

# RANK FOUR VECTOR BUNDLES WITHOUT THETA DIVISOR OVER A CURVE OF GENUS TWO

CHRISTIAN PAULY

ABSTRACT. We show that the locus of stable rank four vector bundles without theta divisor over a smooth projective curve of genus two is in canonical bijection with the set of theta-characteristics. We give several descriptions of these bundles and compute the degree of the rational theta map.

## 1. INTRODUCTION

Let  $C$  be a complex smooth projective curve of genus 2 and let  $\mathcal{M}_r$  denote the coarse moduli space parametrizing semi-stable rank- $r$  vector bundles with trivial determinant over the curve  $C$ . Let  $\Theta \subset \text{Pic}^1(C)$  be the Riemann theta divisor in the degree 1 component of the Picard variety of  $C$ . For any  $E \in \mathcal{M}_r$  we consider the locus

$$\theta(E) = \{L \in \text{Pic}^1(C) \mid h^0(C, L \otimes E) > 0\},$$

which is either a curve linearly equivalent to  $r\Theta$  or  $\theta(E) = \text{Pic}^1(C)$ , in which case we say that  $E$  has no theta divisor. We obtain thus a rational map, the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|,$$

between varieties having the same dimension  $r^2 - 1$ . We denote by  $\mathcal{B}_r$  the closed subvariety of  $\mathcal{M}_r$  parametrizing semi-stable bundles without theta divisor. It is known [R] that  $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$  and that  $\mathcal{B}_r \neq \emptyset$  for  $r \geq 4$ .

It was recently shown that  $\theta$  is generically finite; see [B1] Theorem A. Moreover the cases of low ranks  $r$  have been studied in the past: if  $r = 2$  the theta map is an isomorphism  $\mathcal{M}_2 \cong \mathbb{P}^3$  [NR] and if  $r = 3$  the theta map realizes  $\mathcal{M}_3$  as a double covering of  $\mathbb{P}^8$  ramified along a sextic hypersurface [O].

In this note we study the next case  $r = 4$  and give a complete description of the locus  $\mathcal{B}_4$ . Our main result is the following

**Theorem 1.1.** *Let  $C$  be a curve of genus 2.*

- (1) *The locus  $\mathcal{B}_4$  is of dimension 0, reduced and of cardinality 16.*
- (2) *There exists a canonical bijection between  $\mathcal{B}_4$  and the set of theta-characteristics of  $C$ . Let  $E_\kappa \in \mathcal{B}_4$  denote the stable vector bundle associated with the theta-characteristic  $\kappa$ . Then*

$$\Lambda^2 E_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 E_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha,$$

where  $S(\kappa)$  is the set of 2-torsion line bundles  $\alpha \in J[2]$  such that  $\kappa\alpha \in \Theta \subset \text{Pic}^1(C)$ .

- (3) *If  $\kappa$  is odd, then  $E_\kappa$  is a symplectic bundle. If  $\kappa$  is even, then  $E_\kappa$  is an orthogonal bundle with non-trivial Stiefel-Whitney class.*
- (4) *The 16 vector bundles  $E_\kappa$  are invariant under the tensor product with the group  $J[2]$ .*

---

2000 *Mathematics Subject Classification.* Primary 14H60, 14D20.

The 16 vector bundles  $E_\kappa$  already appeared in Raynaud's paper [R] as Fourier-Mukai transforms and were further studied in [Hi] and [He] — see section 2.2. We note that Theorem 1.1 completes the main result of [Hi] which describes the restriction of  $\mathcal{B}_4$  to *symplectic* rank-4 bundles. The method of this paper is different and is partially based on [P].

As an application of Theorem 1.1 we obtain the degree of the theta map for  $r = 4$ . We refer to [BV] for a geometric interpretation of the general fiber of  $\theta$  in terms of certain irreducible components of a Brill-Noether locus of the curve  $\theta(E) \subset \text{Pic}^1(C)$ .

**Corollary 1.2.** *The degree of the rational theta map  $\theta : \mathcal{M}_4 \dashrightarrow |4\Theta|$  equals 30.*

*Acknowledgements:* I am grateful to George Hitching and Olivier Serman for useful discussions.

