ON KHINTCHINE EXPONENTS AND LYAPUNOV EXPONENTS OF CONTINUED FRACTIONS

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Abstract. Assume that $x \in [0,1)$ admits its continued fraction expansion $x = [a_1(x), a_2(x), \cdots]$. The Khintchine exponent $\gamma(x)$ of x is defined by $\gamma(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \log a_j(x)$ when the limit exists. Khintchine spectrum $\dim E_\xi$ is fully studied, where $E_\xi := \{x \in [0,1) : \gamma(x) = \xi\}$ $(\xi \geq 0)$ and dim denotes the Hausdorff dimension. In particular, we prove the remarkable fact that the Khintchine spectrum $\dim E_\xi$, as function of $\xi \in [0,+\infty)$, is neither concave nor convex. This is a new phenomenon from the usual point of view of multifractal analysis. Fast Khintchine exponents defined by $\gamma^\varphi(x) := \lim_{n\to\infty} \frac{1}{\varphi(n)}\sum_{j=1}^n \log a_j(x)$ are also studied, where $\varphi(n)$ tends to the infinity faster than n does. Under some regular conditions on φ, it is proved that the fast Khintchine spectrum $\dim(\{x \in [0,1] : \gamma^\varphi(x) = \xi\})$ is a constant function. Our method also works for other spectra like the Lyapunov spectrum and the fast Lyapunov spectrum.

1. Introduction and Statements

The continued fraction of a real number can be generated by the Gauss transformation $T : [0,1) \to [0,1)$ defined by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \ (\mathrm{mod}\ 1), \ \text{for } x \in (0,1) \tag{1.1}$$

in the sense that every irrational number x in $[0,1)$ can be uniquely expanded as an infinite expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n(x) + T^n(x)}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \tag{1.2}$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$ are called *partial quotients* of x ($\lfloor x \rfloor$ denoting the integral part of x). For simplicity, we will denote the second term in (1.2) by $[a_1, a_2, \cdots, a_n + T^n(x)]$ and the third term by $[a_1, a_2, a_3, \cdots]$.

It was known to E. Borel [5] (1909) that for Lebesgue almost all $x \in [0,1)$, there exists a subsequence $\{a_{n_r}(x)\}$ of $\{a_n(x)\}$ such that $a_{n_r}(x) \to \infty$. A more explicit result due to Borel-Bernstein (see [2, 5, 6]) is the 0-1 law which hints that for almost all $x \in [0,1]$, $a_n(x) > \varphi(n)$ holds for infinitely many n's or finitely many n's according to $\sum_{n\geq 1} \frac{1}{\varphi(n)}$ diverges or converges. Then it arose a natural question to quantify the exceptional sets in terms of Hausdorff dimension (denoted by dim).

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The first published work on this aspect was due to I. Jarnik [21] (1928) who was concerned with the set E of continued fractions with bounded partial quotients and with the sets $E_2, E_3, \cdots$, where E_α is the set of continued fractions whose partial quotients do not exceed α. He successfully got that the set E is of full Hausdorff dimension, but he didn't find the exact dimensions of $E_2, E_3, \cdots$. Later, many works are done to estimate $\dim E_2$, including those of I. J. Good [16], R. Bumby [9], D. Hensley [19, 20], O. Jenkinson and M. Pollicott [22], R. D. Mauldin, M. Urbański [30] and references therein. Up to now, the optimal approximation on $\dim E_2$ is the result given by O. Jenkinson [23] (2004):

$$\dim E_2 = 0.531280506277205141624468647368471785493059109018398779\cdots$$

which is claimed to be accurate to 54 decimal places.

In the present paper, we study the Khintchine exponents and the Lyapunov exponents of continued fractions. For any $x \in [0,1)$ with its continued fraction (1.2), we define its *Khintchine exponent* $\gamma(x)$ and *Lyapunov exponent* $\lambda(x)$ respectively by

$$\gamma(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n}\log\left|(T^n)'(x)\right| = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\left|T'(T^j(x))\right|,$$

if the limits exist. The Khintchine exponent of x stands for the average (geometric) growth rate of the partial quotients $a_n(x)$, and the Lyapunov exponent which is extensively studied from dynamical system point of view, stands for the expanding rate of T. Their common feature is that both are Birkhoff averages.

Let $\varphi : \mathbb{N} \to \mathbb{R}_+$. Assume that $\lim_{n\to\infty} \frac{\varphi(n)}{n} = \infty$. The *fast Khintchine exponent* and *fast Lyapunov exponent* of $x \in [0,1]$, relative to φ, are respectively defined by

$$\gamma^\varphi(x) := \lim_{n\to\infty} \frac{1}{\varphi(n)}\sum_{j=1}^{n} \log a_j(x) = \lim_{n\to\infty} \frac{1}{\varphi(n)}\sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda^\varphi(x) := \lim_{n\to\infty} \frac{1}{\varphi(n)}\log\left|(T^n)'(x)\right| = \lim_{n\to\infty} \frac{1}{\varphi(n)}\sum_{j=0}^{n-1} \log\left|T'(T^j(x))\right|.$$

It is well known (see [4, 37]) that the transformation T is measure preserving and ergodic with respect to the Gauss measure μ_G defined as

$$d\mu_G = \frac{dx}{(1+x)\log 2}.$$

An application of Birkhoff ergodic theorem yields that for Lebesgue almost all $x \in [0,1)$,

$$\gamma(x) = \xi_0 = \int \log a_1(x) d\mu_G = \frac{1}{\log 2}\sum_{n=1}^{\infty} \log n \cdot \log\Big(1 + \frac{1}{n(n+2)}\Big) = 2.6854...$$

$$\lambda(x) = \lambda_0 = \int \log|T'(x)| d\mu_G = \frac{\pi^2}{6\log 2} = 2.37314....$$

Here ξ_0 is called the *Khintchine constant* and λ_0 the *Lyapunov constant*. Both constants are relative to the Gauss measure.

For real numbers $\xi, \beta \geq 0$, we are interested in the level sets of Khintchine exponents and Lyapunov exponents:

$$E_\xi := \{x \in [0,1) : \gamma(x) = \xi\},$$
$$F_\beta := \{x \in [0,1) : \lambda(x) = \beta\}.$$

We are also interested in the level sets of fast Khintchine exponents and fast Lyapunov exponents:

$$E_\xi(\varphi) := \{x \in [0,1) : \gamma^\varphi(x) = \xi\},$$
$$F_\beta(\varphi) := \{x \in [0,1) : \lambda^\varphi(x) = \beta\}.$$

The *Khintchine spectrum* and the *Lyapunov spectrum* are the dimensional functions:

$$t(\xi) := \dim E_\xi \qquad \tilde{t}(\xi) := \dim F_\xi.$$

The following two functions

$$t^\varphi(\xi) := \dim E_\xi(\varphi) \qquad \tilde{t}^\varphi(\xi) := \dim F_\xi(\varphi)$$

are called the *fast Khintchine spectrum* and the *fast Lyapunov spectrum* relative to φ.

M. Pollicott and H. Weiss [36] initially studied the level set of F_β and obtained some partial results about the function $t(\xi)$. In the present work, we will give a complete study on the Khintchine spectrum and the Lyapunov spectrum. Fast Khintchine spectrum and fast Lyapunov spectrum are considered here for the first time. We shall see that both functions $t^\varphi(\xi)$ and $\tilde{t}^\varphi(\xi)$ are equal.

We start with the statement of our results on fast spectra.

Theorem 1.1. *Suppose* $(\varphi(n+1) - \varphi(n)) \uparrow \infty$ *and* $\lim_{n\to\infty} \frac{\varphi(n+1)}{\varphi(n)} := b \geq 1$. *Then* $E_\xi(\varphi) = F_{2\xi}(\varphi)$ *and* $\dim E_\xi(\varphi) = 1/(b+1)$ *for all* $\xi \geq 0$.

In order to state our results on the Khintchine spectrum, let us first introduce some notation. Let

$$D := \{(t,q) \in \mathbb{R}^2 : 2t - q > 1\}, \qquad D_0 := \{(t,q) \in \mathbb{R}^2 : 2t - q > 1, 0 \leq t \leq 1\}.$$

For $(t,q) \in D$, define

$$P(t,q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} \omega_j^q([\omega_j, \cdots, \omega_n + x])^{2t} \right).$$

It will be proved that $P(t,q)$ is an analytic function in D (Proposition 4.6).

Moreover, for any $\xi \geq 0$, there exists a unique solution $(t(\xi), q(\xi)) \in D_0$ to the equation

$$\begin{cases} P(t,q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t,q) = \xi. \end{cases}$$

(Proposition 4.13).

Theorem 1.2. *Let* $\xi_0 = \int \log a_1(x) d\mu_G(x)$. *For* $\xi \geq 0$, *the set* E_ξ *is of Hausdorff dimension* $t(\xi)$. *Furthermore, the dimension function* $t(\xi)$ *has the following properties:*

1) $t(\xi_0) = 1$ *and* $t(+\infty) = 1/2$;

2) *$t'(\xi) < 0$ for all $\xi > \xi_0$, $t'(\xi_0) = 0$, and $t'(\xi) > 0$ for all $\xi < \xi_0$;*
3) *$t'(0+) = +\infty$ and $t'(+\infty) = 0$;*
4) *$t''(\xi_0) < 0$, but $t''(\xi_1) > 0$ for some $\xi_1 > \xi_0$, so $t(\xi)$ is neither convex nor concave.*

See Figure 1 for the graph of $t(\xi)$.

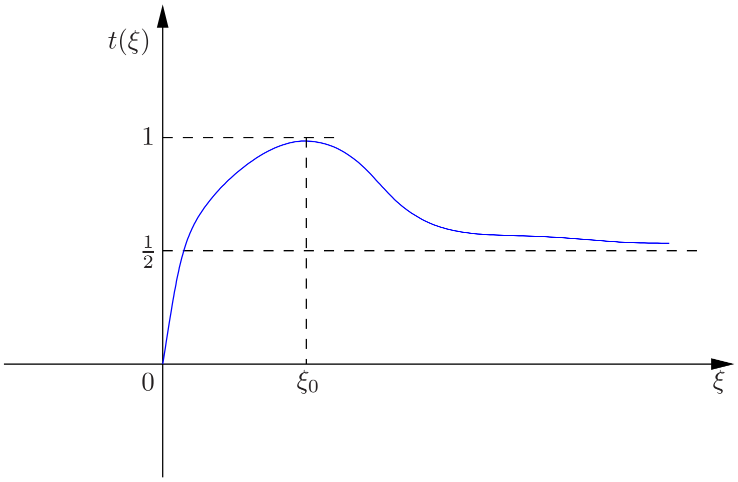

Figure 1. Khintchine spectrum

It should be noticed that the above fourth property of $t(\xi)$, i.e. the non-convexity, shows a new phenomenon for the multifractal analysis in our settings.

Let

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\} \qquad \tilde{D}_0 := \{(\tilde{t}, q) : \tilde{t} - q > 1/2, 0 \le \tilde{t} \le 1\}.$$

For $(\tilde{t}, q) \in \tilde{D}$, define

$$P_1(\tilde{t}, q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\omega_j, \cdots, \omega_n + x])^{2(\tilde{t}-q)} \right).$$

In fact, $P_1(\tilde{t}, q) = P(\tilde{t} - q, 0)$, thus $P_1(\tilde{t}, q)$ is analytic in $\tilde{D}$.

Denote $\gamma_0 := 2\log\frac{1+\sqrt{5}}{2}$. For any $\beta \in (\gamma_0, \infty)$, the system

$$\begin{cases} P_1(\tilde{t}, q) = q\beta, \\ \dfrac{\partial P_1}{\partial q}(\tilde{t}, q) = \beta \end{cases}$$

admits a unique solution $(\tilde{t}(\beta), q(\beta)) \in \tilde{D}_0$ (Proposition 6.3).

Theorem 1.3. *Let $\lambda_0 = \int \log|T'(x)| d\mu_G$ and $\gamma_0 = 2\log\frac{1+\sqrt{5}}{2}$. For any $\beta \in [\gamma_0, \infty)$, the set F_β is of Hausdorff dimension $\tilde{t}(\beta)$. Furthermore the dimension function $\tilde{t}(\xi)$ has the following properties:*
1) *$\tilde{t}(\lambda_0) = 1$ and $\tilde{t}(+\infty) = 1/2$;*
2) *$\tilde{t}'(\beta) < 0$ for all $\beta > \lambda_0$, $\tilde{t}'(\lambda_0) = 0$, and $\tilde{t}'(\beta) > 0$ for all $\beta < \lambda_0$;*
3) *$\tilde{t}'(\gamma_0+) = +\infty$ and $\tilde{t}'(+\infty) = 0$;*

4) $\tilde{t}''(\lambda_0) < 0$*, but* $\tilde{t}''(\beta_1) > 0$ *for some* $\beta_1 > \lambda_0$*, i.e.,* $\tilde{t}(\beta)$ *is neither convex nor concave.*

See Figure 2 for the graph of $\tilde{t}(\beta)$.

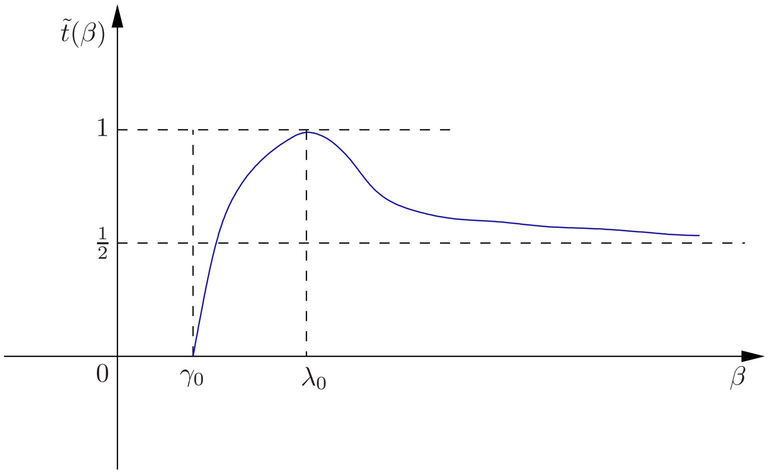

Figure 2. Lyapunov spectrum

The last two theorems are concerned with special Birkhoff spectra. In general, let (X, T) be a dynamical system (T being a map from a metric space X into itself). The Birkhoff average of a function $\phi : X \to \mathbb{R}$, defined by

$$\overline{\phi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)) \qquad x \in X$$

(if the limit exists) is widely studied. From the point of view of multifractal analysis, one is often interested in determining the Hausdorff dimension of the set $\{x \in X : \overline{\phi}(x) = \alpha\}$ for a given $\alpha \in \mathbb{R}$. The function

$$f(\alpha) := \dim\big(\{x \in X : \overline{\phi}(x) = \alpha\}\big)$$

is called the *Birkhoff spectrum* for the function ϕ. When X is compact, T and ϕ are continuous, the Birkhoff spectrum are well studied (see [1, 14, 15] and the references therein. See also the book of Y. B. Pesin [35]).

