

# Ten penalisation results of Brownian motion involving its one-sided supremum until first and last passage times, VIII

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**Abstract** We penalise Brownian motion by a function of its one-sided supremum considered up to the last zero before  $t$ , resp. first zero after  $t$ , of that Brownian motion. This study presents some analogy with penalisation by the longest length of Brownian excursions, up to time  $t$ .

**Key words** Penalisations of Brownian paths, one-sided supremum, last and first zeroes before and after a fixed time  $t$ .

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# 1 Introduction

## 1.1 Notations

Throughout this work,  $(\Omega, (X_t, \mathcal{F}_t)_{t \geq 0}, \mathcal{F}_\infty = \vee_{t \geq 0} \mathcal{F}_t, P_x(x \in \mathbb{R}))$  denotes the canonical realization of one-dimensional Wiener process.  $\Omega = \mathcal{C}(\mathbb{R}_+ \rightarrow \mathbb{R})$  is the space of continuous functions,  $(X_t, t \geq 0)$  the coordinate process on this space,  $(\mathcal{F}_t, t \geq 0)$  its natural filtration and  $(\mathbb{P}_x, x \in \mathbb{R})$  the family of Wiener measures on  $(\Omega, \mathcal{F}_\infty)$  with  $\mathbb{P}_x(X_0 = x) = 1$ . When  $x = 0$ , we write simply  $\mathbb{P}$  for  $\mathbb{P}_0$ .

For every  $t \geq 0$ , let  $g_t := \sup\{s \leq t; X_s = 0\}$  denote the last zero before  $t$  and  $d_t := \inf\{s \geq t; X_s = 0\}$  the first zero after  $t$ . Thus  $d_t - g_t$  is the duration of the excursion which straddles  $t$ .

For every  $t \geq 0$ , let  $S_t := \sup_{s \leq t} X_s$ . The increasing process  $(S_t, t \geq 0)$  is the one-sided supremum process associated with  $X$ .

We also denote :  $X_t^* := \sup_{s \leq t} |X_s|$ .

For every  $a \in \mathbb{R}$ ,  $T_a := \inf\{t; X_t = a\}$  denotes the first hitting time of level  $a$  by the process  $(X_t, t \geq 0)$ .

We denote by  $(L_t, t \geq 0)$  the (continuous) local time process at level 0 for  $(X_t, t \geq 0)$  and  $(\tau_l, l \geq 0)$  its right continuous inverse :

$$\tau_l := \inf\{s; L_s > l\}, \quad l \geq 0 \quad (1.1)$$

$b(\mathcal{F}_t)$  is the space of bounded and  $\mathcal{F}_t$  measurable r.v.'s.

For every  $t \geq 0$ , we denote by  $\theta_t$  the operator of time translation of the Brownian trajectory :

$$X_s \circ \theta_t = X_{s+t} \quad (s, t \geq 0)$$

To any real number  $a$ , we associate  $a^+ = \sup(0, a)$  and  $a^- = -\inf(0, a)$ .

## 1.2 Some useful martingales

Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a probability density, i.e,  $\varphi$  is a Borel function with integral equal to 1. We denote by  $\Phi$  the primitive of  $\varphi$  which is equal to 0 at  $x = 0$  :

$$\Phi(x) := \int_0^x \varphi(y) dy \quad (1.2)$$

In this work, we shall use in an important manner the Azéma-Yor martingale  $(M_t^\varphi, t \geq 0)$  defined by :

$$M_t^\varphi := \varphi(S_t)(S_t - X_t) + 1 - \Phi(S_t) \quad (1.3)$$

(see, for example, [RVY, II]).

We also note that for every predictable and bounded process  $(H_s, s \geq 0)$  the process  $(H_{g_s} \cdot X_s, s \geq 0)$  is a martingale since, from the balayage formula (cf. [RY]) :

$$H_{g_s} \cdot X_s = H_0 \cdot X_0 + \int_0^s H_{g_u} \cdot dX_u \quad (1.4)$$

In particular, for every pair of real numbers  $\alpha$  and  $\beta$  :

$$\left( \alpha \varphi(S_{g_s}) \cdot X_s + \beta M_s^\varphi, s \geq 0 \right) \text{ is a martingale} \quad (1.5)$$

and this martingale is positive as soon  $0 \leq \alpha \leq \beta$ .

### 1.3 Penalisation by $(\varphi(S_t), t \geq 0)$

Let  $\varphi$  a probability density on  $\mathbb{R}_+$ . In [RVY, II], we studied the penalisation of the Wiener measure by  $(\varphi(S_t), t \geq 0)$ . More precisely we obtained the following theorem :

**Theorem 0.**

1) For every  $s \geq 0$  and  $\Lambda_s \in b(\mathcal{F}_s)$  :

$$\lim_{t \rightarrow \infty} \frac{E[\Lambda_s \varphi(S_t)]}{E[\varphi(S_t)]} = E[\Lambda_s M_s^\varphi] := Q^\varphi(\Lambda_s) \quad (1.6)$$

where  $(M_s^\varphi, s \geq 0)$  is the positive martingale defined in (1.3).

2) Under  $Q^\varphi$ , the probability on  $(\Omega, \mathcal{F}_\infty)$  induced by (1.6), the canonical process  $(X_t, t \geq 0)$  satisfies :

- i)  $S_\infty < \infty$  a.s and  $S_\infty$  admits  $\varphi$  as probability density,
- ii) Let  $g := \sup\{s \geq 0; S_s < S_\infty\}$ . Then  $g < \infty$  a.s and
  - $(X_s, s \leq g)$  and  $(X_g - X_{g+s}, s \geq 0)$  are independent
  - $(X_g - X_{g+s}, s \geq 0)$  is a 3-dimensional Bessel process starting from 0.
  - Conditionally on  $S_\infty = y$ ,  $(X_s, s \leq g)$  is a Brownian motion stopped when it first reaches level  $y$ .

In particular,  $X_t \xrightarrow[t \rightarrow \infty]{} -\infty$   $Q^\varphi$  a.s.

On the other hand, we have (see [RVY, II]) :

$$\lim_{t \rightarrow +\infty} \sqrt{\frac{\pi t}{2}} E[\varphi(S_t)] = \int_0^\infty \varphi(y) dy \quad (1.7)$$

### 1.4 The aim of this paper

When we penalize Wiener measure by  $(\varphi(S_t), t \geq 0)$ , with  $\varphi$  integrable, then, as "  $\varphi$  is small at  $+\infty$ ", the trajectories such that  $S_t$  is small are favored. Then it is not so astonishing to notice and this is explained and made precise with Theorem 0, that, under the probability  $Q^\varphi$ , one has :  $X_t \xrightarrow[t \rightarrow \infty]{} -\infty$  and  $S_\infty < \infty$  a.s. Consequently, at least heuristically, the following may happen (and again, this will be justified precisely in the sequel of this paper) :

#### 1.4.1

Penalizing by  $(\varphi(S_t)\mathbf{1}_{X_t < 0}, t \geq 0)$  should not differ much from the penalisation from  $(\varphi(S_t), t \geq 0)$ , since either factor  $\varphi(S_t)$  and  $\mathbf{1}_{X_t < 0}$  favor trajectories for which  $S_t$  is small, or  $X_t$  belongs to  $\mathbb{R}_-$ .

#### 1.4.2

On the other hand, at this point, penalizing by  $(\varphi(S_t)\mathbf{1}_{X_t > 0}, t \geq 0)$  seems to be less easy to understand a priori since the factor  $\varphi(S_t)$  favors the trajectories for which  $S_t$  is small, while the factor  $\mathbf{1}_{X_t > 0}$  favors trajectories such that  $S_t$  is large.

How can one describe the penalisation effect precisely?

#### 1.4.3

In order to answer this question, and also to organize our discussion properly, we recall an analogous phenomenon, studied in [RVY, VII], and also in [RY, M1] in which we studied three penalisations L1, L2, L3 of Wiener measure, related to excursions lengths.

L1 : Penalisation related to the length of the longest excursion before  $g_t$ .

L2 : Penalisation related to the length of the longest excursion before  $t$ .

L3 : Penalisation related to the length of the longest excursion before  $d_t$ .

We then noticed that the penalisations L3 and L2 lead to the same limiting probability, whilst L1 yields to a very different limiting probability.

The aim of this paper is to understand what happens when lengths of excursions are replaced by their heights, i.e : when we study penalisations by  $(\varphi(S_{g_t}), t \geq 0)$ ,  $(\varphi(S_t), t \geq 0)$ ,  $(\varphi(S_{d_t}), t \geq 0)$ .

#### 1.4.4

The above considerations led us to the study of the 9 following penalisations with respect to the weight processes :

$$(\varphi(S_t), t \geq 0); (\varphi(S_t)\mathbf{1}_{X_t < 0}, t \geq 0); (\varphi(S_t)\mathbf{1}_{X_t > 0}, t \geq 0) \quad (1.8)$$

$$(\varphi(S_{g_t}), t \geq 0); (\varphi(S_{g_t})\mathbf{1}_{X_t < 0}, t \geq 0); (\varphi(S_{g_t})\mathbf{1}_{X_t > 0}, t \geq 0) \quad (1.9)$$

$$(\varphi(S_{d_t}), t \geq 0); (\varphi(S_{d_t})\mathbf{1}_{X_t < 0}, t \geq 0); (\varphi(S_{d_t})\mathbf{1}_{X_t > 0}, t \geq 0) \quad (1.10)$$

#### 1.4.5

Below, we find that, as for the penalisations by the longest length of excursions, penalisations in (1.8) and (1.10) yield to the same limiting probabilities, and that these probabilities differ from those obtained from the penalisations in (1.9).

### 1.4.6

Finally, here is a 10<sup>th</sup> penalisation study, which fits naturally with the 9 previous ones : let  $I_t := \inf_{s \leq t} X_s$  and  $X_t^* := \sup_{s \leq t} |X_s| = S_t \vee -I_t$ . In [RVY, X], we studied the penalisation by a function of  $X_t^*$ , and more generally by certain functions of  $S_t$  and  $I_t$ . Here, we complete this study with the penalisation by the process  $(\varphi(X_{gt}^*, t \geq 0)$ , and compare the result with the 9 preceding studies.

### 1.5 Our results

They are presented in the form of a Table, where  $\varphi$  denotes a Borel function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  and the following quantities play a role :

$$I_\varphi = \int_{\mathbb{R}_+} \varphi(x) dx \quad \text{and} \quad J_\varphi = \int_{\mathbb{R}_+} \varphi(x) x^2 dx$$

This table summarizes the following generic theorem :

**Generic theorem.** For  $i = 1, 2, \dots, 10$ , let  $(F_t^{(i)}, t \geq 0)$  denote the  $i^{\text{th}}$  process of Row 1 of the Table, where Row 2 indicates the hypotheses on  $\varphi$ . Then :

For every  $s \geq 0$  and every  $\Lambda_s \in b(\mathcal{F}_s)$  :

$$\lim_{t \rightarrow \infty} \frac{E[\Lambda_s F_t^{(i)}]}{E[F_t^{(i)}]} = E[\Lambda_s M_s^{(i)}] := Q^{(i)}(\Lambda_s) \quad (1.11)$$

where  $Q^{(i)}$ , resp.  $(M_s^{(i)}, s \geq 0)$ , is the probability on  $(\Omega, \mathcal{F}_\infty)$ , resp. the  $((\mathcal{F}_s, s \geq 0), \mathbb{P})$  martingale, found on the  $i^{\text{th}}$  line of Row 4, resp. Row 5, of the Table.

#### Remark 1

- 1) We have chosen our notations for the Table, in a mnemonic manner : for example,  $(g,+)Q^\varphi$  and  $(g,+)M_s^\varphi, s \geq 0)$  are the probability and the martingale obtained by penalisation by  $(\varphi(S_{gt})1_{X_t > 0}, t \geq 0)$ ;  $(d,-)Q$  and  $(d,-)M_s^\varphi, s \geq 0)$  are the probability and the martingale obtained by penalisation by  $(\varphi(S_{dt})1_{X_t < 0}, t \geq 0)$ , and so on...
- 2) For lines 1, 2, 3, 5, 7, 8 and 9, the canonical process  $(X_t, t \geq 0)$  under the limiting probability  $\mathbb{Q}$  is described by Theorem 0, since this probability  $\mathbb{Q}$  is of the form  $\mathbb{Q}^\theta$ , with  $\theta = \varphi$  for lines 1, 2, 5, 7 and 8 and  $\theta = \psi$  for lines 3 and 9 (cf (1.3)).
- 3) Row 3 gives the equivalent of  $E(F_t^{(i)})$  for  $i = 1, \dots, 10$  when  $t \rightarrow \infty$ . We note that this equivalent is of the form  $Ct^{-\frac{1}{2}}$  for all lines, except lines 3 and 9 where this equivalent is  $Ct^{-\frac{3}{2}}$ .
- 4) Line 1 of the Table is the object of Theorem 0 and its proof is found in [RVY, II], to which we refer the reader.
- 5) Line 2 of the Table answers the question of subsection 1.4.1 : indeed, the penalisations by  $(\varphi(S_t), t \geq 0)$  and  $(\varphi(S_t)1_{X_t < 0}, t \geq 0)$  generate the same limiting probabilities.