*Notations:* If  $E$  is a vector bundle over  $C$ , we will write  $H^i(E)$  for  $H^i(C, E)$  and  $h^i(E)$  for  $\dim H^i(C, E)$ . We denote the slope of  $E$  by  $\mu(E) := \frac{\deg E}{\text{rk} E}$ , the canonical bundle over  $C$  by  $K$  and the degree  $d$  component of the Picard variety of  $C$  by  $\text{Pic}^d(C)$ . We denote by  $J := \text{Pic}^0(C)$  the Jacobian of  $C$  and by  $J[n]$  its group of  $n$ -torsion points. The divisor  $\Theta_\kappa \subset J$  is the translate of the Riemann theta divisor  $C \cong \Theta \subset \text{Pic}^1(C)$  by a theta-characteristic  $\kappa$ . The line bundle  $\mathcal{O}_J(2\Theta_\kappa)$  does not depend on  $\kappa$  and will be denoted by  $\mathcal{O}_J(2\Theta)$ .

## 2. PROOF OF THEOREM 1.1

**2.1. The 16 vector bundles  $E_\kappa$ .** We first show that the set-theoretical support of  $\mathcal{B}_4$  consists of 16 stable vector bundles  $E_\kappa$ , which are canonically labelled by the theta-characteristics of  $C$ .

We note that  $\mathcal{B}_4 \neq \emptyset$  by [R], see also [P] Theorem 1.1. We consider a vector bundle  $\mathcal{E} \in \mathcal{B}_4$ . Since  $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$ , we deduce that  $\mathcal{E}$  is stable. We introduce  $\mathcal{E}' = \mathcal{E}^* \otimes K$ . Then  $\mu(\mathcal{E}') = 2$  and since  $\mathcal{E} \in \mathcal{B}_4$ , we obtain that  $h^0(\mathcal{E}' \otimes \lambda^{-1}) = h^1(\mathcal{E} \otimes \lambda) = h^0(\mathcal{E} \otimes \lambda) > 0$  for any  $\lambda \in \text{Pic}^1(C)$ . In particular for any  $x \in C$  we have  $h^0(\mathcal{E}' \otimes \mathcal{O}_C(-x)) > 0$ . On the other hand stability of  $\mathcal{E}$  implies that  $h^0(\mathcal{E}) = h^1(\mathcal{E}') = 0$ . Hence  $h^0(\mathcal{E}') = 4$  by Riemann-Roch. Thus we obtain that the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} \mathcal{E}'$$

is not of maximal rank. Let us denote by  $I := \text{im } ev$  the subsheaf of  $\mathcal{E}'$  given by the image of  $ev$ . Then clearly  $h^0(I) = 4$ . The cases  $\text{rk } I \leq 2$  are easily ruled out using stability of  $\mathcal{E}'$ . Hence we conclude that  $\text{rk } I = 3$ . We then consider the natural exact sequence

$$(1) \quad 0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} I \longrightarrow 0,$$

where  $L$  is the line bundle such that  $L^{-1} := \ker ev$ .

**Proposition 2.1.** *We have  $h^0(I^*) = 0$ .*

*Proof.* Suppose on the contrary that there exists a non-zero map  $I \rightarrow \mathcal{O}_C$ . Its kernel  $S \subset I$  is a rank-2 subsheaf of  $\mathcal{E}'$  and by stability of  $\mathcal{E}'$  we obtain  $\mu(S) < \mu(\mathcal{E}') = 2$ , hence  $\deg S \leq 3$ . Moreover  $h^0(S) \geq h^0(I) - 1 = 3$ .

Assume that  $\deg S = 3$ . Then  $S$  is stable and  $S$  can be written as an extension

$$0 \longrightarrow \mu \longrightarrow S \longrightarrow \nu \longrightarrow 0,$$

with  $\deg \mu = 1$  and  $\deg \nu = 2$ . The condition  $h^0(S) \geq 3$  then implies that  $\mu = \mathcal{O}_C(x)$  for some  $x \in C$ ,  $\nu = K$  and that the extension has to be split, i.e.,  $S = K \oplus \mathcal{O}_C(x)$ . This contradicts stability of  $S$ .