The main tool of our study is the Ruelle-Perron-Frobenius operator with potential function

$$\Phi_{t,q}(x) = t \log |T'(x)| + q \log a_1(x), \quad \Psi_t(x) = t \log |T'(x)|,$$

where (t, q) are suitable parameters. The classical way to obtain the spectrum through Ruelle theory usually fixes q and finds $T(q)$ as the solution of $P(T(q), q) = 0$. (Here $P(t, q)$ is the pressure corresponding to the potential function of two parameters.) By focusing on the curve $T(q)$, one can only get some partial results ([36]). In the present paper, we look for multifractal information from the whole two dimensional surface defined by the pressure $P(t, q)$ rather than the single curve $T(q)$. This leads us to obtain complete graphs of the Khintchine spectrum and Lyapunov spectrum.

For the Gauss dynamics, there exist several works on pressure functions associated to different potentials. For a detailed study on pressure function associated to one potential function, we refer to the works of D. Mayer [32, 33, 34], and for pressure functions associated to two potential functions, we refer to the works of M. Pollicott and H. Weiss [36], of P. Walters [38, 39] and of P. Hanus, R. D. Mauldin and M. Urbanski [17]. We will use the theory developed in [17].

The paper is organized as follows. In Section 2, we collect and establish some basic results that will be used later. Section 3 is devoted to proving the results about the fast Khintchine spectrum and fast Lyapunov spectrum (Theorem 1.1). In Section 4, we present a general Ruelle operator theory developed in [17] and then apply it to the Gauss transformation. Based on Section 4, we establish Theorem 1.2 in Section 5. The last section is devoted to the study of Lyapunov spectrum (Theorem 1.3).

The present paper is a part of the second author's Ph. D. thesis.

2. PRELIMINARY

In this section, we collect some known facts and establish some elementary properties of continued fractions that will be used later. For a wealth of classical results about continued fractions, see the books by J. Cassels [10], G. Hardy and E. Wright [18]. The books by P. Billingsley [4], I. Cornfeld, S. Fomin and Ya. Sinai [11] contain an excellent introduction to the dynamics of the Gauss transformations and its connection with Diophantine approximation.

2.1. **Elementary properties of continued fractions.** Denote by p_n/q_n the usual n-th *convergent* of continued fraction $x = [a_1(x), a_2(x), \cdots] \in [0,1) \setminus \mathbb{Q}$, defined by

$$\frac{p_n}{q_n} := [a_1(x), \cdots, a_n(x)] := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x)}}}.$$

It is known (see [26] p.9) that p_n, q_n can be obtained by the recursive relation:

$$\begin{aligned} p_{-1} = 1,\ p_0 = 0,\ p_n = a_n p_{n-1} + p_{n-2} \qquad (n \geq 2), \\ q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2} \qquad (n \geq 2). \end{aligned}$$

Furthermore, we have

Lemma 2.1 ([28] p.5). *Let* $\varepsilon_1, \cdots, \varepsilon_n \in \mathbb{R}^+$. *Define inductively*

$$Q_{-1} = 0,\ Q_0 = 1,\ Q_n(\varepsilon_1, \cdots, \varepsilon_n) = \varepsilon_n Q_{n-1}(\varepsilon_1, \cdots, \varepsilon_{n-1}) + Q_{n-2}(\varepsilon_1, \cdots, \varepsilon_{n-2}).$$

(Q_n is commonly called a continuant.) Then we have

(i) $Q_n(\varepsilon_1, \cdots, \varepsilon_n) = Q_n(\varepsilon_n, \cdots, \varepsilon_1)$;

(ii) $q_n = Q_n(a_1, \cdots, a_n)$, $p_n = Q_{n-1}(a_2, \cdots, a_n)$.

As consequences, we have the following results.

Lemma 2.2 ([26]). *For any* $a_1, a_2, \cdots, a_n, b_1, \cdots, b_m \in \mathbb{N}$, *let* $q_n = q_n(a_1, \cdots, a_n)$ *and* $p_n = p_n(a_1, \cdots, a_n)$. *We have*

(i) $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$;

(ii) $q_{n+m}(a_1, \cdots, a_n, b_1, \cdots, b_m) = q_n(a_1, \cdots, a_n) q_m(b_1, \cdots, b_m) + q_{n-1}(a_1, \cdots, a_{n-1}) p_{m-1}(b_1, \cdots, b_{m-1})$;

(iii) $q_n \geq 2^{\frac{n-1}{2}}, \quad \prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n} (a_k+1).$

Lemma 2.3 ([41])**.** *For any* $a_1, a_2, \cdots, a_n, b \in \mathbb{N}$,

$$\frac{b+1}{2} \leq \frac{q_{n+1}(a_1, \cdots, a_j, b, a_{j+1}, \cdots, a_n)}{q_n(a_1, \cdots, a_j, a_{j+1}, \cdots, a_n)} \leq b+1 \qquad (\forall 1 \leq j < n).$$

For any $a_1, a_2, \cdots, a_n \in \mathbb{N}$, let

$$I_n(a_1, a_2, \cdots, a_n) = \{x \in [0,1) : a_1(x) = a_1, a_2(x) = a_2, \cdots, a_n(x) = a_n\} \tag{2.1}$$

which is called an *n-th order cylinder*.

Lemma 2.4 ([28] p.18)**.** *For any* $a_1, a_2, \cdots, a_n \in \mathbb{N}$*, the n-th order cylinder* $I_n(a_1, a_2, \cdots, a_n)$ *is the interval with the endpoints* p_n/q_n *and* $(p_n + p_{n-1})/(q_n + q_{n-1})$*. As a consequence, the length of* $I_n(a_1, \cdots, a_n)$ *is equal to*

$$\Big|I_n(a_1, \cdots, a_n)\Big| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{2.2}$$

We will denote $I_n(x)$ the n-th order cylinder that contains x, i.e. $I_n(x) = I_n\big(a_1(x), \cdots, a_n(x)\big)$. Let $B(x, r)$ denotes the ball centered at x with radius r. For any $x \in I_n(a_1, \cdots, a_n)$, we have the following relationship between the ball $B(x, |I_n(a_1, \cdots, a_n)|)$ and $I_n(a_1, \cdots, a_n)$, which is called the *regular property* in [7].

Lemma 2.5 ([7])**.** *Let* $x = [a_1, a_2, \cdots]$*. We have:*

(i) *if* $a_n \neq 1$, $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_n(a_1, \cdots, a_n + j)$*;*

(ii) *if* $a_n = 1$ *and* $a_{n-1} \neq 1$, $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_{n-1}(a_1, \cdots, a_{n-1} + j)$*;*

(iii) *if* $a_n = 1$ *and* $a_{n-1} = 1$, $B(x, |I_n(x)|) \subset I_{n-2}(a_1, \cdots, a_{n-2})$.

The Gauss transformation T admits the following *Jacobian estimate*.

Lemma 2.6. *There exists a positive number* $K > 1$ *such that for all irrational number* x *in* $[0, 1)$*, one has*

$$0 < \frac{1}{K} \leq \sup_{n \geq 0} \sup_{y \in I_n(x)} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq K < \infty.$$

Proof. Assume $x = [a_1, \cdots, a_n, \cdots] \in [0, 1) \setminus \mathbb{Q}$. For any $n \geq 0$ and $y \in I_n(x) = I_n(a_1, \cdots, a_n)$, by the fact that $T'(x) = -\frac{1}{x^2}$ we get

$$\sum_{j=0}^{n-1} \Big| \log\big|T'(T^j(x))\big| - \log\big|T'(T^j(y))\big| \Big| = 2\sum_{j=0}^{n-1} \Big| \log T^j(x) - \log T^j(y) \Big|.$$

Applying the mean-value theorem, we have

$$\big|\log T^j(x) - \log T^j(y)\big| = \left| \frac{T^j(x) - T^j(y)}{T^j(z)} \right| \leq \frac{a_{j+1}}{q_{n-j}(a_{j+1}, \cdots, a_n)},$$

where the assertion follows from the fact that all three points $T^j(x), T^j(y)$ and $T^j(z)$ belong to $I_{n-j}(a_{j+1}, \cdots, a_n)$. By Lemma 2.2, we have

$$\sum_{j=0}^{n-1} \big|\log T^j(x) - \log T^j(y)\big| \leq \sum_{j=0}^{n-1} \frac{1}{q_{n-j-1}(a_{j+2}, \cdots, a_n)} \leq \sum_{j=0}^{n-1} \left(\frac{1}{2}\right)^{n-j-2} \leq 4.$$

Thus the result is proved with $K = e^4$. □

The above Jacobian estimate property of T enables us to control the length of $I_n(x)$ by $|(T^n)'(x)|^{-1}$, through the fact that $\int_{I_n(x)} |(T^n)'(y)|dy = 1$.

Lemma 2.7. *There exist a positive constant* $K > 0$ *such that for all irrational numbers* x *in* $[0, 1)$,

$$\frac{1}{K} \leq \frac{|I_n(x)|}{|(T^n)'(x)|^{-1}} \leq K.$$

We remark that from Lemma 2.4 and Lemma 2.7, we have

$$\frac{1}{2K} q_n^2(x) \leq \Big|(T^n)'(x)\Big| \leq K q_n^2(x).$$

So the Lyapunov exponent $\lambda(x)$ is nothing but the growth rate of $q_n(x)$ up to a multiplicative constant 2:

$$\lambda(x) = \lim_{n\to\infty} \frac{2}{n} \log q_n(x).$$

For any irrational number x in $[0, 1)$, let $N_n(x) := \{j \leq n : a_j(x) \neq 1\}$. Set

$$A := \Big\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \log q_n(x) = \frac{\gamma_0}{2}\Big\},$$
$$B := \Big\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = 0\Big\},$$
$$C := \Big\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sharp N_n(x) = 0\Big\},$$

where $\sharp$ stands for the cardinal of a set. Then we have the following relationship.

Lemma 2.8. *With the notations given above, we have*

$$A = B \subset C.$$

Proof. It is clear that $A \subset C$ and $B \subset C$. Let us prove $A = B$. First observe that, by Lemma 2.3, we have

$$\begin{aligned}\frac{1}{n} \log q_n(x) &\geq \frac{1}{n} \sum_{j\in N_n(x)} \log \frac{a_j(x) + 1}{2} + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1) \\ &\geq \frac{1}{n} \sum_{j\in N_n(x)} \log a_j(x) - \frac{1}{n} \sum_{j\in N_n(x)} \log 2 + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1).\end{aligned}$$

Assume $x \in A$. Since $A \subset C$, we have

$$-\frac{1}{n} \sum_{j\in N_n(x)} \log 2 + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1) \longrightarrow 0 + \frac{\gamma_0}{2} \qquad (n \to \infty).$$

Now by the assumption $x \in A$, it follows

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \frac{1}{n} \sum_{j\in N_n(x)} \log a_j(x) = 0.$$

Therefore we have proved $A \subset B$. For the inverse inclusion, notice that

$$\frac{1}{n} \log q_n(x) \leq \frac{1}{n} \sum_{j\in N_n(x)} \log(a_j(x) + 1) + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1).$$

Let $x \in B$. Since $B \subset C$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log q_{n-\sharp N_n}(1,\ldots,1) = \frac{\gamma_0}{2}.$$

Therefore by the assumption $x \in B$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log q_n(x) \le \frac{\gamma_0}{2}.$$

Thus $B \subset A$. $\square$

2.2. **Exponents $\gamma(x)$ and $\lambda(x)$.** In this subsection, we make a quick examination of the Khintchine exponent $\gamma(x)$ and compare it with the Lyapunov exponent $\lambda(x)$. Our main concern is the possible values of both exponent functions.

A first observation is that for any $x \in [0,1)$, $\gamma(x) \ge 0$ and $\lambda(x) \ge \gamma_0 = 2\log\frac{\sqrt{5}+1}{2}$. By the Birkhoff ergodic theorem, we know that the Khintchine exponent $\gamma(x)$ attains the value ξ_0 for almost all points x with respect to the Lebesgue measure. We will show that every positive number is the Khintchine exponent $\gamma(x)$ of some point x.

Proposition 2.9. *For any $\xi \ge 0$, there exists a point $x_0 \in [0,1)$ such that $\gamma(x_0) = \xi$.*

Proof. Assume $\xi > 0$ (for $\xi = 0$, we take $x_0 = \frac{1+\sqrt{5}}{2}$ corresponding to $a_n \equiv 1$.) Take an increasing sequence of integers $\{n_k\}_{k\ge 1}$ satisfying

$$n_0 = 1, \quad n_{k+1} - n_k \to \infty, \text{ and } \frac{n_k}{n_{k+1}} \to 1, \text{as } \ k \to \infty.$$

Let $x_0 \in (0,1)$ be a point whose partial quotients satisfy

$$e^{(n_k - n_{k-1})\xi} \le a_{n_k} \le e^{(n_k - n_{k-1})\xi} + 1; \quad a_n = 1 \text{ otherwise}.$$

Since for $n_k \le n < n_{k+1}$,

$$\frac{1}{n_{k+1}} \sum_{i=1}^{k} \log e^{(n_i - n_{i-1})\xi} \le \frac{1}{n} \sum_{j=1}^{n} \log a_j \le \frac{1}{n_k} \sum_{i=1}^{k} \log(e^{(n_i - n_{i-1})\xi} + 1),$$

we have

$$\gamma(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \xi.$$

$\square$

In the following, we will show that the set E_ξ and F_λ are never equal. So it is two different problems to study $\gamma(x)$ and $\lambda(x)$. However, as we will see, $E_\xi(\phi) = F_{2\xi}(\phi)$ when ϕ is faster than n.

