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Generation and Recognition of Digital Plane using Multi-dimensional Continued Fractions

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Abstract. This paper provides a multi-dimensional generalization of pattern recognition technics for generation or recognition of digital lines. More precisely, we show how the connection between chain codes of digital lines and continued fractions can be generalized by a connection between tilings and multi-dimensional continued fractions. This leads to a new approach for generating and recognizing digital hyperplanes.

Introduction

Discrete (or digital) geometry deals with discrete sets considered to be digitized objects of the Euclidean space. A challenging problem is to decompose a huge complicated discrete set into elementary ones, which could be easily stored and from which one can easily reconstruct the original discrete set. Good candidates for such elementary discrete sets are digitizations of Euclidean hyperplanes, in particular *arithmetic discrete hyperplanes*, as defined in [7]. This led to search efficient algorithms to go from the parameters of an Euclidean hyperplane to its digitization (generation) and conversely (recognition).

In the particular case of digitizations of lines, among other technics, so-called *linguistic technics* provide a nice connections with words theory and continued fractions. Let us briefly detail this. A digital line made of horizontal or vertical unit segments can be coded by a two-letter word, called *chain code*. For example, if a horizontal (resp. vertical) unit segment is coded by 0 (resp. 1), then a segment of slope 1 can be coded by a word of the form $10\ldots10 = (10)^k$. Then, basic transformations on words correspond to basic operations on slopes of the segments they code. For example, replacing each 0 by 01 and each 1 by 0 in the previous word leads to the word $(001)^k$, which codes a segment of slope $1/2$. Many algorithms use this approach for both recognition and generation of digital lines, and continued fraction expansions of slopes of segments turn out to play a central role in this context (see *e.g.* [6] and references therein).

In higher dimensions, there is also various technics for generation or recognition of digital hyperplane as, for example, linear programming, computational geometry or preimage technics (see *e.g.* [3] and references therein). However, none of these approaches provides a natural extension of the above connection with words theory and continued fractions. The aim of this paper is to introduce such an approach.

The paper is organized as follows. In Sec. 1, we introduce *stepped planes* and *dual maps*. Stepped planes, that can also be seen as tilings, are digitizations of Euclidean hyperplane (see [9]). They play for hyperplanes the role played by chain codes for lines. Dual maps act on stepped planes and generalize the basic transformations on chain codes mentioned above (see [1, 5]). Then, in Sec. 2, we briefly describe the *Brun algorithm*, which is one of existing multi-dimensional continued fraction algorithms (see [8]). The Brun algorithm computes so-called *Brun expansions* of real vectors. Note that the choice of this algorithm is rather arbitrary: many similar algorithms could play the same role in this paper. We also introduce particular dual maps which allow the Brun algorithm to act over stepped planes. This leads, in Sec. 3, to a method for generating a piece of a stepped plane which suffices to generate the whole stepped plane (Th. 2). In Sec. 4, we describe a method to compute so-called *Brun expansions* of stepped planes, by grabing information from local configurations of this stepped plane (namely *runs*). It turns out that the Brun expansion of a stepped plane is nothing but the Brun expansion of its normal vector. This method is then extended in Sec. 5 to compute Brun expansions of so-called *binary functions*, which generalize stepped planes. In particular, let us stress that the notion of normal vectors does not make any more sense for binary functions. Thus, it is rather unclear whether a real vector and a binary function have identical Brun expansions. However, we show in Sec. 6 that Brun expansions of binary functions can be used for recognizing stepped planes among binary functions. More precisely, we give an algorithm computing the paramaters of Euclidean hyperplanes whose digitizations are stepped planes containing a given binary function (Th. 3).

1 Stepped planes and dual maps

We work in the $\mathbb{Z}$-module of functions from $\mathbb{Z}^d \times \{1,\ldots,d\}$ to $\mathbb{Z}$, denoted by $\mathfrak{F}_d$. More precisely, we are especially interested in the following functions of $\mathfrak{F}_d$:

Definition 1. *A* binary function *is a function of $\mathfrak{F}_d$ which takes values in $\{0,1\}$. The* size *of a binary function $\mathcal{B}$, denoted by $|\mathcal{B}|$, is the cardinal of its* support*, that is, the subset of $\mathbb{Z}^d \times \{1,\ldots,d\}$ where $\mathcal{B}$ takes value one. We denote by $\mathfrak{B}_d$ the set of binary functions. For $\boldsymbol{x} \in \mathbb{Z}^d$ and $i \in \{1,\ldots,d\}$, the* face of type i located in $\boldsymbol{x}$ *is the binary function denoted by $(\boldsymbol{x}, i^*)$ whose support is $\{(\boldsymbol{x}, i)\}$.*

Note that any function of $\mathfrak{B}_d$ (resp. $\mathfrak{F}_d$) can be seen as a possibly infinite sum of faces (resp. weighted sum of faces). Let us give a geometric interpretation of binary functions. Let $(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d)$ denote the canonical basis of $\mathbb{R}^d$. The geometric interpretation of a face $(\boldsymbol{x}, i^*)$ is the closed subset of $\mathbb{R}^d$ defined by (see Fig. 1):

$$\{\boldsymbol{x} + \boldsymbol{e}_i + \sum_{j \neq i} \lambda_j \boldsymbol{e}_j \mid 0 \leq \lambda_j \leq 1\}.$$

This subset is nothing but a hyperface of the unit cube of $\mathbb{R}^d$ whose lowest vertex is $\boldsymbol{x}$. Then, the geometric interpretation of a binary function, that is, of a sum of faces, is the union of the geometrical interpretations of these faces. Note that such a geometric interpretation cannot be naturally extended over the whole $\mathfrak{F}_d$. Let us now introduce stepped planes.

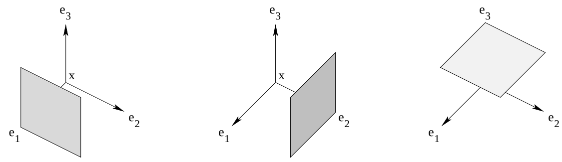

Fig. 1. Geometrical interpretations of faces $(\boldsymbol{x}, i^*)$, for $i = 1, 2, 3$ (from left to right).

