(X_2) = 0$ is the optimum of the profit function for such X_2 values. □

Lemma A.3 *For $Y_{22} \leq X_2$, the optimal solution is given by*

$$Q^*_{22}(X_2) = 0 \;\; and \;\; S^*_2(X_2) = X_2 - Y_{22}. \tag{A.15}$$

Proof. For $Y_{22} \leq X_2$, one finds that

$$\frac{\partial \Pi_2(X_2, 0, X_2 - Y_{22})}{\partial Q_{22}} < 0 \text{ and } \frac{\partial \Pi_{22}(X_2, 0, X_2 - Y_{22})}{\partial S_2} = 0 \tag{A.16}$$

which induces, by concavity, that the solution $Q^*_{22}(X_2) = 0$ and $S^*_2(X_2) = X_2 - Y_{22}$ is the optimum of the profit function for such X_2 values. □

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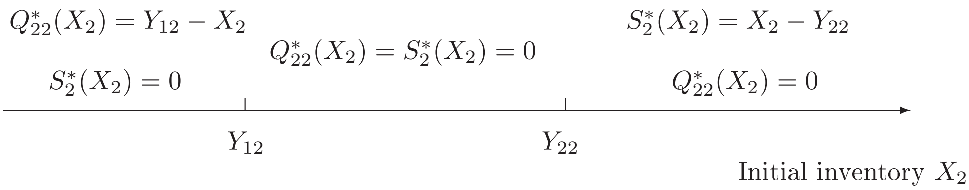

Figure A.1: Structure of the optimal policy of the second period

A.2 Partial derivatives of the total expected objective function with respect to the three decision variables of the first period

$$\begin{aligned}\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}} &= c_{22} - c_{11} \\ &+ (b_2 + c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}) \\ &+ (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}) \\ &- (b_2 + c_{33} + h_2 - s_3) \\ &\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x)dx\end{aligned} \tag{A.17}$$

$$\begin{aligned}\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}} &= c_{22} - c_{11} + b_1 - (b_1 + h_1)F_1(Q_{11} - S_1 + X_1) \\ &+ (b_2 + c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}) \\ &+ (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}) \\ &- (b_2 + c_{33} + h_2 - s_3) \\ &\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x)dx\end{aligned} \tag{A.18}$$

$$\begin{aligned}\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1} &= -c_{22} + s_1 - b_1 + (b_1 + h_1)F_1(Q_{11} - S_1 + X_1) \\ &- (b_2 + c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}) \\ &- (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}) \\ &+ (b_2 + c_{33} + h_2 - s_3) \\ &\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x)dx\end{aligned} \tag{A.19}$$

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A.3 Proof of Lemma 4.3

The hessian of $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ with respect to Q_{11}, Q_{12} and S_1 is given by

$$\begin{aligned}\nabla^2\Pi(X_1, Q_{11}, Q_{12}, S_1) &= -(b_1+h_1)f_1(Q_{11}-S_1+X_1)\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}\\ &- (b_2+c_{33}+h_2-s_3)G(X_1, Q_{11}, Q_{12}, S_1)\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{bmatrix},\end{aligned} \tag{A.20}$$

with

$$G(X_1, Q_{11}, Q_{12}, S_1) = \int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}} f_1(x)f_2(Q_{02}+Q_{11}+Q_{12}-S_1+X_1-x)dx \tag{A.21}$$

For each vector $\mathbf{V} = \begin{pmatrix}V_1\\ V_2\\ V_3\end{pmatrix}$, where $(V_1; V_2; V_3) \in \mathbb{R}^3$, we find

$$\begin{aligned}V^T\left\{\nabla^2\Pi(X_1, Q_{11}, Q_{12}, S_1)\right\}V &= -(b_1+h_1)f_1(Q_{11}-S_1+X_1)(V_1-V_3)^2\\ &- (b_2+c_{33}+h_2-s_3)G(X_1, Q_{11}, Q_{12}, S_1)(V_1+V_2-V_3)^2.\end{aligned} \tag{A.22}$$

From the model assumptions (section 4.2.2), one has $s_3 < c_{33}$, which means that $(b_2+c_{33}+h_2-s_3) > 0$. On one hand, from equation (4.26) one has $Y_{12} < Y_{22}$. On the other hand $f_1(\cdot)$ and $f_2(\cdot)$ are two probability density functions, and therefore two positive functions. Hence the function $G(X_1, Q_{11}, Q_{12}, S_1)$ is positive. One could conclude that

$$V^T\left\{\nabla^2\Pi(X_1, Q_{11}, Q_{12}, S_1)\right\}V \le 0,$$

which proves that the matrix $\nabla^2\Pi(X_1, Q_{11}, Q_{12}, S_1)$ is semi-definite negative. Consequently, the objective function $\Pi(X_1, Q_{11}, Q_{12}, S_1)$ is jointly concave with respect to Q_{11}, Q_{12} and S_1. □

A.4 Proof of Lemma 4.4

It could be easily seen that equations (4.31) and (4.33) can not be satisfied together for the same values of Q_{11}^* and S_1^*.

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Indeed, $\forall\ Q_{12}^*$, suppose that the first optimality equation, namely (4.31) is satisfied for a given value $Q_{11}^* > 0$. Thus if one replaces Q_{11}^* by its value that satisfies equation (4.31) in the partial derivative of the total expected objective function with respect to S_1 (A.19), one gets

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, S_1^*) = s_1 - c_{11}. \tag{A.23}$$

From model assumptions (section 4.2.2), one has $s_1 < c_{11}$. Hence

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, S_1^*) < 0.$$

By concavity, and by the constraint of non-negativity of the decision variables, one gets $S_1^* = 0$.

Now, suppose that $\forall\ Q_{12}^*$, the third optimality equation, namely (4.33) is satisfied for a given value $S_1^* > 0$. Thus if one replaces S_1^* by its value, that satisfies equation (4.33), in the partial derivative of the total expected objective function with respect to Q_{11} given in (A.17), one gets

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}}(Q_{11}^*, Q_{12}^*, S_1^*) = s_1 - c_{11}. \tag{A.24}$$

For the same reasons as in the first case, one gets $Q_{11}^* = 0$, which completes the proof. □

A.5 Proof of Lemma 4.5

If $Q_{12}^* = 0$, this property is satisfied.

Now, suppose that $Q_{12}^* > 0$. Thus if one replaces Q_{12}^* by its positive value in the partial derivative of the total expected objective function with respect to S_1, given in (A.19), one gets

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, S_1^*) = s_1 - c_{12} - b_1 + (h_1 + b_1)F_1(Q_{11}^* - S_1^* + X_1). \tag{A.25}$$

Since $F_1(\cdot)$ is a PDF, therefore $F_1(\cdot) \leq 1$. Hence on gets

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1}(Q_{11}^*, Q_{12}^*, S_1^*) \leq s_1 - c_{12} - b_1 + (h_1 + b_1) = s_1 - c_{12} + h_1.$$

By assumption of Lemma 4.5, one has $c_{12} > s_1 + h_1$. Hence

$$\frac{\partial\Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_{12}}(Q_{11}^*, Q_{12}^*, S_1^*) < 0.$$

By concavity, and by the constraint of non-negativity of the decision variables, one gets $S_1^* = 0$.

