On Solutions of a Non linear Wave Equation Derived From Brain Activity Modeling

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Abstract

The paper deals with a boundary value problem governed by a damped non-linear wave equation. It generalizes the equation modelling cerebral activity that has been proposed by Jirsa and Haken [9]. Using a Galerkin approximation scheme, we prove existence of global solutions. In the 1-D case, we show the continuous dependence of solutions on the initial data and derive uniqueness.

Key words: Neural fields, wave equation, Galerkin method, boundary value problem
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1 Introduction

Non-invasive techniques such as functional Magnetic Resonance Imaging, ElectroEncephaloGraphy (EEG) and MagnetoEncephaloGraphy (MEG) provide entry points to human brain dynamics for clinical purposes, as well as the study of human behavior and cognition. Each of these observation technologies provides spatiotemporal information about the ongoing neural activity in the cortex. Unfortunately measures are generally noisy, and it is difficult to identify the equation which governs the dynamics of neural activity. Several physicists [23,16,9,7] have formulated continuous models called neural fields to predict neural activity using brain anatomy.
Jirsa and Haken's model [9], which we will analyze in section 4 of this paper, generalizes the models [23,16] and leads to the following evolution problem that we investigate in this work:

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$$u_{tt} - \alpha\Delta u = a(u,p)u_t + b(u,p,p_t), \quad \text{in } [0,+\infty)\times\Omega \tag{1.1}$$
$$u(0,\cdot) = u_0 \quad u_t(0,\cdot) = u_1. \tag{1.2}$$
$$u = 0, \quad \text{on } [0,+\infty)\times\partial\Omega \tag{1.3}$$

where Ω is an open bounded domain in $\mathbb{R}^n (n \leq 4)$ with sufficiently smooth boundary $\partial\Omega$. The notation u_t (respectively u_{tt}) stands for the first (respectively second) partial derivative of u with respect to time variable t, Δ is the Laplace operator and p is a smooth given function modelling the external stimulus (input). For simplicity, we assume that $\alpha = 1$.
Although equation (1.1) has a simple form, the existence and uniqueness of the solution is (to our knowledge) still an open problem, if we do not assume additional conditions on a, b or u_0, u_1. In addition, the existence of a **global** solution is not always assured (see for example [5,28,25]).
Equation (1.1) belongs to a class of non-linear damped wave equations that have been widely studied by many mathematicians and engineers. Existence and uniqueness under conditions (1.2) and (1.3) has been considered using various methods such as semi-group theory [17,14,27,3] and fixed point methods [10,2]. In [6,4,18,19,26], the authors prove existence and uniqueness of global solutions under conditions (1.2) and (1.3) using Galerkin methods.
Zhou [28] has studied a particular case of equation (1.1), without external input p. The function a is assumed to be constant and b is defined by

$$b(u) = |u|^{m-1}u$$

with condition (1.2) and $\Omega = \mathbb{R}^n$; he proved that, if $1 < m < \frac{n+2}{n}$, the solution blows up in finite time. Similarly, authors in [17,24,28,29,11,12,21,22,5,15,25] discussed the case when the solution blows up in finite time.
The closest equation to the one we investigate here has been studied by Zhijian [26], who considered:

$$\begin{aligned} u_{tt} - \Delta u - \Delta u_t = &\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}) + \beta_i(u_{x_{it}})) \\ &+ F(u, u_t, \nabla u, \nabla u_t) \quad \text{on } [0,\infty)\times\Omega \end{aligned} \tag{1.4}$$

with initial and boundary conditions (1.2) and (1.3). He proved that this problem has a unique classical solution, assuming the following about the initial data:

$$\|\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\sigma_i(u_{x_i}(0)) + \beta_i(v_{x_i}(0))) + F(u(0), v(0), \nabla u(0), \nabla v(0))\|_{H^k} < \delta \tag{1.5}$$

$$\|v(0)\|^2_{H^{k+1}} + \|v(0)\|_{H^k} + \frac{3}{2}\|u(0)\|_{H^k} + < u(0), v(0) >_{H^k} \quad < \gamma \tag{1.6}$$

where δ and γ are two real numbers. In the present section, H^k stands for the usual Sobolev space and $\|\cdot\|_{H^k}$ denotes the H^k-norm. This notation will

be changed in the sequel, where we shall use H^1 and H^2. In addition, it was supposed that

$$|D^\beta F| \leq B(|\lambda_1|^{\gamma_1} + |\lambda_2|^{\gamma_2} + |\tilde{\lambda_1}|^{\gamma_3} + |\tilde{\lambda_2}|^{\gamma_4} + 1) \text{ and } F(0) = 0. \tag{1.7}$$

The RHS of (1.1) is a particular case of the RHS in (1.4): it involves an additional (parameter) function p that we call external input. This function physically describes a signal generated at the brain surface and transmitted to the neocortex. We have to mention here that the term Δu_t on the LHS of equation (1.4) is crucial in the proof estimates: the solution is bounded with respect to the H^2 norm.
Now, the question is: if the initial data does not satisfy (1.5), (1.6) or (1.7) and if no "regularizing" term Δu_t appears, does problem (1.1)–(1.3) admit a global solution? If yes, is the problem mathematically well posed: is the solution unique and continuous with respect to initial data?