Definition 2. *Let* $\boldsymbol{\alpha} \in \mathbb{R}^d_+ \backslash \{\mathbf{0}\}$ *and* $\rho \in \mathbb{R}$. *The* stepped plane *of* normal vector $\boldsymbol{\alpha}$ *and* intercept $\rho \in \mathbb{R}$, *denoted by* $\mathcal{P}_{\boldsymbol{\alpha},\rho}$, *is the binary function defined by :*

$$\mathcal{P}_{\boldsymbol{\alpha},\rho}(\boldsymbol{x}, i) = 1 \;\Leftrightarrow\; \langle \boldsymbol{x} | \boldsymbol{\alpha} \rangle < \rho \leq \langle \boldsymbol{x} + \boldsymbol{e}_i | \boldsymbol{\alpha} \rangle,$$

where $\langle | \rangle$ *is the canonical dot product. We denote by* $\mathfrak{P}_d$ *the set of stepped planes.*

Fig. 2 depicts the geometrical interpretation of a stepped plane. It is not hard to check that the vertices of a stepped plane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$, that is, the integers vectors which belong to its geometrical interpretation, form a *standard arithmetic discrete plane of parameters* $(\boldsymbol{\alpha}, \rho)$, as defined in [7]. Moreover, one checks that the orthogonal projection along $\boldsymbol{e}_1 + \ldots + \boldsymbol{e}_d$ maps the geometrical representation of a stepped plane onto a tiling of $\mathbb{R}^{d-1}$ whose tiles are projections of geometrical representations of faces (see also Fig. 2).

Binary functions and, among them, stepped planes, are the objects studied in this paper. Let us now introduce the tools used to study them. First, let us recall some basic definitions and notations. We denote by F_d the free group generated by the alphabet $\{1, \ldots, d\}$, with the concatenation as a composition rule and the empty word as unit. An endomorphism of F_d is a *substitution* if it maps any letter to a non-empty concatenation of letters with positive powers. The *parikh mapping* is the map $\boldsymbol{f}$ from F_d to $\mathbb{Z}^d$ defined on $w \in F_d$ by:

$$\boldsymbol{f}(w) = (|w|_1, \ldots, |w|_d),$$

where $|w|_i$ is the sum of the powers of the occurences of the letter i in w. Then, the *incidence matrix* of an endomorphism σ of F_d, denoted by M_σ, is the $d \times d$ integer matrix whose i-th column is the vector $\boldsymbol{f}(\sigma(i))$. Last, an endomorphism of F_d is said to be *unimodular* if its incidence matrix has determinant ± 1.

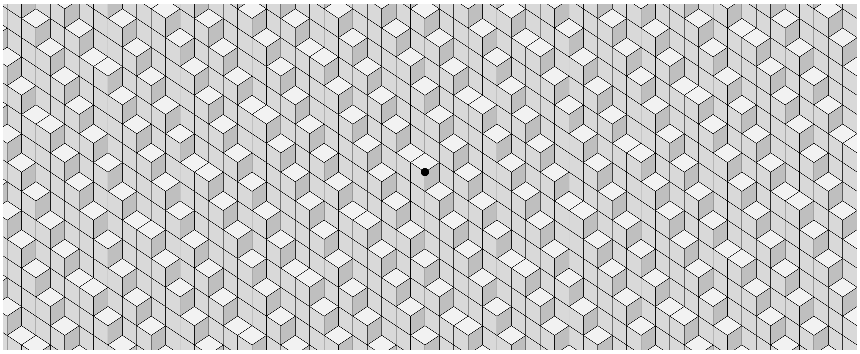

Fig. 2. Geometrical interpretation of the stepped plane $\mathcal{P}_{(24,9,10),0}$ (highlighted origin). This can be seen either as faces of unit cubes, or as a lozenge tiling of the plane.

Example 1. Let σ be the endormorphism of F_3 defined by $\sigma(1) = 12$, $\sigma(2) = 13$ and $\sigma(3) = 1$. Note that σ is a substitution (often called *Rauzy* substitution). One computes, for example, $\sigma(1^{-1}2) = \sigma(1)^{-1}\sigma(2) = 2^{-1}1^{-1}13 = 2^{-1}3$. This substitution is unimodular since its incidence matrix (below) has determinant 1:

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We are now in a position to define *dual maps*:

Definition 3. *The* dual map *of a unimodular endomorphism σ of F_d, denoted by $E_1^*(\sigma)$, maps any function $\mathcal{F} \in \mathfrak{F}_d$ to the function $E_1^*(\sigma)(\mathcal{F})$ defined by:*

$$E_1^*(\sigma)(\mathcal{F}) : (\boldsymbol{x}, i) \mapsto \sum_{j|\sigma(i)=p\cdot j\cdot s} \mathcal{F}(M_\sigma \boldsymbol{x} + \boldsymbol{f}(p), j) - \sum_{j|\sigma(i)=p\cdot j^{-1}\cdot s} \mathcal{F}(M_\sigma \boldsymbol{x} + \boldsymbol{f}(p) - \boldsymbol{e}_j, j).$$

Note that the value of $E_1^*(\sigma)(\mathcal{F})$ in $(\boldsymbol{x}, i)$ is finite since it depends only on the values of $\mathcal{F}$ over a finite subset of $\mathbb{Z}^d \times \{1, \ldots, d\}$. This yields that $E_1^*(\sigma)$ is an endomorphism of $\mathfrak{F}_d$.

Example 2. The dual map of the substitution σ introduced in Ex. 1 satisfies:

$$E_1^*(\sigma) \ : \ \begin{cases} (\mathbf{0}, 1^*) \mapsto (\mathbf{0}, 1^*) + (\mathbf{0}, 2^*) + (\mathbf{0}, 3^*), \\ (\mathbf{0}, 2^*) \mapsto (-\boldsymbol{e}_3, 1^*), \\ (\mathbf{0}, 3^*) \mapsto (-\boldsymbol{e}_3, 2^*). \end{cases}$$

The image of any function of $\mathfrak{F}_d$, that is, of a weighted sum of faces, can then be easily computed by linearity. Fig. 3 illustrates this.

The following theorem, proved in [2], connects dual maps and stepped planes:

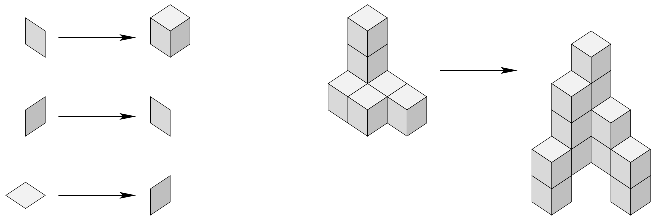

Fig. 3. Action of the dual map of Ex. 2 on faces (left) and on a binary function (right). Let us stress that the image of a binary function is not necessarily binary (unlike here).

Theorem 1 ([2]). *Let σ be a unimodular endomorphism of F_d. Let $\boldsymbol{\alpha} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}$ and $\rho \in \mathbb{R}$. If $M_\sigma^\top \boldsymbol{\alpha} \in \mathbb{R}_+^d$, then the image of the stepped plane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ by the dual map $E_1^*(\sigma)$ is the stepped plane $\mathcal{P}_{M_\sigma^\top \boldsymbol{\alpha},\rho}$. Otherwise, this image is not binary.*

In particular, note that if σ is a substitution, then one has $M_\sigma^\top \boldsymbol{\alpha} \in \mathbb{R}_+^d$ for any $\boldsymbol{\alpha} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}$. Thus, the image of a stepped plane by the dual map of a substitution is always a stepped plane. Let us stress that the the image by $E_1^*(\sigma)$ of a face is a weighted sum of faces: this is only thanks to cancellations between images of different faces that the image by $E_1^*(\sigma)$ of a stepped plane turns out to be, finally, a stepped plane.