The assumption  $\deg S \leq 2$  similarly leads to a contradiction. We leave the details to the reader.  $\square$

Now we take the cohomology of the dual of the exact sequence (1) and we obtain — using  $h^0(I^*) = 0$  — an inclusion  $H^0(\mathcal{E}')^* \subset H^0(L)$ . Hence  $h^0(L) \geq 4$ , which implies  $\deg L \geq 5$ . On the other hand  $\deg L = \deg I$  and by stability of  $\mathcal{E}'$ , we have  $\mu(I) < 2$ , i.e.,  $\deg L \leq 5$ . So we can conclude that  $\deg L = 5$ , that  $H^0(\mathcal{E}')^* = H^0(L)$  and that  $I = E_L$ , where  $E_L$  is the *evaluation bundle* associated to  $L$  defined by the exact sequence

$$(2) \quad 0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0.$$

Moreover the subsheaf  $E_L \subset \mathcal{E}'$  is of maximal degree, hence  $E_L$  is a subbundle of  $\mathcal{E}'$  and we have an exact sequence

$$(3) \quad 0 \longrightarrow E_L \longrightarrow \mathcal{E}' \longrightarrow K^4L^{-1} \longrightarrow 0,$$

with extension class  $e \in \text{Ext}^1(K^4L^{-1}, E_L) = H^1(E_L \otimes K^{-4}L) = H^0(E_L^* \otimes K^5L^{-1})^*$ . Using Riemann-Roch and stability of  $E_L$  (see e.g. [Bu]) one shows that

$$h^0(E_L^* \otimes K^5L^{-1}) = 7, \quad h^0(E_L^* \otimes K^5L^{-1}(-x)) = 4, \quad h^0(E_L^* \otimes K^5L^{-1}(-x-y)) = 1$$

for *general* points  $x, y \in C$ . In that case we denote by  $\mu_{x,y} \in \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$  the point determined by the 1-dimensional subspace  $H^0(E_L^* \otimes K^5L^{-1}(-x-y))$ . We also denote by

$$\mathbb{S} \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$$

the linear span of the points  $\mu_{x,y}$  when  $x$  and  $y$  vary in  $C$  and by  $H_e \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$  the hyperplane determined by the non-zero class  $e$ .

Tensoring the sequence (3) with  $K^{-4}L(x+y)$  and taking cohomology one shows that  $\mu_{x,y} \in H_e$  if and only if  $h^0(\mathcal{E}' \otimes K^{-4}L(x+y)) > 0$ . Since we assume  $\mathcal{E} \in \mathcal{B}_4$ , we obtain

$$\mathbb{S} \subset H_e.$$

We consider a *general* point  $x \in C$  such that  $h^0(E_L^* \otimes K^5L^{-1}(-x)) = 4$  and denote for simplicity

$$A := E_L^* \otimes K^5L^{-1}(-x).$$

Then  $A$  is stable with  $\mu(A) = \frac{7}{3}$ . We consider the evaluation map of global sections

$$ev_A : \mathcal{O}_C \otimes H^0(A) \longrightarrow A$$

and consider the set  $S_A$  of points  $p \in C$  for which  $(ev_A)_p$  is not surjective, i.e.

$$S_A = \{p \in C \mid h^0(A(-p)) \geq 2\}.$$

Then we have the following

**Lemma 2.2.** *We assume that  $x$  is general.*

- (1) *If  $L^2 \neq K^5$ , then the set  $S_A$  consists of the 2 distinct points  $p_1, p_2$  determined by the relation  $\mathcal{O}_C(p_1 + p_2) = K^4L^{-1}(-x)$ .*
- (2) *If  $L^2 = K^5$ , then the set  $S_A$  consists of the 2 distinct points  $p_1, p_2$  introduced in (1) and the conjugate  $\sigma(x)$  of  $x$  under the hyperelliptic involution  $\sigma$ .*

*Proof.* Given a point  $p \in C$ , we tensorize the exact sequence (2) with  $K^5L^{-1}(-x-p)$  and take cohomology:

$$0 \longrightarrow H^0(A(-p)) \longrightarrow H^0(L) \otimes H^0(K^5L^{-1}(-x-p)) \longrightarrow H^0(K^5(-x-p)) \longrightarrow \dots$$

We note that  $h^0(K^5L^{-1}(-x-p)) = 2$ . We distinguish two cases.

