Fictitious domain methods to solve convection-diffusion problems with general boundary conditions

Isabelle Ramière[1,2,*] , Philippe Angot[1,†] , Michel Belliard[2,‡]

[1] *Centre de Mathématiques et d'Informatique*
Laboratoire d'Analyse, Topologie et Probabilités - UMR 6632
39, rue Joliot Curie- 13453 Marseille cedex 13 - FRANCE
ramiere@cmi.univ-mrs.fr, angot@cmi.univ-mrs.fr

[2] *CEA-Cadarache,*
DEN/DTN/SMTM/LMTR,
13108 St-Paul-Lez-Durance - FRANCE
isabelle.ramiere@cea.fr, michel.belliard@cea.fr

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Since a few years, *fictitious domain methods* **have been arising for Computational Fluid Dynamics. The main idea of these methods consists in immersing the original physical domain in a geometrically bigger and simply-shaped other one called fictitious domain. As the spatial discretization is then performed in the fictitious domain, simple** *structured meshes* **can be used.**
The aim of this paper is to solve convection-diffusion problems with fictitious domain methods which can easily simulate free-boundary with possibly deformations of the boundary without increasing the computational cost. Two fictitious domain approaches performing either a spread interface or a thin interface are introduced. These two approaches require neither the modification of the numerical scheme near the immersed interface nor the use of Lagrange multipliers. Several ways to impose general embedded boundary conditions (Dirichlet, Robin or Neumann) are presented.
The spread interface approach is computed using a finite element method as a finite volume method is used for the thin interface approach. The numerical schemes conserve the first-order accuracy with respect to the discretization step as observed in the numerical results reported here. The spread interface approach is then combined with a local adaptive mesh refinement algorithm in order to increase the precision in the vicinity of the immersed boundary. The results obtained are full of promise, more especially as convection-diffusion equations are the core of the resolution of Navier-Stokes equations.

*Ph.D. student, corresponding author
†Associate Professor
‡Engineer in CEA

Nomenclature

$\tilde{.}$	:	superscript invoking data of the physical problem
$\tilde{\Omega}$	:	original domain
Ω	:	fictitious domain
Ω_e	:	exterior domain
Σ	:	immersed interface
$\tilde{\Gamma}$	=	$\partial\tilde{\Omega} \cap \partial\Omega$
Γ_e	:	exterior fictitious boundary
$\omega_{h,\Sigma}$	:	spread approximated interface
Σ_h	:	thin approximated interface
Σ_h^i	=	$\Sigma_h \backslash \partial\Omega$
η	:	penalization coefficient
ϵ	:	characteristic parameter used to impose the immersed flux
Subscript		
h	:	discretization index

I. Introduction

A. Fictitious domain approach

Let $\tilde{\Omega}$ be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a boundary $\partial\tilde{\Omega}$ sufficiently regular. We consider the resolution of a given problem $(\tilde{\mathcal{P}})$ in $\tilde{\Omega}$ with different kinds of boundary conditions (B.C.). When the shape of $\partial\tilde{\Omega}$ is geometrically not simple, classical methods involving structured or unstructured boundary-fitted meshes induce a loss of efficiency and rapidity of numerical solvers in comparison with Cartesian meshes. Moreover in case of moving boundaries, the cost of the mesh generation and re-meshing can be very important. In the fictitious domain approach,[1,2] the *original domain* $\tilde{\Omega}$ is embedded in a geometrically bigger and simply-shaped other one Ω, called *fictitious domain* (cf. Figure 1).

The spatial discretization is now performed in Ω, independently of the shape of the original domain $\tilde{\Omega}$. The original domain and the computational one are uncoupled. Numerical methods involving structured and Cartesian meshes can be used. The advantages of these methods are well known : natural tensor formulation, easy implementation for fast solvers (based for instance on finite volume methods with Cartesian grids) and for multi-level methods,[3] good convergence properties...Consequently, the resolution of the new problem in Ω will be fast and simple.

The main issue of fictitious domain method lies in the choice of the problem $(\mathcal{P})$ solved in the fictitious domain Ω and in the numerical scheme used for the resolution. These two choices have to be linked in order to handle the original boundary conditions on $\partial\tilde{\Omega}$. Indeed, the B.C. on the original boundary $\partial\tilde{\Omega}$ must still be enforced such that the solution u of the extended problem $(\mathcal{P})$ can match the solution $\tilde{u}$ of the original problem $(\tilde{\mathcal{P}})$ in the original domain $\tilde{\Omega}$. Since a few years, fictitious domain methods have arisen in different fields : computational fluid dynamics,[4] medical simulation.[5,6] Numerically, there are mainly two approaches to deal with the embedded boundary conditions on the immersed boundary:

- "Thin" interface approaches : the original boundary is approximated without being enlarged in the normal direction. The original boundary and the approximated one lie in the same $\mathbb{R}^{d-1}$ space. For example, in this group, we find truncated domains methods (e.g. Ref. 7,8), immersed interface methods (I.I.M.) (e.g. Ref. 9), penalty methods (e.g. Ref. 1, 2, 10), fictitious domain methods with Lagrange multipliers (e.g. Ref. 11, 12).

- "Spread" interface approaches : the support of the approximated interface is larger than the original one. The approximated interface has one dimension more than the original one. For example, the spread interface is a ring containing the immersed interface. This kind of approach can be found in Ref. 2,13, in fluid/structure applications with the interface boundary method (I.B.M.) (e.g. Ref. 5,14), and more recently with the fat boundary method.[15]

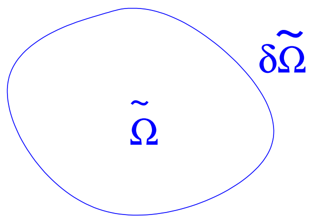
A lot of papers are dedicated to embedded Dirichlet or Neumann B.C., e.g. Ref. 1, 2, 9, 12, 16 and the references herein. Only few studies are devoted to embedded Fourier B.C. (Ref. 2, 13, 17, 18).
In this work, we use our recent works on fictitious domain methods for elliptic problems[18–22] and adapt them to solve convection-diffusion problems. These fictitious domain approaches deal with Dirichlet, Robin or Neumann B.C. on an immersed interface without requiring neither the modification of the numerical scheme near the immersed interface nor the use of Lagrange multipliers. Since the fictitious problem $(\mathcal{P})$ is not a saddle-point problem, no inf-sup condition must be verified (e.g. Ref. 16). Moreover, only one discretization grid is used, a structured regular mesh over the fictitious domain. Thus, these fictitious domain approaches easily simulate free-boundary problems with possibly deformations of the boundary without increasing the computational cost. We use a boundary non-conforming mesh which conserves the first-order accuracy. A local mesh refinement can be implemented in the vicinity of the immersed interface in order to improve the ratio of the obtained precision over the resulting cost (or CPU time).

B. Presentation of the study

For sake of clarity we choose to focus on 2D problems, but the formulations can be extended to 3D problems without more difficulty. We study the resolution of a **convection-diffusion problem** $(\tilde{\mathcal{P}})$ in the original domain $\tilde{\Omega}$.

Let us consider the following model problem :
For $\tilde{\mathbf{a}} \in (L^\infty(\tilde{\Omega}))^{d\times d}$, $\tilde{\mathbf{v}} \in (L^\infty(\tilde{\Omega}))^d$, $\tilde{b} \in (L^\infty(\tilde{\Omega}))$ and $\tilde{f} \in L^2(\tilde{\Omega})$, find $\tilde{u} \in H^1(\tilde{\Omega})$ such that :

$$(\tilde{\mathcal{P}}) \qquad \begin{cases} -div\,(\tilde{\mathbf{a}}\,.\nabla\tilde{u}) + div\,(\tilde{\mathbf{v}}\tilde{u}) + \tilde{b}\tilde{u} \;=\; \tilde{f} & \text{in } \tilde{\Omega} \\ \text{B.C.} & \text{on } \partial\tilde{\Omega} \end{cases}$$



where B.C. represents several types of boundary conditions :

- a Dirichlet condition : $\tilde{u} = u_D$ with $u_D \in H^{1/2}(\partial\tilde{\Omega})$
- a Robin (or Fourier) condition : $-(\tilde{\mathbf{a}}\,.\nabla\tilde{u}).\mathbf{n} = \alpha_R\,\tilde{u} + g_R$, with $\alpha_R \in L^\infty(\partial\tilde{\Omega})$; $\alpha_R \geq 0$, and $g_R \in L^2(\partial\tilde{\Omega})$ (with $\mathbf{n}$ the outward unit normal vector on $\partial\tilde{\Omega}$)
  Remark: a Neumann condition, $-(\tilde{\mathbf{a}}\,.\nabla\tilde{u}).\mathbf{n} = g$, is considerate as a particular Robin condition where $\alpha_R \equiv 0$ and $g_R \equiv g$.