Proposition 2.10. *For any $\xi \ge 0$ and $\lambda \ge 2\log\frac{\sqrt{5}+1}{2}$, we have $E_\xi \ne F_\lambda$.*

Proof. Given $\xi \ge 0$. It suffices to construct two numbers with same Khintchine exponent ξ but different Lyapunov exponents.

For the first number, take just the number x_0 constructed in the proof of Proposition 2.9. We claim that

$$\lambda(x_0) = 2\xi + 2\log\frac{\sqrt{5}+1}{2}. \tag{2.3}$$

In fact, by Lemma 2.3 we have

$$\prod_{j=1}^{k}(\frac{a_{n_j}+1}{2})q_{n_k-k}(1,\cdots,1) \le q_{n_k}(a_1,\cdots,a_{n_k}) \le \prod_{j=1}^{k}(a_{n_j}+1)q_{n_k-k}(1,\cdots,1). \tag{2.4}$$

Then by the assumption on n_k, we have

$$\lambda(x_0) = \lim_{n\to\infty}\frac{2}{n}\log q_n(x_0) = 2(\xi + \log\frac{\sqrt{5}+1}{2}).$$

Construct now the second number. Fix $k \ge 1$. Define $x_1 = [\varsigma_1, \cdots, \varsigma_n, \cdots]$ where

$$\varsigma_n = \left(\underbrace{\overbrace{1,\cdots,1,\lfloor e^{k\xi}\rfloor}^{k},\cdots,1,\cdots,1,\lfloor e^{k\xi}\rfloor}_{kn}, \left\lfloor(\frac{e^{(k+1)\xi}}{[e^{k\xi}]})^n\right\rfloor\right).$$

Notice that there are n small vectors $(1,\cdots,1,\lfloor e^{k\xi}\rfloor)$ in ς_n and the length of ς_n is equal to $N_k := kn+1$. We can prove

$$\gamma(x_1) = \xi, \quad \lambda(x_1) = \lambda\left(\left[\overline{1,\cdots,\lfloor e^{k\xi}\rfloor}\right]\right) + 2\xi - \frac{2}{k}\log\lfloor e^{k\xi}\rfloor,$$

by the same arguments as in proving the similar result for x_0. It is clear that $\lambda(x_0) \ne \lambda(x_1)$ for large $k \ge 1$. □

It is evident that Proposition 2.9 and the formula (2.3) yield the following result due to M. Pollicott and H. Weiss [36].

Corollary 2.11 ([36]). *For any* $\lambda \ge 2\log\frac{\sqrt{5}+1}{2}$, *there exists a point* $x_0 \in [0,1)$ *such that* $\lambda(x_0) = \lambda$.

2.3. **Pointwise dimension.**

We are going to compare the pointwise dimension and the Markov pointwise dimension (corresponding to continued fraction system) of a Borel probability measure.

Let μ be a Borel probability measure on $[0,1)$. Define the pointwise dimension and the Markov pointwise dimension respectively by

$$d_\mu(x) := \lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}, \quad \delta_\mu(x) := \lim_{n\to\infty}\frac{\log\mu(I_n(x))}{\log|I_n(x)|},$$

if the limits exist, where $B(x,r)$ is the ball centered at x with radius r.

For two series $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$, we write $u_n \asymp v_n$ which means that there exist absolute positive constants c_1, c_2 such that $c_1 v_n \le u_n \le c_2 v_n$ for n large enough. Sometimes, we need the following condition at a point x:

$$\mu(B(x,|I_n(x)|)) \asymp \mu(I_n(x)). \tag{2.5}$$

We have the following relationship between $\delta_\mu(x)$ and $d_\mu(x)$.

Lemma 2.12. *Let* μ *be a Borel measure.*
(a) Assume (2.5). If $d_\mu(x)$ *exists then* $\delta_\mu(x)$*exists and* $\delta_\mu(x) = d_\mu(x)$.
(b) If $\delta_\mu(x)$ *and* $\lambda(x)$ *both exist, then* $d_\mu(x)$*exists and* $\delta_\mu(x) = d_\mu(x)$.

Proof. (a) If the limit defining $d_\mu(x)$ exists, then the limit

$$\lim_{n\to+\infty}\frac{\log\mu(B(x,|I_n(x)|))}{\log|I_n(x)|}$$

exists and equals to $d_\mu(x)$. Thus by (2.5), the limit defining $\delta_\mu(x)$ also exists and equals to $d_\mu(x)$.

(b) Since $\lambda(x)$ exists, by Lemma 2.7 we have

$$\lim_{n\to\infty} \frac{\log|I_n(x)|}{\log|I_{n+1}(x)|} = \lim_{n\to\infty} \frac{1}{n}\log|I_n(x)| \Big/ \frac{1}{n+1}\log|I_{n+1}(x)| = 1. \tag{2.6}$$

For any $r > 0$, there exists an n such that $|I_{n+1}(x)| \le r < |I_n(x)|$. Then by Lemma 2.5, we have $I_{n+1}(x) \subset B(x,r) \subset I_{n-2}(x)$. Thus

$$\frac{\log\mu(I_{n-2}(x))}{\log|I_{n+1}(x)|} \le \frac{\log\mu(B(x,r))}{\log r} \le \frac{\log\mu(I_{n+1}(x))}{\log|I_n(x)|}. \tag{2.7}$$

Combining (2.6) and (2.7) we get the desired result. □

Let us give some measures for which the condition (2.5) is satisfied. These measures will be used in the subsection 5.1. The existence of these measures $\mu_{t,q}$ will be discussed in Proposition 4.6 and the subsection 5.1.

Lemma 2.13. *Suppose $\mu_{t,q}$ is a measure satisfying*

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q))|I_n(x)|^t \prod_{j=1}^n a_j^q,$$

where $P(t,q)$ is a constant. Then (2.5) is satisfied by $\mu_{t,q}$.

Proof. Notice that when $a_n(x) = 1$, $\mu_{t,q}(I_n(x)) \asymp \mu_{t,q}(I_{n-1}(x))$. Then in the light of Lemma 2.5, we can show that (2.5) is satisfied by $\mu_{t,q}$. □

3. Fast growth rate: proof of Theorem 1.1

3.1. **Lower bound.** We start with the mass distribution principle (see [12], Proposition 4.2), which will be used to estimate the lower bound of the Hausdorff dimension of a set.

Lemma 3.1 ([12]). *Let $E \subset [0,1)$ be a Borel set and μ be a measure with $\mu(E) > 0$. Suppose that*

$$\liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \ge s, \quad \forall x \in E$$

where $B(x,r)$ denotes the open ball with center at x and radius r. Then $\dim E \ge s$.

Next we give a formula for computing the Hausdorff dimension for a class of Cantor sets related to continued fractions.

Lemma 3.2. *Let $\{s_n\}_{n\ge1}$ be a sequence of positive integers tending to infinity with $s_n \ge 3$ for all $n \ge 1$. Then for any positive number $N \ge 2$, we have*

$$\dim\{x \in [0,1) : s_n \le a_n(x) < Ns_n \ \ \forall\, n \ge 1\} = \liminf_{n\to\infty} \frac{\log(s_1 s_2 \cdots s_n)}{2\log(s_1 s_2\cdots s_n) + \log s_{n+1}}.$$

Proof. Let F be the set in question and s_0 be the lim inf in the statement. We call

$$J(a_1, a_2, \cdots, a_n) := Cl \bigcup_{a_{n+1}\ge s_{n+1}} I_{n+1}(a_1, \cdots, a_n, a_{n+1})$$

a *basic* CF-*interval* of order n with respect to F (or simply basic interval of order n), where $s_k \leq a_k < Ns_k$ for all $1 \leq k \leq n$. Here Cl stands for the closure. Then it follows that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{s_k \leq a_k < Ns_k, 1\leq k \leq n} J(a_1, \cdots, a_n). \tag{3.1}$$

By Lemma 2.4, we have

$$J(a_1, \cdots, a_n) = \left[\frac{p_n}{q_n}, \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}}\right] \text{ or } \left[\frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}}, \frac{p_n}{q_n}\right] \tag{3.2}$$

according to n is even or odd. Then by Lemma 2.4, Lemma 2.2 and the assumption on a_k that $s_k \leq a_k < Ns_k$ for all $1 \leq k \leq n$, we have

$$\frac{1}{2N^n}\frac{1}{s_{n+1}(s_1\cdots s_n)^2} \leq \Big|J(a_1, \cdots, a_n)\Big| = \frac{1}{q_n(s_{n+1}q_n + q_{n-1})} \leq \frac{1}{s_{n+1}(s_1\cdots s_n)^2}. \tag{3.3}$$

Since $s_k \to \infty$ as $k \to \infty$, then

$$\lim_{n\to\infty} \frac{\log s_1 + \cdots + \log s_n}{n} = \infty.$$

This, together with the definition of s_0, implies that for any $s > s_0$, there exists a sequence $\{n_\ell : \ell \geq 1\}$ such that for all $\ell \geq 1$,

$$(N-1)^{n_\ell} < \big(s_{n_\ell+1}(s_1\cdots s_{n_\ell})^2\big)^{\frac{s-s_0}{2}}, \qquad \prod_{k=1}^{n_\ell} s_k \leq \big(s_{n_\ell+1}(s_1\cdots s_{n_\ell})^2\big)^{\frac{s+s_0}{2}}.$$

Then, by (3.1), together with (3.3), we have

$$\begin{aligned} H^s(F) &\leq \liminf_{\ell\to\infty} \sum_{s_k\leq a_k < Ns_k, 1\leq k\leq n_\ell} \Big|J(a_1, \cdots, a_{n_\ell})\Big|^s \\ &\leq \liminf_{\ell\to\infty} \left((N-1)^{n_\ell}\prod_{k=1}^{n_\ell} s_k\right)\left(\frac{1}{s_{n_\ell+1}(s_1\cdots s_{n_\ell})^2}\right)^s \leq 1. \end{aligned}$$

Since $s > s_0$ is arbitrary, we have $\dim F \leq s_0$.

For the lower bound, we define a measure μ such that for any basic CF-interval $J(a_1, a_2, \cdots, a_n)$ of order n,

$$\mu(J(a_1, a_2, \cdots, a_n)) = \prod_{j=1}^{n} \frac{1}{(N-1)s_j}.$$

By the Kolmogorov extension theorem, μ can be extended to a probability measure supported on F. In the following, we will check the mass distribution principle with this measure.

Fix $s < s_0$. By the definition of s_0 and the fact that $s_k \to \infty$ $(k \to \infty)$ and that N is a constant, there exists an integer n_0 such that for all $n \geq n_0$,

$$\prod_{k=1}^{n}(N-1)s_k \geq \left(s_{n+1}(\prod_{k=1}^{n} Ns_k)^2\right)^s. \tag{3.4}$$

We take $r_0 = \dfrac{1}{2N^{n_0}}\dfrac{1}{s_{n_0+1}(s_1\cdots s_{n_0})^2}$.

For any $x \in F$, there exists an infinite sequence $\{a_1, a_2, \cdots\}$ with $s_k \le a_k < Ns_k, \forall k \ge 1$ such that $x \in J(a_1, \cdots, a_n)$, for all $n \ge 1$. For any $r < r_0$, there exists an integer $n \ge n_0$ such that

$$|J(a_1, \cdots, a_{n+1})| \le r < |J(a_1, \cdots, a_n)|.$$

We claim that the ball $B(x, r)$ can intersect only one n-th basic interval, which is just $J(a_1, \cdots, a_n)$. We establish this only at the case that n is even, since for the case that n is odd, the argument is similar.

Case (1): $s_n < a_n < Ns_n - 1$. The left and right adjacent n-th order basic intervals to $J(a_1, \cdots, a_n)$ are $J(a_1, \cdots, a_n - 1)$ and $J(a_1, \cdots, a_n + 1)$ respectively. Then by (3.2) and the condition that $s_n \ge 3$, the gap between $J(a_1, \cdots, a_n)$ and $J(a_1, \cdots, a_n - 1)$ is

$$\frac{p_n}{q_n} - \frac{s_{n+1}(p_n - p_{n-1}) + p_{n-1}}{s_{n+1}(q_n - q_{n-1}) + q_{n-1}} = \frac{s_{n+1} - 1}{q_n\Big(s_{n+1}(q_n - q_{n-1}) + q_{n-1}\Big)} \ge \Big|J(a_1, \cdots, a_n)\Big|.$$

Hence $B(x, r)$ can not intersect $J(a_1, \cdots, a_n - 1)$. On the other hand, the gap $J(a_1, \cdots, a_n)$ and $J(a_1, \cdots, a_n + 1)$ is

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1} - 1}{(q_n + q_{n-1})(s_{n+1}q_n + q_{n-1})} \ge \Big|J(a_1, \cdots, a_n)\Big|.$$

Hence $B(x, r)$ can not intersect $J(a_1, \cdots, a_n + 1)$ either.

Case (2): $a_n = s_n$. The right adjacent n-th order basic interval to $J(a_1, \cdots, a_n)$ is $J(a_1, \cdots, a_n + 1)$. The same argument as in the case (1) shows that $B(x, r)$ can not intersect $J(a_1, \cdots, a_n + 1)$. On the other hand, the gap between the left endpoint of $J(a_1, \cdots, a_n)$ and that of $I_{n-1}(a_1, \cdots, a_{n-1})$ is

$$\frac{p_n}{q_n} - \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} = \frac{s_n - 1}{(q_{n-1} + q_{n-2})q_n} \ge \Big|J(a_1, \cdots, a_n)\Big|.$$

It follows that $B(x, r)$ can not intersect any n-th order CF-basic intervals on the left of $J(a_1, \cdots, a_n)$. In general, $B(x, r)$ can intersect no other n-th order CF-basic intervals than $J(a_1, \cdots, a_n)$.