- (a) The pencil  $|K^5L^{-1}(-x-p)|$  has a base-point, i.e. there exists a point  $q \in C$  such that  $K^5L^{-1}(-x-p) = K(q)$ , or equivalently  $K^4L^{-1}(-x) = \mathcal{O}_C(p+q)$ . Since  $x$  is general, we have  $h^0(K^4L^{-1}(-x)) = 1$ , which determines  $p$  and  $q$ , i.e.,  $\{p, q\} = \{p_1, p_2\}$ . In this case  $|K^5L^{-1}(-x-p)| = |K(q)| = |K|$  and  $h^0(A(-p)) = h^0(K^{-1}L) = 2$ . This shows that  $p_1, p_2 \in S_A$ .

(b) The pencil  $|K^5L^{-1}(-x-p)|$  is base-point-free. By the base-point-free-pencil-trick, we have  $H^0(A(-p)) \cong H^0(L^2K^{-5}(x+p))$ . Since  $\deg L^2K^{-5}(x+p) = 2$ , we have  $h^0(L^2K^{-5}(x+p)) = 2$  if and only if  $L^2K^{-5}(x+p) = K$ , or equivalently  $\mathcal{O}_C(p) = K^6L^{-2}(-x)$ . If  $K^6L^{-2} \neq K$ , then for general  $x \in C$  the line bundle  $K^6L^{-2}(-x)$  is not of the form  $\mathcal{O}_C(p)$ . If  $K^6L^{-2} = K$ , then for any  $x \in C$ ,  $K^6L^{-2}(-x) = \mathcal{O}_C(\sigma(x))$ , which implies that  $\sigma(x) \in S_A$ .

This shows the lemma.  $\square$

**Proposition 2.3.** *If  $L^2 \neq K^5$ , then  $\mathbb{S} = \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$ .*

*Proof.* We consider a general point  $x \in C$  and the rank-3 bundle  $A$ . Let  $B \subset A$  denote the subsheaf given by the image of  $ev_A$ . By Lemma 2.2 (1) we have  $\deg B = \deg A - 2 = 5$ . Moreover  $H^0(B) = H^0(A)$  and there is an exact sequence

$$(4) \quad 0 \longrightarrow M^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(B) \xrightarrow{ev_A} B \longrightarrow 0,$$

with  $M \in \text{Pic}^5(C)$ . It follows that the rational map

$$\phi_x : C \dashrightarrow \mathbb{P}H^0(B) = \mathbb{P}H^0(A) = \mathbb{P}^3, \quad y \mapsto \mu_{x,y}$$

factorizes through

$$C \xrightarrow{\varphi_M} |M|^* \longrightarrow \mathbb{P}H^0(B),$$

where  $\varphi_M$  is the morphism given by the linear system  $|M|$  and the second map is linear and identifies with the projectivization of the dual of  $\delta$ , which is given by the long exact sequence obtained from (4) by dualizing and taking cohomology:

$$0 \longrightarrow H^0(B^*) \longrightarrow H^0(B)^* \xrightarrow{\delta} H^0(M) \longrightarrow H^1(B^*) \longrightarrow \dots$$

We obtain that the linear span of  $\text{im } \phi_x$  is non-degenerate if and only if  $h^0(B^*) = 0$ .

We now show that  $h^0(B^*) = 0$ . Suppose on the contrary that there exists a non-zero map  $B \rightarrow \mathcal{O}_C$ . Its kernel  $S \subset B$  is a rank-2 subsheaf of  $A$  with  $\deg S \geq \deg B = 5$ , hence  $\mu(S) \geq \frac{5}{2}$ , which contradicts stability of  $A$  — recall that  $\mu(A) = \frac{7}{3}$ .