The tensor of diffusion $\tilde{\mathbf{a}} \equiv (\tilde{a}_{ij})_{1\leq i,j\leq d}$ and the reaction coefficient $\tilde{b}$ verify the classical ellipticity assumptions. We suppose that u_D, α_R, u_R and g_R are constant. The non constant case can be treated without more difficulty. The non-constant data are then replaced by their extension in $\omega_{h,\Sigma} \cup \Omega_{e,h}$ such that the trace on Σ of the extensions are equal to the original data on Σ.

In a fictitious domain approach, the original domain $\tilde{\Omega}$ is embedded inside a fictitious domain Ω such that $\Omega = \tilde{\Omega} \cup \Sigma \cup \Omega_e$, where Ω_e is the external fictitious domain and Σ the common interface between $\tilde{\Omega}$ and Ω_e (see Figure 1 and 6(b)). This original interface Σ is called *immersed interface*. The fictitious domain Ω is chosen to be geometrically simple (rectangular for example). If $\partial\tilde{\Omega} \cap \partial\Omega \neq \emptyset$ (see Figure 6(b)), the boundary of $\tilde{\Omega}$ is defined by $\partial\tilde{\Omega} = \tilde{\Gamma} \cup \Sigma$, and the boundary of Ω by $\partial\Omega = \tilde{\Gamma} \cup \Gamma_e$. Otherwise (see Figure 1), $\tilde{\Gamma} \equiv \emptyset$, so

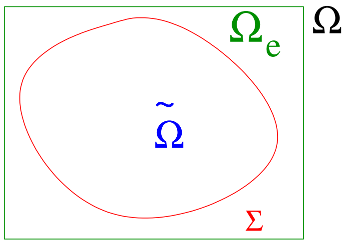


Figure 1. Example of an original domain $\tilde{\Omega}$ immersed in a fictitious rectangular domain Ω: $\Omega = \tilde{\Omega} \cup \Sigma \cup \Omega_e$

$\partial\tilde{\Omega} \equiv \Sigma$ and $\partial\Omega \equiv \Gamma_e$.
The aim of this work is to introduce the fictitious problem $(\mathcal{P})$ to be solved all over the fictitious domain Ω in order to get the restriction of the fictitious solution u over the original domain $\tilde{\Omega}$ equal (or at least similar) to the solution $\tilde{u}$ of the original problem $(\tilde{\mathcal{P}})$: $u|_{\tilde{\Omega}} \simeq \tilde{u}$. The restriction of the problem $(\mathcal{P})$ over the original domain $\tilde{\Omega}$ is chosen to be similar to $(\tilde{\mathcal{P}})$. Appropriate transmissions conditions on Σ, data in the external domain Ω_e and B.C. on $\partial\Omega$ have to be determined in order to handle the original embedded B.C. of $(\tilde{\mathcal{P}})$.

In the further sections, two fictitious domain methods to deal with embedded boundary conditions are presented. These methods are based on respectively a spread and a thin interface approach. The corresponding fictitious domain problems to solve convection-diffusion problems are introduced. The last section is dedicated to the numerical resolution. A Finite Element as well as a Finite Volume Scheme are implemented. Moreover, for the spread interface approach, an example of the possibility to combine Fictitious Domain methods and local Adaptive Mesh Refinement methods is reported. Several results illustrate the accuracy of these Fictitious Domain methods.

II. The fictitious domain method with spread interface

A. Fictitious problem with spread interface $(\mathcal{P}_s)$

The fictitious domain method with spread interface has been introduced by Ramière, Angot and Belliard[19] for elliptic problems. This approach is based on a "spread" approximation (see also Ref. 14) of the immersed interface Σ. The computational domain Ω is uniformly meshed with a family $\mathcal{T}_h = \{K\}$ of disjointed rectangular cells K such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$. The approximated spread interface $\omega_{h,\Sigma}$ is defined as $\omega_{h,\Sigma} = \overline{\omega}_{h,\Sigma} \backslash \partial\overline{\omega}_{h,\Sigma}$ where $\overline{\omega}_{h,\Sigma} = \bigcup_{K \in \mathcal{T}_h} \{\overline{K}, K \cap \Sigma \neq \emptyset\}$ (see Figure 2).

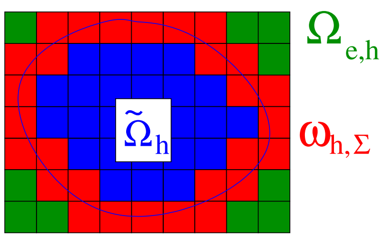


Figure 2. Discretization of the fictitious domain Ω with a spread interface approach: $\Omega = \tilde{\Omega}_h \cup \omega_{h,\Sigma} \cup \Omega_{e,h}$

With an original convection-diffusion problem $(\tilde{\mathcal{P}})$ in $\tilde{\Omega}$, the problem $(\mathcal{P}_s)$ solved in the fictitious domain

Ω has the following generic form : $0 < \eta$ being a real parameter precised later, find u_η^h (depending on the mesh $\mathcal{T}_h$) $\in H^1(\Omega)$ such that

$$(\mathcal{P}_s) \quad \begin{cases} -div\,(\mathbf{a}\,.\nabla u_\eta^h) + div(\mathbf{v}\,u_\eta^h) + b\,u_\eta^h = f & \text{in } \Omega \\ \text{original B.C. of } (\tilde{\mathcal{P}}) & \text{on } \tilde{\Gamma} \\ \text{suitable B.C. for } u_\eta^h & \text{on } \Gamma_e \end{cases}$$

where $\mathbf{a} \in (L^\infty(\Omega))^{d^2}$, $\mathbf{v} \in (L^\infty(\Omega))^d$, $b \in (L^\infty(\Omega))$, and $f \in L^2(\Omega)$ such that:

$$\mathbf{a}|_{\tilde{\Omega}_h} = \tilde{\mathbf{a}}|_{\tilde{\Omega}_h}, \qquad \mathbf{v}|_{\tilde{\Omega}_h} = \tilde{\mathbf{v}}|_{\tilde{\Omega}_h}, \qquad b|_{\tilde{\Omega}_h} = \tilde{b}|_{\tilde{\Omega}_h} \qquad f|_{\tilde{\Omega}_h} = \tilde{f}|_{\tilde{\Omega}_h},$$

and $\mathbf{a}$ and b satisfy classical ellipticity assumptions in Ω.
Remark: the B.C. on Γ_e must be chosen such that the problem $\mathcal{P}_s$ is well-posed.

For each kind of boundary conditions lying on the immersed boundary Σ, different possibilities to enforce these conditions using a spread approximation of the immersed interface Σ are introduced in the next sections. No variant modifies the numerical scheme or introduces local unknowns. We expect that: $u_\eta^h|_{\tilde{\Omega}_h} \simeq \tilde{u}|_{\tilde{\Omega}_h}$.

B. Treatment of the original B.C.

1. *Embedded Dirichlet B.C.*

The Dirichlet B.C. are treated by **penalization** (e.g. Ref. 10). Let $0 < \eta << 1$ be a real penalty parameter which is likely to tend to zero. We propose to compare the **penalization of the spread interface** to the **penalization of the exterior domain**. The penalization of the spread interface (resp. the exterior domain) consists in setting $b = \dfrac{1}{\eta}$ ($b \to +\infty$ at $\eta \to 0$) and $f = \dfrac{1}{\eta} u_D$ in $\omega_{h,\Sigma}$ (resp. Ω_e). These approaches allow to impose $u_\eta^h \simeq u_D$ in $\omega_{h,\Sigma}$ (resp. $\Omega_{e,h}$). It's the $\mathbf{L^2}$ **penalty**.[23] When the coefficient a is also equal to $\dfrac{1}{\eta}$ ($\mathbf{a} = \dfrac{1}{\eta}\mathbf{Id}$) in $\omega_{h,\Sigma}$ (resp. $\Omega_{e,h}$), we obtain the $\mathbf{H^1}$ **penalty**.[23] Elsewhere, the coefficients of the problem ($\mathcal{P}_s$) are arbitrary extensions in Ω of the original coefficients lying on $\tilde{\Omega}$ (see Ref. 19).
Concerning the B.C. on Γ_e, for the spread interface penalization, the Dirichlet B.C. $u_h = u_D$ must only be imposed on $\Gamma_e \cap \partial\omega_{h,\Sigma}$. The B.C. on the rest of Γ_e have no influence on the solution obtained in the physical domain. To penalize the exterior domain, the B.C. on the whole exterior boundary Γ_e must be Dirichlet B.C. $u_h|_{\Gamma_e} = u_D$.