2 Brun expansions of real vectors

Definition 4. *The* Brun map T *is the map from* $[0,1]^d \backslash \{\mathbf{0}\}$ *to* $[0,1]^d$ *defined on* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ *by:*

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where $i = \min\{j \mid \alpha_j = ||\boldsymbol{\alpha}||_\infty\}$. *Then, the* Brun expansion *of a vector* $\boldsymbol{\alpha} \in [0,1]^d$ *is the sequence* $(a_n, i_n)_{n \geq 0}$ *of* $\mathbb{N}^* \times \{1, \ldots, d\}$ *defined, while* $T^n(\boldsymbol{\alpha}) \neq \mathbf{0}$, *by:*

$$a_n = \left\lfloor ||T^n(\boldsymbol{\alpha})||_\infty^{-1} \right\rfloor \quad \textit{and} \quad i_n = \min\{j \mid \langle T^n(\boldsymbol{\alpha}) | \boldsymbol{e}_j \rangle = ||T^n(\boldsymbol{\alpha})||_\infty\}.$$

Let us stress that, in the $d = 1$ case, the Brun map T is nothing but the classic Gauss map, and if $(a_n, i_n)_{n \geq 0}$ is the Brun expansion of $\boldsymbol{\alpha} \in [0,1]$, then $(a_n)_n$ is the continued fraction expansion of $\boldsymbol{\alpha}$, while, for all n, $i_n = 1$.

Example 3. The Brun expansion of $(3/8, 5/12)$ is $(2,2), (1,1), (2,2), (4,1), (1,2)$.

Let us mention that, as in the case of continued fractions, it turns out that a vector has a finite Brun expansion if and only if it has only rational entries. Let us now give a matrix viewpoint of the Brun map T. For $(a, i) \in \mathbb{N} \times \{1, \ldots, d\}$, one introduces the following $(d+1) \times (d+1)$ symmetric matrix:

$$B_{a,i} = \begin{pmatrix} a & & 1 & \\ & I_{i-1} & & \\ 1 & & 0 & \\ & & & I_{d-i} \end{pmatrix}, \qquad (1)$$

where I_p stands for the $p \times p$ identity matrix. Note that the determinant of $B_{a,i}$ is equal to -1. Then, consider a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in [0,1]^d \backslash \{\mathbf{0}\}$. A simple computation shows that, with $i = \min\{j \mid \alpha_j = ||\boldsymbol{\alpha}||_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$, one has:

$$(1, \boldsymbol{\alpha}) = ||\boldsymbol{\alpha}||_\infty B_{a,i}(1, T(\boldsymbol{\alpha})), \tag{2}$$

where, for any vector $\boldsymbol{u}$, $(1, \boldsymbol{u})$ stands for the vector obtained by adding to $\boldsymbol{u}$ a first entry equal to 1. Note that since $B_{a,i}$ is invertible, one can rewrite the previous equation as follows:

$$(1, T(\boldsymbol{\alpha})) = ||\boldsymbol{\alpha}||_\infty^{-1} B_{a,i}^{-1}(1, \boldsymbol{\alpha}). \tag{3}$$

To conclude this section, let us show that this matrix viewpoint allows to connect Brun expansions with the stepped planes and dual maps introduced in the previous section. Let us introduce *Brun substitutions*:

Definition 5. *Let $a \in \mathbb{N}^*$ and $i \in \{1, \dots, d\}$. The* Brun substitution *$\beta_{a,i}$ is the endomorphism of F_{d+1} defined by:*

$$\beta_{a,i}(1) = 1^a \cdot (i+1), \quad \beta_{a,i}(i+1) = 1, \quad \beta_{a,i}(j \notin \{1, i+1\}) = j.$$

One checks that the incidence matrix of $\beta_{a,i}$ is the matrix $B_{a,i}$ previously defined. Thus, $\beta_{a,i}$ is unimodular. Note also that $\beta_{a,i}$ is invertible since one computes:

$$\beta_{a,i}^{-1}(1) = (i+1), \quad \beta_{a,i}^{-1}(i+1) = (i+1)^{-a} \cdot 1, \quad \beta_{a,i}^{-1}(j \notin \{1, i+1\}) = j.$$

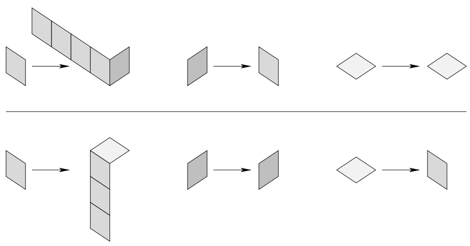

Fig. 4. Action on faces of the dual maps $E_1^*(\beta_{4,1})$ (top) and $E_1^*(\beta_{3,2})$ (bottom).

One then can consider dual maps of Brun substitutions (see Fig. 4). One deduces from Th. 1 that Eq. (2) and (3) respectively yield[1] :

$$E_1^*(\beta_{a,i})(\mathcal{P}_{||\boldsymbol{\alpha}||_\infty(1,T(\boldsymbol{\alpha})),\rho}) = \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}, \tag{4}$$

$$\mathcal{P}_{||\boldsymbol{\alpha}||_\infty(1,T(\boldsymbol{\alpha})),\rho} = E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}). \tag{5}$$

[1] note that $B_{a,i}^\top = B_{a,i}$

3 Generation of stepped planes

This section shows how dual maps and Brun expansions can be combined to easily generate *rational stepped planes*, that is, stepped planes whose normal vectors have rational entries. Indeed, one proves:

Theorem 2. *Let* $\boldsymbol{\alpha} \in [0,1]^d \cap \mathbb{Q}^d$ *with the finite Brun expansion* $(a_n, i_n)_{0 \leq n \leq N}$ *and* $\rho \in \mathbb{R}$*. Let* $\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho}$ *be the binary function defined by:*

$$\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho} = E_1^*(\beta_{a_0,i_0}) \circ \ldots \circ E_1^*(\beta_{a_N,i_N})(\lfloor \rho \rfloor \boldsymbol{e}_1, 1^*),$$

and $L_{(1,\boldsymbol{\alpha}),\rho}$ *be the lattice of rank* d *of* $\mathbb{Z}^{d+1}$ *defined by:*

$$L_{(1,\boldsymbol{\alpha}),\rho} = B_{a_0,i_0}^{-1} \ldots B_{a_N,i_N}^{-1} \sum_{k=2}^{d+1} \mathbb{Z}\boldsymbol{e}_k.$$

Then, the geometrical interpretation of the stepped plane $\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}$ *is the union of all the translations along* $L_{(1,\boldsymbol{\alpha}),\rho}$ *of the geometrical interpretation of* $\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho}$.