This shows that  $\text{im } \phi_x$  spans  $\mathbb{P}H^0(A) \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$  for general  $x \in C$ . We now take 2 general points  $x, x' \in C$  and deduce from  $\dim H^0(A) \cap H^0(A') = \dim H^0(E_L^* \otimes K^5L^{-1}(-x-x')) = 1$  that the linear span of the union  $\mathbb{P}H^0(A) \cup \mathbb{P}H^0(A')$  equals the full space  $\mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$ . This shows the proposition.  $\square$

We deduce from the proposition that the line bundle  $L$  satisfies the relation  $L^2 = K^5$ , i.e.

$$L = K^2\kappa$$

for some theta-characteristic  $\kappa$  of  $C$ . In that case we note that  $H^0(E_L^* \otimes K^5L^{-1})$  equals  $H^0(E_L^* \otimes L)$  and we can consider the exact sequence

$$0 \longrightarrow H^0(E_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0,$$

obtained from (2) by tensoring with  $L$  and taking cohomology. We also note that there is a natural inclusion  $\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L)$ , see e.g. [P] section 2.1. More precisely we can show

**Proposition 2.4.** *The linear span  $\mathbb{S}$  equals*

$$\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L).$$

*Proof.* Using the standard exact sequences and the base-point-free-pencil-trick, one easily works out that for general points  $x, y \in C$

$$\mu_{x,y} = \mathbb{P}\Lambda^2 H^0(L(-x-y)) \subset \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L).$$

This implies that  $\mathbb{S} \subset \mathbb{P}\Lambda^2 H^0(L)$ . In order to show equality one chooses 4 general points  $x_i \in C$  such that their images  $C \rightarrow |L|^* = \mathbb{P}^3$  linearly span the  $\mathbb{P}^3$ . We denote by  $s_i \in H^0(L)$  the global

section vanishing on the points  $x_j$  for  $j \neq i$  and not vanishing on  $x_i$ . Then one checks that for any choice of the indices  $i, j, k, l$  such that  $\{i, j, k, l\} = \{1, 2, 3, 4\}$  one has  $s_i \wedge s_j = \mu_{x_k, x_l}$ . Since the 6 tensors  $s_i \wedge s_j$  are a basis of  $\Lambda^2 H^0(L)$ , we obtain equality.  $\square$

The hyperplane  $\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L)$  determines a unique (up to a scalar) non-zero extension class  $e \in H^0(E_L^* \otimes L)^*$  by  $\mathbb{S} = H_e$ , which in turn determines a unique stable vector bundle  $\mathcal{E} \in \mathcal{B}_4$ , which we will denote by  $E_\kappa$ .

This shows that  $\mathcal{B}_4$  is of dimension 0 and of cardinality 16.

**2.2. The Raynaud bundles.** In this subsection we recall the construction of the Raynaud bundles introduced in [R] as Fourier-Mukai transforms. We refer to [Hi] section 9.2 for the details and the proofs.

The rank-4 vector bundle  $\mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*$  over  $J$  admits a canonical  $J[2]$ -linearization and descends therefore under the duplication map  $[2] : J \rightarrow J$ , i.e., there exists a rank-4 vector bundle  $M$  over  $J$  such that

$$[2]^* M \cong \mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*.$$

**Proposition 2.5.** *For any theta-characteristic  $\kappa$  of  $C$  there exists an isomorphism*

$$\xi_\kappa : M \xrightarrow{\sim} M^* \otimes \mathcal{O}_J(\Theta_\kappa).$$

*Moreover if  $\kappa$  is even (resp. odd), then  $\xi_\kappa$  is symmetric (resp. skew-symmetric).*