In this paper, we prove that under "realistic" assumptions, problem (1.1)–(1.3) has global weak solutions. Moreover, if the problem is a unidimensional one, the solutions depend continuously on the intial data; therefore we get uniqueness. Required assumptions are motivated by the underlying physical cerebral activity model that leads to a particular case of equation (1.1).

The outline of the paper is the following: existence results are stated in Section 2. Uniqueness and stability are discussed in Section 3. Finally, in Section 4, we present and analyze the cerebral activity model and we apply the general result.

2 Existence of weak global solutions

We let $\|\cdot\|$ denote the $L^2(\Omega)$ norm, $\|\cdot\|_\infty$ the L^∞-norm, $\|\cdot\|_1$ the $H^1(\Omega)$ norm, and $(\cdot,\cdot)$ the L^2-inner product.

Definition 2.1 *Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, and $T > 0$ be given. We shall say that u is a weak solution to problem (1.1)–(1.3) if*

$$u \in L^2([0,T), H_0^1(\Omega)), u_t \in L^2([0,T), H^1(\Omega)), u_{tt} \in L^2([0,T), L^2(\Omega))$$

and for almost every $t \geq 0$, $\forall \omega \in H_0^1(\Omega)$

$$(u_{tt}(t), \omega) + (\nabla u(t), \nabla \omega) = (a(u,p)(t)u_t(t) + b(u,p,p_t)(t), \omega) \tag{2.1}$$

and

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 \tag{2.2}$$

From now on, we do not indicate the dependence on t in order to make the paper more readable. First we give an existence result of weak solutions to problem (1.1)–(1.3).

Theorem 2.1 *Assume that for some $T > 0$,*
(A_1) $a \in C^1(\mathbb{R}^2, \mathbb{R})$ *and* $b \in C^1(\mathbb{R}^3, \mathbb{R})$, *where* a, ∇a *and* ∇b *are (uniformly) bounded functions.*
(A_2) $|b(u,p,q)(t,x)| \leq C_1|u(t,x)| + C_2$ *a.e. on* $[0,T) \times \Omega$, *where* C_1 *and* C_2 *are nonnegative constants.*
(A_3) $p \in H^2([0,T), L^\infty(\Omega))$.
(A_4) $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$
Then the problem (1.1)–(1.3) admits (at least) a weak solution u on $[0,T)$ and

$$u \in H^2([0,T), L^2(\Omega)) \cap H^1([0,T), H_0^1(\Omega)).$$

Proof - We need several steps to prove this result. We use the Galerkin method to construct a solution. Let $\{\lambda_k\}_{k=1}^\infty$ be a sequence of eigenvalues of $-\Delta$ in Ω. Let $\{\omega_k\}_{k=1}^\infty \in H_0^1(\Omega) \cap H^2(\Omega)$ be the associated eigenfunctions such that $\{\omega_k\}_{k=1}^\infty$ is a complete orthonormal system of $L^2(\Omega)$. We construct approximate solutions u^n as

$$u^n(t) = \sum_{k=1}^n d^{n,k}(t)\omega_k$$

where the functions $d^{n,k}$ are determined by the system of ordinary differential equations

$$(u_{tt}^n, \omega_k) + (\nabla u^n, \nabla \omega_k) = (a(u^n,p)u_t^n + b(u^n,p,p_t), \omega_k) \quad k = 1, \dots n. \tag{2.3}$$

$$d^{n,k}(0) = u_0^k, \quad d_t^{n,k}(0) = u_1^k \tag{2.4}$$

with $u_0^k, u_1^k \in \mathcal{D}(\Omega)$, such that $(u_0^k, u_1^k) \to (u_0, u_1)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. Here $\mathcal{D}(\Omega)$ denotes the space of $\mathcal{C}^\infty$ functions with compact support in Ω.
Now, (2.3) and (2.4) are equivalent to:

$$d_{tt}^{n,k} + \lambda_k d^{n,k} = f(d^{n,k}, d_t^{n,k}, t) \quad k = 1, \dots n, \tag{2.5}$$

$$d^{n,k}(0) = u_0^k, \quad d_t^{n,k}(0) = u_1^n \tag{2.6}$$

where

$$f(d^{n,k}, d_t^{n,k}, t) := (a(u^n,p)u_t^n + b(u^n,p,p_t), \omega_k) \qquad \text{a.e. } \ \mathrm{t} \geq 0\ . \tag{2.7}$$

For every $n \geq 1$, (2.5), (2.6) is a second-order $n \times n$ Cauchy system of differential equations with continuous nonlinearities. So, it follows from the Cauchy-Peano Theorem [20] that the system has at least one solution $d^{n,k}$ defined on

$[0, T_n]$ for some $T_n > 0$. Moreover, for every k such that $1 \leq k \leq n$, we get $d^{n,k} \in C^2([0, T_n])$.

Lemma 2.1 *The functions (u_n) satisfy*

$$\|u_t^n(t)\| + \|u^n(t)\|_1 \leq M(T)$$

where $M(T)$ only depends on T. In particular, $u^n(t)$ can be extended to $[0, T)$.