Example 4. Fig. 5 shows the generation of the binary function $\mathcal{D}_{(1,3/8,5/12),0}$ by the dual maps of the Brun substitutions associated with the Brun expansion of the vector $(3/8, 5/12)$ (recall Ex. 3). One also computes:

$$L_{(1,3/8,5/12),0} = \mathbb{Z}(\boldsymbol{e}_1 + 4\boldsymbol{e}_2 - 6\boldsymbol{e}_3) + \mathbb{Z}(2\boldsymbol{e}_1 - 2\boldsymbol{e}_2 - 3\boldsymbol{e}_3).$$

Thus, according to Th. 2, the geometrical interpretation of the rational stepped plane $\mathcal{P}_{(1,3/8,5/12),0} = \mathcal{P}_{(24,9,10),0}$ (see Fig. 2) is the union of all the translations along $L_{(1,3/8,5/12),0}$ of the geometrical interpretation of $\mathcal{D}_{(1,3/8,5/12),0}$.

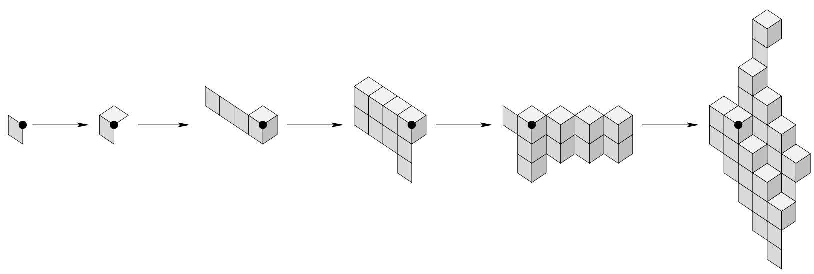

Fig. 5. Generation of $\mathcal{D}_{(1,3/8,5/12),0}$ by applications of the dual maps $E_1^*(\beta_{1,2})$, $E_1^*(\beta_{4,1})$, $E_1^*(\beta_{2,2})$, $E_1^*(\beta_{1,1})$ and $E_1^*(\beta_{2,2})$ (from left ro right – highlighted origin). According to Th. 2, the stepped plane $\mathcal{P}_{(1,3/8,5/12),0}$ can be generated by translating $\mathcal{D}_{(1,3/8,5/12),0}$.

To conclude this section, let us mention that one can show that the binary function $\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho}$ has minimal size among the binary functions which allow to generate the stepped plane $\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}$ by translations along a lattice.

4 Brun expansions of stepped planes

Definition 6. *An (i,j)-run of a stepped plane $\mathcal{P}$ is a binary function less or equal to $\mathcal{P}$, maximal, and which can be written as follows:*

$$\sum_{k \in I} (\boldsymbol{x} + k\boldsymbol{e}_j, i^*),$$

where $\boldsymbol{x} \in \mathbb{Z}^d$ and I is an interval of $\mathbb{Z}$.

In other words, an (i,j)-run of a stepped plane $\mathcal{P}$ is a non-empty sequence of contiguous faces of type i, aligned with the direction $\boldsymbol{e}_j$, whose geometric interpretation is included in the one of $\mathcal{P}$. For example, the stepped plane depicted on Fig. 2 has $(1,2)$-runs and $(1,3)$-runs of size 2 or 3, and $(3,2)$-runs of size 1 or 2. The infimum and the supremum of the sizes of the (i,j)-runs of $\mathcal{P}$ are respectively denoted by $a^-_{i,j}(\mathcal{P})$ and $a^+_{i,j}(\mathcal{P})$. The following proposition shows that runs contain information about the normal vector of a stepped plane:

Proposition 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_+ \backslash \{0\}$ and $\rho \in \mathbb{R}$. Then, for $\alpha_j \neq 0$:*

$$a^-_{i,j}(\mathcal{P}_{\boldsymbol{\alpha},\rho}) = \max(\lfloor \alpha_i/\alpha_j \rfloor, 1) \qquad \textit{and} \qquad a^+_{i,j}(\mathcal{P}_{\boldsymbol{\alpha},\rho}) = \max(\lceil \alpha_i/\alpha_j \rceil, 1),$$

where the floor and the ceiling of $x \in \mathbb{R}$ are respectively denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$.

In particular, let us show that runs contain enough information to compute Brun expansions of normal vectors of so-called *expandable* stepped planes:

Definition 7. *A stepped plane $\mathcal{P} \in \mathfrak{P}_{d+1}$ is said to be* expandable *if one has:*

$$\max_{1 \le i \le d} a^+_{i+1,1}(\mathcal{P}) = 1 \quad \textit{and} \quad \min_{1 \le i \le d} a^-_{1,i+1}(\mathcal{P}) < \infty.$$

In this case, we define:

$$i(\mathcal{P}) = \min_{1 \le i \le d} \{ i \mid \max_{1 \le j \le d} a^+_{j+1,i+1}(\mathcal{P}) \le 1 \} \quad \textit{and} \quad a(\mathcal{P}) = a^-_{1,i(\mathcal{P})+1}(\mathcal{P}).$$

One deduces from Prop. 1 that a stepped plane is expandable if and only if its normal vector is of the form $(1, \boldsymbol{\alpha})$, with $\boldsymbol{\alpha} \in [0,1]^d \backslash \{0\}$. Then, one has:

$$i(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}) = \min\{ i \mid \alpha_i = ||\boldsymbol{\alpha}||_\infty \} \quad \text{and} \quad a(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}) = \lfloor ||\boldsymbol{\alpha}||^{-1}_\infty \rfloor. \tag{6}$$

This leads to the following definition:

Definition 8. *Let $\tilde{T}$ be the map defined over expandable stepped planes by:*

$$\tilde{T}(\mathcal{P}) = E^*_1(\beta^{-1}_{a(\mathcal{P}),i(\mathcal{P})})(\mathcal{P}).$$

In particular, $\tilde{T}$ has values in $\mathfrak{P}_{d+1}$. More precisely, Eq. (4) yields:

$$\tilde{T}(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}) = \mathcal{P}_{(1,T(\boldsymbol{\alpha})),\rho}. \tag{7}$$

Thus, the Brun expansion of a vector $\boldsymbol{\alpha}$ can be computed on a stepped plane $\mathcal{P}$ of normal vector $(1, \boldsymbol{\alpha})$, since it is nothing but the sequence $(a(\tilde{T}^n(\mathcal{P})), i(\tilde{T}^n(\mathcal{P})))_n$.

