The Connectivity Order of Links

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Abstract — We associate at each link a connectivity space which describes its splittability properties. Then, the notion of order for finite connectivity spaces results in the definition of a new numerical invariant for links, their connectivity order. A section of this short paper presents a theorem which asserts that every finite connectivity structure can be realized by a link : the Brunn-Debrunner-Kanenobu Theorem.

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1 Order of Finite Connectivity Spaces

Let us recall the definition of a connectivity space [1, 4].

Definition 1 (Connectivity spaces) *A* connectivity space *is a couple* $\mathbf{X} = (X, \mathcal{K})$ *where* X *is a set and* $\mathcal{K}$ *is a set of nonempty subsets of* X *such that*

$$\forall \mathcal{I} \in \mathcal{P}(\mathcal{K}), \bigcap_{K\in\mathcal{I}} K \neq \emptyset \Longrightarrow \bigcup_{K\in\mathcal{I}} K \in \mathcal{K}.$$

The set X *is called the* support *of the space* $\mathbf{X}$*, the set* $\mathcal{K}$ *is its* connectivity structure*. The elements of* $\mathcal{K}$ *are called the* connected subsets *of* $\mathbf{X}$*. The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets. A connectivity space is called* integral *if every singleton subset is connected. A connectivity space is called* finite *if the number of its points is finite.*

Definition 2 (Irreducible connected subsets) *Let* $\mathbf{X} = (X, \mathcal{K})$ *be a connectivity space. A connected subset* $K \in \mathcal{K}$ *is called* reducible *if there are two connected subsets* $A \subsetneq K$ *and* $B \subsetneq K$ *such that*

$$K = A \cup B \text{ and } A \cap B \neq \emptyset.$$

A connected subset is said to be irreducible *if it is not reducible.*

In the sequel, all connectivity spaces will be supposed to be integral and finite.

Definition 3 (Order of irreducible connected subsets) *Let* $\mathbf{X} = (X, \mathcal{K})$ *be a (finite and integral) connectivity space. We define by induction the* order *of each irreducible subset. Singleton subsets are said to be of order 0. The order* $\omega(L)$ *of an irreducible subset* L *which has more than one point is defined by*

$$\omega(L) = 1 + \max_{K \in \mathcal{S}(L)} \omega(K)$$

where $\mathcal{S}(L)$ *is the set of irreducible connected subsets which are strictly included in* L*.*

Definition 4 (Order of a connectivity space) *The order* $\omega(\mathbf{X})$ *of a (finite and integral) nonempty connectivity space* $\mathbf{X} = (X, \mathcal{K})$ *is the maximum of the order or its irreducible connected subsets.*

Instead of $\omega(\mathbf{X})$, we generally write just $\omega(X)$.

Remark. It is usefull to represent the structure of a (finite and integral) connectivity space $\mathbf{X}$ by a graph G whose vertices are the irreducible connected subsets of $\mathbf{X}$ and edges express the inclusion relations : if a and b are irreducible connected subsets of $\mathbf{X}$, (a, b) is an edge of G iff $a \subseteq b$ and there is no irreducible connected subset c such that $a \subsetneqq c \subsetneqq b$. In [5], I called this graph the *generic graph* of the connectivity space $\mathbf{X}$; the vertices of this graph, *i.e.* the irreducible connected subsets of $\mathbf{X}$, are the *generic points* of $\mathbf{X}$. The order of a space is then the maximal length of paths in the generic graph.

Proposition 1 *The order* $\omega(X)$ *of a nonempty finite integral connectivity space is always less than the number* $\sharp X$ *of its points :* $\omega(X) \leq \sharp X - 1$*. The order* $\omega(X)$ *is zero iff the space is totally disconnected, that is the only connected subsets are the singletons.*

Example. Let $n \geq 1$ an integer, and $(A_n, \mathcal{A}_n$ the connectivity space defined by $A_n = \{1, \cdots, n\}$ and

$$\mathcal{A}_n = \{\{1\}, \{2\}, \cdots, \{n\}\} \cup \{\{1,2\}, \{1,2,3\}, \cdots, \{1,2,\cdots,n\}\}.$$

Then $\omega(A_n) = n - 1$. Indeed, in this space each connected subset is irreducible, and its order is, by induction, equal to its cardinal minus one. It is, up to isomorphism, the only connectivity space with n points which is of order $n - 1$.

2 The Brunn-Debrunner-Kanenobu Theorem

At each (tame) link L, we can associate a connectivity space $\mathbf{X}_L$ taking the components of the link L as points of $\mathbf{X}_L$ and nonsplittable sublinks of L (one considers knots, *i.e.* sublinks with only one component, as nonsplittable links) defining the connected nonempty subsets of $\mathbf{X}_L$.

Definition 5 *The connectivity structure of the connectivity space* $\mathbf{X}_L$ *associated to a link* L *is called the* splittability structure *of* L*.*

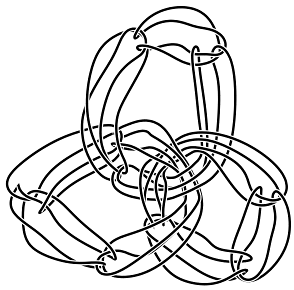

Figure 1: A Borromean ring of borromean rings.

Example. The splittability structure of the Borromean ring is (isomorphic to) $\{\{1\},\{2\},\{3\},\{1,2,3\}\}$.

In [4, 5], I asked whether every finite connectivity space can be represented by a link, that is whether exists (at least) one link whose connectivity structure is (isomorphic to) the one given. It appears that in 1892, Brunn [2] first asked this question, without formalizing the concept of a connectivity space. His answer was positive, and he gave the idea of a proof based on a construction using some of the links now called "brunnian". In 1964, Debrunner [3], rejecting the Brunnian "proof", gave another construction, proving it but only for n-dimensional links with $n \geq 2$. In 1985, Kanenobu [6, 7] seems to be the first to give a proof of the possibility to represent every finite connectivity structure by a classical link, a result which has not been, up to now, very wellknown. The key idea of those different constructions is already in the Brunn's article : for him, what we call "Brunnian links" are not so interesting in themselves, but for the constructions they allow to make, that is the representation of *all* finite connectivity strutures by links.

Theorem 2 (Brunn-Debrunner-Kanenobu) *Every finite connectivity structure is the splittability structure of a link in* $\mathbf{R}^3$*.*

3 The Connectivity Order of Links

At each link L, we associate its connectivity order $\omega(L)$, *i.e.* the order of the connectivity space associated to L.

Examples. The link pictured on the figure 1 has a connectivity order 2. The one of the figure 2 has a connectivity order 8, which is the maximal order for a link with 9 components. The way this very asymetrical link splits when a component is erased or cut highly depends on the position of this component in the link, as shows its connectivity structure, which is the one we called $\mathcal{A}_9$.

Remark. The connectivity order is not a Vassiliev finite type invariant for links. For example, it is easy to check that the connectivity order of the singular link

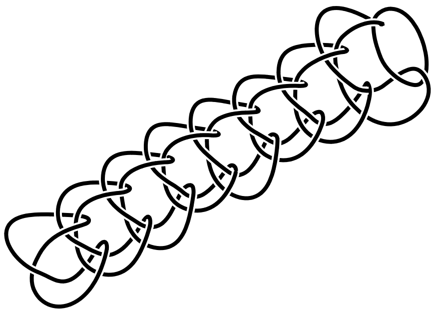

Figure 2: A link with a connectivity order 8.

with two components, a circle and another component crossing this circle at $2n$ double-points, is greater than 2^n.

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