2. *Embedded Robin or Neumann B.C.*

We consider the **transmission problem** between $\tilde{\Omega}$ and Ω_e with continuity of the solution on Σ. For ψ in Ω denoting $\mathbf{a}$, $\mathbf{v}$, b or f, we define $\hat{\psi} = \begin{cases} \tilde{\psi} & \text{in } \tilde{\Omega} \\ \psi_e & \text{in } \Omega_e \end{cases}$. The addition of the two weak formulations of the two convection-diffusion subproblems defined on respectively $\tilde{\Omega}$ and Ω_e leads to the following problem:
Find $u \in H^1(\Omega)$ such that $\forall \psi \in H^1(\Omega)$

$$\begin{aligned} \int_\Omega \hat{\mathbf{a}}.\nabla u.\nabla\psi\,\mathrm{dx} - \int_{\partial\Omega} (\hat{\mathbf{a}}.\nabla u).\mathbf{n}\,\psi\,\mathrm{ds} \quad &- \quad \int_\Omega (\hat{\mathbf{v}}u).\nabla\psi\mathrm{dx} + \int_{\partial\Omega} (\hat{\mathbf{v}}.\mathbf{n}\,u)\psi\mathrm{ds} + \int_\Omega \hat{b}\,u\,\psi\,\mathrm{dx} \\ &= \quad \int_\Omega \hat{f}\,\psi\,\mathrm{dx} - < [\![(\hat{\mathbf{a}}.\nabla u).\mathbf{n}]\!]_\Sigma \delta_\Sigma, \psi > + < [\![(\hat{\mathbf{v}}.\mathbf{n})]\!]_\Sigma\, u\,\delta_\Sigma, \psi > \end{aligned} \tag{1}$$

$$\text{where} \begin{cases} \mathbf{n} \text{ denotes the outward normal unit vector either on } \partial\Omega \text{ or on } \Sigma \text{ (oriented from } \tilde{\Omega} \text{ to } \Omega_e) \\ [\![(\hat{\mathbf{a}}.\nabla u).\mathbf{n}]\!]_\Sigma = (\hat{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma^+ - (\hat{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma^- = (\mathbf{a}_e.\nabla u).\mathbf{n}|_\Sigma - (\tilde{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma \\ [\![(\hat{\mathbf{v}}.\mathbf{n})]\!]_\Sigma = \mathbf{v}_e.\mathbf{n}|_\Sigma - \tilde{\mathbf{v}}.\mathbf{n}|_\Sigma \\ \delta_\Sigma \text{ means the Dirac delta measure supported by } \Sigma \end{cases}$$

In the distribution sense, we obtain the following equation:

$$-div\,(\hat{\mathbf{a}}.\nabla u) + div(\hat{\mathbf{v}}\,u) + \hat{b}u = \hat{f} - [\![(\hat{\mathbf{a}}.\nabla u).\mathbf{n}]\!]_\Sigma\,\delta_\Sigma + [\![(\hat{\mathbf{v}}.\mathbf{n})]\!]_\Sigma\,u\,\delta_\Sigma \tag{2}$$

The jump of fluxes (diffusion and convection) across Σ can be interpreted as source terms carried by Σ.

In our case, we want

$$-(\hat{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma^- = -(\tilde{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma = \alpha_R\,u|_\Sigma + g_R$$

We choose to impose $-(\hat{\mathbf{a}}.\nabla u).\mathbf{n}|_\Sigma^+ = 0$ by $\mathbf{a}_e = \eta$ on Ω_e, so that

$$[\![(\hat{\mathbf{a}}.\nabla u).\mathbf{n}]\!]_\Sigma = \alpha_R\,u|_\Sigma + g_R$$

Moreover, we impose $\hat{\mathbf{v}}|_{\Omega_e} = \mathbf{v}_e = \mathbf{0}$ in order to let the solution free on the exterior domain Ω_e. If $\mathbf{v}_e \neq \mathbf{0}$, as $u|_\Sigma^- = u|_\Sigma^+$, the resolution of the exterior problem influences the embedded Robin condition.

On Ω, then we have:

$$-div\,(\hat{\mathbf{a}}.\nabla u_\eta) + div\,(\hat{\mathbf{v}}\,u_\eta) + \hat{b}\,u_\eta = \hat{f} - [\alpha_R\,u_\eta + g_R + (\hat{\mathbf{v}}.\mathbf{n})^-\,u_\eta]\,\delta_\Sigma \tag{3}$$

However, with a Cartesian mesh on Ω, the support of Σ is not exactly defined. We introduce a characteristic parameter ϵ in order to approximate the measure δ_Σ supported by Σ by mollifiers[24] on the spread interface $\omega_{h,\Sigma}$. The term $\delta_{\omega_{h,\Sigma}}$ is a discrete Dirac function on the spread interface. Here, this discrete delta function is roughly approximated by a crenel function in Ω ($\delta_{\omega_{h,\Sigma}} = 1$ in $\omega_{h,\Sigma}$ and 0 elsewhere) whereas a smoothed approximation using interaction equations is performed in the I.B.M., see e.g. Ref. 5, 14.
The principle is the following:

$$\int_\Sigma \{\alpha_R\,u_\eta + g_R + (\hat{\mathbf{v}}.\mathbf{n})^-\,u_\eta\}\,\mathrm{ds} = \int_{\omega_{h,\Sigma}} \frac{\alpha_R\,u_\eta^h + g_R + (\mathbf{v}.\mathbf{n})\,u_\eta^h}{\epsilon}\,\mathrm{dx} \tag{4}$$

where $\mathbf{v}$ is an extension of the velocity field $\tilde{\mathbf{v}}$ in the spread interface $\omega_{h,\Sigma}$ and $\mathbf{n}$ denotes either the outward unit vector on Σ or its extension on $\omega_{h,\Sigma}$.

Then, the coefficients of the fictitious problem $(\mathcal{P}_s)$ are easily set extending the coefficients of the original domain $(\tilde{\mathcal{P}})$ in $\tilde{\Omega}_h \cup \omega_{h,\Sigma}$ (see Table 1).
Only the B.C. on $\Gamma_e \cap \partial\omega_{h,\Sigma}$ has an effect on the solution obtained in the physical domain. These B.C. must be homogeneous Neumann B.C. in order to have an external diffusion flux equal to zero. The B.C. on the rest of Γ_e can be arbitrarily chosen as long as the whole problem $(\mathcal{P}_s)$ in Ω is well-posed.

The parameter ϵ can be estimated by several ways (see Ref. 19, 22). If h is the discretization step, Angot[25] showed that ϵ is in $\mathcal{O}(h)$. We present here the three approximations of ϵ introduced in Ref. 19:

- A coarse *global approximation* of ϵ in Equation (4) holds :

$$\int_\Sigma \mathrm{ds} = \int_{\omega_{h,\Sigma}} \frac{1}{\epsilon}\,\mathrm{dx} \tag{5}$$

- ⋆ In a first approach, we suppose that ϵ is constant all over $\omega_{h,\Sigma}$.

$$\epsilon = \frac{meas(\omega_{h,\Sigma})}{meas(\Sigma)} \tag{6}$$

- ⋆ In the second approach, the value of ϵ is given element by element. In the equation (4) the integration in $\omega_{h,\Sigma}$ is weighted by a coefficient τ. This coefficient represents the presence rate of the original domain in each element K crossed by the boundary Σ ($K \subset \omega_{h,\Sigma}$). By construction, τ is constant on each K :

$$\tau_K = \frac{\text{volume of } \tilde{\Omega} \text{ included in } K}{\text{volume of the element } K}$$

By this way, the right hand side of (5) is integrated only in the original domain included in $\omega_{h,\Sigma}$. We get :

$$\epsilon_K = \frac{\sum_K [\tau_K.meas(K)]}{\tau_K.meas(\Sigma)} \tag{7}$$

- A *local approximation* of ϵ consists in calculating ϵ in each cell $K \subset \omega_{h,\Sigma}$:

$$\int_{\Sigma_K} \mathrm{ds} = \int_K \frac{1}{\epsilon}\,\mathrm{dx} \tag{8}$$

with $\Sigma_K = \Sigma \cap K$
As Σ_K is not simply defined, the boundary Σ is piecewise linear approximated by a segment $\Sigma_{l,K}$ in each cell $K \subset \omega_{h,\Sigma}$ (see Figure 3).