5 Brun expansions of binary functions

Note that the notion of runs can be naturally extended from stepped planes to binary functions. More precisely (see also Fig. 6):

Definition 9. *An (i,j)-run of a binary function $\mathcal{B}$ is a binary function less or equal to $\mathcal{B}$, maximal, and which can be written as follows:*

$$\sum_{k\in I}(\boldsymbol{x}+k\boldsymbol{e}_j, i^*),$$

where $\boldsymbol{x}\in\mathbb{Z}^d$ and I is an interval of $\mathbb{Z}$. Such an (i,j)-run is said to be right-closed *if I has a maximum d such that $\mathcal{B}(\boldsymbol{x}+d\boldsymbol{e}_j, j^*)=1$, and* left-closed *if I has a minimum g such that $\mathcal{B}(\boldsymbol{x}+(g-1)\boldsymbol{e}_j+\boldsymbol{e}_i, j^*)=1$.*

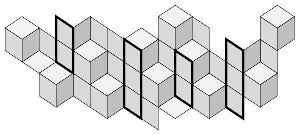

Fig. 6. This binary function has every type of $(1,3)$-runs: left-closed, right-closed, closed and open (framed runs, from left to right). It is moreover recognizable, with $(a,i)=(2,2)$ (see definition below).

One still denotes respectively by $a^-_{i,j}(\mathcal{B})$ and $a^+_{i,j}(\mathcal{B})$ the infimum and the supremum of the sizes of the (i,j)-runs of a binary function $\mathcal{B}$. Thus, runs can be used over binary functions for grabing information, although interpreting this information is not so easy, since the notion of normal vector generally does not make sense for a binary function. However, let us show that this allows to naturally extend the definition of $\tilde{T}$ over so-called *recognizable* binary functions:

Definition 10. *Let $\mathcal{B}\in\mathfrak{B}_{d+1}$ be a binary function. Consider the set:*

$$\left\{i\in\{1,\ldots,d\}\mid a^+_{1,i+1}(\mathcal{B})\geq 2 \text{ and } \min_{1\leq j\leq d} a^+_{i+1,j+1}(\mathcal{B})\geq 2\right\}.$$

If it is not empty, let us denote by $i(\mathcal{B})$ its minimum. Moreover, let us define $a(\mathcal{B})=a^+_{1,i(\mathcal{B})+1}(\mathcal{B})-1$. If $a(\mathcal{B})$ turns out to be the size of a smallest closed $(1,i(\mathcal{B})+1)$-run of $\mathcal{B}$, then $\mathcal{B}$ is said to be recognizable.

It is not hard to deduce from Prop. 1 that if a recognizable binary function $\mathcal{B}$ satisfies, for $\boldsymbol{\alpha}\in\mathbb{R}^d_+$ and $\rho\in\mathbb{R}$, $\mathcal{B}\leq\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}$, then recognizability conditions ensure that $\boldsymbol{\alpha}\in[0,1]^d$, $i(\mathcal{B})=i(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho})$ and $a(\mathcal{B})=a(\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho})$. Thus, the formula defining $\tilde{T}$ over stepped planes can still be used to define $\tilde{T}$ over recognizable binary functions (recall Def. 8). This leads to define the Brun expansion of a recognizable binary function $\mathcal{B}$ as the sequence $(a(\tilde{T}^n(\mathcal{B})), i(\tilde{T}^n(\mathcal{B})))_n$, for n such that $\tilde{T}^n(\mathcal{B})$ is a recognizable binary function.

6 Recognition of stepped planes

We are here interested in, given a binary function $\mathcal{B} \in \mathfrak{B}_{d+1}$, deciding whether the following subset of $\mathbb{R}^{d+1}$ is empty or not:

$$P(\mathcal{B}) = \{(\boldsymbol{\alpha}, \rho) \in [0,1]^d \backslash \{\mathbf{0}\} \times \mathbb{R} \mid \mathcal{B} \leq \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}\}.$$

Note that it is not hard to check that this subset is a convex polytope. The idee is that if the map $\tilde{T}$ previously defined would satisfy, for any $\mathcal{B} \in \mathfrak{B}_{d+1}$:

$$0 \leq \mathcal{B} \leq \mathcal{P} \;\Leftrightarrow\; 0 \leq \tilde{T}(\mathcal{B}) \leq \tilde{T}(\mathcal{P}), \tag{8}$$

then, $P(\mathcal{B})$ would be not empty if and only if computing the sequence $(\tilde{T}^n(\mathcal{B}))_{n\geq 0}$ would lead to a binary function of the form $\sum_{\boldsymbol{x}\in X}(\boldsymbol{x}, 1^*)$, with the vectors of X having all the same first entries (such a binary function is easily recognizable).

However, Eq. (8) does not always hold. Indeed, recall that $\tilde{T}$ is defined only over stepped planes and recognizable binary functions. Note that this problem generally appears only for small binary functions, because their runs do not contain enough information. The following problem seems more tedious: the image by $\tilde{T}$ of a recognizable binary function less or equal to a stepped plane $\mathcal{P}$ is nether necessarily less or equal to $\tilde{T}(\mathcal{P})$, nor even always a binary function. Let us first consider this problem. We introduce three rules acting over binary functions (see Fig. 7, and also Fig. 8, left):

Definition 11. *Let $a \in \mathbb{N}^*$ and $i \in \{1, \ldots, d\}$. The rule $\phi_{a,i}$ left-extends any right-closed and left-open $(1, i+1)$-run into a run of size a; the rule $\psi_{a,i}$ right-closes any right-open $(1, i+1)$-run of size greater than a; the rule χ_i removes any left-closed and right-open $(1, i+1)$-run.*

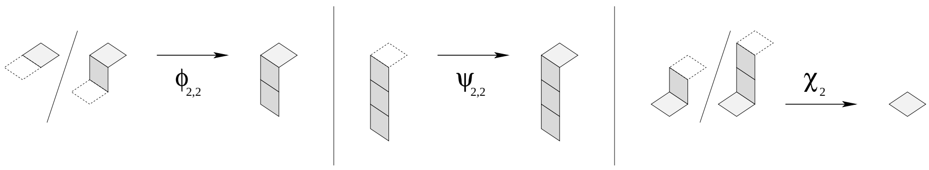

Fig. 7. The rules $\phi_{2,2}$, $\psi_{2,2}$ and χ_2 (dashed edges represent missing faces).