Proof - We multiply equation (2.3) by $d_t^{n,k}$ and sum over k=1,..., n. By assumptions A_1 and A_2 we obtain, for every $t \in [0, T)$,

$$\begin{aligned}(u_{tt}^n, u_t^n) + (\nabla u^n, \nabla u_t^n) &= (a(u^n, p)u_t^n + b(u^n, p, p_t), u_t^n) \\ &\leq C(|u_t^n| + C_1|u^n| + C_2, |u_t^n|).\end{aligned}$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\frac{d}{dt}(\|u_t^n(t)\|^2 + \|\nabla u^n(t)\|^2) &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|\|u_t^n(t)\| + \|u_t^n(t)\|) \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2 + \|u_t^n(t)\|^2 + \|u_t^n(t)\|^2 + 1) \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2 + 1) \\ &\leq C(\|u_t^n(t)\|^2 + \|\nabla u^n(t)\|^2 + 1)\end{aligned}$$

by the Poincaré inequality, where C denotes (here and in the sequel) a generic constant. Applying the Gronwall inequality yields

$$\|u_t^n(t)\| + \|\nabla u^n(t)\| \leq M(T); \tag{2.8}$$

then, by the Poincaré inequality,

$$\|u_t^n(t)\| + \|u^n(t)\|_1 \leq M(T) \tag{2.9}$$

and

$$\begin{aligned}|f(d^{n,k}, d_t^{n,k}, t)| &= |(a(u^n, p)u_t^n + b(u^n, p, p_t), \omega_k)| \qquad \text{a.e. } t \geq 0 \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2) \leq C.\end{aligned}$$

So $f(d^{n,k}, d_t^{n,k}, \cdot)$ is a bounded function, and the solution can be extended to $[0, T)$ [20] . □

Lemma 2.2 *The sequence of approximated solutions u^n satisfies the following:*

(1) u^n is bounded in $L^\infty([0, T), H_0^1(\Omega))$ (and in $L^2([0, T), H_0^1(\Omega))$).
(2) u_t^n is bounded in $L^\infty([0, T), H_0^1(\Omega))$ (and in $L^2([0, T), H_0^1(\Omega))$).

(3) u^n_{tt} is bounded in $L^\infty([0,T),L^2(\Omega))$ (and in $L^2([0,T),L^2(\Omega))$).

Proof - The first item is an immediate consequence of Lemma 2.1. Differentiating (2.3) with respect to t gives, for $k=1,\dots n$,

$$\begin{aligned}(u^n_{ttt},\omega_k)+(\nabla u^n_t,\nabla\omega_k)=&(a_u(u^n,p)(u^n_t)^2+a_p(u^n,p)p_tu^n_t\\&+a(u^n,p)u^n_{tt}+b_u(u^n,p,p_t)u^n_t+b_p(u^n,p,p_t)p_t\\&+b_q(u^n,p,p_t)p_{tt},\omega_k)\quad k=1,\dots n\\&\le C((u^n_t)^2+|u^n_t|+|u^n_{tt}|+|u^n_t|+M,|\omega_k|),\end{aligned}$$

with assumptions A_1, A_2 and A_3. Here, a_u and a_p denote the partial derivatives of $a:(u,p)\mapsto a(u,p)$. Similarly, $b_u,\ b_p$ and b_q are the partial derivatives of $b:(u,p,q)\mapsto b(u,p,q)$.

The method is similar to the one used in the previous Lemma. Replace now ω_k by $u^n_{tt}(t)$:

$$\begin{aligned}\frac{d}{dt}(\|u^n_{tt}(t)\|^2+\|\nabla u^n_t(t)\|^2)\le&\, C\Big(\|(u^n_t(t))^2\|+\|u^n_t(t)\|+\|u^n_{tt}(t)\|+M\Big)\|u^n_{tt}(t)\|\\\le&\, C(\|(u^n_t(t))^2\|\|u^n_{tt}(t)\|+\|u^n_t(t)\|\|u^n_{tt}(t)\|+\\&\qquad\|u^n_{tt}(t)\|^2+\|u^n_{tt}(t)\|)\\\le&\, C(\|(u^n_t(t))^2\|^2+\|u^n_{tt}(t)\|^2+\|u^n_t(t)\|^2+\\&\qquad\|u^n_{tt}(t)\|^2+\|u^n_{tt}(t)\|^2+\|u^n_{tt}(t)\|^2+1)\\\le&\, C(\|(u^n_t(t))^2\|^2+\|u^n_{tt}(t)\|^2+\|u^n_t(t)\|^2+1)\end{aligned}$$

Using the Sobolev-Poincaré inequality [1],

$$\|u^n_t(t)\|_{L^4}\le C\|u^n_t(t)\|_1,$$

and the Poincaré inequality, we get

$$\begin{aligned}\frac{d}{dt}(\|u^n_{tt}(t)\|^2+\|\nabla u^n_t(t)\|^2)\le&\, C(\|u^n_{tt}(t)\|^2+\|u^n_t(t)\|_1^2+1)\\\le&\, C(\|u^n_{tt}(t)\|^2+\|\nabla u^n_t(t)\|^2+1)\ .\end{aligned}$$