An answer to the question of subsection 1.4.2 is obtained by Lines 1 and 3 of the Table, which show that the penalisations by  $(\varphi(S_t), t \geq 0)$  and  $(\varphi(S_t)1_{X_t > 0}, t \geq 0)$  lead to different limiting probabilities. However they do not differ so much, the two limiting martingales are

of the form  $(M_s^\theta, s \geq 0)$  (cf (1.3)) with  $\theta = \varphi$  in the first case and  $\theta = \psi$ , with  $\psi(x) = \varphi(x)x^2 + 2 \int_x^\infty \varphi(y)ydy$  in the second case. Thus, from Theorem 0, the limiting probabilities  $Q^\varphi$  and  $(+)^Q^\varphi$  differ essentially by the fact that

$$Q^\varphi(S_\infty \in dx) = \varphi(x)dx \quad \text{whereas} \quad : \quad (+)^Q^\varphi(S_\infty \in dx) = \psi(x)dx.$$

	Weight process : $(F_t, t \geq 0)$	Condition on $\varphi$	Equivalent of $D_t := E(F_t)$	Symbol for $Q$	Martingale density : $\frac{dQ}{dP} \Big _{\mathcal{F}_s} = M_s$	References
1)	$(\varphi(S_t), t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$Q^\varphi$	$M_s^\varphi = [\varphi(X_s)(S_s - X_s) + \int_{S_s}^\infty \varphi(x) dx] \frac{1}{I_\varphi}$	[RVY, II], (1.6)
2)	$(\varphi(S_t)1_{X_t < 0}, t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(-)Q^\varphi = Q^\varphi$	$(-)M_s^\varphi = M_s^\varphi$	Relation (4.4)
3)	$(\varphi(S_t)1_{X_t > 0}, t \geq 0)$	$I_\varphi + J_\varphi < \infty$	$\frac{3}{2} \sqrt{\frac{2}{\pi t^3}} J_\varphi$	$(+)Q^\varphi = Q^\psi$	$(+)M_s^\varphi = M_s^\psi$ , with $\psi(x) = \varphi(x)x^2 + 2 \int_x^\infty \varphi(y)y dy$	Section 4.4
4)	$(\varphi(S_{g_t}), t \geq 0)$	$I_\varphi < \infty$	$2\sqrt{\frac{2}{\pi t}} I_\varphi$	$(g)Q^\varphi$	$(g)M_s^\varphi = \frac{1}{2I_\varphi} \varphi(S_{g_s})X_s + M_s^\varphi$	Theorem 1
5)	$(\varphi(S_{g_t})1_{X_t < 0}, t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(g,-)Q^\varphi = Q^\varphi$	$(g,-)M_s^\varphi = M_s^\varphi$	Relation (4.4)
6)	$(\varphi(S_{g_t})1_{X_t > 0}, t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(g,+)Q^\varphi$	$(g,+)M_s^\varphi = 2(g)M_s^\varphi - M_s^\varphi$ $= \frac{1}{I_\varphi} (\varphi(S_{g_s})X_s) + M_s^\varphi$	Section 4.3
7)	$(\varphi(S_{d_t}), t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(d)Q^\varphi = Q^\varphi$	$(d)M_s^\varphi = M_s^\varphi$	Theorem 4
8)	$(\varphi(S_{d_t})1_{X_t < 0}, t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(d,-)Q^\varphi = Q^\varphi$	$(d,-)M_s^\varphi = M_s^\varphi$	Relation (4.4)
9)	$(\varphi(S_{d_t})1_{X_t > 0}, t \geq 0)$	$I_\varphi + J_\varphi < \infty$	$\frac{3}{2} \sqrt{\frac{2}{\pi t^3}} J_\varphi$	$(d,+)Q^\varphi = (+)Q^\varphi$ $= Q^\psi$	$(d,+)M_s^\varphi = M_s^\psi = (+)M_s^\varphi$ with $\psi(x) = \varphi(x)x^2 + 2 \int_x^\infty \varphi(y)y dy$	Section 4.5
10)	$(\varphi(X_{g_t}^*), t \geq 0)$	$I_\varphi < \infty$	$\sqrt{\frac{2}{\pi t}} I_\varphi$	$(g,*)Q^\varphi$	$\frac{1}{I_\varphi} \{ \varphi(X_{g_s}^*)X_s + \varphi(X_s^*)(X_s^* -  X_s ) + \int_{X_s^*}^\infty \varphi(y)dy \}$	Section 5

## 1.6 Organization of this paper

- Section 2 is devoted to the precise statement and the proof of Theorem 1 ; this Theorem corresponds to Line 4 of the Table, i.e penalisations by  $(\varphi(S_{g_t}), t \geq 0)$ .
- Section 3 is devoted to the statement and precise proof of Theorem 4, which corresponds to Line 7 of the Table, i.e penalisations by  $(\varphi(S_{d_t}), t \geq 0)$ .
- Section 4 is devoted to the proof of the 6 penalisation, represented in the Table in lines 2, 3, 5, 6, 8 and 9.
- Line 10 of the Table is developed in Theorem 6, which is enounced and proved in Section 5.
- At the end of this work, in a short Section 6, we explain the position of this paper within our studies of penalisations, which we have undertaken since 2002.

## 2 Penalisation by $(\varphi(S_{g_t}), t \geq 0)$

Recall that  $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$  is a probability density on  $\mathbb{R}_+$  :

$$\int_0^\infty \varphi(y)dy = 1 \quad (2.1)$$

and define

$$\Phi(x) := \int_0^x \varphi(y)dy. \quad (2.2)$$

**Theorem 1.** *Under the preceding hypothesis :*

- 1) *For every  $s \geq 0$  and  $\Lambda_s \in b(\mathcal{F}_s)$  :*

$$\lim_{t \rightarrow \infty} \frac{E(\Lambda_s \varphi(S_{g_t}))}{E(\varphi(S_{g_t}))} = E[\Lambda_s \text{}^{(g)}M_s^\varphi] \quad (2.3)$$

with :

$$\text{}^{(g)}M_s^\varphi := \frac{1}{2} \varphi(S_{g_s})|X_s| + \varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s) = \frac{1}{2} \varphi(S_{g_s})X_s + M_s^\varphi \quad (2.4)$$

Moreover,  $(\text{}^{(g)}M_s^\varphi, s \geq 0)$  is a positive martingale, which writes :

$$\text{}^{(g)}M_s^\varphi = 1 + \frac{1}{2} \int_0^s \varphi(S_{g_u}) \text{sgn } X_u dX_u - \int_0^s \varphi(S_u) 1_{X_u > 0} dX_u. \quad (2.5)$$

- 2) *The formula :*

$$\text{}^{(g)}Q^\varphi[\Lambda_s] := E(\Lambda_s \text{}^{(g)}M_s^\varphi) \quad (\Lambda_s \in \mathcal{F}_s) \quad (2.6)$$

induces a probability  $\text{}^{(g)}Q^\varphi$  on  $(\Omega, \mathcal{F}_\infty)$ . Under  $\text{}^{(g)}Q^\varphi$ , the canonical process  $(X_t, t \geq 0)$  satisfies the following :

- i) *let  $g := \sup\{t ; X_t = 0\}$ . Then :*

$$\text{}^{(g)}Q^\varphi\{0 < g < \infty\} = 1 \quad (2.7)$$

ii) the couple  $(L_g, S_g) \equiv (L_\infty, S_g)$  admits the density :

$$f_{L_g, S_g}^{(g)Q^\varphi}(v, c) = \frac{1}{4} \frac{v}{c^2} e^{-\frac{v}{2c}} \varphi(c) 1_{v>0, c>0} \quad (2.8)$$

In particular,  $S_g$  admits  $\varphi$  as its density,  $\frac{1}{2} \frac{L_g}{S_g}$  is a gamma variable, with parameter 2

(i.e, with probability density equal to  $e^{-z} z 1_{z \geq 0}$ ) and  $S_g$  and  $\frac{L_g}{S_g}$  are independent ;

iii)  ${}^{(g)}Q^\varphi\{S_\infty = \infty\} = \frac{1}{2}$  and, conditionally on  $S_\infty < \infty$ ,  $S_\infty$  admits  $\varphi$  as its density.

**3)** Under  ${}^{(g)}Q^\varphi$  :

i)  $(X_{g+t}, t \geq 0) \stackrel{(law)}{=} (\epsilon R_t, t \geq 0)$  where  $\epsilon$  is a symmetric Bernoulli r.v. and  $R$  a 3 dimensional Bessel process started at 0 ;  $(X_t, t \geq g)$  ;  $\epsilon$  and  $R$  are independent.

ii) conditionally upon  $L_g = v$  and  $S_g = c$ , the process  $(X_t, t \leq g)$  is a Brownian motion stopped at  $\tau_v$  and conditioned upon  $S_{\tau_v} = c$ .

**4)** Under  ${}^{(g)}Q^\varphi$ ,  $(|X_t| + L_t, t \geq 0)$  is a 3-dimensional Bessel process, independent from  $(S_g, L_g)$

## 2.1

For the proof of Theorem 1, we need the following preliminary results, any of which is classically found in the Brownian literature ; nonetheless for ease of reading, we shall give a complete proof of these results.

**Proposition 2.** Under Wiener measure  $P$ , the 5 following properties hold :

$$1 \quad \text{For any } t \geq 0 : S_{gt} \stackrel{(law)}{=} \frac{1}{2} \sqrt{t} |N|, \quad \text{with } N \text{ a reduced Gaussian r.v.} \quad (2.9)$$

$$2 \quad \text{For any } a > 0, S_{gT_a} \text{ is uniform on } [0, a]. \quad (2.10)$$

$$3 \quad \text{For any } l > 0, S_{T_l} \text{ admits as density : } f_{S_{T_l}}(c) = \frac{l}{2c^2} e^{-\frac{l}{2c}} 1_{c \geq 0}. \quad (2.11)$$

$$4 \quad \text{Let } a < 0. S_{T_a} \text{ admits as density : } f_{S_{T_a}}(c) = -\frac{a}{(c-a)^2} 1_{c > 0}. \quad (2.12)$$

$$\text{i.e. } S_{T_a} \stackrel{(law)}{=} a - \frac{a}{U} \text{ where } U \text{ is uniform on } [0, 1].$$

$$5 \quad \text{Let } a > 0. \text{ Under } P_a, S_{T_0} \stackrel{(law)}{=} \frac{a}{U}, \quad \text{where } U \text{ is uniform on } [0, 1]. \quad (2.13)$$

From this point 5, we deduce that, for every Borel function  $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$  :

$$E[\psi(S_{d_t}) | \mathcal{F}_t] = \int_0^1 \psi(S_t \vee \frac{X_t^+}{u}) du \quad (2.14)$$

An equivalent form of (2.14) is :

$$E(\psi(S_{d_t})|\mathcal{F}_t) = \psi(S_t) \left(1 - \frac{X_t^+}{S_t}\right) + X_t^+ \int_{S_t}^{\infty} \frac{\psi(v)}{v^2} dv \quad (2.15)$$

**Proof of Proposition 2.**

1) Thanks to the scaling property of Brownian motion :  $S_{g_t} \stackrel{(law)}{=} \sqrt{t} S_{g_1}$ .

On the other hand, for  $\alpha > 0$ , one has :  $P(S_{g_1} < \alpha) = P(g_1 < T_\alpha) = P(1 < d_{T_\alpha})$ .

Now,

$$\begin{aligned} d_{T_\alpha} &= T_\alpha + T_0 \circ \theta_{T_\alpha} \stackrel{(law)}{=} T_\alpha + T'_\alpha \text{ (where } T'_\alpha \text{ is an independent copy of } T_\alpha) \\ &\stackrel{(law)}{=} T_{2\alpha}. \end{aligned}$$

Now, by the reflection principle :

$$P(S_{g_1} < \alpha) = P(1 < T_{2\alpha}) = P(S_1 < 2\alpha) = P\left(\frac{1}{2}|N| < \alpha\right).$$

2) We have :

$$\begin{aligned} P(S_{g_{T_a}} \geq x) &= P(g_{T_a} \geq T_x) \\ &= P(\text{after } T_x, X \text{ reaches 0 before } a) \\ &= 1 - \frac{x}{a} \end{aligned} \quad (2.16)$$

3) We get, successively, for  $c > 0$  :

$$P(S_{\tau_l} < c) = P(\tau_l < T_c) = P(l < L_{T_c})$$

It is well known that  $L_{T_c}$  is exponentially distributed with parameter  $\frac{1}{2c}$ .