On the other hand, if $S_1^* = 0$, the property is also satisfied. Now, suppose that $S_1^* > 0$, then replace S_1^* by this value in equation (A.18), one gets

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$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}}(Q_{11}^*, Q_{12}^*, S_1^*) = s_1 - c_{11}. \tag{A.26}$$

For the same reasons as in the first case, one gets $Q_{12}^* = 0$, which completes the proof. □

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Appendix B

Appendix of Chapter 5

In this appendix, we give the complementary analysis and proofs of Chapter 5. We present the proof of Lemma 5.1 in Appendix B.1. In Appendix B.2 we provide the partial derivatives of the total expected objective function with respect to the first period decision variables. Finally we provide the proofs of Lemma 5.2 and Lemma 5.3 in Appendix B.3 and Appendix B.4 respectively.

B.1 Proof of Lemma 5.1

The hessian of $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))$ with respect to Q_{11}, Q_{12} and S_1 is given by

$$\begin{aligned}
&\nabla^2\Pi\left(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)\right) = \\
-\quad &(b_1+h_1)f_1(Q_{11}-S_1+X_1)\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}\\
-\quad &(b_2+c_{33}+h_2-s_3)\left\{\Omega(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))\right.\\
+\quad &\left.\Psi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))\right\}\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{bmatrix},
\end{aligned} \tag{B.1}$$

with

$$\begin{aligned}
&\Omega(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)) = \\
&\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}} f_1(x)f_2(Q_{02}+Q_{11}+Q_{12}-S_1+X_1-x)dx,
\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}
&\Psi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)) = \\
&\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}+K_{22}}^{+\infty} f_1(x)f_2(Q_{02}+Q_{11}+Q_{12}-S_1+X_1-x+K_{22})dx.
\end{aligned} \tag{B.3}$$

For each vector $\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$, where $(V_1; V_2; V_3) \in \mathbb{R}^3$, we find

$$V^T \left\{ \nabla^2 \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)) \right\} V = \tag{B.4}$$
$$-(b_1 + h_1) f_1(Q_{11} - S_1 + X_1)(V_1 - V_3)^2$$
$$-(b_2 + c_{33} + h_2 - s_3) \{\Omega(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)) +$$
$$\Psi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))\} (V_1 + V_2 - V_3)^2.$$

By the model assumptions (given in Chapter 4), one has $s_3 < c_{33}$, which means that $(b_2 + c_{33} + h_2 - s_3) > 0$. As, from (5.17), one has $Y_{12} < Y_{22}$ and as $f_1(\cdot)$ and $f_2(\cdot)$ are positive functions, we deduce that $\Omega(X_1, Q_{02}, Q_{11}, Q_{12}, S_1)$ and $\Psi(X_1, Q_{02}, Q_{11}, Q_{12}, S_1, K_{22})$ are nonnegative functions. We then find that

$$V^T \left\{ \nabla^2 \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2)) \right\} V \leq 0,$$

which proves that the matrix $\nabla^2 \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))$ is semi-definite negative. Consequently, the objective function $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))$ is jointly concave with respect to Q_{11}, Q_{12} and S_1. □

B.2 Total expected objective function partial derivatives

Assume that $\alpha = Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}$, $\beta = Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}$ and $\gamma = K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}$.

Consider the three partial derivatives of $\Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))$ with respect to Q_{11}, Q_{12} and S_1 respectively given by

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial Q_{11}} = \tag{B.5}$$
$$-c_{11} + b_1 - (b_1 + h_1) F_1(X_1 + Q_{11} - S_1)$$
$$+c_{22} F_1(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$-c_{22} F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$+s_2 F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22})$$
$$+ \int_\alpha^\beta f_1(D_1)$$
$$(b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3) F_2(Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1)) \, dD_1$$
$$+ \int_\gamma^{+\infty} f_1(D_1)$$
$$(b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3) F_2(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1)) \, dD_1,$$

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$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial Q_{12}} = \tag{B.6}$$
$$-c_{12} + c_{22}F_1(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$-c_{22}F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$+s_2F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22})$$
$$+\int_{\alpha}^{\beta} f_1(D_1)$$
$$(b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1))\ dD_1$$
$$+\int_{\gamma}^{+\infty} f_1(D_1)$$
$$(b_2 + c_{33} - (b_2 + c_{33} + h_2 - s_3)F_2(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1))\ dD_1$$

and

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial S_1} = \tag{B.7}$$
$$s_1 - b_1 + (b_1 + h_1)F_1(X_1 + Q_{11} - S_1)$$
$$-c_{22}F_1(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$+c_{22}F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12})$$
$$-s_2F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22})$$
$$+\int_{\alpha}^{\beta} f_1(D_1)$$
$$(-b_2 - c_{33} + (b_2 + c_{33} + h_2 - s_3)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1))\ dD_1$$
$$+\int_{\gamma}^{+\infty} f_1(D_1)$$
$$(-b_2 - c_{33} + (b_2 + c_{33} + h_2 - s_3)F_2(K_{22} + Q_{02} + Q_{11} + Q_{12} - S_1 - D_1 + X_1))\ dD_1$$

B.3 Proof of Lemma 5.2

Using Lemma 5.1, one could use the first order optimality criterion to characterize the first period optimal decision variables. This induces the following optimality equations

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial Q_{11}}(Q_{11}^*, Q_{12}^*, S_1^*) = 0, \tag{B.8}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial Q_{12}}(Q_{11}^*, Q_{12}^*, S_1^*) = 0, \tag{B.9}$$

and

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q_{22}^*(X_2), S_1, S_2^*(X_2))}{\partial S_1}(Q_{11}^*, Q_{12}^*, S_1^*) = 0. \tag{B.10}$$

Regarding equations (B.5) and (B.7) one could easily see that equations (B.8) and (B.10) can not be

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satisfied simultaneously for the same values of $Q^*_{11}(X_1)$ and $S^*_1(X_1)$.

For any given Q_{12} and S_1 values satisfying $0 \leq Q_{12}$ and $0 \leq S_1 \leq X_1$, assume that the first optimality equation, namely (B.8) is satisfied for a given value $Q^*_{11}(X_1) > 0$. Then replacing $Q^*_{11}(X_1)$ by its value that satisfies equation (B.8) in equation (B.7), one gets

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q^*_{22}(X_2), S_1, S^*_2(X_2))}{\partial S_1}(Q^*_{11}(X_1), Q_{12}, S_1(X_1)) = s_1 - c_{11}. \tag{B.11}$$

From the model assumptions (given in Chapter 4), one has $s_1 < c_{11}$. Hence

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q^*_{22}(X_2), S_1, S^*_2(X_2))}{\partial S_1}(Q^*_{11}(X_1), Q_{12}, S_1) < 0.$$

By concavity, and by the constraint of non-negativity of the decision variables, one gets $S^*_1(X_1) = 0$.