Then by the Gronwall inequality we obtain

$$\|u^n_t(t)\|_1+\|u^n_{tt}(t)\|\le M(T)\qquad \text{a.e. } \mathrm{t}\in[0,\mathrm{T}[\ , \tag{2.10}$$

and the lemma is proved. □

Thanks to estimations (2.9) and (2.10), we may extract subsequences of $\{u^n\}$ (denoted similarly in the sequel) such that

$$u^n \rightharpoonup u \quad \text{star-weakly in } L^\infty([0,T), H_0^1(\Omega)) \text{ and weakly in } L^2([0,T), H_0^1(\Omega)) \ ,$$

$$u_t^n \rightharpoonup u_t \quad \text{star-weakly in } L^\infty([0,T), H_0^1(\Omega)) \text{ and weakly in } L^2([0,T), H_0^1(\Omega)) \ ,$$
$$u_{tt}^n \rightharpoonup u_{tt} \quad \text{star-weakly in } L^\infty([0,T), L^2(\Omega)) \text{ and weakly in } L^2([0,T), L^2(\Omega)) \ .$$
By classical compactness results [13],

$$u^n \to u \quad \text{strongly in} \quad L^2([0,T), L^2(\Omega)). \tag{2.11}$$

With the above convergence results, we can extract subsequences $\{u^n\}$ such that

$$u^n(t,x) \to u(t,x) \qquad \text{a.e. } (\mathrm{t,x}) \in [0,\mathrm{T}) \times \Omega \tag{2.12}$$
$$u^n(t,x) \leq g(t,x) \qquad \text{a.e. } (\mathrm{t,x}) \in [0,\mathrm{T}) \times \Omega \tag{2.13}$$
$$u_t^n(t) \to u_t(t) \qquad \text{weakly in } H_0^1(\Omega), \quad \text{a.e. } \mathrm{t} \in [0,\mathrm{T}). \tag{2.14}$$

Now, by continuity of a and b, $(A1)$, and (2.12), we obtain

$$a^4(u^n(t,x), p(t,x)) \to a^4(u(t,x), p(t,x)) \tag{2.15}$$
$$b(u^n(t,x), p(t,x), p_t(t,x)) \to b(u(t,x), p(t,x), p_t(t,x)) \tag{2.16}$$

for almost every $(t,x) \in [0,T) \times \Omega$. Using Lebesgue's theorem, by (2.15) and (A_1) we obtain

$$a(u^n(t), p(t)) \longrightarrow a(u(t), p(t)) \,\mathrm{in}\, \mathrm{L}^4(\Omega) \quad \text{a.e. } \mathrm{t} \in [0,\mathrm{T}) \tag{2.17}$$

and by (2.16), (2.13) and (A_2) we obtain

$$b(u^n(t), p(t), p_t(t)) \to b(u(t), p(t), p_t(t)) \,\mathrm{in}\, \mathrm{L}^2(\Omega) \quad \text{a.e. } \mathrm{t} \in [0,\mathrm{T}) \tag{2.18}$$

On the other hand, for all $v \in L^2$, we have

$$|(a(u^n,p)u_t^n - a(u,p)u_t, v)| = |(a(u^n,p)u_t^n - a(u,p)u_t^n + a(u,p)u_t^n - a(u,p)u_t, v)|$$

$$\begin{aligned} &\leq |(a(u^n,p) - a(u,p))u_t^n, v)| + |(a(u,p)(u_t^n - u_t), v)| \\ &\leq \|a(u^n,p) - a(u,p)\|_{L^4} \|u_t^n\|_{L^4} \|v\| + |(u_t^n - u_t, a(u,p)v)| \\ &\leq C\|a(u^n,p) - a(u,p)\|_{L^4} \|u_t^n\|_1 \|v\| + |(u_t^n - u_t, a(u,p)v)|. \end{aligned}$$

Using (2.17), (2.10) and (2.14), we obtain for almost every $t \in [0,T)$

$$a(u^n(t), p(t))u_t^n(t) \to a(u(t), p(t))u_t(t) \text{ weakly in } L^2(\Omega). \tag{2.19}$$

Then, using (2.18) and (2.19), and letting $n \to \infty$ in (2.3), we obtain that the limiting function $u \in H^2([0,T), L^2(\Omega)) \cap H^1([0,T), H_0^1(\Omega))$ and for almost every $t \in [0,T)$,

$$(u_{tt}, \omega) + (\nabla u, \nabla \omega) = (a(u,p)u_t + b(u,p,p_t), \omega) \quad \forall\, \omega \quad \text{in } H_0^1(\Omega) \tag{2.20}$$

$$u(0) = u_0 \text{ in } \mathrm{H}_0^1(\Omega), \quad \mathrm{u_t}(0) = \mathrm{u_1} \text{ in } \mathrm{L}^2(\Omega) \tag{2.21}$$

□

Corollary 2.1 *Assume that the conditions of theorem 2.1 are fulfilled for every $T > 0$. Then u is a global weak solution of problem (1.1)–(1.3).*

3 The 1-D case

In this part we suppose $n = 1$ and give additional properties of solutions that we are not able to prove if $n \geq 2$. Indeed, we use the Sobolev inclusion $H_0^1(\Omega) \subset \mathcal{C}_0(\Omega)$ (space of bounded continuous functions) which is valid only if the space dimension is 1. The main result is a uniqueness result which is a direct consequence of a "local stability" result, namely the continuous dependence of weak solutions on the initial data.