Indeed, for any  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  bounded with integral equal to 1, the process  $\left(h(L_t)X_t^+ - \frac{1}{2}H(L_t), t \geq 0\right)$  is a martingale, with  $H(x) := \int_0^x h(y)dy$ . Hence :

$$E[h(L_{T_c})c] = \frac{1}{2}E[H(L_{T_c})] \quad (2.17)$$

i.e.  $L_{T_c}$  is an exponential r.v. with parameter  $\frac{1}{2c}$ .

4) For  $c > 0$  and  $a < 0$ , we get :

$$P(S_{T_a} < c) = P(T_a < T_c) = \frac{c}{c-a}, \quad \text{hence : } f_{S_{T_a}}(c) = -\frac{a}{(c-a)^2} 1_{c \geq 0}$$

5) The first assertion of point 5 may be proven similarly to the previous point. Let us show (2.14) and (2.15).

Since  $d_t = t + T_0 \circ \theta_t$ , we get :

$$\begin{aligned} E[\psi(S_{d_t})|\mathcal{F}_t] &= E[\psi(S_t \vee S_{[t, t+T_0 \circ \theta_t]})|\mathcal{F}_t] \quad (\text{with } S_{[t, t+T_0 \circ \theta_t]} := \sup_{u \in [t, t+T_0 \circ \theta_t]} X_u) \\ &= \widehat{E}[\psi(S_t \vee (X_t + \sup_{0 \leq u \leq \widehat{T}_- X_t} \widehat{B}_u))] \end{aligned}$$

where  $(\widehat{B}_u, u \geq 0)$  denotes a Brownian motion starting from 0 and independent from  $\mathcal{F}_t$ . In the preceding expression, the r.v.'s  $S_t$  and  $X_t$  are frozen and the expectation bears upon  $\widehat{B}$ . Therefore :

$$\begin{aligned} E[\psi(S_{d_t})|\mathcal{F}_t] &= \widehat{E}\left[\psi\left(S_t \vee \frac{X_t^+}{\widehat{U}}\right)\right] \quad (\text{from (2.13)}) \\ &= \int_0^1 \psi\left(S_t \vee \frac{X_t^+}{u}\right) du = \int_0^{\frac{X_t^+}{S_t}} \psi\left(\frac{X_t^+}{u}\right) du + \int_{\frac{X_t^+}{S_t}}^1 \psi(S_t) du \\ &= X_t^+ \int_{S_t}^{\infty} \psi(v) \frac{dv}{v^2} + \psi(S_t) \left(1 - \frac{X_t^+}{S_t}\right) \end{aligned}$$

i.e. (2.15) has been proven. ■

## 2.2 Proof of Theorem 1.

1) We first show that, for every  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  integrable, we have :

$$\sqrt{t}E[\psi(S_{g_t})] \underset{t \rightarrow \infty}{\uparrow} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \psi(x) dx \quad (2.18)$$

Indeed, from (2.9) :

$$\sqrt{t}E[\psi(S_{g_t})] = \sqrt{\frac{2t}{\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}} \psi\left(\frac{\sqrt{t}x}{2}\right) dx = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} e^{-\left(\frac{x^2}{t}\right)} \psi(x) dx \underset{t \rightarrow \infty}{\uparrow} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \psi(x) dx$$

In particular :

$$E[\psi(S_{g_t})] \underset{t \rightarrow \infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^{\infty} \psi(x) dx$$

2) Let us prove that :

$$\frac{E[\varphi(S_{g_t})|\mathcal{F}_s]}{E[\varphi(S_{g_t})]} \underset{t \rightarrow \infty}{\xrightarrow{(g)} M_s^\varphi} \quad \text{a.s., where } (g)M_s^\varphi \text{ is defined by (2.4).}$$

We already know, from the preceding point, that  $E[\varphi(S_{g_t})] \underset{t \rightarrow \infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}}$ . We then write :

$$\begin{aligned} N_t := E[\varphi(S_{g_t})|\mathcal{F}_s] &= E[\varphi(S_{g_t}) 1_{(g_t < s)}|\mathcal{F}_s] + E[\varphi(S_{g_t}) 1_{(g_t > s)}|\mathcal{F}_s] \\ &:= (1)_t + (2)_t \end{aligned} \quad (2.19)$$

and we shall study successively the asymptotic behaviors, as  $t \rightarrow \infty$  of  $(1)_t$  and  $(2)_t$ .

2) a. Asymptotic behavior of (1)<sub>t</sub>.

$$\begin{aligned} (1)_t &= E[\varphi(S_{g_t})1_{(g_t < s)}|\mathcal{F}_s] = E[\varphi(S_{g_s})1_{(g_t < s)}|\mathcal{F}_s] \\ &= \varphi(S_{g_s})E[1_{(g_t < s)}|\mathcal{F}_s] \end{aligned} \quad (2.20)$$

since  $g_t = g_s$  if  $g_t < s$ . However, since  $1_{(g_t < s)} = 1_{(d_s > t)} = 1_{s+T_0 \circ \theta_s > t}$ , we get :

$$\begin{aligned} E[1_{(g_t < s)}|\mathcal{F}_s] &= E[1_{T_0 \circ \theta_s > t-s}|\mathcal{F}_s] = P_{|X_s|}[T_0 > t-s] \\ &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{|X_s|}{\sqrt{t-s}} \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{|X_s|}{\sqrt{t}} \end{aligned} \quad (2.21)$$

Hence, gathering (2.20), (2.21) and (2.18), we obtain :

$$\frac{E[\varphi(S_{g_t})1_{(g_t < s)}|\mathcal{F}_s]}{E(\varphi(S_{g_t}))} \xrightarrow{t \rightarrow \infty} \frac{1}{2} \varphi(S_{g_s})|X_s| \quad \text{a.s.} \quad (2.22)$$

2) b. Asymptotic behavior of (2)<sub>t</sub>.

$$\begin{aligned} (2)_t &= E[\varphi(S_{g_t})1_{(g_t > s)}|\mathcal{F}_s] = E[\varphi(S_s \vee S_{[s, g_t]})1_{(d_s < t)}|\mathcal{F}_s] \\ &\quad (\text{with } S_{[s, g_t]} := \sup_{u \in [s, g_t]} X_u) \\ &= \widehat{E}[\varphi(S_s \vee (X_s + \widehat{S}_{\widehat{g}_t - X_s}^{-X_s}))1_{s + \widehat{T}_{-X_s} < t}] \end{aligned} \quad (2.23)$$

with :  $g_t^{(a)} := \sup\{u \leq t, X_u = a\}$  and where, in (2.23), the expressions without hats are frozen, whereas those with hats are being integrated. So, we have to estimate :

$$\widehat{E}[\psi(\widehat{S}_{\widehat{g}_t^{(a)}} 1_{\widehat{T}_a < t}] \quad \text{as } t \rightarrow \infty \quad (2.24)$$

where we shall replace  $t$  by  $t - s$  and, we denote :

$$\psi(c) := \varphi(S_s \vee (X_s + c)) \quad (2.25)$$

Note that  $\psi$  is integrable over  $\mathbb{R}_+$ . Thus, we now estimate the asymptotic behavior, as  $t \rightarrow \infty$ , of  $E(\widehat{\psi}(S_{g_t^{(a)}})1_{T_a < t})$  for  $\widehat{\psi}$  integrable over  $\mathbb{R}_+$  (we deleted the hats, which are no longer useful).

We denote by  $X'$  the Brownian motion independent from  $\mathcal{F}_{T_a}$  defined by :

$$X'_u = X_{T_a+u} - a, u \geq 0$$

Case 1 :  $a \geq 0$  Then with obvious notation :

$$S_{g_t^{(a)}} = S_{[T_a, g_t^{(a)}}] = a + S'_{g_t - T_a}$$

Hence :

$$E[\widehat{\psi}(S_{g_t^{(a)}})1_{T_a < t}] = E[\widehat{\psi}(a + S'_{g_t - T_a})1_{T_a < t}]$$

But, from (2.18) :

$$\sqrt{t - T_a} E[\psi(a + S'_{g_t - T_a})|\mathcal{F}_{T_a}] \xrightarrow{t \rightarrow +\infty} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \psi(a+x) dx$$

On the other hand, from (2.18) again :

$$1_{T_a < t} \sqrt{t} E[\widehat{\psi}(a + S'_{g'_t - T_a}) | \mathcal{F}_{T_a}] \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{\frac{t}{t - T_a}} \cdot 1_{T_a < t}$$

and the family of r.v  $\left(\frac{2\sqrt{2}}{\sqrt{\pi}} \cdot \sqrt{\frac{t}{t - T_a}} \cdot 1_{T_a < t}, t \geq 1\right)$  is uniformly integrable. Indeed, it is easy to prove, using the explicit form of the density of  $T_a$  :

$$f_{T_a}(s) = \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}} 1_{s \geq 0}$$

that :

$$\sup_{t \geq 1} E\left(1_{T_a < t} \left(\sqrt{\frac{t}{t - T_a}}\right)^b\right) < \infty$$

for every  $b \in ]1, 2[$  (see [RVY, I], p. 212, for a similar argument). Hence, if  $a \geq 0$  :

$$E[\widehat{\psi}(S_{g_t^{(a)}}) 1_{T_a < t}] \underset{t \rightarrow \infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^\infty \widehat{\psi}(a + x) dx \quad (2.26)$$

Thus, plugging this estimate (2.26) into (2.23), (and choosing there  $\widehat{\psi}$  as given by (2.25)), we obtain :

$$\begin{aligned} \frac{E[\varphi(S_{g_t}) 1_{(g_t > s)} 1_{X_s < 0} | \mathcal{F}_s]}{E[\varphi(S_{g_t})]} &\xrightarrow{t \rightarrow \infty} 1_{X_s < 0} \int_0^\infty \varphi(S_s \vee (X_s - X_s + x)) dx \\ &= 1_{X_s < 0} \int_0^\infty \varphi(S_s \vee x) dx \\ &= 1_{X_s < 0} \left[ \int_0^{S_s} \varphi(S_s) dx + \int_{S_s}^\infty \varphi(x) dx \right] \\ &= 1_{X_s < 0} [\varphi(S_s) \cdot S_s + 1 - \Phi(S_s)] \end{aligned} \quad (2.27)$$

**Case 2 :  $a < 0$**  We have :

$$S_{g_t^{(a)}} = S_{T_a} \vee (a + S'_{g'_t - T_a})$$

and :

$$E[\psi(S_{g_t^{(a)}}) 1_{T_a < t}] = E[\psi(S_{T_a} \vee (a + S'_{g'_t - T_a})) 1_{T_a < t}] \quad (2.28)$$

The same method as in Case 1 leads to :

$$E[\psi(S_{g_t^{(a)}}) 1_{T_a < t}] \underset{t \rightarrow \infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}} E\left(\int_0^\infty \psi(S_{T_a} \vee (a + x)) dx\right) \quad (2.29)$$

from (2.18). But, we have :

$$\begin{aligned} \Delta : &= E\left(\int_0^\infty \psi(S_{T_a} \vee (a + x)) dx\right) = E\left(\int_a^\infty \psi(S_{T_a} \vee y) dy\right) \\ &= E\left(-a\psi(S_{T_a}) + \int_0^\infty \psi(S_{T_a} \vee y) dy\right) \\ &= a^2 \int_0^\infty \psi(c) \frac{dc}{(c - a)^2} + (-a) \int_0^\infty \frac{dc}{(c - a)^2} \left[ c\psi(c) + \int_c^\infty \psi(y) dy \right] \end{aligned}$$

(from (2.12))

$$= \int_0^\infty \psi(c) \left[ \frac{a^2}{(c-a)^2} - \frac{ac}{(c-a)^2} + \frac{c}{c-a} \right] dc = \int_0^\infty \psi(c) dc \quad (2.30)$$

Thus, from (2.29) and (2.30) :

$$E[\psi(S_{g_t^{(a)}}) 1_{T_a < t}] \underset{t \rightarrow \infty}{\sim} 2\sqrt{\frac{2}{\pi t}} \int_0^\infty \psi(c) dc \quad (2.31)$$