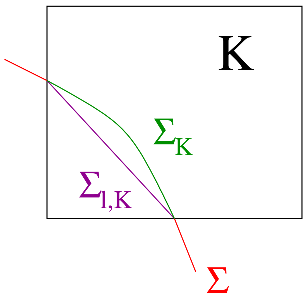


Figure 3. Linear approximation of Σ in a rectangular cell $K \subset \omega_{h,\Sigma}$

Then,

$$\epsilon_K = \frac{meas(K)}{meas(\Sigma_{l,K})} \tag{9}$$

Here again, the value of ϵ depends on the element $K \subset \omega_{h,\Sigma}$ under consideration.
This approach induces a local piecewise linear reconstruction of the interface in each cell K of the mesh.

For each embedded B.C. variant, Table 1 gives the parameters of interest using a spread interface approach.

	Parameters in $\omega_{h,\Sigma}$	Parameters in $\Omega_{e,h}$
Dirichlet B.C. *Spread interface penalization*	$\mathbf{a} = \begin{cases} \mathbf{Id} \ (L^2 \text{ penalty}) \\ \frac{1}{\eta}\mathbf{Id} \ (H^1 \text{ penalty}) \end{cases}, \ \mathbf{v} = \tilde{\mathbf{v}},$ $b = \frac{1}{\eta}, \ f = \frac{1}{\eta} u_D$	$\mathbf{a} = \mathbf{Id}, \ \mathbf{v} = \mathbf{0},$ $b = 0, \ f = 0$
Dirichlet B.C. *Exterior penalization*	$\mathbf{a} = \tilde{\mathbf{a}}, \ \mathbf{v} = \tilde{\mathbf{v}},$ $b = \tilde{b}, \ f = \tilde{f}$	$\mathbf{a} = \begin{cases} \mathbf{Id} \ (L^2 \text{ penalty}) \\ \frac{1}{\eta}\mathbf{Id} \ (H^1 \text{ penalty}) \end{cases}, \ \mathbf{v} = \mathbf{0},$ $b = \frac{1}{\eta}, \ f = \frac{1}{\eta} u_D$
Robin B.C. *with different approximations of ϵ*	$\mathbf{a} = \tilde{\mathbf{a}}, \ \mathbf{v} = \tilde{\mathbf{v}},$ $b = \tilde{b} + \frac{\alpha_R}{\epsilon} + \frac{\mathbf{v}.\mathbf{n}}{\epsilon}, \ f = \tilde{f} - \frac{g_R}{\epsilon}$	$\mathbf{a} = \eta \mathbf{Id}, \ \mathbf{v} = \mathbf{0},$ $b = 0, \ f = 0$

Table 1. Parameters in $\omega_{h,\Sigma}$ and in $\Omega_{e,h}$ for the spread interface approach

III. The fictitious domain approach with thin interface

A. Fictitious problem with thin interface $(\mathcal{P}_t)$

The fictitious domain method with thin interface uses a recent fracture model for elliptic problems with flux and solution jumps.[20] The fictitious domain model for convection-diffusion problems introduced here handle jumps of diffusion and convection fluxes as well as jumps of solution through Σ. As in Ref 20, this model includes *algebraic transmission conditions* linking both normal diffusion flux $-(\mathbf{a}.\nabla u).\mathbf{n}$ and the solution u jumps through the immersed interface Σ. We use an uniform Cartesian mesh $\mathcal{T}_h = \{K\}$ of the fictitious domain Ω where the disjointed open rectangular control volumes K are such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$. As this Cartesian mesh $\mathcal{T}_h$ does not generally fit the immersed interface Σ, an approximated interface Σ_h lying on sides of control volumes is defined by $\Sigma_h = \partial\tilde{\Omega}_h \backslash \tilde{\Gamma}$. The approximated original domain $\tilde{\Omega}_h$ can be chosen for example like $\tilde{\Omega}_h = \cup_{K\in\mathcal{T}_h}\{\overline{K}, K \cap \tilde{\Omega} \neq \emptyset\}$ (see figure 4). Other choices are reported in section IV.

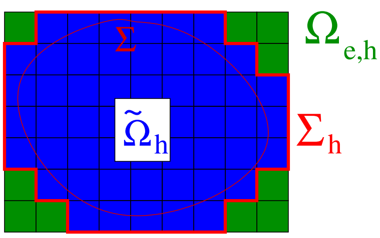


Figure 4. Example of a thin approximation Σ_h of the immersed boundary Σ

The fictitious domain Ω is chosen small in order to have a cheap computational cost. Then, $\Sigma_h \cap \partial\Omega$ may not be empty (see Figure 4). We note $\Sigma_h^i = \Sigma_h \backslash \partial\Omega$ the part of Σ_h strictly immersed inside the fictitious

domain Ω. Let $\mathbf{n}$ be the outward unit normal vector on Σ_h^i oriented from $\tilde{\Omega}_h$ to $\Omega_{e,h}$. For a function $\psi \in H^1(\tilde{\Omega}_h \cup \Omega_{e,h})$, let ψ^- and ψ^+ be respectively the traces of $\psi|_{\tilde{\Omega}_h}$ and $\psi|_{\Omega_{e,h}}$ on each side of Σ_h^i. We define the arithmetic mean of traces of ψ as $\overline{\psi}|_{\Sigma_h^i} = (\psi^+ + \psi^-)/2$, and the jump of traces on Σ_h^i oriented by $\mathbf{n}$ as $[\![\psi]\!]_{\Sigma_h^i} = (\psi^+ - \psi^-)$. The fracture model with immersed transmissions conditions on Σ_h^i for convection-diffusion problems reads :
Find $u_\eta^h \in H^1(\tilde{\Omega}_h \cup \Omega_{e,h})$ (depending on the mesh $\mathcal{T}_h$ and on a parameter η precised later) such that :

$$(\mathcal{P}_t) \quad \begin{cases} -div\,(\mathbf{a}\,.\nabla u_\eta^h) + div(\mathbf{v}\,u_\eta^h) + b\,u_\eta^h = f & \text{in } \Omega \\ \text{original B.C. of } (\tilde{\mathcal{P}}) & \text{on } \tilde{\Gamma} \\ \text{suitable B.C. for } u_\eta^h & \text{on } \Gamma_e \\ [\![(\mathbf{a}.\nabla u_\eta^h)\,.\,\mathbf{n}\,]\!]_{\Sigma_h^i} = \alpha\,\overline{u}_\eta^h|_{\Sigma_h^i} - h & \text{on } \Sigma_h^i \\ \overline{(\mathbf{a}.\nabla u_\eta^h)\,.\,\mathbf{n}}|_{\Sigma_h^i} = \beta\,[\![u_\eta^h]\!]_{\Sigma_h^i} - g & \text{on } \Sigma_h^i \end{cases}$$

where $\mathbf{a}$, $\mathbf{v}$, b, and the transfer coefficients $\alpha, \beta \geq 0$ on Σ_h^i are measurable and bounded functions verifying classical ellipticity assumptions. $f \in L^2(\Omega)$, g and h are given in $L^2(\Sigma_h^i)$. Moreover,

$$\mathbf{a}|_{\tilde{\Omega}_h} = \tilde{\mathbf{a}}|_{\tilde{\Omega}_h}, \qquad \mathbf{v}|_{\tilde{\Omega}_h} = \tilde{\mathbf{v}}|_{\tilde{\Omega}_h}, \qquad b|_{\tilde{\Omega}_h} = \tilde{b}|_{\tilde{\Omega}_h} \qquad f|_{\tilde{\Omega}_h} = \tilde{f}|_{\tilde{\Omega}_h}$$

Remarks: ⋆ When $\alpha = g = h = 0$ and $\beta \to \infty$, the perfect transmission problem is recovered (see Ref.17).
⋆ Here again the choice of the B.C. on Γ_e must defined a well-posed problem $(\mathcal{P}_t)$.