Theorem 3.1 *Assume that $n = 1$ and that the assumptions of Theorem 2.1 are satisfied for some $T > 0$. Let u, v be two solutions of problem (1.1)–(1.3) corresponding (respectively) to initial data (u_0, u_1) and $(v_0, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Then for almost every $t \in [0,T)$,*

$$\|u_t(t) - v_t(t)\|^2 + \|u(t) - v(t)\|_1^2 \leq M(T)(\|u_1 - v_1\|^2 + \|u_0 - v_0\|_1^2),$$

where $M(T)$ is a positive constant depending on T.

Proof - Let u and v be two solutions of problem (1.1)–(1.3) corresponding to initial data (u_0, u_1) and (v_0, v_1) respectively. We set $\omega = u - v$ and $\omega_i = u_i - v_i,\ i = 0, 1$. Then, for every $\varphi \in H_0^1(\Omega)$ we get

$$\begin{aligned}(\omega_{tt}, \varphi) + (\nabla \omega, \nabla \varphi) &= (a(u,p)u_t - a(v,p)v_t + b(u,p,p_t) - b(v,p,p_t), \varphi)\\ \omega &= 0 \text{ on } [0, \mathrm{T}) \times \partial\Omega\\ \omega(0,x) &= \omega_0, \quad \omega_t(0,x) = \omega_1 \text{ in } \Omega.\end{aligned}$$

Replacing φ by $2\omega_t$ (u_t and $v_t \in H_0^1(\Omega)$) gives

$$2(\omega_{tt}, \omega_t) + 2(\nabla \omega, \nabla \omega_t) = (a(u,p)u_t - a(v,p)v_t + b(u,p,p_t) - b(v,p,p_t), 2\omega_t)$$

and

$$\frac{d}{dt}(\|\omega_t(t)\|^2 + \|\nabla\omega(t)\|^2) =$$
$$(\frac{\partial(a(u,p)u_t)}{\partial u}(\beta)\omega + \frac{\partial(a(u,p)u_t)}{\partial u_t}(\beta)\omega_t, 2\omega_t) + (\frac{\partial b}{\partial u}(\gamma)\omega, 2\omega_t) \ ,$$

where $\beta(t,x) \mapsto (\beta_1(t,x), \beta_2(t,x), p(t,x))$ and γ are vector functions between $(u, u_t, p), (v, v_t, p)$ and $(u, p, p_t), (v, p, p_t)$ respectively. By (A_1), (A_2) and the Cauchy-Schwartz inequality,

$$\frac{d}{dt}(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2) \leq C(\|\omega_t(t)\|^2 + \|\omega_t(t)\|\|\omega(t)\|) + |(\beta_2\omega, 2\omega_t))|$$

$$\leq C(\|\omega_t(t)\|^2 + \|\omega_t(t)\|\|\omega(t)\| + \|\beta_2(t)\|\|\omega_t(t)\|\|\omega(t)\|_\infty).$$

On the other hand, as in the proof of (2.10) we have

$$\|u_t(t)\|_1 \quad \text{and} \quad \|v_t(t)\|_1 \leq M \quad \text{a.e. on} \quad [0, \mathrm{T}) \ . \tag{3.1}$$

Therefore, as $\beta_2(t,x)$ is between $u_t(t,x)$ and $v_t(t,x)$, $\|\beta_2(t)\| \leq M$.

Using the Sobolev embedding theorem [1], we get

$$\|\omega(t)\|_\infty \leq C\|\omega(t)\|_1,$$

since we have supposed $n = 1$. So

$$\frac{d}{dt}(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2) \leq M(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2), \quad \text{a.e. t} \in [0, \mathrm{T}) \ . \tag{3.2}$$

Applying the Gronwall inequality to (3.2), we get

$$\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2 \leq M(T)(\|\omega_t(0,x)\|^2 + \|\omega(0,x)\|_1^2) \quad \text{a.e. t} \in [0, \mathrm{T}). \tag{3.3}$$

This achieves the proof. □

Corollary 3.1 *Assume the assumptions of Theorem 2.1 are fulfilled and* $n = 1$. *Then problem (1.1)–(1.3) admits a unique weak solution*

$$u \in H^2([0,T), L^2(\Omega)) \cap H^1([0,T), H_0^1(\Omega)) \ .$$

4 Application to cerebral activity model

We now quickly present a cerebral activity model stated by Jirsa and Haken [9], which we justify mathematically. This model equation is a particular case of equation (1.1).