Bringing this estimate into (2.23), with  $\psi$  defined by (2.25), we obtain :

$$\begin{aligned} & \frac{E[\varphi(S_{g_t}) 1_{(g_t > s)} 1_{X_s > 0} | \mathcal{F}_s]}{E[\varphi(S_{g_t})]} \underset{t \rightarrow \infty}{\longrightarrow} 1_{X_s > 0} \int_0^\infty \varphi(S_s \vee (X_s + c)) dc \\ & = 1_{X_s > 0} \int_{X_s}^\infty \varphi(S_s \vee y) dy = 1_{X_s > 0} [\varphi(S_s)(S_s - X_s) + (1 - \Phi(S_s))] \end{aligned} \quad (2.32)$$

Finally, gathering (2.31), (2.27) and (2.22) leads to :

$$\begin{aligned} \frac{E[\varphi(S_{g_t}) | \mathcal{F}_s]}{E[\varphi(S_{g_t})]} & \underset{t \rightarrow \infty}{\longrightarrow} \frac{1}{2} \varphi(S_{g_s}) |X_s| + 1_{X_s < 0} [\varphi(S_s) S_s + 1 - \Phi(S_s)] \\ & + 1_{X_s > 0} [\varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s)] \\ & = \frac{1}{2} \varphi(S_{g_s}) |X_s| + \varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s) = {}^{(g)}M_s^\varphi \end{aligned} \quad (2.33)$$

**3)** We may now finish the proof of point 1 in Theorem 1

The Itô-Tanaka formula and the balayage formula (see [RY], Chap. VI, §4) imply :

$${}^{(g)}M_s^\varphi = 1 + \int_0^s \left( \frac{1}{2} \varphi(S_{g_u}) \operatorname{sgn} X_u - \varphi(S_u) 1_{X_u > 0} \right) dX_u. \quad (2.34)$$

It follows (see [RVY, II] for similar arguments), that  $({}^{(g)}M_s^\varphi, s \geq 0)$  is a martingale and that in particular  $E({}^{(g)}M_s^\varphi) = 1$ . Thus, from this latter relation and since  $\frac{E[\varphi(S_{g_t}) | \mathcal{F}_s]}{E[\varphi(S_{g_t})]} \underset{t \rightarrow \infty}{\longrightarrow} {}^{(g)}M_s^\varphi$  a.s., this last convergence holds equally in  $L^1$  (it is a particular case of Scheffé's lemma, cf [M], T. 21, p. 37). Point 1 of Theorem 1 follows immediately.

## 2.3 Proofs of ${}^{(g)}Q^\varphi[S_\infty = \infty] = \frac{1}{2}$ and of point 2iii of Theorem 1.

We have, for all  $s > 0$  and  $a > 0$  :

$${}^{(g)}Q^\varphi(S_a > s) = {}^{(g)}Q^\varphi(T_s < a) = E[1_{T_s < a} {}^{(g)}M_a^\varphi] = E[1_{T_s < a} {}^{(g)}M_{T_s}^\varphi]$$

from Doob's optional stopping theorem. Thus, letting  $a \rightarrow +\infty$ , we obtain :

$$\begin{aligned} {}^{(g)}Q^\varphi[S_\infty > s] & = E[{}^{(g)}M_{T_s}^\varphi] = E\left\{ \frac{1}{2} \varphi(S_{g_{T_s}}) s + 1 - \Phi(s) \right\} \\ & = \frac{s}{2s} \int_0^s \varphi(x) dx + \int_s^\infty \varphi(x) dx \end{aligned}$$

from point 2 of Proposition 2, hence :

$${}^{(g)}Q^\varphi[S_\infty > s] \underset{s \rightarrow \infty}{\longrightarrow} {}^{(g)}Q^\varphi[S_\infty = \infty] = \frac{1}{2} \int_0^\infty \varphi(x) dx = \frac{1}{2} \quad (2.35)$$

Point 2iii is an easy consequence of the previous formulae.

## 2.4 Proof that $g := \sup\{t : X_t = 0\}$ is ${}^{(g)}Q^\varphi$ a.s. finite.

Let  $0 < a < t$ . We have :

$${}^{(g)}Q^\varphi[g_t > a] = {}^{(g)}Q^\varphi[d_a < t] = E[1_{d_a < t} \cdot {}^{(g)}M_t^\varphi] = E[1_{d_a < t} {}^{(g)}M_{d_a}^\varphi]$$

Hence, since  $g_\infty = g$  and letting  $t \rightarrow +\infty$ , we obtain :

$$\begin{aligned} {}^{(g)}Q^\varphi[g > a] &= \lim_{t \rightarrow \infty} E[1_{d_a < t} {}^{(g)}M_{d_a}^\varphi] = E[{}^{(g)}M_{d_a}^\varphi] \\ &= E[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})] \end{aligned} \quad (2.36)$$

We shall show now that  ${}^{(g)}Q^\varphi[g > a] \xrightarrow{a \rightarrow \infty} 0$ , which proves that  $g$  is  ${}^{(g)}Q^\varphi$  a.s. finite.

But :  $E[1 - \Phi(S_{d_a})] \xrightarrow{a \rightarrow \infty} 0$  from the dominated convergence Theorem.

On the other hand, from (2.15) :

$$\begin{aligned} E[\varphi(S_{d_a})S_{d_a}] &= E\left[\varphi(S_a)(S_a - X_a^+) + X_a^+ \int_{S_a}^{\infty} \frac{\varphi(v)}{v} dv\right] \\ &\leq E[\varphi(S_a)S_a] + E\left\{\frac{X_a^+}{S_a} \int_{S_a}^{\infty} \varphi(v)dv\right\} \\ &\leq E\left\{[\varphi(S_a)S_a] + 1 - \Phi(S_a)\right\} \end{aligned}$$

But :  $E(1 - \Phi(S_a)) \xrightarrow{a \rightarrow \infty} 0$  from the dominated convergence Theorem and :

$$\begin{aligned} E[\varphi(S_a) \cdot S_a] &= \sqrt{\frac{2}{\pi a}} \int_0^{\infty} \varphi(x)x e^{-\frac{x^2}{2a}} dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \varphi(x) \left[\frac{x}{\sqrt{a}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{a}}\right)^2}\right] dx \xrightarrow{a \rightarrow \infty} 0 \end{aligned}$$

because  $\frac{x}{\sqrt{a}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{a}}\right)^2}$  is uniformly bounded and converges to 0 as  $a \rightarrow \infty$ .

Thus, from (2.36) :

$${}^{(g)}Q^\varphi[g = \infty] = \lim_{a \rightarrow \infty} {}^{(g)}Q^\varphi(g > a) = \lim_{a \rightarrow \infty} E[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})] = 0 \quad (2.37)$$

## 2.5 Computation of Azéma's supermartingale $Z_t := {}^{(g)}Q^\varphi(g > t | \mathcal{F})$ .

In order to complete the proof of Theorem 1, we shall use the enlargement of filtration technique, i.e. : we shall work within the filtration  $(\mathcal{G}_t, t \geq 0)$ , where  $(\mathcal{G}_t, t \geq 0)$  is the smallest filtration with contains  $(\mathcal{F}_t, t \geq 0)$  and such that  $g := \sup\{t ; X_t = 0\}$  becomes a  $(\mathcal{G}_t, t \geq 0)$  stopping time. To apply the enlargement formula, we need to compute the Azéma supermartingale :  $Z_t := {}^{(g)}Q^\varphi(g > t | \mathcal{F}_t)$

**Lemma 3.**

$$1) Z_t := {}^{(g)}Q^\varphi(g > t | \mathcal{F}_t) = \frac{\varphi(S_t)(S_t - X_t^+) + 1 - \Phi(S_t)}{{}^{(g)}M_t^\varphi} = 1 - \frac{1}{2} \frac{\varphi(S_{g_t})|X_t|}{{}^{(g)}M_t^\varphi} \quad (2.38)$$

2) For every positive,  $(\mathcal{F}_s)$  predictable process  $(K_s, s \geq 0)$  one has :

$$E_{{}^{(g)}Q^\varphi}[K_g] = \frac{1}{2} E\left(\int_0^{\infty} K_s \varphi(S_s) dL_s\right) \quad (2.39)$$

**Proof of Lemma 3.**

1) Since  $g := \sup\{t \geq 0 ; X_t = 0\}$ , we get :

$$\begin{aligned} {}^{(g)}Q^\varphi[g > t | \mathcal{F}_t] &= E_{{}^{(g)}Q^\varphi}[1_{d_t < \infty} | \mathcal{F}_t] = \frac{1}{{}^{(g)}M_t^\varphi} E[{}^{(g)}M_{d_t}^\varphi | \mathcal{F}_t] \quad \text{by Doob's optional stopping Theorem} \\ &= \frac{1}{{}^{(g)}M_t^\varphi} E[(\varphi(S_{d_t})S_{d_t} + 1 - \Phi(S_{d_t})) | \mathcal{F}_t] \end{aligned} \quad (2.40)$$

Applying (2.15) to the function  $\psi(x) := \varphi(x)x + 1 - \Phi(x)$ , an elementary computation leads to :

$$E[\varphi(S_{d_t})S_{d_t} + 1 - \Phi(S_{d_t}) | \mathcal{F}_t] = \varphi(S_t)(S_t - X_t^+) + 1 - \Phi(S_t)$$

hence (2.38), by using (2.40).

2) From (2.38), we deduce by approximation, that for every bounded  $(\mathcal{F}_t, t \geq 0)$  stopping time  $T$  :

$$\begin{aligned} E_{{}^{(g)}Q^\varphi}[1_{[0,T]}(g)] &= E_{{}^{(g)}Q^\varphi}\left[\frac{1}{2} \frac{\varphi(S_{g_T} | X_T)}{{}^{(g)}M_T^\varphi}\right] \\ &= \frac{1}{2} E[\varphi(S_{g_T}) | X_T] \quad (\text{from (2.4)}) \\ &= \frac{1}{2} E\left(\int_0^\infty 1_{(s \leq T)} \varphi(S_s) dL_s\right) \end{aligned} \quad (2.41)$$

from the balayage formula. Then, we extend the equality from the elementary predictable processes  $1_{[0,T]}(s)$  to every positive  $(\mathcal{F}_s)$ , predictable process  $(K_s)$  by using the monotone class theorem. Thus :

$$E_{{}^{(g)}Q^\varphi}[K_g] = \frac{1}{2} E\left(\int_0^\infty K_s \varphi(S_s) dL_s\right) \quad \blacksquare$$

**2.6 Proofs of points 2ii and 3ii of Theorem 1.**

1. Applying (2.39) with  $K_s = f_1(L_s)f_2(S_s)$ , with  $f_1, f_2$  Borel and positive, we obtain :

$$\begin{aligned} E_{{}^{(g)}Q^\varphi}[f_1(L_g)f_2(S_g)] &= \frac{1}{2} E\left(\int_0^\infty f_1(L_s)f_2(S_s)\varphi(S_s)dL_s\right) \\ &= \frac{1}{2} E\left(\int_0^\infty f_1(l)f_2(S_\tau)\varphi(S_\tau)dl\right) \end{aligned}$$

(after making the change of variables  $L_s = l$ ).

$$= \frac{1}{2} \int_0^\infty \int_0^\infty f_1(l)f_2(c)\varphi(c) \frac{l}{2c^2} e^{-\frac{l}{2c}} dc dl \quad (2.42)$$

with the help of point 3 of Proposition 2. Thus, the density of the r.v.  $(L_g, S_g)$  under  ${}^{(g)}Q^\varphi$  equals :

$$f_{L_g, S_g}^{(g)Q^\varphi}(l, c) = \frac{1}{4} \frac{l}{c^2} e^{-\frac{l}{2c}} \varphi(c) 1_{l \geq 0} 1_{c \geq 0} \quad (2.43)$$

Point 2ii of Theorem 1 follows easily from this formula (with the help of (2.35)).

2. To show point 3ii of Theorem 1, we use (2.39) with :

$K_s := F(X_u, u \leq s) f_1(L_s) f_2(S_s)$ . We obtain :

$$\begin{aligned} E_{(g)Q^\varphi} [F(X_u, u \leq g) f_1(L_g) f_2(S_g)] &= \frac{1}{2} E \left( \int_0^\infty F(X_u, u \leq s) f_1(L_s) f_2(S_s) dL_s \right) \\ &= \frac{1}{2} E \left( \int_0^\infty F(X_u, u \leq \tau_l) f_1(l) f_2(S_{\tau_l}) \varphi(S_{\tau_l}) dl \right) \\ &\text{(after making the change of variables : } L_s = l) \\ &= \frac{1}{2} \int_0^\infty E(F((X_u, u \leq \tau_l)) | S_{\tau_l} = c) f_1(l) f_2(c) \varphi(c) \frac{l}{2c^2} e^{-\frac{l}{2c}} dc dl \end{aligned} \quad (2.44)$$

Of course, we recover (2.43) by making  $F = 1$  in (2.44). But, it also holds that :

$$\begin{aligned} E_{(g)Q^\varphi} [F(X_u, u \leq g) f_1(L_g) f_2(S_g)] \\ = \int_0^\infty \int_0^\infty E_{(g)Q^\varphi} [F(X_u, u \leq g) | L_g = l, S_g = c] f_1(l) f_2(c) \frac{l}{4c^2} e^{-\frac{l}{2c}} \varphi(c) dc dl \end{aligned} \quad (2.45)$$

Hence, comparing (2.44) and (2.45), we obtain :

$$E_{(g)Q^\varphi} [F(X_u, u \leq g) | L_g = l, S_g = c] = E(F(X_u, u \leq \tau_l) | S_{\tau_l} = c) \quad (2.46)$$

which is point 3ii of Theorem 1.