B. Treatment of the original B.C.

1. Embedded Dirichlet B.C.

We present two manners to deal with a Dirichlet condition.
The first one consists in **penalizing** the exterior domain at u_D using a L^2 orH^1 **penalty**[23] ($a|_{\Omega_{e,h}} = \mathbf{Id}$ or $a|_{\Omega_{e,h}} = \frac{1}{\eta}\mathbf{Id}$, $b|_{\Omega_{e,h}} = \frac{1}{\eta}$ and $f|_{\Omega_{e,h}} = \frac{1}{\eta}u_D$) such that $\lim_{\eta\to 0} u_\eta^h|_{\Sigma_h^i}^+ = u_D$. To impose $u_\eta^h|_{\Sigma_h^i}^- \simeq u_D$, we set $\beta = \frac{1}{\eta}$ on Σ_h^i (where $0 < \eta << 1$ is a penalty parameter), thus $[\![u_\eta^h]\!]_{\Sigma_h^i} = 0$. The other transfer coefficients are equal to zero.
In the second approach, a **surface penalization** on the approximated interface Σ_h^i is performed using the transmission equations. We set $\beta = \frac{1}{\eta}$ in order to have $[\![u_\eta^h]\!]_{\Sigma_h^i} = 0$ and then $\overline{u_\eta^h}|_{\Sigma_h^i} = u_\eta^h|_{\Sigma_h^i}^- = u_\eta^h|_{\Sigma_h^i}^+ = u_\eta^h|_{\Sigma_h^i}$. In this case, penalizing also $\alpha = \frac{1}{\eta}$ and $h = \frac{1}{\eta}u_D$ induces $u_\eta^h|_{\Sigma_h^i} \simeq u_D$ on Σ_h^i. The exterior domain has no influence on the solution obtained in the physical domain.
Concerning the B.C. on Γ_e, in these two approaches, only the B.C. on $\Gamma_e \cap \Sigma_h$ are of interest : $u_\eta^h|_{\Gamma_e\cap\Sigma_h} = u_D$.

2. Embedded Robin or Neumann B.C.

We want that $-(\mathbf{a}.\nabla\mathbf{u}_\eta^\mathbf{h}).\mathbf{n}|_{\Sigma_h}^- = \alpha_R\,u_\eta^h|_{\Sigma_h}^- + g_R$ on Σ_h.
There are many ways to deal with the transfer coefficients in order to impose a Robin B.C. on Σ_h^i (see Ref. 18). One of particular interest is the one **without control on the exterior domain**. In this case, the elimination of $-(\mathbf{a}.\nabla\mathbf{u}_\eta^\mathbf{h}).\mathbf{n}|_{\Sigma_h^i}^+$ and $u_\eta^h|_{\Sigma_h^i}^+$ in the transmissions equations[18,21] yields :

$$-(\mathbf{a}.\nabla u_\eta^h).\mathbf{n}|_{\Sigma_h^i}^- = \frac{\alpha}{2}\,u_\eta^h|_{\Sigma_h^i}^- + g - \frac{h}{2} \qquad \text{with } \beta = \frac{\alpha}{4}$$

This formulation enables us to deduce transmission coefficients (see Table 2).
On $\Gamma_e \cap \Sigma_h$, we set the Robin B.C. of the original problem $(\tilde{\mathcal{P}})$. The B.C. on $\Gamma_e \backslash \Sigma_h$ are free (without effect on the solution on the physical domain as long as $(\mathcal{P}_t)$ is well-posed).

Remarks:
⋆ In this case, no penalization parameter η is required. The solution of the fictitious problem $(\mathcal{P}_t)$ depends only on the mesh $\mathcal{T}_h$.
⋆ A Dirichlet B.C. can be considered as a Robin B.C. where $\alpha_R = \dfrac{1}{\eta}$ $(\alpha_R \to \infty$ as $\eta \to 0)$ and $g_R = -\dfrac{1}{\eta}\, u_D$.

Moreover, when the interface Σ is roughly approximated by the mesh into Σ_h, a correction is required for a Robin or Neumann condition. A **characteristic parameter** ϵ is introduced in order to respect the following equality :

$$\int_\Sigma \{\alpha_R\, \tilde{u} + g_R + (\tilde{\mathbf{v}}.\mathbf{n})\, \tilde{u}\}\, \mathrm{ds} = \int_{\Sigma_h} \frac{\alpha_R\, u_\eta^{h-} + g_R + (\mathbf{v}.\mathbf{n})^-\, u_\eta^h}{\epsilon}\, \mathrm{ds} \tag{10}$$

On Σ_h, the discrete data α_R and g_R must then be divided by ϵ.

We propose two manners to approximate this characteristic parameter ϵ :

- A *global correction* :

$$\int_\Sigma \mathrm{ds} \quad = \quad \int_{\Sigma_h} \frac{1}{\epsilon}\, \mathrm{dx} \tag{11}$$

$$\epsilon \quad = \quad \frac{meas(\Sigma_h)}{meas(\Sigma)} \tag{12}$$

- A *local correction* :

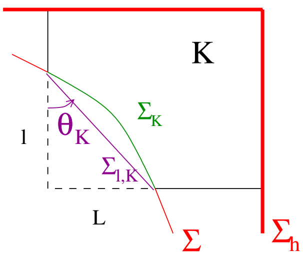


Figure 5. Local parameters in a cell K crossed by the immersed interface Σ

This correction takes account of the relative surfaces considered in each cell K of the mesh crossed by the immersed interface Σ. In 2D, with a piecewise linear approximation Σ_l of Σ composed by a segment $\Sigma_{l,K}$ in each K crossed by Σ (see figure 5), we get:

$$\epsilon_K = \frac{l + L}{\mathrm{meas}\,(\Sigma_{l,K})} = \cos\theta_K + \sin\theta_K \qquad 0 \le \theta_K \le \frac{\pi}{2} \tag{13}$$

Table 2 recapitulates the parameters of the fictitious problem $(\mathcal{P}_t)$ on the immersed approximated interface Σ_h^i and in the exterior domain $\Omega_{e,h}$. We remind that the coefficients of $(\mathcal{P}_t)$ in the approximated original domain $\tilde{\Omega}_h$ are the same than the original problem $(\tilde{\mathcal{P}})$ ones.

	Parameters on $\Sigma_h^i = \Sigma_h \backslash \partial\Omega$	Parameters in $\Omega_{e,h}$
Dirichlet B.C. *volumic exterior penalization*	$\alpha = g = h = 0$ $\beta = \frac{1}{\eta}$	$\mathbf{a} = \begin{cases} \mathbf{Id} & L^2 \text{ penalty} \\ \frac{1}{\eta}\mathbf{Id} & H^1 \text{ penalty} \end{cases}, \mathbf{v} = 0,$ $b = \frac{1}{\eta},\ f = \frac{1}{\eta} u_D$
Dirichlet B.C. *surface penalization*	$\alpha = \frac{1}{\eta},\ h = \frac{1}{\eta}$ $\beta = \frac{1}{\eta},\ g = 0$	$\mathbf{a} = \mathbf{Id},\ \mathbf{v} = 0,$ $b = f = 0$
Robin B.C. *no exterior control*	$\alpha = 2\frac{\alpha_R}{\epsilon},\ \beta = \frac{\alpha_R}{2\epsilon},$ $g - \frac{h}{2} = \frac{g_R}{\epsilon}$	$\mathbf{a} = \mathbf{Id},\ \mathbf{v} = 0,$ $b = f = 0$

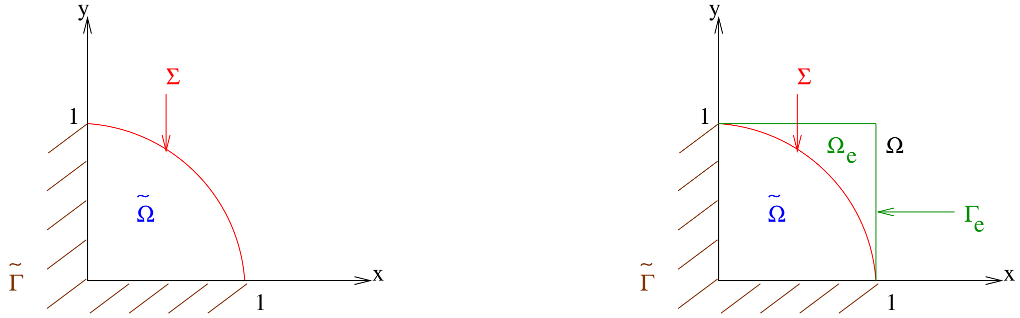
Table 2. Parameters on $\Sigma_h^i = \Sigma_h \backslash \partial\Omega$ and in $\Omega_{e,h}$ for the thin interface approach

IV. Numerical Resolution

A. Description of the test problem

We solve a 2D boundary value convection-diffusion problem in $\tilde{\Omega}$ that is a quarter of the unit disk with symmetry conditions on $\tilde{\Gamma}$ (see Figure 6(a)). The problem $(\tilde{\mathcal{P}})$ under study writes:

$$(\tilde{\mathcal{P}}) \quad \begin{cases} -\triangle \tilde{u} + div\,(\tilde{\mathbf{v}}\,\tilde{u}) = \tilde{f} & \text{in } \tilde{\Omega} \\ \nabla\tilde{u}.\mathbf{n} = 0 & \text{on } \tilde{\Gamma} \\ \tilde{u} = u_D \text{ or } \; -\nabla\tilde{u}.\mathbf{n} = \alpha_R\,\tilde{u} + g_R & \text{on } \Sigma \end{cases}$$