Now, for any Q_1 and Q_{12} values satisfying $0 \leq Q_{11}$ and $0 \leq Q_{12}$, assume that the third optimality equation, namely (B.10), is satisfied for a given $S^*_1(X_1) > 0$. Replace then $S^*_1(X_1)$ by its value that satisfies equation (B.10) in equation (B.5). The following equation is obtained

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, Q^*_{22}(X_2), S_1, S^*_2(X_2))}{\partial Q_{11}}(Q_{11}, Q_{12}, S^*_1) = s_1 - c_{11}. \tag{B.12}$$

For the same reasons as in the first case, one gets $Q^*_{11} = 0$. □

B.4 Proof of Lemma 5.3

Assuming that $Q^*_{12} > 0$ implies that there exists a $Q^*_{12} > 0$ that verifies (B.9). Replace then $Q^*_{12}(X_1, Q^*_{11}, S^*_1)$ obtained by solving (B.9) by its value in equations (B.5) and (B.7). The obtained system is a two-variable system defined as follows

$$\frac{\partial \Pi\left(X_1, Q_{11}, Q^*_{12}(X_1, Q^*_{11}, S^*_1), Q^*_{22}(X_2), S_1, S^*_2(X_2)\right)}{\partial Q_{11}} = \\ c_{12} - c_{11} + b_1 - (b_1 + h_1)F_1(X_1 + Q_{11} - S_1), \tag{B.13}$$

and

$$\frac{\partial \Pi\left(X_1, Q_{11}, Q^*_{12}(X_1, Q^*_{11}, S^*_1), Q^*_{22}(X_2), S_1, S^*_2(X_2)\right)}{\partial S_1} = \\ -c_{12} + s_1 - b_1 + (b_1 + h_1)F_1(X_1 + Q_{11} - S_1). \tag{B.14}$$

The system constituted by equations (B.13) and (B.14) is similar to the system studied in Appendix A.1. Then by using the same approach as in Appendix A.1 one gets the two-threshold optimal policy defined by Y_{11} and Y_{21} given in (5.25). □

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Appendix C

Appendix of Chapter 6

This appendix completes the analysis of Chapter 6, where we provide in Appendix C.1 the partial derivatives of the expected objective function with respect to the three decision variables of the first period.

C.1 Partial derivatives of the expected objective function with respect to the three decision variables of the first period

$$\begin{aligned}
\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{11}} &= c_{22} - c_{11} + b_1 - (b_1 + h_1)F_1(Q_{11} - S_1 + X_1) \qquad \text{(C.1)}\\
+ & \int_0^{\infty} \Big[(b_2 + c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}(i))\\
+ & (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}(i))\\
- & (b_2 + c_{33} + h_2 - s_3)\\
& \int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}(i)}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}(i)} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x|i)dx\Big] g(i)\, di
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial Q_{12}} &= c_{22} - c_{12} \qquad \text{(C.2)}\\
+ & \int_0^{\infty} \Big[(b_2 + c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}(i))\\
+ & (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}(i))\\
- & (b_2 + c_{33} + h_2 - s_3)\\
& \int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}(i)}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}(i)} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x|i)dx\Big] g(i)\, di
\end{aligned}$$

$$\frac{\partial \Pi(X_1, Q_{11}, Q_{12}, S_1)}{\partial S_1} = -c_{22} + s_1 - b_1 + (b_1 + h_1)F_1(Q_{11} - S_1 + X_1) \tag{C.3}$$

$$+ \int_0^{\infty} \Big[(-b_2 - c_{33} - c_{22})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{12}(i))$$

$$- (s_2 - b_2 - c_{33})F_1(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - Y_{22}(i))$$

$$+ (b_2 + c_{33} + h_2 - s_3)$$

$$\int_{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{22}(i)}^{Q_{02}+Q_{11}+Q_{12}-S_1+X_1-Y_{12}(i)} f_1(x)F_2(Q_{02} + Q_{11} + Q_{12} - S_1 + X_1 - x|i)dx\Big]g(i)\, di$$

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Appendix D

Appendix of Chapter 7

In this appendix we complete the analysis of Chapter 7. In Appendix D.1 we provide the second decision stage expected optimal objective function. The proof of Lemma 7.2 is given in Appendix D.2. Finally, the first decision stage expected objective function partial derivatives are given in Appendix D.3.

D.1 Second decision stage expected optimal objective function

$$\begin{aligned}
&E_{(c_e;s_1)}\left[E_i\left[E_{D|i}\left[\Pi_1^*(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]\right] = \\
&\quad \beta\Big\{\int_0^{U_2^L(Q_0)}\left[E_{D|i}\left[\Pi_{11}^{*L}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{U_2^L(Q_0)}^{U_1^L(Q_0)}\left[E_{D|i}\left[\Pi_{12}^{*L}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{U_1^L(Q_0)}^{V_1^L(Q_T)}\left[E_{D|i}\left[\Pi_{13}^{*L}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{V_1^L(Q_T)}^{\infty}\left[E_{D|i}\left[\Pi_{14}^{*L}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di\Big\} \\
&\quad +(1-\beta)\Big\{\int_0^{U_2^H(Q_0)}\left[E_{D|i}\left[\Pi_{11}^{*H}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{U_2^H(Q_0)}^{U_1^H(Q_0)}\left[E_{D|i}\left[\Pi_{12}^{*H}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{U_1^H(Q_0)}^{V_1^H(Q_T)}\left[E_{D|i}\left[\Pi_{13}^{*H}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di \\
&\qquad +\int_{V_1^H(Q_T)}^{\infty}\left[E_{D|i}\left[\Pi_{14}^{*H}(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]g(i)di\Big\}
\end{aligned} \tag{D.1}$$

D.2 Proof of Lemma 7.2

The different parts that define the expected optimal objective function of the second decision stage, given in (7.17) are jointly concave with respect to the decision variables Q_0 and Q_T and the information i.

Indeed, let us prove the concavity of the $E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ and therefore the concavity of the other three functions could be easily shown using the same methodology.

The hessian of $E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ with respect to q_1, S_1 and i is given by

$$\nabla^2 E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = -\frac{p+b-s_2}{\delta_0^2}\rho^2\sigma_0^2 h\left(Y_2(i)\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For each vector $\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$, where $(V_1; V_2; V_3) \in \mathbb{R}^3$, we find

$$V^T\left\{\nabla^2 E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]\right\} V = -\frac{p+b-s_2}{\delta_0^2}\rho^2\sigma_0^2 h\left(Y_2(i)\right) V_3^2. \tag{D.2}$$

From the model assumptions (section 7.2.2), one has $p + b > s_2$, which means that

$$V^T\left\{\nabla^2 E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]\right\} V \leq 0,$$

which proves that the matrix $\nabla^2 E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ is semi-definite negative. Consequently, the objective function $E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ is jointly concave with respect to q_1, S_1 and i.

Note that if the functions $E_{D|i}\left[\Pi_{11}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$, $E_{D|i}\left[\Pi_{12}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$, $E_{D|i}\left[\Pi_{13}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ and $E_{D|i}\left[\Pi_{14}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ are all defined for each i value, then an alternative to define $E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ is (see (Bassok and Anupindi, 1997))

$$E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right] = \max_{1\leq i\leq 4}\left(E_{D|i}\left[\Pi_{1i}^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]\right). \tag{D.3}$$

Theorem 4.13 in (Avriel, 1976) ensures that a function that is defined as the maximum (pointwise) of several concave functions is concave, which proves that the function $E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]$ is jointly concave with respect to Q_0, Q_T and i.

It is well known that the weighted non-negative sum (or integral) of concave functions is a concave one (Boyd and Vandenberghe, 2004).