## 2.7 End of the proof of Theorem 1 with the help of enlargement formulae.

From Girsanov's theorem (cf [RY], chap. VIII, §3), using the expression (2.5) of  ${}^{(g)}M_t^\varphi$  as a stochastic integral, we know that there exists a  $((\mathcal{F}_t)_{t \geq 0}, {}^{(g)}Q^\varphi)$  Brownian motion  $(\beta_t, t \geq 0)$  such that :

$$X_t = \beta_t + \int_0^t \frac{\frac{1}{2} \varphi(S_{g_s}) \operatorname{sgn}(X_s) - \varphi(S_s) 1_{X_s > 0}}{({}^{(g)}M_s^\varphi)} ds \quad (2.47)$$

We denote by  $(\mathcal{G}_t, t \geq 0)$  the smallest filtration which contains  $(\mathcal{F}_t, t \geq 0)$  and which makes  $g$  a  $(\mathcal{G}_t, t \geq 0)$  stopping time. The enlargement formulae (see [J], [JY], or [MY]) imply the existence of a  $((\mathcal{G}_t, t \geq 0), {}^{(g)}Q^\varphi)$  Brownian motion  $(\tilde{\beta}_t, t \geq 0)$  such that :

$$\begin{aligned} X_t &= \tilde{\beta}_t + \int_0^t \frac{\frac{1}{2} \varphi(S_{g_s}) \operatorname{sgn}(X_s) - \varphi(S_s) 1_{X_s > 0}}{({}^{(g)}M_s^\varphi)} ds \\ &+ \int_0^{t \wedge g} \frac{d \langle Z, X \rangle_s}{Z_s} - \int_{t \wedge g}^t \frac{d \langle Z, X \rangle_s}{1 - Z_s} \end{aligned} \quad (2.48)$$

In order to make (2.48) more explicit, we need to compute the martingale part  $d \langle Z, X \rangle_s$ . From Itô's formula and (2.38), we get (to simplify we write  $M_t$  for  ${}^{(g)}M_t^\varphi$ ) :

$$\begin{aligned} dZ_t &= - \frac{\varphi(S_t)(S_t - X_t^+) + (1 - \Phi(S_t))}{M_t^2} \left[ \frac{1}{2} \varphi(S_{g_t}) \operatorname{sgn} X_t dX_t - \varphi(S_t) 1_{X_t > 0} dX_t \right] \\ &- \frac{1}{M_t} \varphi(S_t) 1_{X_t > 0} dX_t + d(\text{bounded variation terms}) \end{aligned} \quad (2.49)$$

Thus, we obtain :

$$d \langle Z, X \rangle_t = -\frac{\varphi(S_t)(S_t - X_t^+) + (1 - \Phi(S_t))}{M_t^2} \left[ \frac{1}{2} \varphi(S_{g_t}) \operatorname{sgn} X_t - \varphi(S_t) 1_{X_t > 0} \right] dt - \frac{1}{M_t} \varphi(S_t) 1_{X_t > 0} dt. \quad (2.50)$$

a computation which may be done indifferently under  $P$  or under  ${}^{(g)}Q^\varphi$ .

Thus, plugging (2.50) in (2.48) (and using (2.38)), we obtain, for all  $t \geq 0$  :

$$\begin{aligned} X_{g+t} &= (\tilde{\beta}_{g+t} - \tilde{\beta}_g) + \int_g^{g+t} \frac{\frac{1}{2} \varphi(S_{g_s}) \operatorname{sgn} X_s - \varphi(S_s) 1_{X_s > 0}}{M_s} ds \\ &+ \int_g^{g+t} \left[ \frac{\varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s)}{M_s^2} \left\{ \frac{1}{2} \varphi(S_{g_s}) \operatorname{sgn} X_s - \varphi(S_s) 1_{X_s > 0} \right\} \right. \\ &\quad \left. + \frac{1}{M_s} \varphi(S_s) 1_{X_s > 0} \right] \cdot \frac{2M_s}{\varphi(S_{g_s})|X_s|} ds. \end{aligned} \quad (2.51)$$

We obtain, after simplification :

$$\begin{aligned} X_{g+t} &= (\tilde{\beta}_{g+t} - \tilde{\beta}_g) + \int_g^{g+t} \frac{1}{M_s \varphi(S_{g_s}) |X_s|} \{ \varphi(S_{g_s}) \operatorname{sgn} X_s M_s - 2\varphi(S_s) 1_{X_s > 0} M_s \} ds \\ &+ \int_g^{g+t} \frac{2\varphi(S_s) 1_{X_s > 0}}{\varphi(S_{g_s}) |X_s|} ds, \quad \text{that is :} \\ X_{g+t} &= (\tilde{\beta}_{g+t} - \tilde{\beta}_g) + \int_0^t \frac{ds}{X_{g+s}}. \end{aligned} \quad (2.52)$$

On the other hand, the sign of  $X_u$  is constant after  $g$  : it is positive with probability  $1/2$ , from point 3ii of Theorem 1.

Thus, we now deduce from equation (2.52) that  $(X_u, u \leq g)$ ,  $\epsilon$ , the sign of  $(X_{g+u}, u \geq 0)$  and  $(|X_{g+u}|, u \geq 0)$  are independent, with  $\epsilon$  a symmetric Bernoulli variable, and  $(|X_{g+u}|, u \geq 0)$  a 3-dimensional Bessel process. ■

We also note that (2.48), written before  $g$ , leads to :

$$X_t = \tilde{\beta}_t - \int_0^t \frac{\varphi(S_s) 1_{X_s > 0}}{\varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s)} ds \quad (2.53)$$

## Remark 2

The penalisation by  $(\varphi(S_{g_t}), t \geq 0)$  is described in Theorem 1. The intuitive content of this theorem is the following :

In Theorem 0, we penalised Brownian motion with  $(\varphi(S_t), t \geq 0)$ , i.e we "favored" Brownian trajectories which are not "too high" and it followed that  $Q^\varphi(S_\infty < \infty) = 1$ ; in fact, under  $Q^\varphi$ , the trajectories go to  $-\infty$  as  $t \rightarrow \infty$ . It is their "response" to that kind of penalisation.

What is happening here ?

We penalise by  $(\varphi(S_{g_t}), t \geq 0)$ , i.e. we favor the trajectories which are not too high before their last zero. How will the trajectories "respond"? Will they decide to remain bounded? Or to have a last zero? In fact, we have shown that the trajectories "decide", under  ${}^{(g)}Q^\varphi$ , the limit probability, to eventually quit 0, forever, so that  $g < \infty$   ${}^{(g)}Q^\varphi$  a.s., hence  $S_g < \infty$  a.s., whereas  $S_\infty = \infty$  with probability  $\frac{1}{2}$ .

## 2.8 Proof of point 4 of Theorem 1.

Since, owing to point 3ii, conditionally on  $L_g = v$  and  $S_g = c$ ,  $(X_t, t \leq g)$  is a (stopped) Brownian motion, from Pitman's theorem (see [P]), the process  $(|X_t| + L_t, t \leq g)$  is a 3-dimensional Bessel process.

From 3ii, the same process, after  $g$ , is also a 3-dimensional Bessel process; note that, for  $t > g$  the differential  $d(|X_t| + L_t) = d(|X_t|)$  since, for  $t \geq g$ ,  $|X_t| > 0$  implies  $dL_t = 0$ . Thus, the entire process  $(|X_t| + L_t, t \geq 0)$  is a 3-dimensional Bessel process independent from  $(S_g, L_g)$  since the conditional law of  $(|X_t| + L_t, t \geq 0)$  does not depend on  $(S_g, L_g)$ .

## 3 Penalisation by $(\varphi(S_{d_t}), t \geq 0)$

Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  denote a probability density, i.e. :

$$\int_0^\infty \varphi(x) dx = 1. \quad (3.1)$$

As previously, we denote :  $\Phi(x) := \int_0^x \varphi(y) dy$  ( $x \geq 0$ ).

We define  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$  via :

$$f(b, a) := \varphi(b) \left(1 - \frac{a_+}{b}\right) + a_+ \int_b^\infty \frac{\varphi(v)}{v^2} dv \quad (3.2)$$

**Theorem 4.** *Under the previous hypothesis (3.1), one has for any  $s \geq 0$ , and  $\Lambda_s \in b(\mathcal{F}_s)$*

$$\lim_{t \rightarrow \infty} \frac{E[\Lambda_s \varphi(S_{d_t})]}{E[\varphi(S_{d_t})]} = \lim_{t \rightarrow \infty} \frac{E[\Lambda_s f(S_t, X_t)]}{E[f(S_t, X_t)]} = \lim_{t \rightarrow \infty} \frac{E[\Lambda_s \varphi(S_t)]}{E[\varphi(S_t)]} \quad (3.3)$$

$$= E[\Lambda_s M_s^\varphi] := Q^\varphi(\Lambda_s) \quad (3.4)$$

$$\text{where } M_s^\varphi := \varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s) \quad (3.5)$$

is a  $(P, (\mathcal{F}_s, s \geq 0))$  positive martingale.

In other terms, the penalisation by  $\varphi(S_{d_t})$  is the same as that by  $\varphi(S_t)$  (see (1.1) and (1.2) above, or [RVY, II]). Thus, we may refer the reader to [RVY, II] for a study of the canonical process  $(X_t, t \geq 0)$  under  $Q^\varphi$ .

### 3.1 Proof of Theorem 4.

1) Recall that, from (2.15) :

$$E[\varphi(S_{d_t}) | \mathcal{F}_t] = \varphi(S_t) \left(1 - \frac{X_t^+}{S_t}\right) + X_t^+ \int_{S_t}^\infty \frac{\varphi(v)}{v^2} dv = f(S_t, X_t) \quad (3.6)$$

which proves the first equality in (3.3).

2) We now study the denominator in (3.3) and we prove that :

$$E[f(S_t, X_t)] \underset{t \rightarrow \infty}{\sim} E[\varphi(S_t)] \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_0^\infty \varphi(x) dx = \sqrt{\frac{2}{\pi t}}. \quad (3.7)$$

To prove (3.7), we study successively the 3 terms which constitute  $E[f(S_t, X_t)]$  :

$$\bullet E[\varphi(S_t)] = \sqrt{\frac{2}{\pi t}} \int_0^\infty \varphi(x) e^{-\frac{x^2}{2t}} dx \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_0^\infty \varphi(x) dx = \sqrt{\frac{2}{\pi t}}. \quad (3.8)$$

We now prove that :

$$\bullet E \left[ \varphi(S_t) \frac{X_t^+}{S_t} \right] = o \left( \frac{1}{\sqrt{t}} \right) \quad (t \rightarrow \infty) \quad \text{and} \quad (3.9)$$

$$\bullet E \left[ X_t^+ \int_{S_t}^\infty \frac{\varphi(v)}{v^2} dv \right] = o \left( \frac{1}{\sqrt{t}} \right) \quad (t \rightarrow \infty) \quad (3.10)$$

(3.9) and (3.10) are obvious consequences of Lemma 5 below in the particular case where  $\sigma = x = 0$ , because :

$$\begin{aligned} \varphi(S_t) \frac{X_t^+}{S_t} &\leq \varphi(S_t) 1_{X_t \geq 0} \quad \text{and} \\ X_t^+ \int_{S_t}^\infty \frac{\varphi(v)}{v^2} dv &\leq \frac{X_t^+}{S_t} \int_{S_t}^\infty \frac{\varphi(v)}{v} dv \leq 1_{X_t \geq 0} \tilde{\varphi}(S_t), \quad \text{with} \\ \tilde{\varphi}(c) &= \int_c^\infty \frac{\varphi(v)}{v} dv ; \tilde{\varphi} \text{ is integrable since :} \\ \int_0^\infty \tilde{\varphi}(c) dc &= \int_0^\infty dc \int_c^\infty \frac{\varphi(v)}{v} dv = \int_0^\infty \frac{\varphi(v)}{v} \int_0^v dc = \int_0^\infty \varphi(v) dv. \end{aligned}$$

**Lemma 5.** *Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be integrable. Then, for any  $\sigma \geq 0$  and  $x \leq \sigma$  :*

$$E_0[h(\sigma \vee (x + S_t)) 1_{x + X_t > 0}] = o \left( \frac{1}{\sqrt{t}} \right) \quad (3.11)$$

**Proof of Lemma 5.**

• For  $\sigma = x = 0$  we have, from the formula (see [KS], p. 95) which gives the law of the pair  $(S_t, X_t)$  :

$$\begin{aligned} E_0[h(S_t) 1_{X_t > 0}] &= \sqrt{\frac{2}{\pi t^3}} \int_0^\infty db h(b) \int_0^b (2b - a) e^{-\frac{(2b-a)^2}{2t}} da \\ &= \sqrt{\frac{2}{\pi t}} \int_0^\infty h(b) db [e^{-\frac{b^2}{2t}} - e^{-\frac{2b^2}{t}}] = o \left( \frac{1}{\sqrt{t}} \right) \end{aligned}$$

from the dominated convergence Theorem.