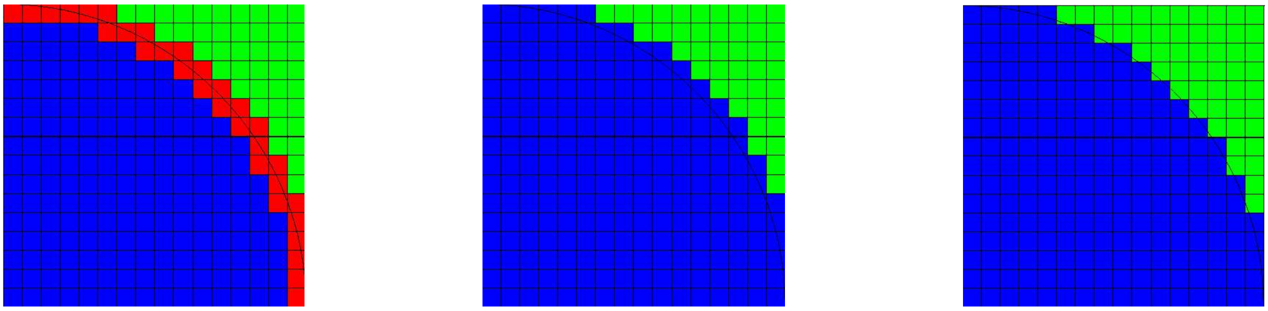


(a) Original domain $\tilde{\Omega}$ (b) Fictitious domain Ω

Figure 6. Immersion of the unit disk in the unit square

The fictitious domain is the unit square $\Omega =]0,1[\times]0,1[$ (see Figure 6(b)). The domain Ω is meshed uniformly with square cells K with a grid step varying from $h = \frac{1}{4}$ to $h = \frac{1}{256}$. This defines the *spread interface* $\omega_{h,\Sigma}$

(see Figure 7(a)) and two thin approximated interfaces : an *exterior* one (see Figure 7(b)) such that $\tilde{\Omega} \subset \tilde{\Omega}_h$ and a *cut* one (see Figure 7(c)) which may cross the physical immersed interface Σ.



(a) Spread interface (b) Exterior thin interface (c) Cut thin interface

Figure 7. Discretization of Ω and approximated interfaces

B. Numerical schemes

1. *A Finite Element (F.E.) method with an Adaptive mesh refinement (A.M.R.) method for the Spread interface approach*

A $\mathcal{Q}_1$ F.E. Scheme The spread interface approach is implemented using a $\mathcal{Q}_1$ F.E. Scheme, where $\mathcal{Q}_k$ stands for the space of polynomials of degree of each variable less than or equal to k. For example, $\mathcal{Q}_1 = span\{1, x, y, xy\}$ in $\mathbb{R}^2$. The $\mathcal{Q}_1$ discretization nodes are located on the vertex of the discrete elements.
We set :

$$\begin{cases} \mathbf{a}_h,\, b_h,\, f_h & \in \mathcal{Q}_0(\Omega_h) \\ u_h,\, \mathbf{v}_h & \in \mathcal{Q}_1(\Omega_h) \end{cases}$$

where the subscript h denotes the F.E. approximation of the original variable, and $u_h = u^h_{\eta,h}$.
A Bi-CGSTAB[26] algorithm is used to solve the linear system. Moreover a diagonal preconditioner is implemented in order to improve the ill-conditionning due to the penalization coefficients.

An A.M.R. algorithm : the Local Defect Correction[27] (L.D.C) method Using approximated interfaces, A.M.R. techniques are necessary to improve the accuracy of the solution near the immersed interface and by the way in the whole physical domain. Most of these techniques are derived from multi-grid methods.[28]
A L.D.C. method[27] is combined with the spread interface approach. This method is a multi-grid method with a defect restriction in the ascent step. As the immersed interface Σ is coarsely approximated by the spread interface $\omega_{h,\Sigma}$, the refinement area is chosen around the spread interface in order to improve the accuracy of the solution. At each level, the local refinement zone is composed by all the elements of the spread interface $\omega_{h,\Sigma}$ and their neighbors. This choice enables to correct the values of all the nodes located on the spread interface.
The combination of a fictitious domain method with an A.M.R algorithm is expected be very efficient with a relatively cheap over-cost.

2. *A cell-centered Finite Volume (F.V.) scheme for the Thin interface approach*

For each control volume $K \in \mathcal{T}_h$, a cell-centered discretization point $x_K \in K$ is chosen. Let $\mathcal{E}$ denote the family of edges σ of control volumes. The set of interior (resp. boundary) edges is denoting by $\mathcal{E}_{int}$ (resp.

$\mathcal{E}_{ext}$), that is $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}\,;\, \sigma \notin \partial\Omega\}$ (resp. $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}\,;\, \sigma \in \partial\Omega\}$). Let $\mathcal{E}_\Sigma$ be the set of edges σ lying on Σ_h^i : $\mathcal{E}_\Sigma = \{\sigma \in \mathcal{E}_{int}; \sigma \subset \Sigma_h\}$. For each $K \in \mathcal{T}_h$ or $\sigma \in \mathcal{E}$, $m(K)$ and $m(\sigma)$ represents the measure of K or σ. For given quantity ψ, the discrete values ψ_K and ψ_σ are defined by the mean value of ψ over K or σ. For each $K \in \mathcal{T}_h$ the main discrete unknown is denoted by $u_K \approx u_\eta^h(x_K)$ and, for any $(K, L) \in \mathcal{T}_h^2$ two auxiliary unknowns $(u_{\sigma,K}, u_{\sigma,L})$ are introduced on $\sigma = K|L \in \mathcal{E}_{int}$. Let $d_{K,\sigma} > 0$ be the distance from x_K to σ and $\mathbf{n}_{K,\sigma}$ be the unit normal vector to σ outward to K.
The cell-centered F.V. numerical scheme implemented is a generalization for convection-diffusion problems of the F.V. scheme proposed in Ref. 20, 21 for elliptic problems, where the numerical convection flux is obtained with an upstream scheme. This F.V. scheme allows jumps of diffusion and convection fluxes as well as jumps of solution on each edge $\sigma \in \mathcal{E}_{int}$ and it reads in the following synthetic form:

$$\sum_{\{\sigma\in\mathcal{E};\sigma\subset\partial K\}} m(\sigma)\,(F_{K,\sigma} + v_{K,\sigma}u_\sigma^{Up}) + m(K)\,b_K\,u_K = m(K)\,f_K \quad \forall K \in \mathcal{T}_h \tag{14}$$

where the numerical diffusion flux $F_{K,\sigma}$ reads (for sake of simplicity we consider an isotropic diffusion tensor $\mathbf{a}$=a(x)$\mathbf{Id}$):

$$F_{K,\sigma} = \begin{cases} -a_K \dfrac{u_{\sigma,K} - u_K}{d_{K,\sigma}} & \text{if } \sigma = K|L \in \mathcal{E}_{int} \\ -a_K \dfrac{u_\sigma - u_K}{d_{K,\sigma}} & \text{if } \sigma \in \mathcal{E}_{ext} \text{ with a Dirichlet B.C. on } \sigma:\ u_\sigma = g^D \\ \lambda u_K + g^R & \text{if } \sigma \in \mathcal{E}_{ext} \text{ with a Robin (or Neumann) B.C. on } \sigma:\ -(a\,.\nabla u).\mathbf{n} = \lambda\,u + g^R \end{cases}$$

and the numerical convection flux $V_{K,\sigma} = v_{K,\sigma}u_\sigma^{Up}$ is obtained with an upstream scheme :

$$v_{K,\sigma} = \frac{1}{m(\sigma)} \int_\sigma \mathbf{v}^-.\mathbf{n}_{K,\sigma}\,\mathrm{ds}$$

and

$$\begin{cases} \text{if } v_{K,\sigma} \geq 0 & u_\sigma^{Up} = u_K \\ \text{otherwise} & u_\sigma^{Up} = \begin{cases} u_{\sigma,K} & \text{if } (u_{\sigma,L} - u_{\sigma,K}) \neq 0 \\ u_L & \text{otherwise} \end{cases} \end{cases}$$

Moreover $\forall\sigma = K|L \in \mathcal{E}_{int}$ transmission conditions on σ yields :

$$F_{L,\sigma} + F_{K,\sigma} = \overline{\alpha}_{\sigma,K}\frac{u_{\sigma,K} + u_{\sigma,L}}{2} - \overline{h}_{\sigma,K} \tag{15}$$

$$\frac{1}{2}(F_{L,\sigma} - F_{K,\sigma}) = \overline{\beta}_{\sigma,K}(u_{\sigma,L} - u_{\sigma,K}) - \overline{g}_{\sigma,K} \tag{16}$$

with continuity of the transmissions coefficients through σ ($\overline{\alpha}_{\sigma,K} = \overline{\alpha}_{\sigma,L}$, $\overline{h}_{\sigma,K} = \overline{h}_{\sigma,L}$, $\overline{\beta}_{\sigma,K} = \overline{\beta}_{\sigma,L}$) except $\overline{g}_{\sigma,K}$ which verifies $\overline{g}_{\sigma,K} = -\overline{g}_{\sigma,L}$.
We can see that the auxiliary unknowns $u_{\sigma,L}, u_{\sigma,K}$ can be eliminated as reported in Ref. 20 and the numerical scheme is then as cheap as the standard scheme without any jump.