We conclude then that the function

$$E_i\left[E_{D|i}\left[\Pi_1^*(Q_1^*, S_1^*|i, Q_0, Q_T)\right]\right]$$

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is jointly concave and consequently the function

$$E_{(c_e;s_1)}\left[E_i\left[E_{D|i}\left[\Pi_1^*(Q_1^*,S_1^*|i,Q_0,Q_T)\right]\right]\right]$$

is also jointly concave with respect to Q_0 and Q_T. □

D.3 First decision stage expected objective function partial derivatives

$$\begin{aligned}
&\frac{\partial \Pi_0(Q_0,Q_T)}{\partial Q_T} = \\
&-c_{op}+\beta\Bigg\{-\int_{V_1^L(Q_T)}^{\infty}\left[c_e^L-p-b+(p+b-s_2)H(K|i)\right]g(i)di \\
&+\frac{\delta_0}{\rho\sigma_0}g(V_1^L(Q_T))\Bigg[(c_e^L-p-b)\left(Q_T-Y_1^L\left(V_1^L(Q_T)\right)\right) \\
&-(p+b-s_2)\Big[Y_1^L\left(V_1^L(Q_T)\right)H\left(Y_1^L\left(V_1^L(Q_T)\right)|i=V_1^L(Q_T)\right) \\
&-Q_TH\left(Q_T|i=V_1^L(Q_T)\right)+\int_0^{Q_T}Dh(D|i=V_1^L(K))dD \\
&-\int_0^{Y_1^L(V_1^L(Q_T))}Dh(D|i=V_1^L(K))dD\Big]\Bigg]\Bigg\} \\
&+(1-\beta)\Bigg\{-\int_{V_1^H(Q_T)}^{\infty}\left[c_e^H-p-b+(p+b-s_2)H(K|i)\right]g(i)di \\
&+\frac{\delta_0}{\rho\sigma_0}g(V_1^H(Q_T))\Bigg[(c_e^H-p-b)\left(Q_T-Y_1^H\left(V_1^H(Q_T)\right)\right) \\
&-(p+b-s_2)\Big[Y_1^H\left(V_1^H(Q_T)\right)H\left(Y_1^H\left(V_1^H(Q_T)\right)|i=V_1^H(Q_T)\right) \\
&-Q_TH\left(Q_T|i=V_1^H(Q_T)\right)+\int_0^{Q_T}Dh(D|i=V_1^H(K))dD \\
&-\int_0^{Y_1^H(V_1^H(Q_T))}Dh(D|i=V_1^H(K))dD\Big]\Bigg]\Bigg\}
\end{aligned} \tag{D.4}$$

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and

$$
\begin{aligned}
&\frac{\partial \Pi_0(Q_0,Q_T)}{\partial Q_0} = \qquad\qquad \text{(D.5)}\\
&-c_0 + c_{op} + \beta\Bigg\{ s_1^L G(U_2^L(Q_0)) - c_e^L G(U_1^L(Q_0)) + c_e^L \\
&+ \int_{U_2^L(Q_0)}^{U_1^L(Q_0)} \left[(s_2 - b - p)H(Q_0|i) + p + b\right] g(i)di \\
&-\frac{\delta_0}{\rho\sigma_0} g(U_1^L(Q_0)) \Big[(c_e^L - b - p)(Q_0 - Y_1^L(U_1^L(Q_0))) \\
&+(p+b-s_2)[(Q_0 - Y_1^L(U_1^L(Q_0)))H(Y_1^L(U_1^L(Q_0))|i = U_1^L(Q_0))] \\
&- \int_0^{Q_0} Dh(D|i = U_1^L(Q_0))dD + \int_0^{Y_1^L(U_1^L(Q_0))} Dh(D|i = U_1^L(Q_0))dD\Big] \\
&-\frac{\delta_0}{\rho\sigma_0} g(U_2^L(Q_0)) \Big[(s_1^L - b - p)(Q_0 - Y_2^L(U_2^L(Q_0))) \\
&+(p+b-s_2)[(Q_0 - Y_2^L(U_2^L(Q_0)))H(Y_2^L(U_2^L(Q_0))|i = U_2^L(Q_0))] \\
&- \int_0^{Q_0} Dh(D|i = U_2^L(Q_0))dD + \int_0^{Y_2^L(U_2^L(Q_0))} Dh(D|i = U_2^L(Q_0))dD\Big] \Bigg\} \\
&(1-\beta)\Bigg\{ s_1^H G(U_2^H(Q_0)) - c_e^H G(U_1^H(Q_0)) + c_e^H + \\
&\int_{U_2^H(Q_0)}^{U_1^H(Q_0)} \left[(s_2 - b - p)H(Q_0|i) + p + b\right] g(i)di \\
&-\frac{\delta_0}{\rho\sigma_0} g(U_1^H(Q_0)) \Big[(c_e^H - b - p)(Q_0 - Y_1^H(U_1^H(Q_0))) \\
&+(p+b-s_2)[(Q_0 - Y_1^H(U_1^H(Q_0)))H(Y_1^H(U_1^H(Q_0))|i = U_1^H(Q_0))] \\
&- \int_0^{Q_0} Dh(D|i = U_1^H(Q_0))dD + \int_0^{Y_1^H(U_1^H(Q_0))} Dh(D|i = U_1^H(Q_0))dD\Big] \\
&-\frac{\delta_0}{\rho\sigma_0} g(U_2^H(Q_0)) \Big[(s_1^H - b - p)(Q_0 - Y_2^H(U_2^H(Q_0))) \\
&+(p+b-s_2)[(Q_0 - Y_2^H(U_2^H(Q_0)))H(Y_2^H(U_2^H(Q_0))|i = U_2^H(Q_0))] \\
&- \int_0^{Q_0} Dh(D|i = U_2^H(Q_0))dD + \int_0^{Y_2^H(U_2^H(Q_0))} Dh(D|i = U_2^H(Q_0))dD\Big] \Bigg\}
\end{aligned}
$$

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Appendix E

Appendix of Chapter 8

This appendix deals with the analysis of Chapter 8. We provide in the following Appendixes the proofs of Lemmas 8.1, 8.2, 8.3, 8.4 and 8.5 respectively.

E.1 Proof of Theorem 8.1

Assuming that $Q^*_{t,t+1} > 0$ implies that there exists a $Q^*_{t,t+1} > 0$ that satisfies (8.25). Replace then $Q^*_{t,t+1}$ by its value in (8.24) to obtain the following single-variable equation

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial y_t}(y_t^{Max}, Q^*_{t,t+1}) = -c_{t,t+1} + \alpha c_t - b_t + (b_t + h_t)F_t(y_t^{Max}) = 0, \tag{E.1}$$

which implies

$$y_t^{Max} = F_t^{-1}\left(\frac{c_{t,t+1} - \alpha c_t + b_t}{b_t + h_t}\right). \tag{E.2}$$

Since, the decision variables y_t is constrained ($y_t \geq x_t$, see (8.2)), then one has

$$y_t^* = \max\left(x_t; y_t^{Max}\right). \tag{E.3}$$

which completes the proof. □

E.2 Proof of Theorem 8.2

Define the following terms:

- $F_{ij}(\cdot)$: cumulative distribution function of the demand from period i to period j,
- $F^R(\cdot) = 1 - F(\cdot)$,
- $Y_1 = (y_1, y_2, y_3, ...y_N)$ and $X_1 = (Q_{1,2}, Q_{2,3}, ..., Q_{N-1,N})$ represent any ordering policy for the hole horizon,

- $Y_2 = (y_2, y_3, ..., y_N)$,
- $P_j(Y)$ the probability that a supplementary unit ordered at the first period (over y_1) with the fast mode (Q_1) to be used at the period j,
- $P_j^* = P_j(y_1^*, y_2^*, ..., y_N^*)$.

Note that

$$P_{N+1} = F_{1N},$$

and

$$\sum_{1=1}^{N} P_i = F_{1N}^R.$$

We will take only a Near-Myopic case, where we look only to the cases where the demand of the first period (D_1) satisfies the following equation

$$y_t^* - D_t > y_{t+1}^*, \tag{E.4}$$

which means that we take only the cases where the units ordered at period t will be used in the following periods. Therefore, we assume that the available inventory at the beginning of the first period is sufficiently high to satisfy the demand of the first period, the second period and so on. The cost induced by the use of each unit ordered at period t and used at a period j ($j > t$), is multiplied by its probability.