Case (3): $a_n = Ns_n - 1$. From the case (1), we know that $B(x, r)$ can not intersect any n-th order CF-basic intervals on the left of $J(a_1, \cdots, a_n)$. While for on the right, the gap between the right endpoint of $J(a_1, \cdots, a_n)$ and that of $I_{n-1}(a_1, \cdots, a_{n-1})$ is

$$\frac{p_{n-1}}{q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1}}{(s_{n+1}q_n + q_{n-1})q_{n-1}} \ge \Big|J(a_1, \cdots, a_n)\Big|.$$

It follows that $B(x, r)$ can not intersect any n-th order CF-basic intervals on the right of $J(a_1, \cdots, a_n)$. In general, $B(x, r)$ can intersect no other n-th order CF-basic intervals than $J(a_1, \cdots, a_n)$.

Now we distinguish two cases to estimate the measure of $B(x, r)$.

Case (i). $|J(a_1, \cdots, a_{n+1})| \le r < |I_{n+1}(a_1, \cdots, a_{n+1})|$. By Lemma 2.5 and the fact $a_{n+1} \ne 1$, $B(x, r)$ can intersect at most five $(n+1)$-th order basic intervals. As a consequence, by (3.4), we have

$$\mu(B(x, r)) \le 5 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k} \le 5 \left(\frac{1}{s_{n+2}(N^{n+1}s_1 \cdots s_{n+1})^2}\right)^s. \tag{3.5}$$

Since

$$r > \Big|J(a_1, \cdots, a_{n+1})\Big| = \frac{1}{q_{n+1}(s_{n+2}q_{n+1} + q_n)} \geq \frac{1}{2s_{n+2}(N^{n+1}s_1 \cdots s_{n+1})^2},$$

it follows that

$$\mu(B(x,r)) \leq 10r^s.$$

Case (ii). $|I_{n+1}(a_1, \cdots, a_{n+1})| \leq r < |J(a_1, \cdots, a_n)|$. In this case, we have

$$I_{n+1}(a_1, \cdots, a_{n+1}) = \frac{1}{q_{n+1}(q_{n+1} + q_n)} \geq \frac{1}{2q_{n+1}^2} \geq \frac{1}{2N^{2(n+1)}} \left(\prod_{k=1}^{n+1} s_k\right)^2.$$

So $B(x,r)$ can intersect at most a number $8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2$ of $(n+1)$-th basic intervals. As a consequence,

$$\begin{aligned}\mu(B(x,r)) &\leq \min\Big\{\mu(J(a_1, \cdots, a_n)),\ 8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k}\Big\} \\ &\leq \prod_{k=1}^{n} \frac{1}{(N-1)s_k} \min\Big\{1,\ 8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}}\Big\}.\end{aligned}$$

By (3.4) and the elementary inequality $\min\{a,\ b\} \leq a^{1-s}b^s$ which holds for any $a, b > 0$ and $0 < s < 1$, we have

$$\begin{aligned}\mu(B(x,r)) &\leq \left(\frac{1}{s_{n+1}(N^n s_1 \cdots s_n)^2}\right)^s \cdot \left(8rN^{2(n+1)}(s_1 \cdots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}}\right)^s \\ &\leq 16Nr^s.\end{aligned}$$

Combining these two cases, together with mass distribution principle, we have $\dim F \geq s_0$. ☐

Let

$$E' = \{x \in [0,1) : e^{\varphi(n)-\varphi(n-1)} \leq a_n(x) \leq 2e^{\varphi(n)-\varphi(n-1)},\ \forall n \geq 1\}.$$

It is evident that $E' \subset E_\xi(\varphi)$. Then applying Lemma 3.2, we have

$$E_\xi(\varphi) \geq \liminf_{n\to\infty} \frac{\varphi(n)}{\varphi(n+1) + \varphi(n)} = \frac{1}{b+1}.$$

3.2. **Upper bound.** We first give a lemma which is a little bit more than the upper bound for the case $b = 1$. Its proof uses a family of Bernoulli measures with an infinite number of states.

Lemma 3.3. *If* $\lim\limits_{n\to\infty} \frac{\varphi(n)}{n} = \infty$, *then* $\dim E_\xi(\varphi) \leq \frac{1}{2}$.

Proof. For any $t > 1$, we introduce a family of Bernoulli measures μ_t:

$$\mu_t(I_n(a_1, \cdots, a_n)) = e^{-nC(t) - t\sum_{j=1}^{n} \log a_j(x)} \tag{3.6}$$

where $C(t) = \log \sum\limits_{n=1}^{\infty} \frac{1}{n^t}$.

Fix $x \in E_\xi(\varphi)$ and $\epsilon > 0$. If n is sufficiently large, we have

$$(\xi - \epsilon)\varphi(n) < \sum_{j=1}^{n} \log a_j(x) < (\xi + \epsilon)\varphi(n). \tag{3.7}$$

So

$$E_\xi(\varphi) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n(\epsilon),$$

where

$$E_n(\epsilon) = \{x \in [0,1) : (\xi - \epsilon)\varphi(n) < \sum_{j=1}^{n} \log a_j(x) < (\xi + \epsilon)\varphi(n)\}.$$

Now let $\mathcal{I}(n,\epsilon)$ be the family of all n-th order cylinders $I_n(a_1, \cdots, a_n)$ satisfying (3.7). For each $N \geq 1$, we select all those cylinders in $\bigcup_{n=N}^{\infty} \mathcal{I}(n,\epsilon)$ which are maximal ($I \in \bigcup_{n=N}^{\infty} \mathcal{I}(n,\epsilon)$ is maximal if there is no other I' in $\bigcup_{n=N}^{\infty} \mathcal{I}(n,\epsilon)$ such that $I \subset I'$ and $I \neq I'$). We denote by $\mathcal{J}(N,\epsilon)$ the set of all maximal cylinders in $\bigcup_{n=N}^{\infty} \mathcal{I}(n,\epsilon)$. It is evident that $\mathcal{J}(N,\epsilon)$ is a cover of $E_\xi(\varphi)$. Let $I_n(a_1, \cdots, a_n) \in \mathcal{J}(N,\epsilon)$, we have

$$\mu_t(I_n(a_1, \cdots, a_n)) = e^{-nC(t) - t\sum\limits_{j=1}^{n} \log a_j} \geq e^{-nC(t) - t(\xi+\epsilon)\varphi(n)}.$$

On the other hand,

$$\Big|I_n(a_1, \cdots, a_n)\Big| \leq e^{-2\log q_n} \leq e^{-2\sum\limits_{j=1}^{n} \log a_j} \leq e^{-2(\xi-\epsilon)\varphi(n)}.$$

Since $\lim\limits_{n\to\infty} \frac{\varphi(n)}{n} = \infty$, for each $s > t/2$ and N large enough, we have

$$\Big|I_n(a_1, \cdots, a_n)\Big|^s \leq \mu_t(I_n(a_1, \cdots, a_n)).$$

This implies $\dim E_\xi(\varphi) \leq 1/2 = \frac{1}{b+1}$. □

Now we return back to the proof of the upper bound.

Case (i) $b = 1$. Since $(\varphi(n+1) - \varphi(n)) \uparrow \infty$, Lemma 3.3 implies immediately $\dim E_\xi(\varphi) \leq \frac{1}{2}$.

Case (ii) $b > 1$. By (3.7), for each $x \in E_\xi(\varphi)$ and n sufficiently large

$$(\xi - \epsilon)\varphi(n+1) - (\xi + \epsilon)\varphi(n) \leq \log a_{n+1}(x) \leq (\xi + \epsilon)\varphi(n+1) - (\xi - \epsilon)\varphi(n).$$

Take

$$L_{n+1} = e^{(\xi-\epsilon)\varphi(n+1) - (\xi+\epsilon)\varphi(n)}, \quad M_{n+1} = e^{(\xi+\epsilon)\varphi(n+1) - (\xi-\epsilon)\varphi(n)}.$$

Define

$$F_N = \{x \in [0,1] : L_n \leq a_n(x) \leq M_n, \forall n \geq N\}.$$

Then we have

$$E_\xi(\varphi) \subset \bigcup_{N=1}^{\infty} F_N.$$

We can only estimate the upper bound of $\dim F_1$. Because F_N can be written as a countable union of sets with the same form as F_1, then by the σ-stability of Hausdorff dimension, we will have $\dim F_N = \dim F_1$. We can further assume that $M_n \geq L_n + 2$.

For any $n \geq 1$, define

$$D_n = \{(\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n : L_k \leq \sigma_k \leq M_k,\ 1 \leq k \leq n\}.$$

It follows that

$$F_1 = \bigcap_{n\geq 1} \bigcup_{(\sigma_1,\cdots,\sigma_n)\in D_n} J(\sigma_1,\cdots,\sigma_n),$$

where

$$J(\sigma_1,\cdots,\sigma_n) := Cl \bigcup_{\sigma\geq L_{n+1}} I(\sigma_1,\cdots,\sigma_n,\sigma)$$

(called an admissible cylinder of order n). For any $n \geq 1$ and $s > 0$, we have

$$\sum_{(\sigma_1,\cdots,\sigma_n)\in D_n} \Big|J(\sigma_1,\cdots,\sigma_n)\Big|^s \leq \sum_{(\sigma_1,\cdots,\sigma_n)\in D_n} \Big|\frac{1}{q_n^2 L_{n+1}}\Big|^s \leq \frac{M_1\cdots M_n}{\Big((L_1\cdots L_n)^2 L_{n+1}\Big)^s}.$$

It follows that

$$\dim F_1 \leq \liminf_{n\to\infty} \frac{\log M_1 + \cdots + \log M_n}{\sum\limits_{k=1}^{n} \log L_k + \sum\limits_{k=1}^{n+1} \log L_k} = \frac{\xi + \epsilon + \frac{2\epsilon}{b-1}}{(\xi-\epsilon)(b+1) - 2\epsilon - \frac{4\epsilon}{b-1}}.$$

Letting $\epsilon \to 0$, we get

$$\dim E_\xi(\varphi) \leq \frac{1}{b+1}.$$

4. Ruelle operator theory

There have been various works on the Ruelle transfer operator for the Gauss dynamics. See D. Mayer [32], [33], [34], O. Jenkinson [23], O. Jenkinson and M. Pollicott [22], M. Pollicott and H. Weiss [36], P. Hanus, R. D. Mauldin and M. Urbanski [17]. In this section we will present a general Ruelle operator theory for conformal infinite iterated function system which was developed in [17] and then apply it to the Gauss dynamics. We will also prove some properties of the pressure function in the case of Gauss dynamics , which will be used later.

4.1. **Conformal infinite iterated function systems.** In this subsection, we present the conformal infinite iterated function systems which were studied by P. Hanus, R. D. Mauldin and M. Urbanski in [17]. See also the book of Mauldin and Urbanski [31].

Let X be a non-empty compact connected subset of $\mathbb{R}^d$ equipped with a metric ρ. Let I be an index set with at least two elements and at most countable elements. An *iterated function system* $S = \{\phi_i : X \to X : i \in I\}$ is a collection of injective contractions for which there exists $0 < s < 1$ such that for each $i \in I$ and all $x, y \in X$,

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x,y). \tag{4.1}$$

Before further discussion, we are willing to give a list of notation.

- $I^n := \{\omega : \omega = (\omega_1, \cdots, \omega_n), \omega_k \in I, 1 \le k \le n\}$,
- $I^* := \cup_{n\ge1} I^n$,
- $I^\infty := \Pi_{i=1}^{\infty} I$,
- $\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$, for $\omega = \omega_1\omega_2\cdots\omega_n \in I^n, n \ge 1$,
- $|\omega|$ denote the length of $\omega \in I^* \cup I^\infty$,
- $\omega|_n = \omega_1\omega_2\ldots\omega_n$, if $\big|w\big| \ge n$,
- $[\omega|_n] = [\omega_1 \ldots \omega_n] = \{x \in I^\infty : \ x_1 = \omega_1, \cdots, x_n = \omega_n\}$,
- $\sigma : I^\infty \to I^\infty$ the shift transformation,
- $\|\phi'_\omega\| := \sup\limits_{x\in X} |\phi'_\omega(x)|$ for $\omega \in I^*$,
- $C(X)$ space of continuous functions on X,
- $|| \cdot ||_\infty$ supremum norm on the Banach space $C(X)$.

For $\omega \in I^\infty$, the set

$$\pi(\omega) = \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton. We also denote its only element by $\pi(\omega)$. This thus defines a coding map $\pi : I^\infty \to X$. The limit set J of the iterated function system is defined by

$$J := \pi(I^\infty).$$

Denote by ∂X the boundary of X and by $\mathrm{Int}(X)$ the interior of X.

We say that the iterated function system $S = \{\phi_i\}_{i\in I}$ satisfies the *open set condition* if there exists a non-empty open set $U \subset X$ such that $\phi_i(U) \subset U$ for each $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for each pair $i, j \in I, i \ne j$.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to be *conformal* if the following are satisfied:

(1) the open set condition is satisfied for $U = \mathrm{Int}(X)$;

(2) there exists an open connected set V with $X \subset V \subset \mathbb{R}^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V;

(3) there exist $h, \ell > 0$ such that for each $x \in \partial X \subset \mathbb{R}^d$, there exists an open cone $\mathrm{Con}(x, h, \ell) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure h and altitude ℓ;

(4) (Bounded Distortion Property) there exists $K \ge 1$ such that $|\phi'_\omega(y)| \le K|\phi'_\omega(x)|$ for every $\omega \in I^*$ and every pair of points $x, y \in V$.

The *topological pressure function* for a conformal iterated function systems $S = \{\phi_i : X \to X : i \in I\}$ is defined as

$$\mathcal{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi'_\omega||^t.$$

The system S is said to be *regular* if there exists $t \ge 0$ such that $\mathcal{P}(t) = 0$.

Let $\beta > 0$. A *Hölder family of functions* of order β is a family of continuous functions $F = \{f^{(i)} : X \to \mathbb{C} : i \in I\}$ such that

$$V_\beta(F) = \sup_{n\ge1} V_n(F) < \infty,$$

where

$$V_n(F) = \sup_{\omega\in I^n} \sup_{x,y\in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\} e^{\beta(n-1)}.$$

A family of functions $F = \{f^{(i)} : X \to \mathbb{R}, i \in I\}$ is said to be *strong* if

$$\sum_{i\in I} ||e^{f^{(i)}}||_\infty < \infty.$$

Define the *Ruelle operator* on $C(X)$ associated to F as

$$\mathcal{L}_F(g)(x) := \sum_{i\in I} e^{f^{(i)}(x)} g(\phi_i(x)).$$

Denote by $\mathcal{L}_F^*$ the dual operator of $\mathcal{L}_F$.