- For  $x \leq \sigma$ ,  $\sigma \geq 0$  :

$$\begin{aligned}
E_0[h(\sigma \vee (x + S_t)) \cdot 1_{x+X_t>0}] &= h(\sigma)P_0[S_t < \sigma - x, X_t > -x] \\
&\quad + E[h(x + S_t)1_{S_t>\sigma-x}1_{X_t>-x}] \\
&= h(\sigma)\sqrt{\frac{2}{\pi t^3}} \int_0^{\sigma-x} db \int_{(-x \wedge b)}^b (2b-a)e^{-\frac{(2b-a)^2}{2t}} da \\
&\quad + \sqrt{\frac{2}{\pi t^3}} \int_{\sigma-x}^{\infty} db h(x+b) \int_{(-x \wedge b)}^b (2b-a)e^{-\frac{(2b-a)^2}{2t}} da \\
&= h(\sigma)\sqrt{\frac{2}{\pi t}} \int_0^{\sigma-x} db [e^{-\frac{b^2}{2t}} - e^{-\frac{(2b-(-x \wedge b))^2}{2t}}] \\
&\quad + \sqrt{\frac{2}{\pi t}} \int_{\sigma-x}^{\infty} h(x+b) db [e^{-\frac{b^2}{2t}} - e^{-\frac{(2b-(-x \wedge b))^2}{2t}}] = o\left(\frac{1}{\sqrt{t}}\right)
\end{aligned}$$

by the dominated convergence Theorem. ■

Lemma 5 is proven.

- 3) We prove that, for fixed  $s$  :

$$E[\varphi(S_{d_t})|\mathcal{F}_s] = E[f(S_t, X_t)|\mathcal{F}_s] \underset{t \rightarrow \infty}{\sim} E[\varphi(S_t)|\mathcal{F}_s]. \quad (3.12)$$

The first equality in (3.12) follows immediately from (3.6). Furthermore, from (3.2) we deduce that :

$$\begin{aligned}
E[f(S_t, X_t)|\mathcal{F}_s] &= E[\varphi(S_t)|\mathcal{F}_s] - E\left[\varphi(S_t)\frac{X_t^+}{S_t}|\mathcal{F}_s\right] + E\left[X_t^+ \int_{S_t}^{\infty} \frac{\varphi(v)}{v^2} dv|\mathcal{F}_s\right] \\
&:= (1)_t - (2)_t + (3)_t
\end{aligned}$$

We know (see [RVY, II]) that :

$$(1)_t = E[\varphi(S_t)|\mathcal{F}_s] \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} [\varphi(S_s)(S_s - X_s) + (1 - \Phi(S_s))] \quad (3.13)$$

and

$$(2)_t = E\left[\varphi(S_t)\frac{X_t^+}{S_t}|\mathcal{F}_s\right] = E\left[\varphi(\sigma \vee (x + S_{t-s}))\frac{(x + X_{t-s})^+}{\sigma \vee (x + S_{t-s})}\right] \quad (3.14)$$

with  $\sigma = S_s$  and  $x = X_s$ , and :

$$(2)_t = o\left(\frac{1}{\sqrt{t-s}}\right) \quad \text{from Lemma 5 and from inequalities used to prove (3.9) and (3.10).} \quad (3.15)$$

The same argument leads to :

$$(3)_t = o\left(\frac{1}{\sqrt{t-s}}\right). \quad (3.16)$$

Finally, gathering (3.14), (3.15) and (3.16) we have obtained :

$$\frac{E[\varphi(S_{d_t})|\mathcal{F}_s]}{E[\varphi(S_{d_t})]} \xrightarrow{t \rightarrow \infty} M_s^\varphi = \varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s) \quad \text{a.s.} \quad (3.17)$$

As already discussed while proving (1.6), Itô's formula allows to show that  $(M_s^\varphi, s \geq 0)$  is a martingale; hence that  $E(M_s^\varphi) = 1$ , which implies, (cf [M], p. 37, t. 21) that the convergence in (3.17) takes place in  $L^1$ , and Theorem 4 follows immediately.

**Remark 3.** By comparison, on one hand, of :

- Theorem 4, Theorem 1 and Theorem 6 in [RVY, II];
- and, on the other hand of Theorem III 1, Theorem IV 1 and Theorem IV 2 of [RVY, VII].

We obtain the informal, but remarkable following analogy :

- Penalisations by  $\varphi(S_{d_t})$  and  $\varphi(S_t)$  are identical and differ from the penalisation by  $\varphi(S_{g_t})$ ;
- Penalisations by  $\varphi(V_{d_t}^{(1)})$  and  $\varphi(V_t^{(1)})$  are identical and differ from the penalisation by  $\varphi(V_{g_t}^{(1)})$ , with :  $V_{g_t}^{(1)} := \sup\{d_s - g_s ; d_s \leq t\}$ ,  $V_{d_t}^{(1)} := \sup\{d_s - g_s ; g_s \leq t\}$  and  $V_t^{(1)} := V_{g_t}^{(1)} \vee (t - g_t)$ .

## 4 On statements of lines 2, 3, 5, 6, 8 and 9 of the Table.

### 4.1 Penalisations by $(\varphi(S_t)1_{X_t < 0}, t \geq 0)$ and by $(\varphi(S_t), t \geq 0)$ induce the same limiting probability.

Indeed, for every  $s \geq 0$  and  $\Lambda_s \in b^+(\mathcal{F}_s)$ , we have :

$$\begin{aligned} E[\varphi(S_t)1_{X_t < 0} \cdot \Lambda_s] &= E[\varphi(S_t)\Lambda_s] - E[\varphi(S_t)1_{X_t > 0} \cdot \Lambda_s] \\ &\underset{t \rightarrow \infty}{\sim} E[\varphi(S_t)\Lambda_s] \end{aligned} \quad (4.1)$$

since :

$$\begin{aligned} E[\varphi(S_t)1_{X_t > 0} \Lambda_s] &\leq \|\Lambda_s\|_\infty \cdot E[\varphi(S_t)1_{X_t > 0}] \\ &= o\left(\frac{1}{\sqrt{t}}\right), \quad (\text{from Lemma 5}) \end{aligned}$$

whereas :

$$E[\varphi(S_t)\Lambda_s] \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \left( \int_0^\infty \varphi(x) dx \right) \cdot E[\Lambda_s M_s^\varphi] \quad (4.2)$$

with

$$M_s^\varphi = \frac{1}{\int_0^\infty \varphi(x) dx} \left[ \varphi(S_s)(S_s - X_s) + \int_{S_s}^\infty \varphi(y) dy \right] \quad (4.3)$$

(see [RVY, II], Theorem 3.6). Hence :

$$\frac{E[\Lambda_s \varphi(S_t)]}{E[\varphi(S_t)]} - \frac{E[\Lambda_s \varphi(S_t)1_{X_t < 0}]}{E[\varphi(S_t)1_{X_t < 0}]} \xrightarrow{t \rightarrow \infty} 0.$$

**4.2 Penalisations by  $(\varphi(S_t)1_{X_t < 0}, t \geq 0)$ ,  $(\varphi(S_{g_t})1_{X_t < 0}, t \geq 0)$  and  $(\varphi(S_{d_t})1_{X_t < 0}, t \geq 0)$  are the same.**

It is a consequence of the following identity :

$$\varphi(S_t)1_{X_t < 0} \equiv \varphi(S_{g_t})1_{X_t < 0} \equiv \varphi(S_{d_t})1_{X_t < 0} \quad (4.4)$$

Thus, since the proof of the statement for line 10 of the Table is postponed to Section 5, it now remains to prove results in lines 6, 3 and 9.

**4.3 Statement of line 6 of the Table.**

We write, for  $\Lambda_s \in b(\mathcal{F}_s)$  :

$$\frac{E(\Lambda_s \varphi(S_{g_t})1_{X_t > 0})}{E[\varphi(S_{g_t})1_{X_t > 0}]} = \frac{E[\Lambda_s \varphi(S_{g_t})] - E[\Lambda_s \varphi(S_t)1_{X_t < 0}]}{E[\varphi(S_{g_t})] - E[\varphi(S_t)1_{X_t < 0}]} \quad (4.5)$$

since :

$$\begin{aligned} \varphi(S_{g_t})1_{X_t < 0} &\equiv \varphi(S_t)1_{X_t < 0} \\ &\underset{t \rightarrow \infty}{\sim} \frac{2\sqrt{\frac{2}{\pi t}} E[\Lambda_s^{(g)} M_s^\varphi] - \sqrt{\frac{2}{\pi t}} E[\Lambda_s M_s^\varphi]}{2\sqrt{\frac{2}{\pi t}} - \sqrt{\frac{2}{\pi t}}} \end{aligned}$$

because, from Theorem 1 and line 2 of the Table, we have :

$$\begin{aligned} E[\Lambda_s \varphi(S_{g_t})] &\underset{t \rightarrow \infty}{\sim} 2\sqrt{\frac{2}{\pi t}} E[\Lambda_s^{(g)} M_s^\varphi] \cdot \int_0^\infty \varphi(x) dx \quad \text{and :} \\ E[\Lambda_s \varphi(S_t)1_{X_t < 0}] &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} E[\Lambda_s M_s^\varphi] \int_0^\infty \varphi(x) dx \end{aligned}$$

Thus, the LHS of (4.5) converges, as  $t \rightarrow \infty$ , toward :

$$E[\Lambda_s(2 \cdot^{(g)} M_s^\varphi - M_s^\varphi)] = E[\Lambda_s(\varphi(S_{g_s})X_s + M_s^\varphi)] \quad (4.6)$$

We note that  $(^{(g,+)} M_s^\varphi = \varphi(S_{g_s})X_s + M_s^\varphi, s \geq 0)$  is a positive martingale, from (1.5) applied with  $\alpha = \beta = 1$ .

**4.4 Statement of line 3 of the Table.**

We have, by the Markov property, for  $t \geq s$  :

$$\begin{aligned} E[\varphi(S_t)1_{X_t > 0} | \mathcal{F}_s] &= E[\varphi(\sigma \vee (x + S_{t-s}))1_{x+X_{t-s} > 0}] \\ &:= \alpha(\sigma, x, t-s) \quad (x < \sigma, \sigma \geq 0) \end{aligned} \quad (4.7)$$

with  $x = S_s$  and  $x = X_s$ . But, from the explicit expression given by Lemma 5, we have :

$$\alpha(\sigma, x, t-s) \underset{t \rightarrow \infty}{\sim} \frac{1}{2} \sqrt{\frac{2}{\pi(t-s)^3}} \left\{ \varphi(\sigma) \int_0^{\sigma-x} q(b, x) db + \int_{\sigma-x}^\infty \varphi(x+b) q(b, x) db \right\}, \quad (4.8)$$

$$\text{with } q(b, x) = -b^2 + (2b - ((-x) \wedge b))^2 \quad (4.9)$$

Hence, we need to calculate :

$$\begin{aligned}\tilde{\alpha}(\sigma, x) &:= \varphi(\sigma) \int_0^{\sigma-x} q(b, x) db + \int_{\sigma-x}^{\infty} \varphi(x+b) q(b, x) db \\ &= \varphi(\sigma) \int_x^{\sigma} q(b-x, x) db + \int_{\sigma}^{\infty} \varphi(b) q(b-x, x) db\end{aligned}$$

Observe that, if  $b \geq 0$ ,  $b-x \geq -x$  and  $q(b-x, x) = 3b^2 - 2bx$  whereas if  $b < 0$ ,  $b-x < -x$  and  $q(b-x, x) = 0$ . Hence :

$$\begin{aligned}\tilde{\alpha}(\sigma, x) &= \begin{cases} \varphi(\sigma) \int_0^{\sigma} b(3b-2x) db + \int_{\sigma}^{\infty} \varphi(b)(3b^2-2bx) db & \text{if } x < 0 \\ \varphi(\sigma) \int_x^{\sigma} b(3b-2x) db + \int_{\sigma}^{\infty} \varphi(b)(3b^2-2bx) db & \text{if } x \geq 0 \end{cases} \\ &= \varphi(\sigma)(\sigma^3 - x\sigma^2) + \int_{\sigma}^{\infty} \varphi(b)(3b^2 - 2bx) db\end{aligned}\tag{4.10}$$

We deduce from these computations, since  $\tilde{\alpha}(0, 0) = 3 \int_0^{\infty} \varphi(b)b^2 db$ , that :

$$\frac{E[\Lambda_s \varphi(S_t) 1_{X_t > 0}]}{E[\varphi(S_t) 1_{X_t > 0}]} \xrightarrow{t \rightarrow \infty} \frac{1}{3} \frac{E[\Lambda_s \tilde{\alpha}(S_s, X_s)]}{\int_0^{\infty} \varphi(b)b^2 db}\tag{4.11}$$

We now define :

$$\psi(\sigma) := \varphi(\sigma)\sigma^2 + 2 \int_{\sigma}^{\infty} \varphi(b)b db \quad (\sigma \geq 0)\tag{4.12}$$

An elementary computation gives :

$$\tilde{\alpha}(\sigma, x) = \psi(\sigma)(\sigma - x) = \int_{\sigma}^{\infty} \psi(y) dy\tag{4.13}$$

Hence, from (4.8) :

$$\frac{E[\Lambda_s \varphi(S_t) 1_{X_t > 0}]}{E[\varphi(S_t) 1_{X_t > 0}]} \xrightarrow{t \rightarrow \infty} E[\Lambda_s M_s^{\psi}]\tag{4.14}$$

which is line 3 of the Table.