The elementary unit of the nervous system is the neuron (Fig. 1), which is

divided into three basic components [7]: dendrites, cell body and axons. A neuron communicates with others at synapses and there are mainly two kinds of synapses [7]: excitatory or inhibitory types. The information transfer between two neurons A and B can be described as follows:

- A pulse (electrical signal) arrives at a dendrite of neuron A (under potential action) and emits neurotransmitters (chemical substances) so that it is transformed into a wave. According to [7], this conversion is a linear operation between neural sheets.
- Then the wave reaches the so-called "trigger zone" (Fig. 1) and is converted to a pulse again. Now the conversion law is sigmoidal .

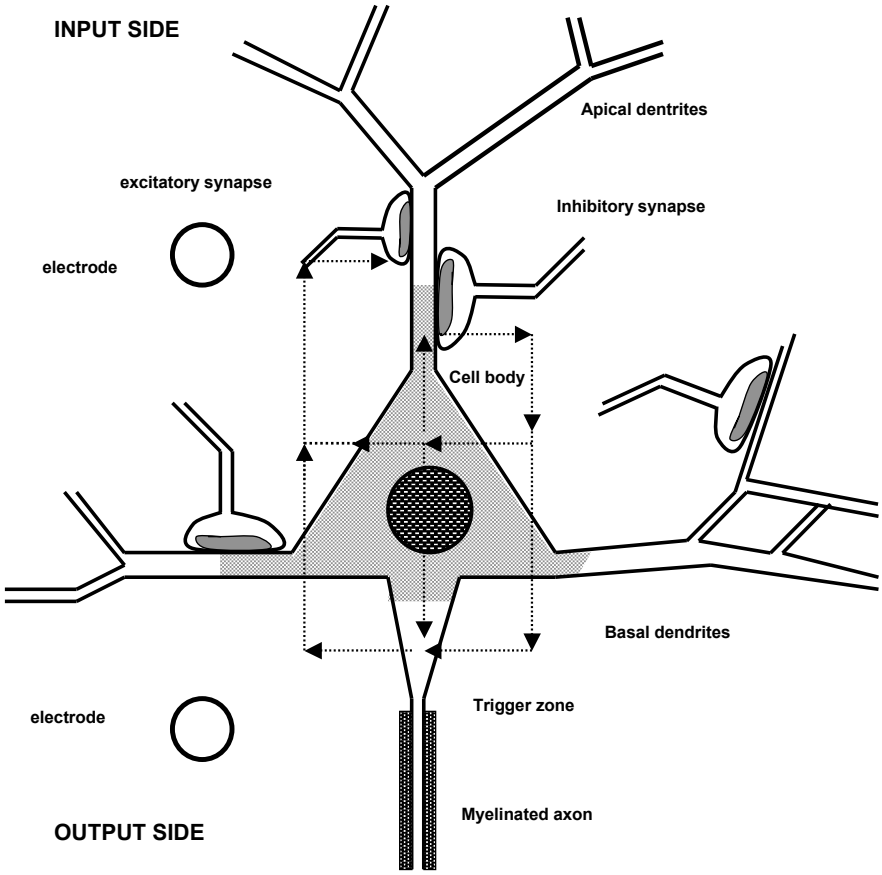

Fig 1. Dendritic trees form thousands of synapses. Excitatory current flows inward at excitatory synapses and outward at the trigger zone. Inhibitory loop current flows in the opposite direction.

The analysis of the conversion operations at the synapses and the trigger zones of neural ensembles allows us to derive a nonlinear partial differential

field equation describing the spatio-temporal behavior of brain activity.
The pulse and wave variables are classified as two sub-variables according to their excitatory or inhibitory characters: We have the excitatory pulse $E(x,t)$ and inhibitory pulse $I(x,t)$, the excitatory wave $\psi_e(x,t)$ and inhibitory wave $\psi_i(x,t)$. We can view the wave as

$$\psi_k(x,t) = \int_\Omega f_k(x,X) H_k(x,X,t)\, dX. \tag{4.1}$$

The functions $H_k(x,X,t),\ k=i,e$ are conversion operation outputs and the functions $f_k(x,X)$ are the corresponding spatial connectivity distributions. Ω denotes the neural sheet set (at the brain surface, which is supposed to be continuous). In what follows $\Omega \subset \mathbb{R}$: we have chosen a 1-D model for the brain geometry as in [8]. We assume that these connectivity functions have the following form:

$$f_k e(x,X) = \frac{1}{2\sigma_e} e^{-\frac{|x-X|}{\sigma_e}} \tag{4.2}$$

where $k=e,i$ and σ_e and σ_i are excitatory and inhibitory connectivity parameters (respectively). Moreover, $\sigma_i \ll 1$, since the corticocortical connections (between two distant neurons via corticocortical fibers) are only excitatory.
On the other hand, the information propagation along corticocortical fibers implies a delay $t = \dfrac{|x-X|}{v_j}$, where v_e and v_i are the excitatory and inhibitory propagation speed, respectively, so that we obtain

$$H_k(x,X,t) = a_k E\Big(X, t - \frac{|x-X|}{v_k}\Big), \tag{4.3}$$

where $k=e,i$ and a_e and a_i are real constants. Substituting equations (4.3) into (4.1), we obtain:

$$\psi_e(x,t) = a_e \int_\Omega f_e(x,X) E\left(X, t - \frac{|x-X|}{v}\right) dX \tag{4.4}$$

$$\begin{aligned}\psi_i(x,t) &= \int_\Omega f_i(x,X) H_i(x,X,t) dX \\ &\approx f_i(x,x) H_i(x,x,t) = a_i I(x,t) \quad (\text{because } \sigma_i \ll 1)\end{aligned} \tag{4.5}$$