The *topological pressure* of F is defined by

$$P(F) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_{x\in X} \sum_{j=1}^{n} f^{\omega_j} \circ \phi_{\sigma^j\omega}(x) \right).$$

A measure ν is called *F-conformal* if the following are satisfied:
(1) ν is supported on J;
(2) for any Borel set $A \subset X$ and any $\omega \in I^*$,

$$\nu(\phi_\omega(A)) = \int_A \exp\left(\sum_{j=1}^{n} f^{(\omega_j)} \circ \phi_{\sigma^j\omega} - P(F)|\omega| \right) d\nu;$$

(3) $\nu(\phi_\omega(X) \cap \phi_\tau(X)) = 0 \quad \omega, \tau \in I^n, \omega \neq \tau, n \geq 1.$

Two functions $\phi, \varphi \in C(X)$ are said to be *cohomologous* with respect to the transformation T, if there exists $u \in C(X)$ such that

$$\varphi(x) = \phi(x) + u(x) - u(T(x)).$$

The following two theorems are due to Hanus, Mauldin and Urbanski [17].

Theorem 4.1 ([17]). *For a conformal iterated function system $S = \{\phi_i : X \to X : i \in I\}$ and a strong Hölder family of functions $F = \{f^{(i)} : X \to \mathbb{C} : i \in I\}$, there exists a unique F-conformal probability measure ν_F on X such that $\mathcal{L}_F^*\nu_F = e^{P(F)}\nu_F$. There exists a unique shift invariant probability measure $\tilde{\mu}_F$ on I^∞ such that $\mu_F := \tilde{\mu}_F \circ \pi^{-1}$ is equivalent to ν_F with bounded Radon-Nikodym derivative. Furthermore, the Gibbs property is satisfied:*

$$\frac{1}{C} \leq \frac{\tilde{\mu}_F([\omega|_n])}{\exp\left(\sum_{j=1}^{n} f^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(F)\right)} \leq C.$$

Let $\Psi = \{\psi^{(i)} : X \to \mathbb{R} : i \in I\}$ and $F = \{f^{(i)} : X \to \mathbb{R} : i \in I\}$ be two families of real-valued Hölder functions. We define the *amalgamated functions* on I^∞ associated to Ψ and F as follows:

$$\tilde{\psi}(\omega) := \psi^{(\omega_1)}(\pi(\sigma\omega)), \qquad \tilde{f}(\omega) := f^{(\omega_1)}(\pi(\sigma\omega)) \qquad \forall \omega \in I^\infty.$$

Theorem 4.2 ([17], see also [31], pp. 43-48). *Let Ψ and F be two families of real-valued Hölder functions. Suppose the sets $\{i \in I : \sup_x(\psi^{(i)}(x)) > 0\}$ and $\{i \in$*

$I : \sup_x(f^{(i)}(x)) > 0\}$ are finite. Then the function $(t, q) \mapsto P(t, q) = P(t\Psi + qF)$, is real-analytic with respect to $(t, q) \in \operatorname{Int}(D)$, where

$$D = \left\{ (t, q) : \sum\nolimits_{i \in I} \exp(\sup_x (t\psi^{(i)}(x) + qf^{(i)}(x))) < \infty \right\}.$$

Furthermore, if $t\Psi + qF$ is a strong Hölder family for $(t, q) \in D$ and

$$\int (|\tilde{f}| + |\tilde{\psi}|) d\tilde{\mu}_{t,q} < \infty,$$

where $\tilde{\mu}_{t,q} := \tilde{\mu}_{t\Psi+qF}$ is obtained by Theorem 4.1, then

$$\frac{\partial P}{\partial t} = \int \tilde{\psi} d\tilde{\mu}_{t,q} \qquad \text{and} \qquad \frac{\partial P}{\partial q} = \int \tilde{f} d\tilde{\mu}_{t,q}.$$

If $t\tilde{\psi} + q\tilde{f}$ is not cohomologous to a constant function, then $P(t, q)$ is strictly convex and

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

is positive definite.

4.2. **Continued fraction dynamical system.** We apply the theory in the precedent subsection to the continued fraction dynamical system. Let $X = [0, 1]$ and $I = \mathbb{N}$. The continued fraction dynamical system can be viewed as an iterated function system:

$$S = \left\{ \psi_i(x) = \frac{1}{i + x} : i \in \mathbb{N} \right\}.$$

Recall that the projection mapping $\pi : I^\infty \to X$ is defined by

$$\pi(\omega) := \bigcap_{n=1}^{\infty} \psi_{\omega|_n}(X), \ \forall \omega \in I^\infty.$$

Notice that $\psi_1'(0) = -1$, thus (4.1) is not satisfied. However, this is not a real problem, since we can consider the system of second level maps and replace S by $\tilde{S} := \{\psi_i \circ \psi_j : i, j \in \mathbb{N}\}$. In fact, for any $x \in [0, 1)$

$$(\psi_i \circ \psi_j)'(x) = \Big(\frac{1}{i + \frac{1}{j+x}} \Big)' = \Big(\frac{1}{i(j + x) + 1} \Big)^2 \leq \frac{1}{4}.$$

In the following, we will collect or prove some facts on the continued fraction dynamical system, which will be useful for applying Theorem 4.1 and 4.2.

Lemma 4.3 ([29]). *The continued fraction dynamical system S is regular and conformal.*

For the investigation in the present paper, our problems are tightly connected to the following two families of Hölder functions.

$$\Psi = \{\log |\psi_i'| : i \in \mathbb{N}\} \text{ and } F = \{-\log i : i \in \mathbb{N}\}.$$

Remark 4.4. *We mention that our method used here is also applicable to other potentials than the two special families introduced here.*

The families Ψ and F are Hölder families and their amalgamated functions are equal to

$$\tilde{\psi}(\omega) = -2\log(\omega_1 + \pi(\sigma\omega)), \quad \tilde{f}(\omega) = -\log \omega_1 \quad \forall \omega \in \mathbb{N}^\infty.$$

For our convenience, we will consider the function $t\Psi - qF$ instead of $t\Psi + qF$.

Lemma 4.5. *Let* $D := \{(t,q) : 2t - q > 1\}$. *For any* $(t,q) \in D$, *we have*

(i) *The family* $t\Psi - qF := \{t \log |\psi_i'| + q \log i : i \in \mathbb{N}\}$ *is Hölder and strong.*

(ii) *The topological pressure* P *associated to the potential* $t\Psi - qF$ *can be written as*

$$P(t,q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1,\cdots,\omega_n} \exp\left(\sup_x \log \prod_{j=1}^n \omega_j^q([\omega_j, \cdots, \omega_n + x])^{2t} \right).$$

Proof. The assertion on the domain D follows from

$$\frac{1}{4^t}\zeta(2t-q) = \mathcal{L}_{t\Psi-qF}1 = \sum_{i=1}^\infty \frac{i^q}{(i+x)^{2t}} \le \sum_{i=1}^\infty i^{q-2t} = \zeta(2t-q).$$

where $\zeta(2t-q)$ is the Riemann zeta function, defined by

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s} \quad \forall s > 1.$$

(i) For $(t,q) \in D$, write $(t\Psi - qF)^{(i)} := t\log|\psi_i'| + q\log i$. Then

$$\sum_{i\in I} \left\| \exp\left\{ (t\Psi - qF)^{(i)} \right\} \right\|_\infty = \sum_{i=1}^\infty \left\| \frac{i^q}{(i+x)^{2t}} \right\|_\infty = \sum_{i=1}^\infty i^{q-2t} = \zeta(2t-q) < \infty.$$

Thus $t\Psi - qF$ is strong.

(ii) It suffices to noticed that

$$\sup_x \left(\sum_{j=1}^n (t|\psi_{\omega_j}'| + q\log\omega_j) \circ \psi_{\sigma^j\omega}(x) \right) = \sup_x \log \prod_{j=1}^n \omega_j^q([\omega_j, \cdots, \omega_n + x])^{2t}.$$

□

Denote by $\mathcal{L}^*_{t\Psi-qF}$ the conjugate operator of $\mathcal{L}_{t\Psi-qF}$. Applying Theorem 4.1 with the help of Lemma 4.3 and Lemma 4.5, we get

Proposition 4.6. *For each* $(t,q) \in D$, *there exists a unique* $t\Psi - qF$*-conformal probability measure* $\nu_{t,q}$ *on* $[0,1]$ *such that* $\mathcal{L}^*_{t\Psi-qF}\nu_{t,q} = e^{P(t,q)}\nu_{t,q}$, *and a unique shift invariant probability measure* $\tilde{\mu}_{t,q}$ *on* $\mathbb{N}^\infty$ *such that* $\mu_{t,q} := \tilde{\mu}_{t,q} \circ \pi^{-1}$ *on* $[0,1]$ *is equivalent to* $\nu_{t,q}$ *and*

$$\frac{1}{C} \le \frac{\tilde{\mu}_{t,q}([\omega|_n])}{\exp\left(\sum_{j=1}^n (t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j\omega)) - nP(t,q)\right)} \le C \qquad \forall \omega \in \mathbb{N}^\infty.$$

Lemma 4.7. *For the amalgamated functions* $\tilde{\psi}(\omega) = -2\log(\omega_1 + \pi(\sigma\omega))$ *and* $\tilde{f}(\omega) = -\log\omega_1$, *we have*

$$-\int \log|T'(x)|\mu_{t,q} = \int \tilde{\psi} d\tilde{\mu}_{t,q} \quad \text{and} \quad \int \log a_1(x) d\mu_{t,q} = -\int \tilde{f} d\tilde{\mu}_{t,q}. \quad (4.2)$$

and $t\tilde{\psi} - q\tilde{f}$ *is not cohomologous to a constant.*

Proof. (i). Assertion (4.2) is just a consequence of the facts

$$-\log|T'(\pi(\omega))| = \tilde{\psi}(\omega), \qquad \log a_1(\pi(\omega)) = -\tilde{f}(\omega) \quad \forall \omega \in I^\infty.$$

Suppose $t\tilde{\psi} - q\tilde{f}$ was not cohomologous to a constant. Then there would be a bounded function g such that $t\tilde{\psi} - q\tilde{f} = g - g \circ T + C$, which implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})(\sigma^j \omega) = \lim_{n\to\infty} \frac{g - g\circ\sigma^n}{n} + C = C$$

for all $\omega \in I^\infty$. On the other hand, if we take $\omega_1 = [1, 1, \cdots,]$, $\omega_2 = [2, 2, \cdots]$ and $\omega_3 = [3, 3, \cdots]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})(\sigma^j \omega_i) = C_i,$$

where

$$C_1 = 2t\log(\frac{\sqrt{5}-1}{2}), \ \ C_2 = 2t\log(\frac{\sqrt{5}-2}{2}) + q\log 2, \ \ C_3 = 2t\log(\frac{\sqrt{5}-3}{2}) + q\log 3.$$

Thus we get a contradiction. ☐

By Theorem 4.2 and the proof of Lemma 4.5, we know that $D = \{(t, q) : 2t - q > 1\}$ is the analytic area of the pressure $P(t, q)$. Applying Lemma 4.7 and Theorem 4.2, we get more:

Proposition 4.8. *On* $D = \{(t, q) : 2t - q > 1\}$*,*

(1) $P(t, q)$ *is analytic, strictly convex.*

(2) $P(t, q)$ *is strictly decreasing and strictly convex with respect to* t*. In other words,* $\frac{\partial P}{\partial t}(t, q) < 0$ *and* $\frac{\partial^2 P}{\partial t^2}(t, q) > 0$*. Furthermore,*

$$\frac{\partial P}{\partial t}(t, q) = -\int \log|T'(x)| d\mu_{t,q}. \tag{4.3}$$

(3) $P(t, q)$ *is strictly increasing and strictly convex with respect to* q*. In other words,* $\frac{\partial P}{\partial q}(t, q) > 0$ *and* $\frac{\partial^2 P}{\partial q^2}(t, q) > 0$*. Furthermore,*

$$\frac{\partial P}{\partial q}(t, q) = \int \log a_1(x) d\mu_{t,q}. \tag{4.4}$$

(4)

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

is positive definite.

At the end of this subsection, we would like to quote some results by D. Mayer [34] (see also M. Pollicott and H. Weiss [36]).

Proposition 4.9 ([34])**.** *Let* $P(t) := P(t, 0)$ *and* $\mu_t := \mu_{t,0}$*, then* $P(t)$ *is defined in* $(1/2, \infty)$ *and we have* $P(1) = 0$ *and* $\mu_1 = \mu_G$*. Furthermore,*

$$P'(t) = -\int \log|T'(x)| d\mu_t(x). \tag{4.5}$$

In particular

$$P'(0) = -\int \log |T'(x)| d\mu_G(x) = -\lambda_0. \tag{4.6}$$

Remark 4.10. *Since $\mu_{1,0} = \mu_1 = \mu_G$, by (4.4), we have*

$$\frac{\partial P}{\partial q}(1,0) = \int \log a_1(x) d\mu_G = \xi_0. \tag{4.7}$$

4.3. **Further study on $P(t,q)$.** We will use the following simple known fact of convex functions.

Fact 4.11. *Suppose f is a convex continuously differentiable function on an interval I. Then $f'(x)$ is increasing and*

$$f'(x) \le \frac{f(y)-f(x)}{y-x} \le f'(y) \qquad x, y \in I, x < y.$$

First we give an estimation for the pressure $P(t,q)$ and show some behaviors of $P(t,q)$ when q tends to $-\infty$ and $2t-1$ (t being fixed).

Proposition 4.12. *For $(t,q) \in D$, we have*

$$-t\log 4 + \log \zeta(2t-q) \le P(t,q) \le \log \zeta(2t-q). \tag{4.8}$$

Consequently,

(1) $P(0,q) = \log \zeta(-q)$, and for any point (t_0, q_0) on the line $2t-q=1$,

$$\lim_{(t,q)\to(t_0,q_0)} P(t,q) = \infty;$$

(2) for fixed $t \in \mathbb{R}$,

$$\lim_{q\to 2t-1} \frac{\partial P}{\partial q}(t,q) = +\infty; \tag{4.9}$$

(3) for fixed $t \in \mathbb{R}$, we have

$$\lim_{q\to-\infty} \frac{P(t,q)}{q} = 0, \quad \lim_{q\to-\infty} \frac{\partial P}{\partial q}(t,q) = 0. \tag{4.10}$$

Proof. Notice that $\frac{1}{\omega_j+1} \le [\omega_j, \cdots, \omega_n + x] \le \frac{1}{\omega_j}$. for $x \in [0,1)$ and $1 \le j \le n$. Thus we have

$$\frac{1}{4^{nt}} \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n \le \sum_{\omega_1,\cdots,\omega_n} \prod_{j=1}^{n} \omega_j^q [\omega_j, \cdots, \omega_n + x]^{2t} \le \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n.$$

Hence by Lemma 4.5 (ii), we get (4.8).