Each time we assume that units from period t will be used in period $t+2$ means that the remaining inventory from period $t+1$ to period $t+2$ is positive with two information: the remaining inventory from period $t+1$ to period $t+2$ is constituted of units ordered at period t; the units of period t are sufficient to satisfy the hole demand of periods t and $t+1$, so we do not need to order any unit in period $t+1$ in order to satisfy the demand D_{t+1}, and consequently the decision variable $Q_{t,t+1}$ is equal to zero.

In this case, the vector $X_1 = (Q_{1,2}, Q_{2,3}, ..., Q_{N-1,N})$ will be equal to zero,

$$X_1 = (Q_{1,2}, Q_{2,3}, ..., Q_{N-1,N}) = (0, 0, ..., 0).$$

After taking into account these information, and rewriting equation (8.21) one gets

$$\begin{aligned} &\frac{\partial \Pi_1(x_1, y_1)}{\partial y_1}(x_1, y_1^*) = L_1'(y_1^*) + \\ &\alpha \int_0^{(y_1^* - y_2^*)^+} \left[L_2'(y_1^* - D_1) + \alpha \int_0^{(y_1^* - D_1 - y_3)^+} \frac{\partial \Pi_3^*(y_1^* - D_1 - D_2)}{\partial y_1} dF_2(D_2) \right] dF_1(D_1) = 0. \end{aligned} \tag{E.5}$$

The interpretation of this assumption is as follows (Morton and Pentico, 1995): a marginal unit ordered above optimal in period t will firstly cause extra marginal holding costs and save penalty costs in period t. That is the first term ($L_1'(y_1^*)$). Secondly, if in the next period one would place a positive order, then one would order one less unit in order to restore the original state in future periods, ending the perturbation to the original optimal solution (as for the case of increasing demand distributions). However, if the next period would have ordered 0, the extra unit will incur same sorts of costs in the next

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period, and for every period up until the first period which would have ordered anyway. In this period, one unit in less will be ordered, to end the perturbation as before. The first term is the myopic term, and hence the second term must be small whenever (E.5) is considered as a near-myopic equation. We continue below to work to estimate this term.

By iteratively substituting (E.5), and noting that $L'_j = -b_j F^R_{jj} + h_j F_{jj}$ we will get the following equation

$$\left[-b_1 P_1^* + h_1 \sum_{i=2}^{N+1} P_i^*\right] + \alpha \left[-b_2 P_2^* + h_2 \sum_{i=3}^{N+1} P_i^*\right] + ... + \alpha^{k-1}\left[-b_k P_k^* + h_k \sum_{i=k+1}^{N+1} P_i^*\right]$$
$$+ ... + \alpha^{N-1}\left[-b_N P_N^* + h_N P_{N+1}^*\right] + \alpha^N \nu P_{N+1}^* = 0. \quad \text{(E.6)}$$

By regrouping terms of the above equation we get

$$\left[\sum_{j=1}^{N} h_j \alpha^{j-1} + \alpha^N \nu\right] P_{N+1}(y_1^*, Y_2^*) - \sum_{i=1}^{N}\left[\alpha^{i-1} b_i - \sum_{j=1}^{i-1} \alpha^{j-1} h_j\right] P_i(y_1^*, Y_2^*) = 0. \quad \text{(E.7)}$$

We assume that $b_i > \alpha b_{i+1} - h_i$ which is equivalent to

$$0 < \alpha < \frac{b_i + h_i}{b_{i+1}}.$$

By using the above assumption, and by substituting it into itself we get the following expression

$$b_1 > \alpha b_2 - h_1 > (\alpha(\alpha b_3 - h_2) - h_1 = \alpha^2 b_3 - \alpha h_2 - h_1) > ... > \alpha^{i-1} b_i - \sum_{j=1}^{i-1} \alpha^{j-1} h_j > ...$$

By replacing the coefficients of $P_i^*,\ i = 1, ..., N$ in (E.7) by $-b_1$, one gets

$$\left[\sum_{j=1}^{N} h_j \alpha^{j-1} + \alpha^N \nu\right] P_{N+1}(y_1^*, Y_2^*) - b_1 \sum_{i=1}^{N} P_i(y_1^*, Y_2^*) \le 0. \quad \text{(E.8)}$$

Now increase y_1^* to $\tilde{y}_1$ ($y_1^* \le \tilde{y}_1$) in order to restore equality, giving

$$\left[\sum_{j=1}^{N} h_j \alpha^{j-1} + \alpha^N \nu\right] P_{N+1}(\tilde{y}_1, Y_2^*) - b_1 \sum_{i=1}^{N} P_i(\tilde{y}_1, Y_2^*) \le 0, \quad \text{(E.9)}$$

which is equivalent to

$$-b_1 F^R_{1N}(\tilde{y}_1, Y_2^*) + h^* F_{1N}(\tilde{y}_1, Y_2^*), \quad \text{(E.10)}$$

with

$$h^* = \sum_{j=1}^{N} h_j \alpha^{j-1} + \alpha^N \nu.$$

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Since Y_2^* has no influence on (E.10), we may reduce it to zero.

We may recognize now $\tilde{y}_1$ as optimal solution to the revised problem with all demand in the first period with $\tilde{y}_1 \geq y_1^*$.

Assume now that $y_{t,N}^{*m}$ is the solution of the revised problem (problem with demand convoluted in period t), then

$$-b_t F_{tN}^R(y_{t,N}^{*m}) + \left(\sum_{j=t}^{N} h_j \alpha^{j-t} + \alpha^{N-t+1}\nu\right) F_{tN}(y_{t,N}^{*m}) = 0, \tag{E.11}$$

which gives,

$$y_{t,N}^{*m} = F_{t,N}^{-1}\left[\frac{b_t}{b_t + \sum_{k=t}^{N} h_k \alpha^{k-t} + \alpha^{N-t+1}\nu}\right], \tag{E.12}$$

and completes the proof. □

E.3 Proof of Theorem 8.3

As we have done for y_t we will write a near-myopic equation for $Q_{t,t+1}$. We will take into account only cases where units from $Q_{t,t+1}$ will be used in the following periods. Thus we assume that demand in period $t+1$ satisfies the following equation

$$Q_{t,t+1}^* + y_t^* - \mu_t - \sum_{k=t+1}^{j} D_k > y_{j+1}^*, \quad j = t+1, ..., N-1, \quad t = 1, ..., N-1. \tag{E.13}$$

Equation (8.33) becomes

$$\begin{aligned} &\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha \int_0^\infty L'_{t+1}(y_t^* + Q_{t,t+1}^* - \mu_t) dF_t(D_t) \\ &\quad + \alpha \int_0^\infty \left[\alpha \int_0^{(Q_{t,t+1}^* + y_t^* - \mu_t - y_{t+2}^*)^+} \right. \\ &\qquad \left. \frac{\partial \Pi_{t+2}^*}{\partial Q_{t,t+1}} (Q_{t,t+1}^* + y_t^* + Q_{t+1,t+2}^* - \mu_t - D_{t+1}) dF_{t+1}\right] dF_t = 0, \end{aligned} \tag{E.14}$$

which is equivalent to

$$\begin{aligned} &\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y_t^* + Q_{t,t+1}^* - \mu_t) \\ &\quad + \alpha^2 \int_0^{(Q_{t,t+1}^* + y_t^* - \mu_t - y_{t+2}^*)^+} \frac{\partial \Pi_{t+2}^*}{\partial Q_{t,t+1}} (Q_{t,t+1}^* + y_t^* + Q_{t+1,t+2}^* - \mu_t - D_{t+1}) dF_{t+1} = 0. \end{aligned} \tag{E.15}$$

Define the following terms:

- $Z_t^* = y_t^* + Q_{t,t+1}^* - \mu_t$,
- $X_t = (Q_{t,t+1}, Q_{t+1,t+2}, ..., Q_{N-1,N})$, the vector of the ordered quantities using the slow mode from

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period t to N,

- $X_t^* = (Q_{t,t+1}^*, Q_{t+1,t+2}^*, ..., Q_{N-1,N}^*)$ the optimal X_t,
- $P_j^{*'}(Z_t^*, Y_{t+1}^*, X_{t+1}^*)$ the probability that a supplementary unit ordered with the slow mode at period t (over $Q_{t,t+1}^*$) to be used at the period j.