The pulse value is calculated from the wave value and the external input in the neural tissue (Fig 1); it has a sigmoidal form:

$$E(x,t) = S_e[\psi_e(x,t) - \psi_i(x,t) + p_e(x,t)]\ ,\ I(x,t) = S_i[\psi_e(x,t) - \psi_i(x,t) + p_i(x,t)] \tag{4.6}$$

where $p_k(x,t)$ is the external input. The sigmoid function S_j is the following:

$$S_j(n_j) = \frac{1}{1+\exp(-v_j n_j)} - \frac{1}{2}$$

When n_j is small enough, we have a third order approximation of S_j:

$$S_j(n_j) \approx n_j - \frac{v_j^3 n_j^3}{48} \tag{4.7}$$

Because $\frac{v_e}{v_i} \approx 100$, we take only the linear part of (4.7) into consideration to obtain:

$$\psi_i(x,t) \approx \frac{a_i v_i}{4 + a_i v_i}(\psi_e(x,t) + p_i(x,t)) \tag{4.8}$$

We finally obtain the following equation for the dynamics of the excitatory synaptic activity:

$$\psi_e(x,t) = a_e \int_\Omega f_e(x,X) \cdot S_e\Big[\widetilde{\rho}\psi_e\Big(X, t - \frac{|x-X|}{v}\Big) + p\Big(X, t - \frac{|x-X|}{v}\Big)\Big] dX$$

where $p(x,t) = p_e(x,t) - \frac{a_i v_i}{4 + a_i v_i} \cdot p_i(x,t)$, and $\widetilde{\rho} = 1 - \frac{a_i v_i}{4 + a_i v_i}$.

We suppose hereafter that $\Omega := (0, L)$ is one-dimensional; that is, we connect all neuron sheets by a line (see [8]), $\psi_e(0,t) = \psi_e(L,t) = 0$ as in [8], and $f_e(x,X)$ is an exponential function depending only on the distance between two neural sheets. Therefore

$$\psi_e(x,t) = a_e \int_\Omega f_e(x,X) \cdot S_e\Big[\widetilde{\rho}\psi_e\Big(X, t - \frac{|x-X|}{v}\Big) + p\Big(X, t - \frac{|x-X|}{v}\Big)\Big] dX \ .$$

Let us mathematicaly justify the computation of [9].
We set $\rho(X,T) = a_e S_e\Big(\widetilde{\rho}\psi(X,T) + p(X,T)\Big)$ and

$$\begin{aligned} \mu : L^1(\mathbb{R} \times \mathbb{R}) &\to \mathbb{R} \\ \varphi \qquad &\mapsto <\mu, \varphi> = \int_{\mathbb{R}} h(x)\varphi(x, \frac{|x|}{v})dx \end{aligned}$$

with $h(x) = 1_{(0,L)}\, e^{-\frac{|x|}{\sigma}}$ and $1_{(0,L)}$ the indicator function of $(0, L)$. Then

$$\begin{aligned} \mu * \rho(x,t) &= \int_{\mathbb{R}} h(X) g(X, \frac{|X|}{v})dX \quad \text{with } g(X,T) = \rho(x - X, t - T) \\ &= \int_{\mathbb{R}} h(X)\rho(x - X, t - \frac{|x|}{v})dX \\ &= \int_\Omega f_e(x,X)\rho(X, t - \frac{|x-X|}{v})dX \end{aligned}$$

and

$$\mu * \rho(x,t) = \psi_e(x,t).$$

The extended Fourier transform gives: $\widehat{\psi_e} = \widehat{\mu}\widehat{\rho}$.

Now, we prove that $\widehat{\mu} = T_f, f \in L^1_{loc}(\mathbb{R}\times\mathbb{R})$. Let $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R})$.

$$\begin{aligned}
<\widehat{\mu},\varphi> = <\mu,\widehat{\varphi}> &= \int_{\mathbb{R}} h(X)\widehat{\varphi}(X,\frac{|X|}{v})dX \\
&= \int_{\mathbb{R}} h(X)\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,t)e^{-2i\pi(xX+t\frac{|X|}{v})}dxdt\Big)dX \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} h(X)e^{-2i\pi(xX+t\frac{|X|}{v})}dX\Big)\varphi(x,t)dxdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,t)\varphi(x,t)dxdt
\end{aligned}$$

with $f(x,t) = \dfrac{1}{2\sigma_e}\displaystyle\int_\Omega e^{-\frac{|X|}{\sigma}}e^{-2i\pi(xX+t\frac{|X|}{v})}dX$.
Then

$$\widehat{\psi_e}(x,t) = f(x,t)\widehat{\rho}(x,t) \quad \forall (x,t)\in\mathbb{R}\times\mathbb{R}. \tag{4.9}$$