We get (1) immediately from (4.8).

Look at (2). For all $q > q_0$, by the convexity of $P(t,q)$ and Fact 4.11, we have

$$\frac{\partial P}{\partial q}(t,q) \ge \frac{P(t,q)-P(t,q_0)}{q-q_0}.$$

Thus

$$\lim_{q\to 2t-1} \frac{\partial P}{\partial q}(t,q) \ge \lim_{q\to 2t-1} \frac{P(t,q_0)-P(t,q)}{q_0-q} = \infty.$$

Here we use the fact that $\lim_{q\to 2t-1} P(t,q) = +\infty$. Hence we get (4.8).

In order to show (3), we consider $P(t,q)/q$ as function of q on $(-\infty, 2t-1)\setminus\{0\}$. Noticed that for fixed $t\in\mathbb{R}$, $\lim_{q\to-\infty}\zeta(2t-q)=1$. Thus

$$\lim_{q\to-\infty}\frac{\log\zeta(2t-q)}{q}=0.$$

Then the first formula in (4.10) is followed from (4.8).

Fix $q_0<2t-1$. Then for all $q<q_0$, by the convexity of $P(t,q)$ and Fact 4.11, we have

$$\frac{\partial P}{\partial q}(t,q)\le\frac{P(t,q_0)-P(t,q)}{q_0-q}.$$

Thus

$$\lim_{q\to-\infty}\frac{\partial P}{\partial q}(t,q)\le\lim_{q\to-\infty}\frac{P(t,q_0)-P(t,q)}{q_0-q}=0.$$

Hence by Proposition 4.8 (3), we get the second formula in (4.10). ☐

4.4. **Properties of** $(t(\xi),q(\xi))$**.** Recall that $\xi_0=\int\log a_1(x)\mu_G$ and $D_0:=\{(t,q): 2t-q>1, 0\le t\le 1\}$.

Proposition 4.13. *For any* $\xi\in(0,\infty)$*, the system*

$$\begin{cases} P(t,q)=q\xi,\\ \dfrac{\partial P}{\partial q}(t,q)=\xi \end{cases}\tag{4.11}$$

admits a unique solution $(t(\xi),q(\xi))\in D_0$*. For* $\xi=\xi_0$*, the solution is* $(t(\xi_0),q(\xi_0))=(1,0)$*. The function* $t(\xi)$ *and* $q(\xi)$ *are analytic.*

Proof. Existence and uniqueness of solution $(t(\xi),q(\xi))$. Recall that $P(1,0)=0$ and $P(0,q)=\log\zeta(-q)$ (Proposition 4.12).

We start with a geometric argument which will followed by a rigorous proof. Consider $P(t,q)$ as a family of function of q with parameter t. It can be seen from the graph (see Figure 3) that for any $\xi>0$, there exists a unique $t\in(0,1]$, such that the line ξq is tangent to $P(t,\cdot)$. This $t=t(\xi)$ can be described as the unique point such that

$$\inf_{q<2t(\xi)-1}\Big(P(t(\xi),q)-q\xi\Big)=0.\tag{4.12}$$

We denote by $q(\xi)$ the point where the infimum in (4.12) is attained. Then the tangent point is $(q(\xi),P(t(\xi),q(\xi)))$ and the derivative of $P(t(\xi),q)-q\xi$ (with respect to q) at $q(\xi)$ equals 0, i.e.,

$$\Big(P(t(\xi),q)-q\xi\Big)'|_{q(\xi)}=0.$$

Thus we have $\frac{\partial P}{\partial q}(t(\xi),q(\xi))=\xi$. By (4.12), we also have $P(t(\xi),q(\xi))-q(\xi)\xi=0$. Therefore $(t(\xi),q(\xi))$ is a solution of (4.11). The uniqueness of $q(\xi)$ follows by the fact that $\frac{\partial P}{\partial q}$ is monotonic with respect to q (Proposition 4.8).

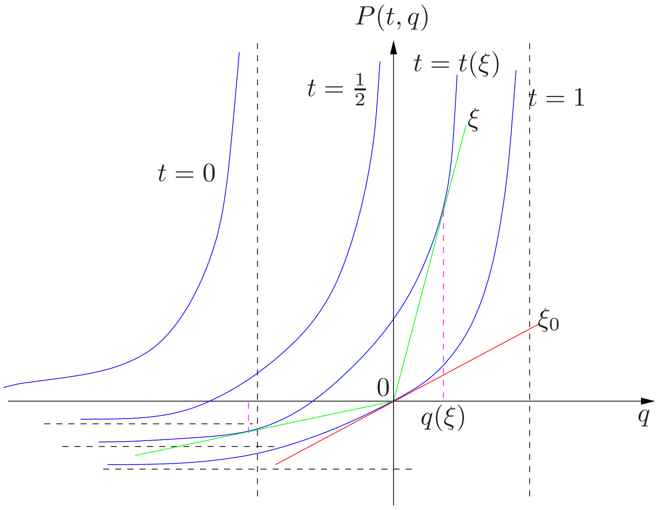

Figure 3. Solution of (4.11)

Let us give a rigorous proof. By (4.9), (4.10) and the mean-value theorem, for fixed $t \in \mathbb{R}$ and any $\xi > 0$, there exists a $q(t,\xi) \in (-\infty, 2t-1)$ such that

$$\frac{\partial P}{\partial q}\big(t, q(t,\xi)\big) = \xi. \tag{4.13}$$

The monotonicity of $\frac{\partial P}{\partial q}$ with respect to q implies the uniqueness of $q(t,\xi)$ (Proposition 4.8).

Since $P(t,q)$ is analytic, the implicit $q(t,\xi)$ is analytic with respect to t and ξ. Fix ξ and set

$$W(t) := P\big(t, q(t,\xi)\big) - \xi q(t,\xi).$$

Since

$$\begin{aligned} W'(t) &= \frac{\partial P}{\partial t}\big(t,q(t,\xi)\big) + \frac{\partial P}{\partial q}\big(t,q(t,\xi)\big)\frac{\partial q}{\partial t}(t,\xi) - \xi\frac{\partial q}{\partial t}(t,\xi) \\ &= \frac{\partial P}{\partial t}\big(t,q(t,\xi)\big) \qquad (\text{by}(4.13)) \\ &< 0 \qquad\qquad (\text{by Proposition 4.8(2)}). \end{aligned}$$

Thus $W(t)$ is strictly decreasing.

Since $P(0,q) = \log\zeta(-q) > 0$ $(q < -1)$, for $\xi > 0$ we have

$$W(0) = P\big(0, q(0,\xi)\big) - \xi q(0,\xi) > 0.$$

Since $P(1,q)$ is convex and $P(1,0) = 0$, by Fact 4.11 we have

$$\frac{P\big(1,q(1,\xi)\big) - 0}{q(1,\xi) - 0} \le \frac{\partial P}{\partial q}\big(1,q(1,\xi)\big) = \xi, \qquad \text{if} \quad q(1,\xi) > 0,$$

and

$$\frac{0 - P\big(1,q(1,\xi)\big)}{0 - q(1,\xi)} \ge \frac{\partial P}{\partial q}\big(1,q(1,\xi)\big) = \xi, \qquad \text{if} \quad q(1,\xi) < 0.$$

If $q(1,\xi)=0$, we have in fact $\xi=\xi_0$ and $P\big(1,q(1,\xi)\big)=0$. Hence, in any case we have

$$P\big(1,q(1,\xi)\big)-\xi q(1,\xi)\le 0. \tag{4.14}$$

Therefore, $W(1)=P\big(1,q(1,\xi)\big)-\xi q(1,\xi)\le 0$.

Thus by the mean-value theorem and the monotonicity of $W(t)$, there exists a unique $t=t(\xi)\in(0,1]$ such that $W(t(\xi))=0$, *i.e.*

$$P\Big(t(\xi),q\big(t(\xi),\xi\big)\Big)=\xi q\big(t(\xi),\xi\big). \tag{4.15}$$

If we write $q\big(t(\xi),\xi\big)$ as $q(\xi)$, both (4.13) and (4.15) show that $\big(t(\xi),q(\xi)\big)$ is the unique solution of (4.11). For $\xi=\xi_0$, the assertion in Proposition 4.9 that $P(0,1)=0=0\cdot\xi_0$ and the assertion of Remark 4.10 that $\frac{\partial P}{\partial q}(1,0)=\xi_0$ imply that $(0,1)$ is a solution of (4.11). Then the uniqueness of the solution to (4.11) implies $\big(t(\xi_0),q(\xi_0)\big)=(0,1)$.

Analyticity of $\big(t(\xi),q(\xi)\big)$. Consider the map

$$F=\begin{pmatrix} F_1\\ F_2\end{pmatrix}=\begin{pmatrix} P(t,q)-q\xi\\ \frac{\partial P}{\partial q}(t,q)-\xi\end{pmatrix}.$$

Then the jacobian of F is equal to

$$J(F)=:\begin{pmatrix} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial q}\\ \frac{\partial F_2}{\partial t} & \frac{\partial F_1}{\partial q}\end{pmatrix}=\begin{pmatrix} \frac{\partial P}{\partial t} & \frac{\partial P}{\partial q}-\xi\\ \frac{\partial^2 P}{\partial t\partial q} & \frac{\partial^2 P}{\partial q^2}\end{pmatrix}.$$

Consequently,

$$\det(J(F))|_{t=t(\xi),q=q(\xi)}=\frac{\partial P}{\partial t}\cdot\frac{\partial^2 P}{\partial q^2}\neq 0.$$

Thus by the implicit function theorem, $t(\xi)$ and $q(\xi)$ are analytic. □

Now let us present some properties on $t(\xi)$. Recall that $\xi_0=\frac{\partial P}{\partial q}(1,0)$.

Proposition 4.14. *$q(\xi)<0$ for $\xi<\xi_0$; $q(\xi_0)=0$; $q(\xi)>0$ for $\xi>\xi_0$.*

Proof. Since $P(1,q)$ is convex and $P(1,0)=0$, by Fact 4.11, we have

$$\frac{P(1,q)-0}{q-0}\ge\frac{\partial P}{\partial q}(1,0)=\xi_0,\quad (q>0);\qquad \frac{0-P(1,q)}{0-q}\le\frac{\partial P}{\partial q}(1,0)=\xi_0,\quad (q<0).$$

Hence for all $q<1$,

$$P(1,q)\ge\xi_0 q. \tag{4.16}$$

We recall that $(t(\xi_0),q(\xi_0))=(1,0)$ is the unique solution of the system (4.11) for $\xi=\xi_0$. By the above discussion of the existence of $t(\xi)$, $t(\xi)=1$ if and only if $\xi=\xi_0$. Now we suppose $t\in(0,1)$. For $\xi>\xi_0$, using (4.16), we have

$$P(t,q)>P(1,q)\ge q\xi_0\ge q\xi\qquad(\forall q\le 0).$$

Thus $q(\xi)>0$. For $\xi<\xi_0$, using (4.16), we have

$$P(t,q)>P(1,q)\ge q\xi_0\ge q\xi\qquad(\forall q\ge 0).$$

Thus $q(\xi)<0$. □

Proposition 4.15. *For $\xi \in (0, +\infty)$, we have*

$$t'(\xi) = \frac{q(\xi)}{\frac{\partial P}{\partial t}(t(\xi), q(\xi))}. \tag{4.17}$$

Proof. Recall that

$$\begin{cases} P(t(\xi), q(\xi)) = q(\xi)\xi, \\ \dfrac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi. \end{cases} \tag{4.18}$$

By taking the derivation with respect to ξ of the first equation in (4.18), we get

$$t'(\xi)\frac{\partial P}{\partial t}(t(\xi), q(\xi)) + q'(\xi)\frac{\partial P}{\partial q}(t(\xi), q(\xi)) = q'(\xi)\xi + q(\xi).$$

Taking into account the second equation in (4.18), we get

$$t'(\xi)\frac{\partial P}{\partial t}(t(\xi), q(\xi)) = q(\xi). \tag{4.19}$$

□

Proposition 4.16. *We have $t'(\xi) > 0$ for $\xi < \xi_0$, $t'(\xi_0) = 0$, and $t'(\xi) < 0$ for $\xi > \xi_0$. Furthermore,*

$$t(\xi) \to 0 \quad (\xi \to 0), \tag{4.20}$$

$$t(\xi) \to 1/2 \quad (\xi \to +\infty). \tag{4.21}$$

Proof. By Propositions 4.14 and 4.15 and the fact $\frac{\partial P}{\partial t} > 0$, $t(\xi)$ is increasing on $(0, \xi_0)$ and decreasing on (ξ_0, ∞). Then by the analyticity of $t(\xi)$, we can obtain two analytic inverse functions on the two intervals respectively. For the first inverse function, write $\xi_1 = \xi_1(t)$. Then $\xi_1'(t) > 0$ and

$$\xi_1(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)).$$

(the equations (4.11) are considered as equations on t). By Proposition 4.14, we have $q(\xi_1(t)) < 0$ then $P(t, q(\xi_1(t))) < 0$. By Proposition 4.12 (1), $\lim\limits_{q \to 2t-1} P(t, q) = \infty$. Thus there exists $q_0(t)$ such that $q_0(t) > q(t)$ and $P(t, q_0(t)) = 0$. Therefore

$$\xi_1(t) = \frac{\partial P}{\partial q}(t, q(t)) < \frac{\partial P}{\partial q}(t, q_0(t)).$$

Since $P(0, q) = \log \zeta(-q)$, we have $\lim_{t \to 0} q_0(t) = \infty$. Thus we get

$$\lim_{t \to 0} \frac{\partial P}{\partial q}(t, q_0(t)) = \lim_{q \to -\infty} \frac{\partial P}{\partial q}(0, q) = 0.$$

Hence by $\xi_1(t) \geq 0$, we obtain $\lim_{t \to 0} \xi_1(t) = 0$ which implies (4.20).