For simplicity reasons we abbreviate $P_j^{*'}(Z_t^*, Y_{t+1}^*, X_{t+1}^*)$ by $P_j^{*'}$.

Note that $P_{N+1}^{*'} = F_{t+1,N}$ and $\sum_{1=t+1}^{N} P_i^{*'} = F_{t+1,N}^R$.

By substituting (E.15) into itself and noting that $L_j' = -b_j F_{jj}^R + h_j F_{jj}$, one gets

$$\begin{aligned}
&\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha\left[-b_{t+1}P_{t+1}^{*'} + h_{t+1}\sum_{i=t+2}^{N+1} P_i^{*'}\right] + \alpha^2\left[-b_{t+2}P_{t+2}^{*'} + h_{t+2}\sum_{i=t+3}^{N+1} P_i^{*'}\right] \\
&+ ... + \alpha^{k-t}\left[-b_k P_k^{*'} + h_k \sum_{i=k+1}^{N+1} P_i^{*'}\right] + ... + \alpha^{N-t}\left[-b_N P_N^{*'} + h_N P_{N+1}^{*'}\right] \\
&+\alpha^{N-t+1}\nu P_{N+1}^{*'} = 0.
\end{aligned} \tag{E.16}$$

By rearranging terms in (E.16), one gets

$$\begin{aligned}
&\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \left[\sum_{j=t+1}^{N} h_j\alpha^{j-t} + \alpha^{N-t+1}\nu\right] P{*'}_{N+1} - \sum_{i=t+1}^{N}\left[b_i\alpha^{i-t} - \sum_{j=t+1}^{i-1} h_j\alpha^{j-t}\right] P{*'}_i \\
&= 0.
\end{aligned} \tag{E.17}$$

Using the following assumption

$$b_t > \alpha b_{t+1} - h_t,$$

and by substituting it into itself one gets

$$b_t > \alpha b_{t+1} - h_t > \alpha(\alpha b_{t+2} - h_{t+1}) - h_t = \alpha^2 b_{t+2} - \alpha h_{t+1} - h_t > ... > \alpha^{k-t} b_k - \sum_{j=t}^{k-1} h_j\alpha^{j-t} > ...,$$

which implies

$$\alpha b_{t+1} > \alpha^{k-t} b_k - \sum_{j=t+1}^{k-1} h_j \alpha^{j-t} > ..., \qquad k > t.$$

If we replace all the coefficients of $P{*'}_i$ by αb_{t+1} in (E.17), we get

$$\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \left[\sum_{j=t+1}^{N} h_j\alpha^{j-t} + \alpha^{N-t+1}\nu\right] P{*'}_{N+1} - (\alpha b_{t+1})\sum_{i=t+1}^{N} P{*'}_i \leq 0. \tag{E.18}$$

Increase $Q_{t,t+1}^*$ enough to restore equality, into $\tilde{Q}_{t,t+1}$. That means increase Z_t^* into $\tilde{Z}_t = y_t^* + \tilde{Q}_{t,t+1} - \mu_t$. That gives

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$$\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \left[\sum_{j=t+1}^{N} h_j\alpha^{j-t} + \alpha^{N-t+1}\nu\right] F_{t+1,N}(\tilde{Z}_t, Y^*_{t+1}, X^*_{t+1})$$
$$-(\alpha b_{t+1})F^R_{t+1,N}(\tilde{Z}_t, Y^*_{t+1}, X^*_{t+1}) = 0. \tag{E.19}$$

Since we have assumed that we will satisfy all demands of period $t+1$ with units ordered with the slow and less expensive mode ($Q_{t,t+1}$), then the vectors Y^*_{t+1} and X^*_{t+1} have no influence on equation (E.19) and can be set to zero. That implies

$$\tilde{Z}_t = F^{-1}_{t+1,N}\left[\frac{\alpha\tilde{c}_{t+1} - \tilde{c}_{t,t+1} + \alpha b_{t+1}}{\left(\sum_{j=t+1}^{N} h_j\alpha^{j-t} + \alpha^{N-t+1}\nu\right) + \alpha b_{t+1}}\right], \tag{E.20}$$

and the upper bound on the optimal $Q_{t,t+1}$ will be

$$\tilde{Q}_{t,t+1} = \left(F^{-1}_{t+1,N}\left[\frac{\alpha\tilde{c}_{t+1} - \tilde{c}_{t,t+1} + \alpha b_{t+1}}{\left(\sum_{j=t+1}^{N} h_j\alpha^{j-t} + \alpha^{N-t+1}\nu\right) + \alpha b_{t+1}}\right] - y^*_t + \mu_t\right)^+, \tag{E.21}$$

which completes the proof. □

E.4 Proof of Theorem 8.4

We rewrite (8.21) taking into account the fact that for the near myopic terms of (8.21) we consider only demands that satisfy

$$0 \le D_t \le (y^*_t - y^*_{t+1})$$

$$\frac{\partial\Pi_t(x_t, y_t, Q_{t,t+1})}{\partial y_t} = L'_t(y_t) + \alpha\int_0^{(y_t - y_{t+1})^+} \frac{\partial\Pi^*_{t+1}(Q_{t,t+1} + y_t - D_t)}{\partial y_t} dF_t(D_t). \tag{E.22}$$

For an initial inventory level $x_t > y^*_t$, one has

$$\frac{\partial\Pi_t(x_t, y_t, Q_{t,t+1})}{\partial y_t} = L'_t(x_t) + \alpha\int_0^{(x_t - y_{t+1})^+} \frac{\partial\Pi^*_{t+1}(Q_{t,t+1} + x_t - D_t)}{\partial y_t} dF_t(D_t). \tag{E.23}$$

Now substitute (E.22) into itself, and consider the optimal values of Y_t, one gets

$$L'_t(y^*_t) + \alpha\int_0^{(y^*_t - y^*_{t+1})^+} \Bigg[\quad L'_{t+1}(y^*_t + Q_{t,t+1} - D_t) +$$
$$\alpha\int_0^{(y^*_t - D_t - y_{t+2})^+} L'_{t+2}(y^*_t + Q_{t,t+1} - D_t - D_{t+1} + Q_{t,t+2})$$
$$\ldots\Bigg] dF_N(D_N)dF_{N-1}(D_{N-1})\ldots dF_t(D_t) = 0. \tag{E.24}$$

Since we have $y^*_t - y^*_{t+1} > D_t$, which is equivalent to $y_t - D_t > y_{t+1}$, the optimal $Q_{t,t+1}$ will be equal

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to zero, $Q^*_{t,t+1} = 0$.