Let  $\gamma_\kappa : C \rightarrow J$  be the Abel-Jacobi map defined by  $\gamma_\kappa(p) = \kappa^{-1}(p)$ . We define the Raynaud bundle

$$R_\kappa := \gamma_\kappa^* M \otimes \kappa^{-1}.$$

Then by [R] the bundle  $R_\kappa \in \mathcal{B}_4$ . Since  $\gamma_\kappa^* \mathcal{O}_J(\Theta_\kappa) = K$  we see that the isomorphism  $\xi_\kappa$  induces an orthogonal (resp. symplectic) structure on the bundle  $R_\kappa$ , if  $\kappa$  is even (resp. odd). In particular the bundle  $R_\kappa$  is self-dual, i.e.,  $R_\kappa = R_\kappa^*$ . The pull-back  $\gamma_{\kappa'}^*(\xi'_\kappa)$  for a theta-characteristic  $\kappa' = \kappa\alpha$  with  $\alpha \in J[2]$  gives an isomorphism

$$R_\kappa \xrightarrow{\sim} R_\kappa^* \otimes \alpha,$$

hence a non-zero section in  $H^0(\Lambda^2 R_\kappa \otimes \alpha)$  (resp.  $H^0(\text{Sym}^2 R_\kappa \otimes \alpha)$ ) if  $h^0(\kappa\alpha) = 1$  (resp.  $h^0(\kappa\alpha) = 0$ ). We deduce that there are isomorphisms

$$(5) \quad \Lambda^2 R_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 R_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha.$$

In particular the 16 bundles  $R_\kappa$  are non-isomorphic. Each  $R_\kappa$  is invariant under tensor product with  $J[2]$ . The isomorphisms (5) can be used to prove the relation

$$(6) \quad R_\kappa \otimes \beta = R_{\kappa\beta^2}, \quad \forall \beta \in J[4].$$

**2.3. Symplectic and orthogonal bundles.** In this subsection we give a third construction of the bundles in  $\mathcal{B}_4$  as symplectic and orthogonal extension bundles. Let  $\kappa$  be a theta-characteristic.

If  $\kappa$  is odd, then  $\kappa = \mathcal{O}_C(w)$  for some Weierstrass point  $w \in C$ . The construction outlined in [P] section 2.2 gives a unique symplectic bundle  $\mathcal{E}_e \in \mathcal{B}_4$  with  $e \in H^1(\text{Sym}^2 G)_+$ . We denote this bundle by  $V_\kappa$ .

If  $\kappa$  is even, there is an analogue construction, which we briefly outline for the convenience of the reader. The proofs are similar to those given in [Hi]. Using the Atiyah-Bott-fixed-point formula one observes that among all non-trivial extensions

$$0 \longrightarrow \kappa^{-1} \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

there are 2 extensions (up to scalar), which are  $\sigma$ -invariant. We take one of them. Then any non-zero class  $e \in H^1(\Lambda^2 G) = H^1(\kappa^{-1})$  determines an orthogonal bundle  $\mathcal{E}_e$ , which fits in the exact sequence

$$(7) \quad 0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

The composite map

$$D_G : \mathbb{P}H^1(\Lambda^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_G : H^1(\Lambda^2 G) \longrightarrow H^0(\text{Pic}^1(C), 4\Theta).$$

Moreover  $\text{im } \widetilde{D}_G \subset H^0(\text{Pic}^1(C), 4\Theta)_-$ , which can be seen as follows. By [Se] Thm 2 the second Stiefel-Whitney class  $w_2(\mathcal{E}_e)$  of an orthogonal bundle  $\mathcal{E}_e$  is given by the parity of  $h^0(\mathcal{E}_e \otimes \kappa')$  for any theta-characteristic  $\kappa'$ . This parity can be computed by taking the cohomology of the exact sequence (7) tensorized with  $\kappa'$  and taking into account that the coboundary map is skew-symmetric. One obtains that  $w_2(\mathcal{E}_e) \neq 0$  and one can conclude the above-mentioned inclusion by [B2] Lemma 1.4.

We now observe that by the Atiyah-Bott-fixed-point-formula  $h^1(\Lambda^2 G)_+ = h^1(\Lambda^2 G)_- = 1$ . By the argument given in [P] section 2.2 we conclude that one of the two eigenspaces  $H^1(\Lambda^2 G)_\pm$  is contained in the kernel  $\ker \widetilde{D}_G$ . We denote the corresponding bundle  $\mathcal{E}_e$  by  $V_\kappa \in \mathcal{B}_4$ .