As

$$\int_0^{+\infty} e^{-(a+i\alpha)\zeta}d\zeta = \frac{1}{a+i\alpha}, \tag{4.10}$$

supposing $\omega_0 = \dfrac{v}{\sigma}$, we obtain

$$\begin{aligned}
f(x,t) &= \frac{1}{2}\Big[\frac{1}{\frac{1}{\sigma_e}+i\Big(\frac{t}{v}-x\Big)} + \frac{1}{\frac{1}{\sigma_e}+i\Big(\frac{t}{v}+x\Big)}\Big] \\
&= \frac{1}{2}\Big[\frac{\omega_0}{\omega_0+i(t-xv)} + \frac{\omega_0}{\omega_0+i(t+xv)}\Big] \\
&= \Big[\frac{\omega_0^2+i\omega_0 t}{(\omega_0+it)^2+x^2v^2}\Big] \in L^1_{loc}(\mathbb{R}\times\mathbb{R}).
\end{aligned}$$

Equation (4.9) becomes

$$\widehat{\psi_e}(x,t) = \Big[\frac{\omega_0^2+i\omega_0 t}{(\omega_0+it)^2+x^2v^2}\Big]\widehat{\rho}(x,t). \tag{4.11}$$

Developing equation (4.11), we obtain

$$\omega_0^2\widehat{\psi_e}(x,t) + 2\omega_0\frac{\widehat{\partial\psi_e}}{\partial t}(x,t) + \frac{\widehat{\partial^2\psi_e}}{\partial t^2}(x,t) - v^2\frac{\widehat{\partial^2\psi_e}}{\partial x^2}(x,t) = \widehat{\omega_0^2+\omega_0\frac{\partial}{\partial t}}\cdot\rho(x,t)$$

Applying the inverse Fourier Transform, we formally obtain the following partial differential equation:

$$\frac{\partial^2\psi_e}{\partial t^2} + (\omega_0^2 - v^2\frac{\partial^2}{\partial x^2})\psi_e + 2\omega_0\frac{\partial\psi_e}{\partial t} = (\omega_0^2+\omega_0\frac{\partial}{\partial t})\cdot\rho \tag{4.12}$$

with

$$\rho(x,t) = a_e S_e\Big(\widetilde{\rho}\psi_e(x,t) + p(x,t)\Big)\ .$$

Developing (4.12) and summing the similar terms gives

$$\frac{\partial^2\psi_e}{\partial t^2} - v^2\frac{\partial^2\psi_e}{\partial x^2} = \Big(a_e\omega_0\widetilde{\rho}S_e'(\widetilde{\rho}\psi_e + p) - 2\omega_0\Big)\frac{\partial\psi_e}{\partial t} \\ - \omega_0^2\psi_e + a_e\omega_0^2 S_e(\widetilde{\rho}\psi_e + p) + a_e\omega_0 S_e'(\widetilde{\rho}\psi_e + p)p_t \ . \qquad (4.13)$$

We complete this equation with $\psi_e(t,0) = \psi_e(t,L) = 0$ and Cauchy initial condition. Thanks to the previous sections' results, we have the following:

Theorem 4.1 *Assume $p \in H^2([0,T), L^\infty(0,L))$. Then equation (4.13) with $\psi_e(t,0) = \psi_e(t,L) = 0$ and Cauchy initial condition, with $(\psi_{e,0}, \psi_{e,1}) \in H_0^1(0,L)\times L^2(0,L)$, has a unique weak solution in $H^2([0,T), L^2(0,L)) \cap H^1([0,T), H_0^1(0,L))$.*

Proof - Equation (4.13) is a particular case of equation (1.1), with

$$a(u,p) = a_e\omega_0\widetilde{\rho}S_e'(\widetilde{\rho}u + p) - 2\omega_0$$

and

$$b(u,p,p_t) = -\omega_0^2 u + a_e\omega_0^2 S_e(\widetilde{\rho}u + p) + a_e\omega_0 S_e'(\widetilde{\rho}u + p)p_t.$$

If we suppose $p \in H^2([0,T), L^\infty(0,L))$, it is easy to see that functions a and b satisfy the conditions of Theorem 2.1 and consequently, by (2.20) and (3.3), the above equation has a unique weak solution. □

5 Conclusion

In this paper, we prove the existence of solutions to a new class of wave equations which is motivated by brain activity modeling. In the 1-D spatial dimension case, the solution depends Lipschitz-continuously on the initial data, and so it is unique. This means that the responses to small perturbations are small as well, which is coherent from a physical point of view. We have simplified the model by assuming that:

- the geometry of the brain surface area is 1-D. The 2-D geometry is more consistent but introduces additional difficulties,
- the connectivity functions (4.2) between the neural sheets are homogeneous. In fact more than 50 % of neural sheets have heterogeneous connections, so we should add another term representing the heterogeneity,
- the external input p belongs to $H^2([0,T), L^\infty(\Omega))$. In reality, p may be less smooth or non-local.

In our future work, we will try to generalize our model in order to satisfy the general constraints which are introduced above.

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