## 4.5 Statement of line 9 of the Table.

### 4.5.1

Since, from formula (2.15), we have :

$$E[\varphi(S_{d_t}) 1_{X_t > 0} | \mathcal{F}_t] = 1_{X_t > 0} \left\{ \varphi(S_t) \left( 1 - \frac{X_t}{S_t} \right) + X_t \int_{S_t}^{\infty} \frac{\varphi(v)}{v^2} dv \right\}\tag{4.15}$$

the penalisation by  $(\varphi(S_{d_t})1_{X_t > 0}, t \geq 0)$  amounts to penalise by  $(f(X_t, S_t), t \geq 0)$  with :

$$f(a, b) := 1_{a > 0} \left\{ \varphi(b) \left( 1 - \frac{a}{b} \right) + a \int_b^\infty \frac{\varphi(v)}{v^2} dv \right\} \quad (4.16)$$

We already observe that, under the hypothesis  $\int_0^\infty \varphi(b)b^2 db < \infty$ , we have :

$$\bar{f} = \int_0^\infty da \int_a^\infty (2b - a) f(a, b) db = \int_0^\infty \varphi(b)b^2 db < \infty$$

Indeed :

$$\begin{aligned} \bar{f} &= \int_0^\infty da \int_a^\infty (2b - a) \left[ \varphi(b) \left( 1 - \frac{a}{b} \right) + a \int_b^\infty \frac{\varphi(v)}{v^2} dv \right] db \\ &= \int_0^\infty \varphi(b) db \int_0^b (2b - a) \left( 1 - \frac{a}{b} \right) da + \int_0^\infty a da \int_a^\infty \frac{\varphi(v)}{v^2} dv \int_a^v (2b - a) db \\ &= \frac{5}{6} \int_0^\infty \varphi(b)b^2 db + \int_0^\infty a da \int_a^\infty \frac{\varphi(v)}{v} (v - a) dv \\ &= \frac{5}{6} \int_0^\infty \varphi(b)b^2 db + \int_0^\infty \frac{\varphi(v)}{v} dv \int_0^v a(v - a) da \\ &= \frac{5}{6} \int_0^\infty \varphi(b)b^2 db + \frac{1}{6} \int_0^\infty \varphi(b)b^2 db = \int_0^\infty \varphi(b)b^2 db \end{aligned} \quad (4.17)$$

We now use Theorem 1.6 from [RVY, III].

This Theorem asserts that, for a positive function  $f$  such that  $\bar{f} < \infty$ , for every  $s \geq 0$  and  $\Lambda_s \in b(\mathcal{F}_s)$  :

$$\lim_{t \rightarrow \infty} \frac{E(\Lambda_s f(X_t, S_t))}{E(f(X_t, S_t))} \xrightarrow{t \rightarrow \infty} E[\Lambda_s M_s^\theta] \quad (4.18)$$

$$\text{with } M_s^\theta := \theta(S_s)(S_s - X_s) + \int_{S_s}^\infty \theta(y) dy \quad (4.19)$$

and where  $\theta$  is defined by :

$$\theta(b) = \frac{1}{f} \left\{ \int_{-\infty}^\infty da \int_{b \vee a^+}^\infty f(a, \eta) d\eta + \int_{-\infty}^b f(a, b)(b - a) da \right\} \quad (4.20)$$

We shall have proven the statement of line 9 of the Table once we have established that :

$$\theta(b) = \frac{\varphi(b)b^2 + 2 \int_b^\infty \varphi(v)v dv}{3 \int_0^{-\infty} \varphi(v)v^2 dv} \quad (4.21)$$

where, in (4.20),  $f$  is given by (4.16).

### 4.5.2 Computation of $\theta$

We have, from (4.20) et (4.16)

$$\begin{aligned}\bar{f}\theta(b) &= \int_0^\infty da \int_{b \vee a}^\infty \left[ \varphi(\eta) \left(1 - \frac{a}{\eta}\right) + a \int_\eta^\infty \varphi(v) \frac{dv}{v^2} \right] d\eta \\ &\quad + \int_0^b (b-a) \left[ \varphi(b) \left(1 - \frac{a}{b}\right) + a \int_b^\infty \varphi(v) \frac{dv}{v^2} \right] da \\ &:= (2) + (1)\end{aligned}\tag{4.22}$$

where :

$$\begin{aligned}(1) &= \int_0^b (b-a) \left[ \varphi(b) \left(1 - \frac{a}{b}\right) + a \int_b^\infty \varphi(v) \frac{dv}{v^2} \right] da \\ &= \varphi(b) \int_0^b (b-a) \left(1 - \frac{a}{b}\right) da + \int_b^\infty \varphi(v) \frac{dv}{v^2} \int_0^b a(b-a) da \\ &= \varphi(b) \frac{b^2}{3} + \frac{b^3}{6} \int_b^\infty \varphi(v) \frac{dv}{v^2}\end{aligned}\tag{4.23}$$

and :

$$\begin{aligned}(2) &= \int_0^\infty da \int_{b \vee a}^\infty \varphi(\eta) \left(1 - \frac{a}{\eta}\right) d\eta + \int_0^\infty a da \int_{b \vee a}^\infty d\eta \int_\eta^\infty \varphi(v) \frac{dv}{v^2} \\ &:= (2') + (2'') \\ (2') &= \int_0^b da \int_b^\infty \varphi(\eta) \left(1 - \frac{a}{\eta}\right) d\eta + \int_b^\infty da \int_a^\infty \varphi(\eta) \left(1 - \frac{a}{\eta}\right) d\eta \\ &= \int_b^\infty \varphi(\eta) d\eta \int_0^b \left(1 - \frac{a}{\eta}\right) da + \int_b^\infty \varphi(\eta) \left( \int_b^\eta \left(1 - \frac{a}{\eta}\right) da \right) d\eta \\ &= \int_b^\infty \varphi(\eta) d\eta \left( b - \frac{b^2}{2\eta} + \eta - b - \frac{\eta^2}{2\eta} + \frac{b^2}{2\eta} \right) = \frac{1}{2} \int_b^\infty \varphi(\eta) \eta d\eta\end{aligned}\tag{4.24}$$

$$\begin{aligned}(2'') &= \int_0^b a da \int_b^\infty d\eta \int_\eta^\infty \varphi(v) \frac{dv}{v^2} + \int_b^\infty a da \int_a^\infty d\eta \left( \int_\eta^\infty \varphi(v) \frac{dv}{v^2} \right) \\ &= \int_0^b a da \int_b^\infty \varphi(v) \frac{dv}{v^2} \int_b^v d\eta + \int_b^\infty a da \int_a^\infty \varphi(v) \frac{dv}{v^2} \int_a^v d\eta \\ &= \int_0^b a da \int_b^\infty \varphi(v) \frac{dv}{v^2} (v-b) + \int_b^\infty a da \int_a^\infty \frac{\varphi(v) dv}{v^2} (v-a) \\ &= \int_b^\infty \varphi(v) \frac{dv}{v^2} \int_0^b (v-b)a da + \int_b^\infty \varphi(v) \frac{dv}{v^2} \int_b^v a(v-a) da \\ &= \int_b^\infty \varphi(v) \frac{dv}{v^2} \left( \frac{v b^2}{2} - \frac{b^3}{2} + \frac{v^3}{2} - \frac{v b^2}{2} - \frac{v^3}{3} + \frac{b^3}{3} \right) \\ &= -\frac{b^3}{6} \int_b^\infty \frac{\varphi(v)}{v^2} dv + \frac{1}{6} \int_b^\infty \varphi(v) v dv\end{aligned}\tag{4.25}$$

Hence, by addition of (4.25), (4.24) and (4.23), we obtain :

$$\begin{aligned}\bar{f}\theta(b) &= \frac{1}{3} \varphi(b)b^2 + \int_b^\infty \varphi(v) \frac{dv}{v^2} \left( \frac{b^3}{6} - \frac{b^3}{6} \right) + \int_b^\infty \varphi(v)v \, dv \left( \frac{1}{2} + \frac{1}{6} \right) \\ &= \frac{1}{3} \varphi(b)b^2 + \frac{2}{3} \int_b^\infty \varphi(v)v \, dv.\end{aligned}$$

hence :

$$\theta(b) = \frac{\varphi(b)b^2 + 2 \int_b^\infty \varphi(v)v \, dv}{3 \int_0^\infty \varphi(v)v^2 \, dv}$$

and it is easy to show that  $\int_0^\infty \theta(b)db = 1$ . This is the statement of line 9 of the Table.

**Remark 4.**

**1.** The transformation  $\varphi \rightarrow \psi$ , with  $\psi(b) = \varphi(b)b^2 + 2 \int_0^\infty \varphi(y)y \, dy$  appears on lines 3 and 9 of the Table. This transformation may be described probabilistically as follows. Assume that  $3 \int_0^\infty \varphi(b)b^2 db = 1$  and let  $Z$  denote a r.v. with density  $f_Z(b) = 3 \varphi(b)b^2$  ( $b \geq 0$ ). Then, the r.v. :

$$Z_p \stackrel{\text{law}}{=} \varepsilon_p Z + (1 - \varepsilon_p)UZ \tag{4.26}$$

with  $p = \frac{1}{3}$ , admits  $\psi$  as density. In (4.26), the r.v.'s  $\varepsilon_p$ ,  $Z$  and  $U$  are independent ;  $U$  is uniform on  $[0, 1]$  and  $\varepsilon_p$  is a Bernoulli r.v. :  $P(\varepsilon_p = 1) = p$ ,  $P(\varepsilon_p = 0) = 1 - p$ .

**2.** Given Theorem 0, together with point 2 of Remark 1, Theorem 1 and Theorem 6, all the limiting probabilities  $Q$  of the Table have been described precisely, excepted probability  $(g,+)Q^\varphi$  which is found on line 6 of the Table. This probability may be studied by taking up again the technique used for the proof of points 2, 3 and 4 of Theorem 1. Details are left to the reader.

**3.** Let  $\varphi_1$  and  $\varphi_2$  denote two Borel functions from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  such that  $\int_0^\infty (\varphi_1 + \varphi_2)(x)dx = 1$ .

The techniques we have just used allow to study the penalisation by  $(\varphi_1(S_{g_t})1_{X_t > 0} + \varphi_2(S_{g_t})1_{X_t < 0}, t \geq 0)$ . We obtain : For every  $s \geq 0$  and  $\Lambda_s \in b(\mathcal{F}_s)$  :

$$\lim_{t \rightarrow \infty} \frac{E[\Lambda_s \{ \varphi_1(S_{g_t})1_{X_t > 0} + \varphi_2(S_{g_t})1_{X_t < 0} \}]}{E[\varphi_1(S_{g_t})1_{X_t > 0} + \varphi_2(S_{g_t})1_{X_t < 0}]} \xrightarrow{t \rightarrow \infty} E[\Lambda_s^{(g)} M_s^{\varphi_1, \varphi_2}] \tag{4.27}$$

where the martingale  $(^{(g)}M_s^{\varphi_1, \varphi_2}, s \geq 0)$  is defined by :

$$(^{(g)}M_s^{\varphi_1, \varphi_2} := \varphi_1(S_{g_s}) \cdot X_s + M_s^{\varphi_1 + \varphi_2} \tag{4.28}$$

(see (1.3) for the definition of  $M_s^{\varphi_1 + \varphi_2}$ ).