The same reasoning is still valid for the following periods.

Define $D_{t,j} = \sum_{k=t}^{j} D_k$, with $D_{t,t-1} = 0$.

Thus equation (E.24) becomes as follows

$$L'_t(y^*_t) + \sum_{j=t}^{N} \alpha^{j+1-t} \left[\left(\prod_{k=t+1}^{j+1} \int_0^{(y^*_t - D_{t,k-2} - y^*_{j+1})^+} \right) \times L'_{j+1}(y^*_t - D_{t,j}) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t) \right] = 0, \tag{E.25}$$

where $\prod(\cdot)$ represents the convolution and $t \neq N+1$ and $y^*_{N+1} = 0$.

Define

$$\begin{aligned} P_{t,j} &= P_{t,j}(y^*_t, y^*_{t+1}, ..., y^*_{j+1}) \\ &= \left(\prod_{k=t+1}^{j+1} \int_0^{(y^*_t - D_{t,k-2} - y^*_{j+1})^+} \right) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t), \end{aligned} \tag{E.26}$$

as the probability that any unit will be ordered from period $t+1$ to period $j+1$.

For $x_t \leq y_t$, define

$$P_t^N = \alpha^{N-t+1} P_{t,N}, \tag{E.27}$$

as the discounted probability that there is no order until the end of the horizon.

Define also

$$T_t^N = \sum_{j=t}^{N-1} \alpha^{j-t+1} P_{t,j} h_{j+1}, \tag{E.28}$$

as the discounted partial expectation of the time from period $t+1$ until the first order in a period ($\geq t+1$) and N times the unit holding cost. This is equivalent to the discounted partial expectation of holding cost of a unit ordered at period t until first order is passed at a period ($\geq t+1$).

In order to develop the lower bound we need to rewrite the second term of the left hand side of (E.25) as follows

$$\begin{aligned} &\sum_{j=t}^{N} \alpha^{j+1-t} \left[\left(\prod_{k=t+1}^{j+1} \int_0^{(y^*_t - D_{t,k-2} - y^*_{j+1})^+} \right) \times L'_{j+1}(y^*_t - D_{t,j}) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t) \right] \\ &= \sum_{j=t}^{N-1} \alpha^{j+1-t} \left[\left(\prod_{k=t+1}^{j+1} \int_0^{(y^*_t - D_{t,k-2} - y^*_{j+1})^+} \right) \times L'_{j+1}(y^*_t - D_{t,j}) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t) \right] + \alpha^{N+1-t} \nu \mathbf{P}_{t,N}. \end{aligned} \tag{E.29}$$

Using (E.26), and noting that $L'_t \leq h_t$, and $L'_{N+1} = f'_{N+1} = \nu$, we get

$$\sum_{j=t}^{N} \alpha^{j+1-t} \left[\left(\prod_{k=t+1}^{j+1} \int_0^{(y_t^* - D_{t,k-2} - y_{j+1}^*)^+} \right) \times \right.$$
$$\left. L'_{j+1}(y_t^* - D_{t,j}) dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t) \right]$$
$$\leq \sum_{j=t}^{N-1} \alpha^{j+1-t} \left[\left(\prod_{k=t+1}^{j+1} \int_0^{(y_t^* - D_{t,k-2} - y_{j+1}^*)^+} \right) \times \right.$$
$$\left. h_{j+1} dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_t(D_t) \right] + \alpha^{N+1-t} \nu \mathbf{P}_{t,N}$$
$$= \sum_{i=t}^{N-1} \alpha^{j+1-t} P_{t,j} h_{j+1} + \alpha^{N-t+1} \nu P_{t,N}$$
$$= T_t^N + \nu P_t^N. \tag{E.30}$$

Substituting the results above in (E.25) we get

$$L'_t(y_t^*) + T_t^N + \nu P_t^N \geq 0. \tag{E.31}$$

Decrease y_t^* to y_t^{*L} to restore equality, then one gets

$$L'_t(y_t^{*L}) + T_t^N + \nu P_t^N = 0, \tag{E.32}$$

which is equivalent to

$$h_t F_t(y_t^{*L}) - b_t + b_t F_t(y_t^{*L}) + T_t^N + \nu P_t^N = 0. \tag{E.33}$$

That gives

$$y_t^{*L} = F_t^{-1} \left[\frac{b_t - T_t^N - \nu P_t^N}{h_t + b_t} \right], \tag{E.34}$$

which completes the proof. □

E.5 Proof of Theorem 8.5

Consider the partial derivative of the expected objective function Π_t with respect to $Q_{t,t+1}$, at the optimal point $(y_t^*, Q_{t,t+1}^*)$ given by

$$\frac{\partial \Pi_t(x_t, y_t, Q_{t,t+1})}{\partial Q_{t,t+1}}(x_t, y_t^*, Q_{t,t+1}^*) =$$
$$\tilde{c}_{t,t+1} - \alpha \tilde{c}_{t+1} + \alpha \int_0^\infty L'_{t+1}(y_t^* + Q_{t,t+1}^* - D_t) dF_t(D_t)$$
$$+\alpha \int_0^\infty \left[\alpha \int_0^\infty \frac{\partial \Pi_{t+2}^*}{\partial Q_{t,t+1}} (Q_{t,t+1}^* + y_t^* + Q_{t+1,t+2}^* - D_t - D_{t+1}) dF_{t+1} \right] dF_t = 0. \tag{E.35}$$

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As we have done for y_t we will write a near-myopic equation for $Q_{t,t+1}$. We will take into account only cases where units from $Q_{t,t+1}$ will be used in the following periods. Thus we assume that the demand in periods $t+1,...,N$ satisfies the following equation

$$Q^*_{t,t+1} + y^*_t - \sum_{k=t}^{j} D_k > y^*_{j+1}, \quad j = t+1,...,N-1, \quad t = 1,...,N-1. \tag{E.36}$$

Equation (E.36) implies that $(Q^*_{t+1,t+2}, Q^*_{t+2,t+3}, ..., Q^*_{N-1,N}) = (0,0,...,0)$.

Equation (E.35) implies the following near-myopic equation

$$\begin{aligned}\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha\int_0^\infty L'_{t+1}(y^*_t + Q^*_{t,t+1} - D_t)dF_t(D_t)&\\ + \alpha\int_0^\infty \Bigg[\alpha\int_0^{(Q^*_{t,t+1}+y^*_t-D_t-y^*_{t+2})^+}&\\ \frac{\partial\Pi^*_{t+2}}{\partial Q_{t,t+1}}(Q^*_{t,t+1} + y^*_t + Q_{t+1,t+2} - D_t - D_{t+1})dF_{t+1}\Bigg]dF_t = 0.&\end{aligned} \tag{E.37}$$

As when deciding for $Q_{t,t+1}$ we do not know the realized value of the demand of period t, namely D_t, and to simplify the development of the upper bound of $Q_{t,t+1}$ we will replace the D_t in (E.37), by the mean of the demand in period t, namely μ_t. We get then the following equation

$$\begin{aligned}&\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y^*_t + Q^*_{t,t+1} - \mu_t)\\ &+\alpha^2\int_0^{(Q^*_{t,t+1}+y^*_t-\mu_t-y^*_{t+2})^+} \frac{\partial\Pi^*_{t+2}}{\partial Q_{t,t+1}}(Q^*_{t,t+1} + y^*_t + Q_{t+1,t+2} - \mu_t - D_{t+1})dF_{t+1}\\ &= 0.\end{aligned} \tag{E.38}$$