The following theorem then shows that one can replace any recognizable binary function $\mathcal{B}$ by a binary function $\tilde{\mathcal{B}}$, which turns out to be suitable under an additional hypothesis (it shall not have open run):

Proposition 2. *Let $\mathcal{B} \in \mathfrak{B}_{d+1}$ be a recognizable binary function and $\tilde{\mathcal{B}}$ be the binary function obtained by successively applying $\phi_{a(\mathcal{B}),i(\mathcal{B})}$, $\psi_{a(\mathcal{B}),i(\mathcal{B})}$ and $\chi_{i(\mathcal{B})}$.*

Then, for any stepped plane $\mathcal{P} \in \mathfrak{P}_{d+1}$*, one has* $\mathcal{B} \leq \mathcal{P}$ *if and only if* $\tilde{\mathcal{B}} \leq \mathcal{P}$*. Moreover, if* $\tilde{\mathcal{B}}$ *does not have open* $(1, i(\mathcal{B})+1)$*-run, then one has :*

$$0 \leq \tilde{\mathcal{B}} \leq \mathcal{P} \;\Leftrightarrow\; 0 \leq \tilde{T}(\tilde{\mathcal{B}}) \leq \tilde{T}(\mathcal{P}).$$

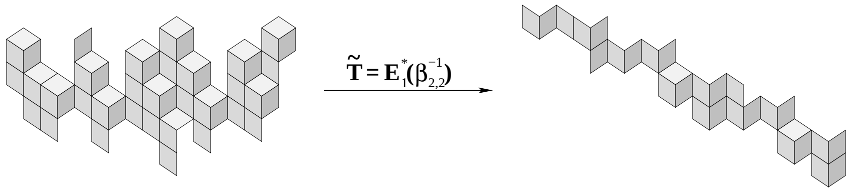

Fig. 8. The recognizable binary function $\mathcal{B}$ of Fig. 6 is transformed by applying the rules of Fig. 7 into a binary function $\tilde{\mathcal{B}}$ (left) such that $0 \leq \mathcal{B} \leq \mathcal{P} \Leftrightarrow 0 \leq \tilde{\mathcal{B}} \leq \mathcal{P}$. Here, since $\tilde{\mathcal{B}}$ does not have open $(1,3)$-run, its image by $\tilde{T}$ (right) is such that, for any stepped plane $\mathcal{P}$, one has: $0 \leq \tilde{\mathcal{B}} \leq \mathcal{P} \Leftrightarrow 0 \leq \tilde{T}(\tilde{\mathcal{B}}) \leq \mathcal{P}$.

Thus, it remains two problems: recognizability does not always hold, and $\tilde{\mathcal{B}}$ can have open runs which make troubles. However, again, let us stress that unrecognizable binary functions as well as remaining open runs are often small. Hence, it could be worth considering a hybrid algorithm. Given a recognizable binary function $\mathcal{B}$, we compute $\tilde{\mathcal{B}}$, remove problematic open runs and apply the map $\tilde{T}$. We iterate this up to obtain an unrecognizable binary function. Then, we use an other existing algorithm to recognize this binary function and also, finally, to refine the recognition by considering the previously removed open runs. More precisely, consider the following algorithm, where `XReco` is an algorithm which computes the set $P(\mathcal{B})$ and $B'_{a,i}$ is the $(d+2)\times(d+2)$ block matrix whose first block is $B_{a,i}$ and the second the 1×1 identity matrix:

`HybridBrunReco`$(\mathcal{B})$

$n \leftarrow 0$;
$\mathcal{B}_0 \leftarrow \mathcal{B}$;
while $\mathcal{B}_n$ is recognizable **do**
    $(a_n, i_n) \leftarrow (a(\mathcal{B}_n), i(\mathcal{B}_n))$;
    compute $\tilde{\mathcal{B}}_n$;
    $L_n \leftarrow$ open runs of $\tilde{\mathcal{B}}_n$;
    $\mathcal{B}_{n+1} \leftarrow E_1^*(\beta_{a_n,i_n}^{-1})(\tilde{\mathcal{B}}_n - L_n)$;
    $n \leftarrow n+1$;
end while;
$P_n \leftarrow$ `XReco`$(\mathcal{B}_n)$;
for $k = n-1$ **downto** k=0 **do**
    $P_k \leftarrow B'_{a_k,i_k} P_{k+1}$;
    $P_k \leftarrow P_k \cap$ `XReco`(L_k);
end for;
return P_0;

One shows:

Theorem 3. *The algorithm* `HybridBrunReco` *with a binary function* $\mathcal{B}$ *as input returns the set* $P(\mathcal{B})$ *in finite time.*

To conclude, let us discuss the computational cost of the above algorithm. Let us first focus on the "Brun" stage of the algorithm, that is, on lines `3–9`. One can shows that each step of this stage can be performed in time $\mathcal{O}(|\mathcal{B}_n|)$ and that $|\mathcal{B}_n|$ strictly decreases. Thus, the whole stage can be performed in quadratic time (in the size of $\mathcal{B}$). However, let us stress that $(|\mathcal{B}_n|)_n$ generally decreases with an exponential rate (this is the case, for example, for any stepped plane), so that this stage is expected, in practice, to be performed in near linear time. Let us now consider the "correction" stage of the algorithm, that is, lines `10–14`. Note that the sum of sizes of inputs of `XReco` is less than $|\mathcal{B}|$. Thus, assuming that `XReco` works in time no more than quadratic (such algorithms do exist!), the bound given for the first stage still holds. We also need to compute intersections of convex polytopes. The complexity of such operations is not trivial in high dimensions, but let us stress that the intersection of k convex polytopes of $\mathbb{R}^3$ can be computed in time $\mathcal{O}(m \ln k)$, where m stands for the total size of these polytopes (see [4]). Moreover, let us recall that the first unrecognizable $\mathcal{B}_n$ as well as the sum of sizes of the L_k's are expected to be much smaller than $\mathcal{B}$. In conclusion, theoretical time complexity bounds are probably much bigger than the practical efficiency of this algorithm, and further expriments should be performed to get a better analysis.

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Appendix

Proposition 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d \backslash \{0\}$ and $\rho \in \mathbb{R}$. Then, for $\alpha_j \neq 0$:*

$$a_{i,j}^-(\mathcal{P}_{\boldsymbol{\alpha},\rho}) = \max(\lfloor \alpha_i/\alpha_j \rfloor, 1) \qquad \textit{and} \qquad a_{i,j}^+(\mathcal{P}_{\boldsymbol{\alpha},\rho}) = \max(\lceil \alpha_i/\alpha_j \rceil, 1),$$

where the floor and the ceiling of $x \in \mathbb{R}$ are respectively denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$.

Proof. Let $\boldsymbol{x} \in \mathbb{Z}^d$ and $I \subset \mathbb{Z}$ such that the following binary function is an (i,j)-run of $\mathcal{P}_{\boldsymbol{\alpha},\rho}$:

$$\mathcal{K} = \sum_{k \in I} (\boldsymbol{x} + k\boldsymbol{e}_j, i^*).$$

Assume that I contains an interval $[a,b]$, of length $b-a+1$. Then, one has:

$$\mathcal{P}_{\boldsymbol{\alpha},\rho}(\boldsymbol{x} + a\boldsymbol{e}_j, i) = 1 \;\Rightarrow\; \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + a\alpha_j < \rho \leq \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + a\alpha_j + \alpha_i,$$

$$\mathcal{P}_{\boldsymbol{\alpha},\rho}(\boldsymbol{x} + b\boldsymbol{e}_j, i) = 1 \;\Rightarrow\; \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + b\alpha_j < \rho \leq \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + b\alpha_j + \alpha_i.$$