## 5 Penalisation by $(\varphi(X_{g_t}^*), t \geq 0)$ .

### 5.1

We note :

$$X_t^* := \sup_{s \leq t} |X_s| \quad (5.1)$$

$$\text{and for } a \geq 0 \quad T_a^* := \inf \{t \geq 0 ; |X_t| = a\} \quad (5.2)$$

As above, we assume that  $\varphi$  is a probability density on  $\mathbb{R}_+$  ; we define :

$$\Phi(x) := \int_0^x \varphi(y) dy, \quad \text{so that : } \Phi(0) = 0, \quad \text{and } \Phi(\infty) = 1.$$

**Theorem 6.** *Under the preceding hypotheses, one has :*

1) *For any  $s \geq 0$  and  $\Lambda_s \in b(\mathcal{F}_s)$*

$$\lim_{t \rightarrow \infty} \frac{E[\Lambda_s \varphi(X_{g_t}^*)]}{E[\varphi(X_{g_t}^*)]} = E(\Lambda_s M_s^{*\varphi}) := Q^{*\varphi}(\Lambda_s) \quad (5.3)$$

with

$$M_s^{*\varphi} := \varphi(X_{g_s}^*)|X_s| + \varphi(X_s^*)(X_s^* - |X_s|) + 1 - \Phi(X_s^*) \quad (5.4)$$

Furthermore,  $(M_s^{*\varphi}, s \geq 0)$  is a positive martingale, which converges to 0 as  $s \rightarrow \infty$ .

2) *Formula (5.3) induces a probability  $Q^{*\varphi}$  on  $(\Omega, \mathcal{F}_\infty)$ . Under  $Q^{*\varphi}$ , the canonical process satisfies :*

$$i) \quad g := \sup\{t, X_t = 0\} \quad \text{is finite a.s.} \quad (5.5)$$

$$ii) \quad X_\infty^* = \infty \quad \text{a.s.} \quad (5.6)$$

iii) *the processes  $(X_t, t < g)$  and  $(X_{g+t}, t \geq 0)$  are independent ;*

iv)  *$(X_{g+t}, t \geq 0)$  is with probability 1/2, a 3-dimensional Bessel process, starting from 0, and with probability 1/2, it is the opposite of a 3-dimensional Bessel process. In other words,  $(X_{g+t}, t \geq 0) = (\varepsilon R_t, t \geq 0)$ , where  $\varepsilon$  is a symmetric Bernoulli r.v. independent of the 3-dimensional Bessel process  $(R_t, t \geq 0)$  started at 0.*

v) *Conditionally on  $L_g = v$  and  $|X_g^*| = c$ , the process  $(X_t, t \leq g)$  is a Brownian motion stopped at  $\tau_v$  and conditioned on  $X_{\tau_v}^* = c$ .*

### 5.2 A Lemma for the proof of Theorem 6.

This proof is close to that of Theorem 1. Hence, we shall not develop it entirely, and we shall only indicate briefly the elements which differ :

**Lemma 7.**

1. *For any real  $a$  and  $\alpha > 0$  :*

i) *If  $\alpha < |a|$ ,*

$$P_a(X_{g_t}^* < \alpha) = 0 \quad (5.7)$$

ii) If  $\alpha > |a|$ ,

$$\begin{aligned} P_a(X_{g_t}^* < \alpha) &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} \alpha \\ P_a(X_{g_t}^* < \alpha, g_t = 0) &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} |a| \\ P_a(X_{g_t}^* < \alpha, g_t > 0) &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} (|\alpha| - a) \end{aligned} \quad (5.8)$$

2. For every Borel integrable function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  we have :

$$E_a(\psi(X_{g_t}^*) 1_{g_t > 0}) \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_{|a|}^{\infty} \psi(x) dx \quad (5.9)$$

$$E_a(\psi(X_{g_t}^*) 1_{g_t = 0}) \underset{t \rightarrow \infty}{\sim} \psi(|a|) |a| \sqrt{\frac{2}{\pi t}} \quad (5.9')$$

**Sketch of the proof of Lemma 7.** (5.7) is obvious. Let us prove (5.8). From the identities :

$$(X_{g_t}^* < \alpha) = (g_t < T_\alpha^*) = (t < d_{T_\alpha^*}) = (t < T_\alpha^* + T_0 \circ \theta_{T_\alpha^*})$$

we deduce :

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P_a(X_{g_t}^* < \alpha) dt &= E_a \left( \int_0^{d_{T_\alpha^*}} e^{-\lambda t} dt \right) = \frac{1}{\lambda} (1 - E_a(e^{-\lambda(T_\alpha^* + T_0 \circ \theta_{T_\alpha^*})})) \\ &= \frac{1}{\lambda} (1 - E_a(e^{-\lambda T_\alpha^*}) \cdot E_a(e^{-\lambda T_0})) \\ &= \frac{1}{\lambda} \left\{ 1 - e^{-\alpha \sqrt{2\lambda}} \frac{\cosh(a\sqrt{2\lambda})}{\cosh(\alpha\sqrt{2\lambda})} \right\} \quad (\text{see [KS], p.100}) \\ &\underset{\lambda \rightarrow 0}{\sim} \frac{\sqrt{2}}{\sqrt{\lambda}} \alpha \end{aligned}$$

Hence (5.8) follows, with the help of the Tauberian Theorem, since  $P_a(X_{g_t}^* < \alpha)$  is a non-decreasing function of  $t$  (see [Fel], vol. 2, p. 442).

On the other hand, we have, if  $|a| < \alpha$

$$\begin{aligned} P_a(X_{g_t}^* < \alpha, g_t = 0) &= P_a(T_0 > t) \\ &= \int_t^\infty \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}} ds \\ &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} |a| \end{aligned}$$

Relations (5.9) and (5.9') follow easily from point 1 of Lemma 7.

### 5.3 We prove that :

$$\frac{E[\varphi(X_{g_t}^*) | \mathcal{F}_s]}{E(\varphi(X_{g_t}^*))} \xrightarrow{t \rightarrow \infty} M_s^* \varphi \quad \text{a.s.} \quad (5.10)$$

We already note that, from (5.9), (5.9') and since  $g_t > 0$ ,  $P_0$  a.s.

$$E(\varphi(X_{g_t}^*)) \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \quad (5.11)$$

On the other hand :

$$\begin{aligned} E[\varphi(X_{g_t}^*)|\mathcal{F}_s] &= E[\varphi(X_{g_t}^*)1_{g_t \leq s}|\mathcal{F}_s] + E[\varphi(X_{g_t}^*)1_{g_t > s}|\mathcal{F}_s] \\ &= (1)_t + (2)_t \end{aligned} \quad (5.12)$$

One has :

$$(1)_t = E[\varphi(X_{g_t}^*)1_{g_t \leq s}|\mathcal{F}_s] = E[\varphi(X_{g_s}^*)1_{g_t < s}|\mathcal{F}_s]$$

since  $g_s = g_t$  when  $g_t \leq s$

$$(1)_t = \varphi(X_{g_s}^*)E[1_{g_t \leq s}|\mathcal{F}_s] \underset{t \rightarrow \infty}{\sim} \varphi(X_{g_s}^*)|X_s| \sqrt{\frac{2}{\pi(t-s)}} \quad (5.13)$$

from (2.21). On the other hand :

$$\begin{aligned} (2)_t &= E[\varphi(X_{g_t}^*)1_{g_t > s}|\mathcal{F}_s] = E[\varphi(X_s^* \vee X_{[s, g_t]}^*)1_{g_t > s}|\mathcal{F}_s] \\ &\quad \left( \text{with } X_{[s, g_t]}^* = \sup_{u \in [s, g_t]} |X_u| \right) \\ &\underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_{|X_s|}^{\infty} \varphi(X_s^* \vee x) dx \quad \left( \text{from (5.9), (5.9') and the Markov property} \right) \\ &= \sqrt{\frac{2}{\pi t}} \left\{ \int_{|X_s|}^{X_s^*} \varphi(X_s^*) dx + \int_{X_s^*}^{\infty} \varphi(x) dx \right\} \\ &= \sqrt{\frac{2}{\pi t}} \left( (X_s^* - |X_s|) \varphi(X_s^*) + 1 - \Phi(X_s^*) \right) \end{aligned} \quad (5.14)$$

Gathering (5.11), (5.13) and (5.14), (5.10) follows immediately. Using similar arguments as in the proof of Theorem 4 point 1, Theorem 6 follows.

#### 5.4 Proof of $Q^{*\varphi}(g < \infty) = 1$ .

We have :

$$\begin{aligned} Q^{*\varphi}(g_t > a) &= Q^{*\varphi}\{d_a < t\} = E[1_{d_a < t} \cdot M_t^{*\varphi}] \\ &= E[1_{d_a < t} M_{d_a}^{*\varphi}] = E\left[1_{d_a < t} [\varphi(X_{d_a}^*)X_{d_a}^* + 1 - \Phi(X_{d_a}^*)]\right] \end{aligned}$$

Hence, letting  $t \rightarrow +\infty$  :

$$Q^{*\varphi}(g > a) = E[\varphi(X_{d_a}^*)X_{d_a}^* + 1 - \Phi(X_{d_a}^*)] \leq 2E[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})]$$

because, with obvious notations,  $X_{d_a}^* = S_{d_a}$  or  $-I_{d_a}$  (with  $I_u := \inf_{s \leq u} X_s$ ) and then

$$Q^{*\varphi}(g = \infty) = \lim_{a \rightarrow \infty} Q^{*\varphi}(g > a) = 0 \quad \left( \text{from (2.37)} \right)$$

### 5.5 Proof of $Q^{*\varphi}(X_\infty^* = \infty) = 1$ .

Indeed, operating as above, with  $a > 0$ , we obtain :

$$\begin{aligned} Q^{*\varphi}[X_\infty^* > a] &= Q^{*\varphi}[T_a^* < \infty] = E[M_{T_a^*}^{*\varphi}] \\ &= E[\varphi(X_{g_{T_a^*}^*}) a + 1 - \Phi(a)] \\ &= \int_a^\infty \varphi(x) dx + \int_0^a \varphi(x) dx = 1 \end{aligned} \tag{5.15}$$

since the r.v.,  $X_{g_{T_a^*}^*}^*$  is uniformly distributed on  $[0, a]$ . Indeed :

$$\begin{aligned} P(X_{g_{T_a^*}^*}^* \geq x) &= P(g_{T_a^*}^* \geq T_x^*) \\ &= P(\text{after } T_x^*, X \text{ reaches } 0 \text{ before } \pm a) \\ &= 1 - \frac{x}{a} \end{aligned}$$

It now remains to let  $a$  tend to  $+\infty$  in (5.15).

We leave to the interested reader the task of completing the proof of Theorem 6. ■

## 6 The relative position of this paper in our penalisation studies.

Since roughly 2002, we have devoted most of our research activities to various kinds of penalisations of Brownian paths; two sets of papers are emerging from these studies : essentially, the first set, with Roman numberings, going from I to X discusses "individual" cases of penalisations, whereas the second set consists of two monographs ([RY, M1] and [NRY, M2]). Let us now discuss a little more in detail the contents of these two sets :

a) "The Roman set" consists in a number of detailed studies of penalisations of Brownian paths with various functionals, including :

- continuous additive functionals such as  $A_t^{(q)} = \int_0^t q(X_s) ds$  ([RVY, I] and [RY, IX]) ; we now call these Feynman-Kac penalisations
- the one sided supremum, or the local time at 0 ([RVY, II], [RVY, III], [RVY, IV]), or the amplitude process ([RVY, X])
- lengths of excursions, ranked in decreasing order ([RVY, VII]).

This latter study led us, at no big extra cost, to work in the set-up of a  $d$ -dimensional Bessel process, for  $0 < d < 2$ , since the Brownian arguments may be extended here in a natural manner ([RVY, V]). We also developed penalisation studies in the context of planar Brownian and its winding process ([RVY, VI]).

The present paper complements [RVY, II].

b) In the monographs M1 and M2, we attempt to develop a global viewpoint about penalisation, e.g. concerning the Feynman-Kac type penalisations, we exhibit some  $\sigma$ -finite measures on path space which "rule" jointly all the penalisations. See also J. Najnudel's thesis [N], which gives some full proofs to certain "meta-theorems" presented in [RY, M1]. A CRAS Note ([NRY]) summarizes our results relative to these  $\sigma$ -finite measures.

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