On the edges $\sigma \notin \mathcal{E}_\Sigma$, we set the discrete transfer coefficients of Equations (15) and (16) in order to respect the local conservativity and the solution continuity properties :

$$F_{K,\sigma} = -F_{L,\sigma}, \quad \text{and} \quad u_{\sigma,K} = u_{\sigma,L}, \quad \text{if } \sigma = K|L \in \mathcal{E}_{int}\backslash\mathcal{E}_\Sigma \tag{17}$$

Hence,

$$\overline{\alpha}_{\sigma,K} = \overline{h}_{\sigma,K} = \overline{g}_{\sigma,K} = 0 \quad \textit{and} \quad \overline{\beta}_{\sigma,K} = \frac{1}{\eta} \to \infty \quad \forall\sigma \notin \mathcal{E}_\Sigma$$

Otherwise, $\forall\sigma \in \mathcal{E}_\Sigma$, the discrete transfer coefficients are equal to the mean value on σ of the transfer coefficients of the fictitious problem $(\mathcal{P}_t)$ lying on Σ_h^i :

$$\overline{\alpha}_{\sigma,K} = \alpha_\sigma,\ \overline{h}_{\sigma,K} = h_\sigma,\ \overline{g}_{\sigma,K} = g_\sigma\,\mathbf{n}.\mathbf{n}_{K,\sigma},\ \overline{\beta}_{\sigma,K} = \beta_\sigma \quad \forall\sigma \in \mathcal{E}_\Sigma$$

where $\mathbf{n}$ denotes the outward unit normal vector of Σ_h^i on σ : $\mathbf{n} = \pm\mathbf{n}_{K,\sigma}$

As in the F.E. resolution, the linear system is solved for the family $(u_K)_{K\in\mathcal{T}_h}$ by the Bi-CGSTAB[26] algorithm with a diagonal preconditionner to improve the ill-conditionning due to the penalization coefficients.

C. Numerical results

The F.E. simulations have been computed thanks to the finite element industrial code PYGENE[29,30] of the Neptune project. This project, co-developed by the CEA and EDF, is dedicated to the simulation of two-phase flows in Nuclear Power Plants.
A F.V. code using a structured approach had been implemented for the scheme presented here.

The test problem $(\tilde{\mathcal{P}})$ has an analytical solution for a source term correctly term. The error between the numerical and analytical solutions are calculated with the discrete L^2-norm in $\tilde\Omega$.

1. *Dirichlet problem*

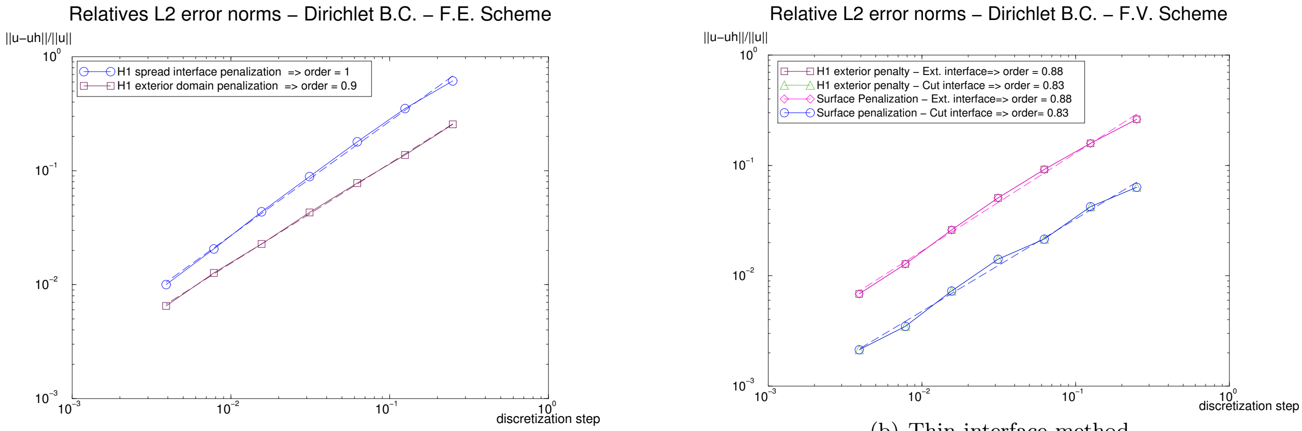
We consider the Dirichlet problem :

$$\begin{cases} -\triangle\tilde u + div(\tilde{\mathbf{v}}\tilde u) &= 4 \quad in\ \tilde\Omega \\ \dfrac{\partial\tilde u}{\partial n} &= 0 \quad on\ \tilde\Gamma \\ \tilde u = u_D &= 0 \quad on\ \Sigma \end{cases}$$

with $\tilde{\mathbf{v}} = \dfrac{r}{2}\mathbf{e_r}$ where $r = \sqrt{x^2+y^2}$ and $\mathbf{e_r}$ is the radial unit vector.
The analytic solution of this problem is :

$$\tilde u = 4\,(1 - \exp(\frac{r^2-1}{4})) \qquad in\ \tilde\Omega$$

The fictitious domain problem $(\mathcal{P}_s)$ with a spread interface approach is solved in Ω with the two Dirichlet embedded B.C. methods : the spread interface penalization and the exterior penalization (see Table 1). For the fictitious domain problem $(\mathcal{P}_t)$ with immersed jumps on the approximated thin interface, the two Dirichlet methods (see Table 2) using either a volumic H^1 penalty in $\Omega_{e,h}$ or a surface penalty on Σ_h without exterior control, are computed with both the exterior and the cut approximated interface Σ_h (see Figures 7(b) and 7(c)). The following results have been performed with $\eta = 10^{-12}$ to get the modelling error negligible compared to the discretization error.
The Figure 8 shows the convergence of the numerical error with respect to the discretization step h.
As excepted since $|\text{meas}(\tilde\Omega_h) - \text{meas}(\tilde\Omega)| = \mathcal{O}(h)$ in both the spread and the thin interface approaches, all the Dirichlet embedded B.C. variants are of first-order for the L^2-norm.
For the spread interface approach implemented with a $\mathcal{Q}_1$ F.E. scheme, the penalization of the exterior domain is more accurate than the penalization of the spread interface. Indeed, by performing the $\mathcal{Q}_1$ F.E. scheme (see in section 1), the reaction coefficient is computed by element ($b_h \in \mathcal{Q}_0$). All the nodes belonging to a penalized element are then penalized. The spread interface penalization induces the penalization at u_D of all the spread interface nodes. Hence, nodes inside the original domain $\tilde\Omega$ are penalized. The exterior penalization imposes $u_h \simeq u_D$ on the whole exterior domain and on the exterior nodes of the spread interface. In the original quarter disk domain, interior penalized nodes are globally farther of the physical interface



(a) Spread interface method (b) Thin interface method

Figure 8. Discretization errors for a Dirichlet B.C. with the two fictitious domain approaches

Σ than the exterior nodes (see Figure 7(a)). However, the accuracy of the different penalization methods strictly depends on the geometry of the original domain $\tilde{\Omega}$.
In the F.V. approach, we can observe that the volumic H^1 penalty of the exterior domain and the surface penalization on Σ_h lead to the same errors. Indeed, performing a H^1 penalty method on $\Omega_{e,h}$, the solution and its gradient are penalized. So $u \simeq u_D$ on the exterior domain $\Omega_{e,h}$ until the approximated interface Σ_h. For the two penalization approaches, the cut approximated interface leads to more accurate results than the exterior one since the cut interface approximates Σ more precisely than the exterior interface.

For the spread interface approach with an exterior volumic H^1 penalization, the Figure 9 shows the difference between the approximated solution and the analytic one. We can observe the main errors are located around the spread interface.