One deduces:

$$(b-a)\alpha_j < \rho - \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle \leq \alpha_i,$$

that is, for $\alpha_j \neq 0$:

$$b - a + 1 < \frac{\alpha_i}{\alpha_j} + 1.$$

This thus gives an upper bounds of the length of I. If I is not empty, let us write $I = [a,b]$. Then, one has:

$$\mathcal{P}_{\boldsymbol{\alpha},\rho}(\boldsymbol{x} + a\boldsymbol{e}_j, i) = 1 \;\Rightarrow\; \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + (a-1)\alpha_j < \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + a\alpha_j < \rho,$$

and one deduces:

$$\mathcal{P}_{\boldsymbol{\alpha},\rho}(\boldsymbol{x} + (a-1)\boldsymbol{e}_j, i) = 0 \;\Rightarrow\; \rho > \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + (a-1)\alpha_j + \alpha_i.$$

Similarly, one shows:

$$\rho \leq \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle + (b+1)\alpha_j + \alpha_i.$$

Finally, one has:

$$(a-1)\alpha_j + \alpha_i < \rho - \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle \leq (b+1)\alpha_j,$$

that is, for $\alpha_j \neq 0$:

$$b - a + 1 > \frac{\alpha_i}{\alpha_j} - 1.$$

This thus gives a lower bounds of the length of I. Moreove, note that if I is empty, then one has:

$$\forall \boldsymbol{x} \in \mathbb{Z}^d,\; \rho - \langle \boldsymbol{x}|\boldsymbol{\alpha}\rangle \notin]0, \alpha_i].$$

It is not hard to see that this yields that $\alpha_j > \alpha_i$, that is, $\mathrm{c}\alpha_i/\alpha_j - 1 < 0$. The above lower bound thus still holds. In conclusion, we shown:

$$\frac{\alpha_i}{\alpha_j} - 1 < a_{i,j}^-(\mathcal{P}_{\boldsymbol{\alpha},\rho}) \leq a_{i,j}^+(\mathcal{P}_{\boldsymbol{\alpha},\rho}) < \frac{\alpha_i}{\alpha_j} + 1.$$

The result follows (recall that, by definition, runs are non-empty). □

Theorem 2. *Let $\boldsymbol{\alpha} \in [0,1]^d \cap \mathbb{Q}^d$ with the finite Brun expansion $(a_n, i_n)_{0 \leq n \leq N}$ and $\rho \in \mathbb{R}$. Let $\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho}$ be the binary function defined by:*

$$\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho} = E_1^*(\beta_{a_0,i_0}) \circ \ldots \circ E_1^*(\beta_{a_N,i_N})(\lfloor \rho \rfloor \boldsymbol{e}_1, 1^*),$$

and $L_{(1,\boldsymbol{\alpha}),\rho}$ be the lattice of rank d of $\mathbb{Z}^{d+1}$ defined by:

$$L_{(1,\boldsymbol{\alpha}),\rho} = B_{a_0,i_0}^{-1} \ldots B_{a_N,i_N}^{-1} \sum_{k=2}^{d+1} \mathbb{Z}\boldsymbol{e}_k.$$

Then, the geometrical interpretation of the stepped plane $\mathcal{P}_{(1,\boldsymbol{\alpha}),\rho}$ is the union of all the translations along $L_{(1,\boldsymbol{\alpha}),\rho}$ of the geometrical interpretation of $\mathcal{D}_{(1,\boldsymbol{\alpha}),\rho}$.

Proof. On the other hand, one easily sees that translations of the geometrical interpretation of $(\lfloor \rho \rfloor \boldsymbol{e}_1, \boldsymbol{e}_1^*)$ along the lattice $\mathbb{Z}\boldsymbol{e}_2 + \ldots + \mathbb{Z}\boldsymbol{e}_{d+1}$ yield the geometrical interpretation of the stepped plane $\mathcal{P}_{(1,\boldsymbol{0}),\rho}$. On the other hand, if $\mathcal{D}$ is a binary function such that the translations along a lattice L of its geometrical interpretation yield the geometrical interpretation of a stepped plane $\mathcal{P}$, then, for any unimodular substitution σ, Th.1 yields that $E_1^*(\sigma)(\mathcal{D})$ is a binary function whose geometrical interpretation, translated along the lattice $M_\sigma^{-1}L$, yields the geometrical interpretation of the stepped plane $E_1^*(\sigma)(\mathcal{P})$. The result follows by considering the unimodular substitution $\sigma = \beta_{a_N,i_N} \circ \ldots \circ \beta_{a_0,i_0}$. □

Proposition 2. *Let $\mathcal{B}$ be a recognizable binary function of $\mathfrak{B}_{d+1}$ and $\tilde{\mathcal{B}}$ the binary function obtained by successively applying $\phi_{a(\mathcal{B}),i(\mathcal{B})}$, $\psi_{a(\mathcal{B}),i(\mathcal{B})}$ and $\chi_{i(\mathcal{B})}$. Then, for any stepped plane $\mathcal{P} \in \mathfrak{P}_{d+1}$, one has $\mathcal{B} \leq \mathcal{P}$ if and only if $\tilde{\mathcal{B}} \leq \mathcal{P}$. Moreover, if $\tilde{\mathcal{B}}$ does not have open $(1, i(\mathcal{B})+1)$-run, then one has :*

$$0 \leq \tilde{\mathcal{B}} \leq \mathcal{P} \;\Leftrightarrow\; 0 \leq \tilde{T}(\tilde{\mathcal{B}}) \leq \tilde{T}(\mathcal{P}).$$

Proof. Let $\mathcal{B}$ be a recognizable binary function. Assume that there is a stepped plane $\mathcal{P}$ such that $\mathcal{B} \leq \mathcal{P}$. Thus, any left-open and right-closed $(1, i+1)$-run of $\mathcal{B}$ is less or equal to a closed $(1, i+1)$-run of $\mathcal{P}$. Since such a run has length at least $a(\mathcal{P}) = a(\mathcal{B})$, this yields that $\phi_{a(\mathcal{B}),i(\mathcal{B})}(\mathcal{B})$ is still less or equal to $\mathcal{P}$. Conversely, if $\phi_{a(\mathcal{B}),i(\mathcal{B})}(\mathcal{B})$ is less or equal to $\mathcal{P}$, then $\mathcal{B}$ also since $\mathcal{B} \leq \phi_{a(\mathcal{B}),i(\mathcal{B})}(\mathcal{B})$. This shows that $\mathcal{B} \leq \mathcal{P}$ if and only if $\phi_{a(\mathcal{B}),i(\mathcal{B})}(\mathcal{B}) \leq \mathcal{P}$. One similarly proceeds for $\psi_{a,i}$ et χ_i, so that, finally, $\mathcal{B} \leq \mathcal{P}$ if and only if $\tilde{\mathcal{B}} \leq \mathcal{P}$.