Write $\xi_2 = \xi_2(t)$ for the second inverse function. Then $\xi_2'(t) < 0$ and

$$\xi_2(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)) > \frac{\partial P}{\partial q}(t, 0) \to \infty \quad (t \to 1/2).$$

This implies (4.21). □

Let us summarize. We have proved that $t(\xi)$ is analytic on $(0, \infty)$, $\lim\limits_{\xi \to 0} t(\xi) = 0$ and $\lim\limits_{\xi \to \infty} t(\xi) = 1/2$. We have also proved that $t(\xi)$ is increasing on $(0, \xi_0)$, decreasing on (ξ_0, ∞) and $t(\xi_0) = 1$.

5. Khintchine spectrum

Now we are ready to study the Hausdorff dimensions of the level set

$$E_\xi = \{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) = \xi\}.$$

Since $\mathbb{Q}$ is countable, we need only to consider

$$\{x \in [0,1)\setminus\mathbb{Q} : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) = \xi\}.$$

which admits the same Hausdorff dimension with E_ξ and is still denoted by E_ξ.

5.1. **Proof of Theorem 1.2 (1) and (2).** Let $(t,q) \in D$ and $\mu_{t,q}, \tilde{\mu}_{t,q}$ be the measures in Proposition 4.6. For $x \in [0,1)\setminus\mathbb{Q}$, let $x = [a_1, \cdots, a_n, \cdots]$ and $\omega = \pi^{-1}(x)$. Then $\omega = a_1 \cdots a_n \cdots \in \mathbb{N}^{\mathbb{N}}$ and

$$\mu_{t,q}(I_n(x)) = \mu_{t,q}(I_n(a_1,\cdots,a_n)) = \tilde{\mu}_{t,q}([\omega|_n]).$$

By the Gibbs property of $\tilde{\mu}_{t,q}$,

$$\tilde{\mu}_{t,q}(\pi([\omega|_n])) \quad \asymp \quad \exp(-nP(t,q)) \prod_{j=1}^{n} \omega_j^q (\omega_j + \pi(\sigma^j\omega))^{-2t}.$$

In other words,

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q)) \prod_{j=1}^{n} a_j^q [a_j, \cdots, a_n, \cdots]^{2t}.$$

By Lemma 2.7, $|I_n(x)| \asymp |(T^n)'(x)|^{-1} = \prod_{j=0}^{n-1} |T^j(x)|^2$. Thus we have the following Gibbs property of $\mu_{t,q}$:

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t,q))|I_n(x)|^t \prod_{j=1}^{n} a_j^q. \tag{5.1}$$

It follows that

$$\delta_{\mu_{t,q}}(x) = \lim_{n\to\infty} \frac{\log \mu_{t,q}(I_n(x))}{\log |I_n(x)|} = t + \lim_{n\to\infty} \frac{q \cdot \frac{1}{n}\sum_{j=1}^{n} \log a_j - P(t,q)}{\frac{1}{n}\log|I_n(x)|}.$$

The Gibbs property of $\tilde{\mu}_{t,q}$ implies that $\mu_{t,q}$ is ergodic. Therefore,

$$\delta_{\mu_{t,q}}(x) = t + \frac{q\int \log a_1(x) d\mu_{t,q} - P(t,q)}{-\int \log|T'(x)| d\mu_{t,q}} \qquad \mu_{t,q} - a.e..$$

Using the formula (4.3) and (4.4) in Proposition 4.8, we have

$$\delta_{\mu_{t,q}}(x) = t + \frac{q\frac{\partial P}{\partial q}(t,q) - P(t,q)}{\frac{\partial P}{\partial t}(t,q)} \qquad \mu_{t,q} - a.e.. \tag{5.2}$$

Moreover, the ergodicity of $\tilde{\mu}_{t,q}$ also implies that the Lyapunov exponents $\lambda(x)$ exist for $\mu_{t,q}$ almost every x in $[0,1)$. Thus by (5.1), Lemma 2.12 and Lemma 2.13, we obtain

$$d_{\mu_{t,q}}(x) = \delta_{\mu_{t,q}}(x) = t + \frac{q\frac{\partial P}{\partial q}(t,q) - P(t,q)}{\frac{\partial P}{\partial t}(t,q)} \qquad \mu_{t,q} - a.e.. \tag{5.3}$$

For $\xi \in (0,\infty)$, choose $(t,q) = (t(\xi), q(\xi)) \in D_0$ be the unique solution of (4.11). Then (5.3) gives

$$d_{\mu_{t(\xi),q(\xi)}}(x) = t(\xi) \qquad \mu_{t,q} - a.e..$$

By the ergodicity of $\tilde{\mu}_{t(\xi),q(\xi)}$ and (4.4), we have for $\mu_{t(\xi),q(\xi)}$ almost every x,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) = \int \log a_1(x) d\mu_{t(\xi),q(\xi)} = \frac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi.$$

So $\mu_{t(\xi),q(\xi)}$ is supported on E_ξ. Hence

$$\dim(E_\xi) \geq \dim \mu_{t(\xi),q(\xi)} = t(\xi). \tag{5.4}$$

In the following we will show that

$$\dim(E_\xi) \leq t \qquad (\forall t > t(\xi)). \tag{5.5}$$

Then it will imply that $\dim(E_\xi) = t(\xi)$ for any $\xi > 0$. For any $t > t(\xi)$, take an $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 < \frac{P(t(\xi), q(\xi)) - P(t, q(\xi))}{q(\xi)} \quad \text{if } q(\xi) > 0,$$

and

$$0 < \epsilon_0 < \frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} \quad \text{if } q(\xi) < 0.$$

(For the special case $q(\xi) = 0$, *i.e.*, $\xi = \xi_0$, we have $\dim E_\xi = 1$ which is a well-known result). Such an ϵ_0 exists, for $P(t,q)$ is strictly decreasing with respect to t. For all $n \geq 1$, set

$$E_\xi^n(\epsilon_0) := \left\{ x \in [0,1) \setminus \mathbb{Q} : \xi - \epsilon_0 < \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0 \right\}.$$

Then we have

$$E_\xi \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_\xi^n(\epsilon_0).$$

Let $\mathcal{I}(n,\xi,\epsilon_0)$ be the collection of all n-th order cylinders $I_n(a_1,\cdots,a_n)$ such that

$$\xi - \epsilon_0 < \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0.$$

Then

$$E_\xi^n(\epsilon_0) = \bigcup_{J \in \mathcal{I}(n,\xi,\epsilon_0)} J.$$

Hence $\{J : J \in \mathcal{I}(n,\xi,\epsilon_0), n \geq 1\}$ is a cover of E_ξ. When $q(\xi) > 0$, by (5.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} |J|^t \\
\leq \; & \sum_{n=1}^{\infty} \sum_{(a_1\cdots a_n) > e^{n(\xi-\epsilon_0)}} \frac{e^{nP(t,q(\xi))}}{(a_1\cdots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1\cdots a_n)^{q(\xi)}}{e^{nP(t,q(\xi))}} \\
\leq \; & C \cdot \sum_{n=1}^{\infty} e^{n(P(t,q(\xi)) - (\xi-\epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} \mu_{t,q(\xi)}(J) < \infty
\end{aligned}$$

where C is a constant. When $q(\xi) < 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} |J|^t \\
\leq\ & \sum_{n=1}^{\infty} \sum_{(a_1 \cdots a_n) < e^{n(\xi+\epsilon_0)}} \frac{e^{nP(t,q(\xi))}}{(a_1 \cdots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1 \cdots a_n)^{q(\xi)}}{e^{nP(t,q(\xi))}} \\
\leq\ & C \cdot \sum_{n=1}^{\infty} e^{n(P(t,q(\xi)) - (\xi+\epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} \mu_{t,q(\xi)}(J) < \infty.
\end{aligned}$$

Hence we get (5.5).

For the special case $\xi = 0$, we need only to show $\dim(E_0) = 0$. This can be induced by the same process. For any $t > 0$, since $\lim_{\xi \to 0} t(\xi) = 0$, there exists $\xi > 0$ such that $0 < t(\xi) < t$. We can also choose $\epsilon_0 > 0$ such that

$$\frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} > \epsilon_0.$$

For $n \geq 1$, set

$$E_0^n(\epsilon_0) := \left\{ x \in [0,1) \setminus \mathbb{Q} : \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0 \right\}.$$

We have

$$E_0 \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_0^n(\epsilon_0).$$

By the same calculation, we get $\dim(E_0) \leq t$. Since t can be arbitrary small, we obtain $\dim(E_0) = 0$.

By the discussion in the preceding subsection, we have proved Theorem 1.2 (1) and (2).

5.2. **Proof of Theorem 1.2 (3) and (4).** We are going to investigate more properties of the functions $q(\xi)$ and $t(\xi)$.

Proposition 5.1. *We have*

$$\lim_{\xi \to 0} q(\xi) = -\infty, \quad \lim_{\xi \to \infty} q(\xi) = 0.$$

Proof. We prove the first limit by contradiction. Suppose there exists a subsequence $\xi_\delta \to 0$ such that $q(\xi_\delta) \to M > -\infty$. Then by (4.20) and Proposition 4.8 (3), we have

$$\lim_{\xi_\delta \to 0} \frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \frac{\partial P}{\partial q}(0, M) > 0.$$

This contradicts with

$$\frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \xi_\delta \to 0.$$

On the other hand, we know that $q(\xi) \geq 0$ when $\xi \geq \xi_0$, and $0 \leq q(\xi) < 2t(\xi) - 1$. Then by (4.21), we have $\lim_{\xi \to \infty} q(\xi) = 0$. □

Apply this proposition and (4.17), combining (4.9) and (4.10). We get

$$\lim_{\xi\to 0} t'(\xi) = +\infty, \qquad \lim_{\xi\to\infty} t'(\xi) = 0.$$

This is the assertion (3) of Theorem 1.2.

Now we will prove the last assertion of Theorem 1.2, *i.e.*, $t''(\xi_0) < 0$ and there exists $\xi_1 > \xi_0$ such that $t''(\xi_1) > 0$, basing on the following proposition.

Proposition 5.2. *For $\xi \in (0, +\infty)$, we have*

$$q'(\xi) = \frac{1 - t'(\xi)\frac{\partial^2 P}{\partial t \partial q}\big(t(\xi), q(\xi)\big)}{\frac{\partial^2 P}{\partial q^2}\big(t(\xi), q(\xi)\big)}; \tag{5.6}$$

$$t''(\xi) = \frac{t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}\big(t(\xi), q(\xi)\big) - q'(\xi)^2 \frac{\partial^2 P}{\partial q^2}\big(t(\xi), q(\xi)\big)}{-\frac{\partial P}{\partial t}\big(t(\xi), q(\xi)\big)}. \tag{5.7}$$

Proof. Taking derivative of (4.19) with respect to ξ, we get

$$t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}\big(t(\xi), q(\xi)\big) + q'(\xi)t'(\xi)\frac{\partial^2 P}{\partial q \partial t}\big(t(\xi), q(\xi)\big) + t''(\xi)\frac{\partial P}{\partial t}\big(t(\xi), q(\xi)\big) = q'(\xi). \tag{5.8}$$

Taking derivative of the second equation of (4.18) with respect to ξ, we get

$$t'(\xi)\frac{\partial^2 P}{\partial t \partial q}\big(t(\xi), q(\xi)\big) + q'(\xi)\frac{\partial^2 P}{\partial q^2}\big(t(\xi), q(\xi)\big) = 1, \tag{5.9}$$

which gives immediately (5.6).

Subtract (5.9) multiplied by $q'(\xi)$ from (5.8), we get (5.7). □

We divide the proof of the assertion (4) of Theorem 1.2 into two parts.

Proof of $t''(\xi_0) < 0$. By Proposition 4.8, $\frac{\partial P}{\partial t}(1, 0) < 0$. Since $q(\xi_0) = 0$, by (4.17) we have $t'(\xi_0) = 0$. Also by Proposition 4.8, we get

$$0 < \frac{\partial^2 P}{\partial t^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial t^2}(1, 0) < +\infty,$$

and

$$0 \le \frac{\partial^2 P}{\partial q^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial q^2}(1, 0) < +\infty.$$

By (5.6) and (5.7), we have

$$t''(\xi) = \frac{t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}(t(\xi), q(\xi))\frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi)) - \Big(1 - t'(\xi)\frac{\partial^2 P}{\partial t \partial q}(t(\xi), q(\xi))\Big)^2}{-\frac{\partial P}{\partial t}(t(\xi), q(\xi))\frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi))}. \tag{5.10}$$

Thus by $t'(\xi_0) = 0$, we have $t''(\xi_0) < 0$. □

Proof of $t''(\xi_1) > 0$. Proposition 5.1 shows $\lim_{\xi\to\infty} q(\xi) = 0$ and we know that $q(\xi_0) = 0$. However, $q(\xi)$ is not always equal to 0, so there exists a $\xi_1 \in [\xi_0, +\infty)$, such that $q'(\xi_1) < 0$. Write

$$H(t, q) := \begin{pmatrix} \frac{\partial^2 P}{\partial t^2} & \frac{\partial^2 P}{\partial t \partial q} \\ \\ \frac{\partial^2 P}{\partial t \partial q} & \frac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

and add (5.9) multiplied by $q'(\xi)$ to (5.8), we get

$$\Big(t'(\xi), q'(\xi)\Big) H(t,q) \big(t'(\xi), q'(\xi)\big)^T + \frac{\partial P}{\partial t}\big(t(\xi), q(\xi)\big) t''(\xi) = 2q'(\xi). \tag{5.11}$$

Since $H(t,q)$ is definite positive, $\frac{\partial P}{\partial t}(t,q) < 0$ and $q'(\xi_1) < 0$, we have $t''(\xi_1) > 0$. This completes the proof. □

6. Lyapunov spectrum

In this last section, we follow the same procedure as in Section 4 and Section 5 to deduce the Lyapunov spectrum of the Gauss map. Kesseböhmer recently pointed out to us that the Lyapunov spectrum was also studied by M. Kesseböhmer and B. Stratmann [25].