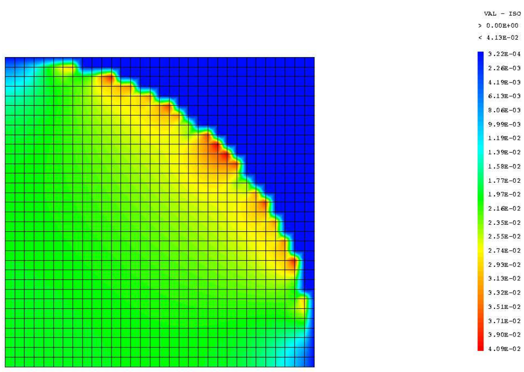


Figure 9. Error distribution for the H^1 exterior penalization - F.E Scheme - Dirichlet case - 32x32 mesh

An adaptive mesh refinement is then performed in this zone. On each local grid, a $\mathcal{Q}_1$ F.E. scheme with an H^1 exterior domain penalization method is computed. A three-grid LDC algorithm (two refinement levels) is applied on each original mesh. The local refinement zone is composed by the elements of the spread interface $\omega_{h,\Sigma}$ and their neighbors. This algorithm converges by three V-cycles.

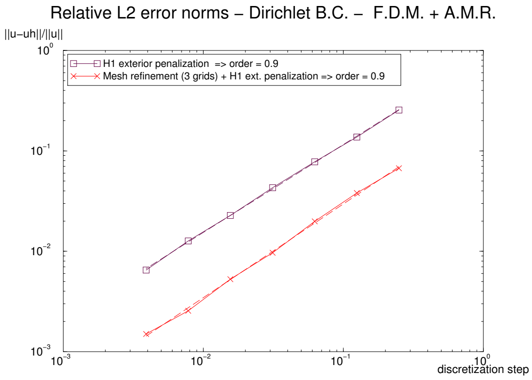


Figure 10. Errors with or without A.M.R. for a Dirichlet embedded B.C.- Spread interface approach with H^1 exterior penalization - F.E. Scheme

As shown in Figure 10, the errors obtained with a L.D.C. algorithm vary like $\mathcal{O}(h_f)$ where h_f is the discretization step of the local finest refinement grid (last level of refinement). The error on the original coarse grid is similar to the error obtained without refinement on a mesh with a discretization step equal to the local finest grid's one. The use of a local A.M.R. combined with the fictitious domain method allows to increase the precision of the solution even if the method remains first-order.

2. Robin problem

We now consider the following Robin problem :

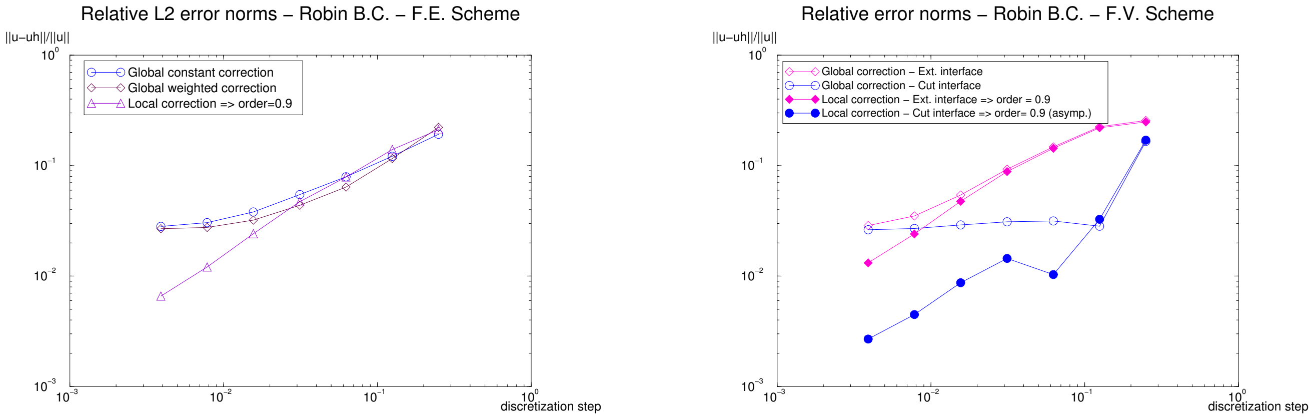
$$\begin{cases} -\triangle\tilde{u} + div(\tilde{\mathbf{v}}\tilde{u}) & = & 16r^2 & in\ \tilde{\Omega} \\ \dfrac{\partial\tilde{u}}{\partial n} & = & 0 & on\ \tilde{\Gamma} \\ \dfrac{\partial\tilde{u}}{\partial n} & = & \tilde{u}+3 & on\ \Sigma\ (\alpha_R = 1,\, g_R = 3) \end{cases}$$

with $\tilde{\mathbf{v}} = 2r^3\mathbf{e_r}$ where $r = \sqrt{x^2+y^2}$ and $\mathbf{e_r}$ is the radial unit vector.
The analytic solution of this problem is :

$$\tilde{u} = 2 - \frac{5}{3}\exp(\frac{r^4-1}{2}) \qquad in\ \tilde{\Omega}$$

The fictitious domain problem $(\mathcal{P}_s)$ is solved in Ω with an embedded Robin B.C. in the spread interface (see Table 1). The results obtained with the three approximations of the characteristic parameter ϵ discussed in section 2 are reported on Figure 11(a).
For the resolution of the fictitious domain problem $(\mathcal{P}_t)$, the method to enforce an embedded Robin B.C. on a thin approximated interface without exterior control (see Table 2) is implemented. In Figure 11(b) we investigate the effect of the two ϵ corrections introduced in section 2 for the two kinds of approximated interface (exterior and cut).

In the two fictitious domain methods, a global approximation of the characteristic parameter ϵ (see Eqs. (6), (7) and (12)) leads to an asymptotic stagnation of the error and then the first-order precision is lost. With a local correction (Eqs. (9) and (13)) no stagnation appears and the asymptotically first-order accuracy is then yielded for the L^2-norm error. For the thin interface approach with a cut approximated interface, the first-order accuracy appears asymptotically for meshes with a discretization step $h \leq \dfrac{1}{32}$. Here again,



(a) Spread interface method (b) Thin interface method

Figure 11. Discretization errors for a Robin B.C. with the two fictitious domain approaches

the fictitious domain method with a cut approximated interface gives a better precision than an exterior approximated interface.
For Robin problems, a local correction is thus required to keep the first-order method.

For the fictitious domain method with spread interface, a local A.M.R. algorithm is performed on the method involving a local epsilon (Eq. (13)). As in the Dirichlet case, we compute a three-grid LDC algorithm, which converges by three V-cycles.

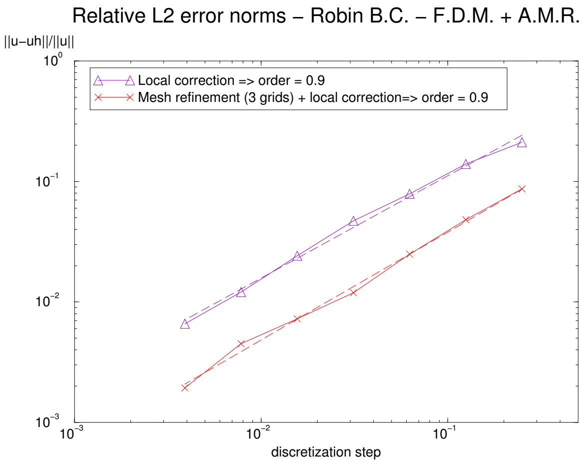


Figure 12. Errors with and without A.M.R. for a Robin embedded B.C. - Spread interface method with a local correction - F.E. Scheme

We can observe on Figure 12, that the combination of the spread interface fictitious domain method with an A.M.R. method leads to a discretization error in $\mathcal{O}(h_f)$ with h_f the discretization step of the local finest grid.

V. Conclusion and Perspectives

Two accurate fictitious domain approaches to solve convection-diffusion problems have been introduced. They are based either on a spread interface or a thin interface approach and handle general embedded boundary conditions (Dirichlet, Robin and Neumann). Two numerical schemes are used to compute these methods : a $\mathcal{Q}_1$ F.E. scheme for the spread interface approach and a F.V. scheme with jumps of fluxes and solutions for the thin interface approach. The main advantage of these methods is their weak costs : the numerical resolution is computed on a single Cartesian mesh without modifying locally the numerical scheme or introducing local unknowns. Even if these methods remain of first order, a cheap adaptive multi-level local mesh refinement algorithm can be easily computed to increase the precision of the solution.

Such fictitious domain methods are full of promise, especially to simulate moving boundaries with a cheap computational cost as no re-meshing is required. The next step will consist in extending these fictitious domain methods to Navier-Stokes equations with moving boundaries problems like two phase flows simulations or fluid/structure interactions.

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