Let us now assume that $\tilde{\mathcal{B}}$ does not have open $(1, i(\mathcal{B})+1)$-run. It is not hard to see that $\tilde{\mathcal{B}}$ can be written as the image by $E_1^*(\beta_{a,i})$ of a binary function, say $\tilde{\mathcal{B}}'$ (actually, this is what led the definition of rules $\phi_{a,i}$, $\psi_{a,i}$ and χ_i). It is also easily seen that $\tilde{\mathcal{B}}$ is, as $\mathcal{B}$, recognizable. In particular, $\tilde{T}(\tilde{\mathcal{B}}) = E_1^*(\beta_{a(\mathcal{B}),i(\mathcal{B})}^{-1})(\tilde{\mathcal{B}})$ is a binary function. Now, assume that there is a stepped plane $\mathcal{P}$ such that $\tilde{\mathcal{B}} \leq \mathcal{P}$ and $\tilde{T}(\mathcal{P}) \geq 0$. Let us introduce the binary function $\mathcal{C} = \mathcal{P} - \tilde{\mathcal{B}}$. The fact that both $\mathcal{P}$ and $\tilde{\mathcal{B}}$ are images by $E_1^*(\beta_{a,i})$ of binary functions yields that it is also the case for $\mathcal{C}$. So, one has: $\mathcal{C} = E_1^*(\beta_{a,i})(\mathcal{C}')$, for some binary function $\mathcal{C}'$. Hence, by applying $\tilde{T} = E_1^*(\beta^{-1}_{a(\mathcal{P}),i(\mathcal{P})})$ on $\mathcal{P}$, one obtains:

$$\tilde{T}(\mathcal{P}) = \tilde{T}(\tilde{\mathcal{B}}) + \tilde{T}(\mathcal{C}) = \tilde{T}(\tilde{\mathcal{B}}) + \mathcal{C}' \geq \tilde{T}(\tilde{\mathcal{B}}) = \tilde{\mathcal{B}}' \geq 0.$$

Thus, we shown that one has, for any stepped plane $\mathcal{P}$:

$$0 \leq \tilde{\mathcal{B}} \leq \mathcal{P} \;\Rightarrow\; 0 \leq \tilde{T}(\tilde{\mathcal{B}}) \leq \tilde{T}(\mathcal{P}).$$

Conversely, assume that $0 \leq \tilde{T}(\tilde{\mathcal{B}}) leq \tilde{T}(\mathcal{P})$ for some stepped plane $\mathcal{P}$. It is easily seen that the subset of positive functions of $\mathfrak{F}$ is stable under dual maps of substitutions. Thus, since $\beta_{a(\mathcal{P}),i(\mathcal{P})}$ is a subsitution, applying $E_1^*(\beta_{a(\mathcal{P}),i(\mathcal{P})})$ yields $0 \leq \tilde{\mathcal{B}} \leq \mathcal{P}$. This concludes the proof. □

Theorem 3. *The algorithm* `HybridBrunReco` *with a binary function* $\mathcal{B}$ *as input returns the set* $P(\mathcal{B})$ *in finite time.*

Proof. Let us first shows that the algorithm finishes, by proving that $|\mathcal{B}_{n+1}|$ is less than $|\mathcal{B}_n|$ (so that, eventually, $\mathcal{B}_n$ is not a recognizable binary function). Let us respectively denote $\boldsymbol{f}(\mathcal{B}_n)$, $\boldsymbol{f}(\tilde{\mathcal{B}}_n - L_n)$ and $\boldsymbol{f}(\mathcal{B}_{n+1})$ by $(x_1, \ldots, x_{d+1})$, $(y_1, \ldots, y_{d+1})$ and $(z_1, \ldots, z_{d+1})$, where $\boldsymbol{f}$ maps any binary function of finite size onto the integer vector whose i-th entry counts the number of faces of type i in this binary function. One checks that the action of dual maps yields:

$$\begin{cases} z_1 = y_{i_n+1}, \\ z_{i_n+1} = y_1 - a_n y_{i_n+1}, \\ z_j = y_j. \end{cases}$$

We also easily deduce from the definition of $\tilde{\mathcal{B}}$:

$$\begin{cases} y_1 = x_1 + a x_{i_n+1} - x_1', \\ y_{i_n+1} = x_{i_n+1} + \frac{1}{a_n+1} x_1'', \\ y_j = x_j, \end{cases}$$

where x_1' (resp. x_1'') is the sum of the sizes of the $(1, i_n+1)$-runs extended by ϕ_{a_n,i_n} (resp. ψ_{a_n,i_n}). One then computes:

$$|\mathcal{B}_{n+1}| = \sum_{j=1}^{d+1} z_j = \sum_{j=1}^{d+1} x_j + \frac{1-a_n}{a_n+1} x_1'' - x_1' = |\mathcal{B}_n| + \frac{1-a_n}{a_n+1} x_1'' - x_1'.$$

Since $a_n \geq 1$, one has $|\mathcal{B}_{n+1}| \leq |\mathcal{B}_n|$, with the inegality being strict except if $x_1' = 0$. But $x_1' = 0$ would mean that there is no right-closed $(1, i_n+1)$-run, and thus that $\mathcal{B}_n$ would not be recognizable. Thus, $x_1' \neq 0$, and one has $|\mathcal{B}_{n+1}| < |\mathcal{B}_n|$.

Let us now prove the correction of the algorithm. We proceed by induction on the number of steps of the “Brun” stage, that is, lines 3–9. If $n = 0$, this follows from the (assumed) correction of `XReco`. Assume that the result holds for n. One checks:

$$\begin{aligned} ((1,\boldsymbol{\alpha}),\rho)\in P(\mathcal{B}_0) &\Leftrightarrow 0\leq \mathcal{B}_0 \leq \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho} \\ &\Leftrightarrow 0\leq \tilde{\mathcal{B}_0} \leq \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho} \\ &\Leftrightarrow 0\leq \tilde{\mathcal{B}_0} - L_0 \leq \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho} \quad \text{et} \quad 0\leq L_0 \leq \mathcal{P}_{(1,\boldsymbol{\alpha}),\rho} \\ &\Leftrightarrow 0\leq \mathcal{B}_1 \leq \mathcal{P}_{B^{-1}_{a_0,i_0}(1,\boldsymbol{\alpha}),\rho} \quad \text{et} \quad ((1,\boldsymbol{\alpha}),\rho)\in \texttt{XReco}(L_0) \\ &\Leftrightarrow (B^{-1}_{a_0,i_0}(1,\boldsymbol{\alpha}),\rho)\in P(\mathcal{B}_1) \quad \text{et} \quad ((1,\boldsymbol{\alpha}),\rho)\in \texttt{XReco}(L_0) \end{aligned}$$

Note that this is Prop. 2 which ensures that we can go from the first to the second lines and from the third one to the fourth one (by applying $E_1^*(\beta^{-1}_{a_0,i_0})$). Finally, one has:

$$P(\mathcal{B}_0) = B'_{a_0,i_0} P(\mathcal{B}_1) \cap \texttt{XReco}(L_0).$$

The correction of the algorithm follows by induction. □