Take

$$F = \Psi = \{\log|\psi_i'| : i \in \mathbb{N}\}.$$

instead of $F = \{-\log i : i \in \mathbb{N}\}$ and $\Psi = \{\log|\psi_i'| : i \in \mathbb{N}\}$. Then the strong Hölder family becomes $(\tilde{t} - q)\Psi$ and D should be changed to

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\}.$$

Here and in the rest of this section we will use $\tilde{t}$ instead of t to distinguish the present situation from that of Khintchine exponents. What we have done in Section 4 is still useful. Denote by $P_1(\tilde{t}, q)$ the pressure $P((\tilde{t} - q)\Psi)$. Then

$$P_1(\tilde{t}, q) = P(\tilde{t} - q), \quad \text{with } P(\cdot) = P(\cdot, 0),$$

where $P(\cdot,\cdot)$ is the pressure function studied in Section 4. Hence $P_1(\tilde{t}, q)$ is analytic and similar equations (4.3) and (4.4) are obtained just with $\log|T'(x)|$ instead of $\log a_1(x)$.

To determine the Lyapunov spectrum, we begin with the following proposition which take the place of Proposition 4.12.

Proposition 6.1. *For $(\tilde{t}, q) \in \tilde{D}$, we have*

$$-(\tilde{t} - q)\log 4 + \log\zeta(2\tilde{t} - 2q) \le P_1(\tilde{t}, q) \le \log\zeta(2\tilde{t} - 2q). \tag{6.12}$$

Consequently,

(1) *for any point $(\tilde{t}_0, q_0)$ on the line $\tilde{t} - q = 1/2$,*

$$\lim_{(\tilde{t},q)\to(\tilde{t}_0,q_0)} P(\tilde{t}, q) = \infty;$$

(2) *for fixed $\tilde{t} \in \mathbb{R}$,*

$$\lim_{q\to\tilde{t}-\frac{1}{2}} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = +\infty;$$

(3) *recalling $\gamma_0 = 2\log\frac{1+\sqrt{5}}{2}$, for fixed $\tilde{t} \in \mathbb{R}$,*

$$\lim_{q\to-\infty} \frac{P_1(\tilde{t}, q)}{q} = \gamma_0, \qquad \lim_{q\to-\infty} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = \gamma_0.$$

Proof. $P_1(\tilde{t}, q)$ is defined as

$$P_1(\tilde{t}, q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\omega_j, \cdots, \omega_n + x])^{2(\tilde{t}-q)} \right).$$

The proofs of (1) and (2) are the same as in the proof of Proposition 4.12.

To get (3), we follow another method. Since $P_1(\tilde{t}, q) = P(\tilde{t} - q)$, we need only to show

$$\lim_{\tilde{t}\to\infty} P'(\tilde{t}) = -\gamma_0, \quad P(\tilde{t}) + \tilde{t}\gamma_0 = o(\tilde{t}) \quad (\tilde{t} \to \infty).$$

By Proposition 4.9, $P(\tilde{t})$ is analytic on $(1/2, \infty)$. Let $E := \{P'(\tilde{t}) : \tilde{t} > 1/2\}$, denote by $\mathrm{Int}(E)$ and $\mathrm{Cl}(E)$ the interior and closure of E. By a result in [24], we have

$$\mathrm{Int}(E) \subset \left\{-\int \log|T'(x)|d\mu : \mu \in \mathcal{M}\right\} \subset \mathrm{Cl}(E),$$

where $\mathcal{M}$ is the set of the invariant measures on $[0, 1]$. By Birkhoff's theorem, for any $\mu \in \mathcal{M}$, we have

$$\int \lambda(x)d\mu = \int \log|T'(x)|d\mu.$$

However, we know that $\lambda(x) \geq \gamma_0 = 2\log\frac{1+\sqrt{5}}{2}$. Thus

$$-\int \log|T'(x)|d\mu \leq -\gamma_0 \qquad \forall \mu \in \mathcal{M}. \tag{6.13}$$

Let $\theta_0 = \frac{\sqrt{5}-1}{2}$. Then $T(\theta_0) = \theta_0$ and the Dirac measure $\mu = \delta_{\theta_0}$ is invariant, and

$$-\int \log|T'(x)|d\delta_{\theta_0} = -\log|T'(\theta_0)| = -\gamma_0.$$

However, by the continuity of P', we know that E is an interval. Therefore $-\gamma_0$ is the right endpoint of E. Since $P'(\tilde{t})$ is increasing, we get

$$\lim_{\tilde{t}\to\infty} P'(\tilde{t}) = -\gamma_0.$$

Let $\{\beta_n\}_{n\geq 1}$ be such that $\beta_n < -\gamma_0$ and $\lim\limits_{n\to\infty} \beta_n = -\gamma_0$. There exist $t_n \in \mathbb{R}$ such that $t_n \to \infty$ and $P'(t_n) = \beta_n$. By the variational principle ([40], see also [34]), there exists an ergodic measure μ_{t_n} such that

$$P(t_n) = h_{\mu_{t_n}} - t_n \int \log|T'|(x)d\mu_{t_n},$$

where $h_{\mu_{t_n}}$ stands for the metric entropy of μ_{t_n}. By the compactness of $\mathcal{M}$ there exists an invariant measure μ_∞ which is the weak limit of μ_{t_n}(more precisely some subsequence of μ_{t_n}. But, without loss of generality, we write it as μ_{t_n}). By the semi-continuity of metric entropy, for any $\epsilon > 0$ we have $h_{\mu_{t_n}} \leq h_{\mu_\infty} + \epsilon$ when t_n is large enough. Thus by (6.13),

$$P(t_n) \leq h_{\mu_\infty} + \epsilon - t_n\gamma_0.$$

We will show that $h_{\mu_\infty} = 0$ (see the next lemma), which will imply

$$P(t_n) \leq \epsilon - t_n\gamma_0.$$

However, by the definition of $P_1(\tilde{t}, q)$, $P(\tilde{t})$ can be written as

$$P(\tilde{t}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\omega_j, \cdots, \omega_n + x])^{2\tilde{t}}\right).$$

Thus if we just take one term in the summation, we have

$$P(\tilde{t}) \geq \lim_{n\to\infty} \frac{1}{n} \log \exp \left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\underbrace{1, \cdots, 1}_{n-j}, 1+x])^{2\tilde{t}} \right) = -\tilde{t}\gamma_0.$$

Hence we get

$$P(\tilde{t}) + \tilde{t}\gamma_0 = o(\tilde{t}) \qquad (\tilde{t} \to \infty).$$

□

Now we are led to show

Lemma 6.2. $h_{\mu_\infty} = 0$.

Proof. Let $h_{\mu_\infty}(x)$ be the local entropy of μ_∞ at x which is defined by

$$h_{\mu_\infty}(x) = \lim_{n\to\infty} \frac{\log \mu_\infty(I_n(x))}{n},$$

if the limit exists. Let $\underline{D}_{\mu_\infty}(x)$ be the lower local dimension of μ_∞ at x which is defined by

$$\underline{D}_{\mu_\infty}(x) := \liminf_{r\to 0} \frac{\log \mu_\infty(B(x,r))}{\log r}.$$

By Shannon-McMillan-Breiman theorem, $h_{\mu_\infty}(x)$ exists μ_∞-almost everywhere. It is also known that $\lambda(x)$ exists almost everywhere (by Birkhoff's theorem). So, by the definitions, we have

$$h_{\mu_\infty}(x) = \underline{D}_{\mu_\infty}(x)\lambda(x) \quad \mu_\infty - a.e..$$

By Birkhoff's theorem and (4.5),

$$\begin{aligned} \int \lambda(x) d\mu_\infty(x) &= \int \log |T'|(x) d\mu_\infty(x) \\ &= \lim_{n\to\infty} \int \log |T'|(x) d\mu_{t_n} \\ &= -\lim_{n\to\infty} P'(t_n) = \gamma_0 < \infty. \end{aligned}$$

Hence $\lambda(x)$ is almost everywhere finite. Recall that [8]

$$h_{\mu_\infty} = \int h_{\mu_\infty}(x) d\mu_\infty(x).$$

Thus it suffices to prove

$$\underline{D}_{\mu_\infty}(x) = 0 \quad \mu_\infty - a.e..$$

That means ([13]) the upper dimension of μ_∞ is zero, i.e., μ_∞ is supported by a zero-dimensional set.

Since $\int \lambda(x) d\mu_\infty(x) = \gamma_0$ and $\lambda(x) \geq \gamma_0$ for any x, we have for μ_∞ almost everywhere $\lambda(x) = \gamma_0$. Thus by Birkhoff's theorem, μ_∞ is supported by the following set

$$\left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j x)| = \gamma_0 \right\}. \tag{6.14}$$

Thus we need only to show that the Hausdorff dimension of this set is zero.

Recall

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log|T'(T^jx)| = 2\lim_{n\to\infty}\frac{1}{n}\log q_n(x).$$

By Lemma 2.8, (6.14) is in fact the following

$$\Big\{x\in[0,1] : \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\log a_j(x)=0\Big\}. \tag{6.15}$$

However, the Hausdorff dimension of (6.15) is nothing but $t(0)$, the special case $\xi=0$ discussed in the subsection 5.1., which was proved to be zero. Thus the proof is completed. $\square$

Recall that $\lambda_0=\int\log|T'(x)|d\mu_G$. Let $\tilde{D}_0 := \{(\tilde{t},q) : \tilde{t}-q>1/2, 0\le\tilde{t}\le 1\}$. We have a proposition similar to Proposition 4.13.

Proposition 6.3. *For any $\beta\in(\gamma_0,\infty)$, the system*

$$\begin{cases} P_1(\tilde{t},q)=q\beta,\\ \dfrac{\partial P_1}{\partial q}(\tilde{t},q)=\beta \end{cases} \tag{6.16}$$

admits a unique solution $(\tilde{t}(\beta),q(\beta))\in\tilde{D}_0$. For $\beta=\lambda_0$, the solution is $(\tilde{t}(\lambda_0),q(\lambda_0))=(1,0)$. The functions $\tilde{t}(\beta)$ and $q(\beta)$ are analytic.

With the same argument, we can prove that $\tilde{t}(\beta)$ is the spectrum of Lyapunov exponent. It is analytic, increasing on $(\gamma_0,\lambda_0]$ and decreasing on (λ_0,∞). It is also neither concave nor convex. In other words, Theorem 1.3 can be similarly proved.

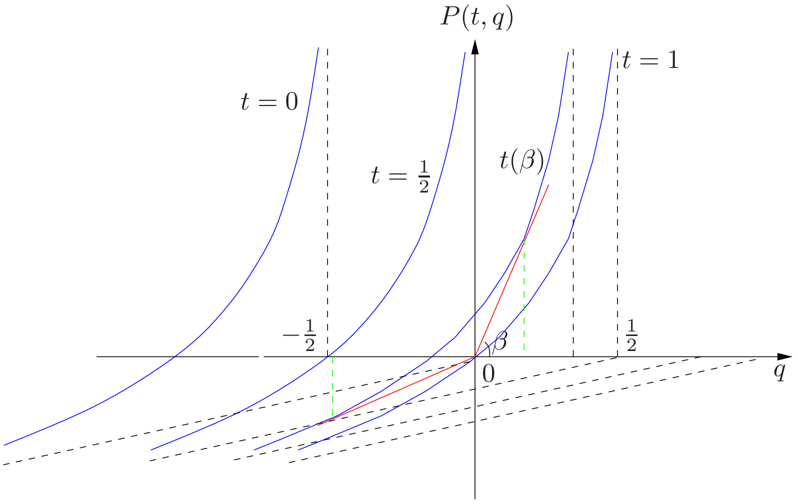

Figure 4. Solution of (6.16)

We finish the paper by the observation that the Lyapunov spectrum can be stated as follows, which is similar to the classic formula, but with the difference that we have to divide the Legendre transform by β.

Proposition 6.4.

$$\tilde{t}(\beta) = \frac{P(-q(\beta))}{\beta} - q(\beta) = \frac{1}{\beta}\inf_q\{P(-q) - q\beta\}. \tag{6.17}$$

Proof. In fact, the family of functions $P_1(\tilde{t}, q)$ with parameter $\tilde{t}$ are just right translation of the function $P(-q)$ with the length $\tilde{t}$. Write the system (6.16) as follows

$$\begin{cases} P(\tilde{t} - q) = q\beta, \\ \frac{dP}{dq}(\tilde{t} - q) = \beta. \end{cases} \tag{6.18}$$

If we denote by μ_q, the Gibbs measure with respect to potential $q\Psi$, then by a left translation the system (6.18) can be written as

$$\begin{cases} P(-q) = (\tilde{t} + q)\beta, \\ \frac{dP}{dq}(-q) = \beta. \end{cases}$$

Thus

$$\begin{cases} \tilde{t} = \frac{P(-q)}{\beta} - q, \\ \frac{dP}{dq}(-q) = \beta. \end{cases}$$

By using the second equation, we can write q as a function of β, hence we get (6.17). ☐

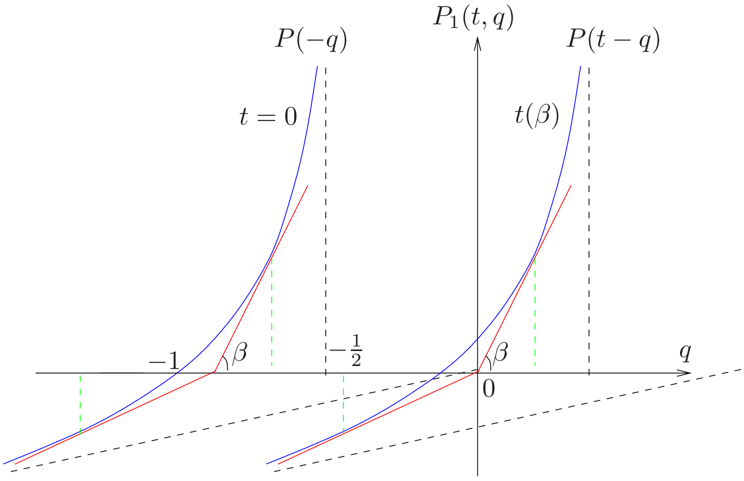

Figure 5. The other way to see $t(\beta)$

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