By substituting (E.38) into itself, we get

$$\begin{aligned}&\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y^*_t + Q^*_{t,t+1} - \mu_t)\\ &+ \sum_{j=t+1}^{N} \alpha^{j+1-t}\Bigg[\left(\prod_{k=t+1}^{j}\int_0^{(Q^*_{t,t+1}+y^*_t-\mu_t-D_{t+1,k-1}-y^*_{j+1})^+}\right)\times\\ &\quad L'_{j+1}(Q^*_{t,t+1} + y^*_t - \mu_t - D_{t+1,j})dF_j(D_j)dF_{j-1}(D_{j-1})...dF_{t+1}(D_{t+1})\Bigg] = 0.\end{aligned} \tag{E.39}$$

Define

$$\begin{aligned}P'_{t,j} &= P'_{t,j}(Q^*_{t,t+1}, y^*_t, y^*_{t+1}, ..., y^*_{j+1})\\ &= \left(\prod_{k=t+1}^{j}\int_0^{(Q^*_{t,t+1}+y^*_t-\mu_t-D_{t+1,k-1}-y^*_{j+1})^+}\right)dF_j(D_j)...dF_{t+1}(D_{t+1}),\end{aligned} \tag{E.40}$$

$$T'^N_{t+1} = \sum_{j=t+1}^{N-1} \alpha^{j+1-t} P'_{t,j} h_{j+1}, \tag{E.41}$$

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and

$$P_t^{'N} = \alpha^{N+1-t} P'_{t,N}. \tag{E.42}$$

Using the fact that $L'_t(\cdot) \leq h_t, \quad t \leq N$ and $L'_{N+1} = \nu$, we can bound the left hand side of (E.39) by

$$\begin{aligned} &\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y_t^* + Q^*_{t,t+1} - \mu_t) \\ &\quad + \sum_{j=t+1}^{N-1} \alpha^{j+1-t} \Bigg[\Bigg(\prod_{k=t+1}^{j} \int_0^{(Q^*_{t,t+1}+y_t^*-\mu_t-D_{t+1,k-1}-y^*_{j+1})^+} \Bigg) \\ &\qquad \times h_{j+1} dF_j(D_j) dF_{j-1}(D_{j-1}) ... dF_{t+1}(D_{t+1}) \Bigg] + \alpha^{N-t+1} P'_{t,N}\nu \geq 0, \end{aligned} \tag{E.43}$$

which is equivalent to

$$\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y_t^* + Q^*_{t,t+1} - \mu_t) + T_{t+1}^{'N} + P_t^{'N}\nu \geq 0. \tag{E.44}$$

If we decrease $Q^*_{t,t+1}$ into $Q^{*L}_{t,t+1}$ in order to restore equality, we get

$$\tilde{c}_{t,t+1} - \alpha\tilde{c}_{t+1} + \alpha L'_{t+1}(y_t^* + Q^{*L}_{t,t+1} - \mu_t) + T_{t+1}^{'N} + P_t^{'N}\nu = 0, \tag{E.45}$$

which gives

$$Q^{*L}_{t,t+1} = \left(F^{-1}_{t+1} \left[\frac{\alpha\tilde{c}_{t+1} - \tilde{c}_{t,t+1} - T_{t+1}^{'N} - P_t^{'N}\nu + \alpha b_{t+1}}{\alpha(h_{t+1} + b_{t+1})} \right] + \mu_t - y_t^* \right)^+, \tag{E.46}$$

which completes the proof. □

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Résumé : Le phénomène d'incertitude, dont les sources sont variées, est rencontré dans plusieurs domaines et on devrait y faire face. Cette incertitude est due essentiellement à notre incapacité à prédire avec exactitude le comportement futur d'une partie ou de la totalité d'un système. Dans les dernières décades, plusieurs techniques mathématiques ont été développées pour maîtriser cette incertitude, afin de réduire son impact négatif, et par conséquent, l'impact négatif de notre méconnaissance.

Dans le domaine du « Supply Chain Management » la source principale d'incertitude est la demande future. Cette demande est, en général, modélisée par des lois de probabilité paramétrées en utilisant des techniques de prévision. L'impact de l'incertitude de la demande sur les performances de la « Supply Chain » est important: par exemple, le taux mondial de rupture de stock, dans l'industrie de distribution était en 2007 de 8.3%. De l'autre côté, le taux mondial de produits invendus, dans la grande distribution, était en 2003 de 1%. Ces deux types de coûts, qui sont dus essentiellement à l'incertitude de la demande, représentent des pertes significatives pour les différents acteurs de la « Supply Chain ».

Dans cette thèse, on s'intéresse au développement de modèles mathématiques de planification de production et de gestion de stock, qui prennent en compte ce phénomène d'incertitude sur la demande, essentiellement pour de produits à court cycle de vie. On propose plusieurs modèles de planification de production, à petit horizon de planification, qui prennent en compte les différents aspects de notre problématique, tels que les capacités de production, la remise à jour des prévisions de la demande, les options de réservation de capacité, et les options de retour « Payback » des produits. On souligne, dans ces modèles, un aspect important qui prend de l'ampleur à cause de la mondialisation, et qui est lié à la différence entre les coûts de production des différents fournisseurs. On propose à la fin de la thèse, un modèle généralisé qui pourrait être appliqué à des produits à long cycle de vie, et qui exploite quelques résultats obtenus pour les produits à court cycle de vie. Tous ces modèles sont résolus analytiquement ou bien numériquement en utilisant la programmation dynamique stochastique.

Mots clefs : Planification de Production, Gestion de Stocks, Demande Aléatoire, Minimisation de Coûts (Maximisation de Profits), Programmation Dynamique Stochastique, Solutions Analytiques et Numériques, Produits à Court Cycle de Vie.

Abstract: The phenomenon of uncertainty is encountered in many domains and should be faced. Even if the sources of this phenomenon are numerous, it is essentially due to our incapacity to predict precisely the future behaviour of a part or the whole of a given system. Many mathematical techniques have been emerged in the few last decades, which permit to master the uncertainty, and therefore to reduce our ignorance of how systems really behave.

In the Supply Chain Management domain, the main source of randomness is the future demand. This later is generally modelled using probability distribution functions, which are developed via different forecasting techniques. The influence of this demand variability on the performance of the Supply Chain is very important: for example, in 2007 the global inventory shortage rate in the retail industry were around 8.3%. On the other hand, in 2003 the global Unsaleable products cost around 1% in the grocery industry. These two types of costs, which are mainly caused by the uncertainty of the future demand, represent important lost for the whole Supply Chain actors.

This Ph.D. dissertation aims at developing mathematical production planning and inventory management models, which take into consideration the randomness of the future demand in order to reduce its economic negative impact, essentially for short life cycle products. We provide many planning models that consider the main issues of the planning problems, such as the production capacities, the information updating processes, the supply contracts and the advanced capacity reservation in a total costs minimization context. We consider in these models some aspects that are not considered in the literature, such as the "Payback" or the return options. We emphasize also on an important issue that characterize the globalization of the industry, which may be resumed in the difference between the procurement costs of the different suppliers. This issue is considered in the most chapters presenting models for short life cycle products and in the last chapter it is generalized to a long life cycle products setting. All the presented models are solved either analytically or numerically using the dynamic stochastic programming.

Keywords: Production Planning, Inventory Management, Stochastic Demand, Costs Minimization (Profits Maximization), Stochastic Dynamic Programming, Analytical and Numerical Resolution, Short Life Cycle Products.