#### 2.4. Three descriptions of the same bundle.

**Proposition 2.6.** *For any theta-characteristic  $\kappa$  the three bundles  $E_\kappa$ ,  $R_\kappa$  and  $V_\kappa$  coincide.*

*Proof.* If  $\kappa$  is odd, this was worked out in detail in [Hi] section 8 and Theorem 29. If  $\kappa$  is even, the proofs are similar.  $\square$

This proposition shows all assertions of Theorem 1.1 except reducedness of  $\mathcal{B}_4$ .

I am grateful to Olivier Serman for giving me the following fourth description of the bundle  $E_\kappa$  for an even theta-characteristic  $\kappa$ . We recall that an even theta-characteristic  $\kappa$  corresponds to a partition of the set of six Weierstrass points of  $C$  into two subsets of three points, which we denote by  $\{w_1, w_2, w_3\}$  and  $\{w_4, w_5, w_6\}$ . With this notation we have

**Proposition 2.7.** *Let  $\kappa$  be an even theta-characteristic. We denote by  $A_\kappa$  (resp.  $B_\kappa$ ) the unique stable rank-2 bundle with determinant  $\kappa$  and which contains the four 2-torsion line bundles  $\mathcal{O}_C$ ,  $\mathcal{O}_C(w_1 - w_2)$ ,  $\mathcal{O}_C(w_1 - w_3)$  and  $\mathcal{O}_C(w_2 - w_3)$  (resp.  $\mathcal{O}_C$ ,  $\mathcal{O}_C(w_4 - w_5)$ ,  $\mathcal{O}_C(w_4 - w_6)$  and  $\mathcal{O}_C(w_5 - w_6)$ ). Then the orthogonal rank-4 vector bundle  $E_\kappa$  is isomorphic to*

$$\text{Hom}(A_\kappa, B_\kappa)$$

*equipped with the quadratic form given by the determinant.*

We refer to [S] section 5.5 for the proof.

**2.5. Reducedness of  $\mathcal{B}_4$ .** We denote by  $\mathcal{L}$  the determinant line bundle over the moduli space  $\mathcal{M}_4$  and recall that the set  $\mathcal{B}_4$  can be identified with the base locus of the linear system  $|\mathcal{L}|$ . This endows the set  $\mathcal{B}_4$  with a natural scheme-structure.

We start with a description of the space of global sections  $H^0(\mathcal{M}_4, \mathcal{L})$ .

**Proposition 2.8.** *For any theta-characteristic  $\kappa$  there is a section  $s_\kappa \in H^0(\mathcal{M}_4, \mathcal{L})$  with zero divisor*

$$\Delta_\kappa := \text{Zero}(s_\kappa) = \{E \in \mathcal{M}_4 \mid h^0(\Lambda^2 E \otimes \kappa) > 0\}.$$

*The 16 sections  $s_\kappa$  form a basis of  $H^0(\mathcal{M}_4, \mathcal{L})$ .*

*Proof.* The Dynkin index of the second fundamental representation  $\rho : \mathfrak{sl}_4(\mathbb{C}) \rightarrow \text{End}(\Lambda^2 \mathbb{C}^4)$  equals 2 (see e.g. [LS] Proposition 2.6). Moreover the bundle  $\Lambda^2 E \otimes \kappa$  admits a  $K$ -valued non-degenerate quadratic form, which allows to construct the Pfaffian divisor  $s_\kappa$ , which is a section of  $\mathcal{L}$  (see [LS]). The space  $H^0(\mathcal{M}_4, \mathcal{L})$  is a representation of level 2 of the Heisenberg group  $Heis(2)$ , which is a central extension of  $J[2]$  by  $\mathbb{C}^*$ . One can work out that the sections  $s_\kappa$  generate the 16 one-dimensional character spaces for the  $Heis(2)$ -action on  $H^0(\mathcal{M}_4, \mathcal{L})$ . This shows that the sections  $s_\kappa$  are linearly independent.  $\square$

Since  $E_\kappa \in \mathcal{B}_4$ , we have  $E_\kappa \in \Delta_{\kappa'}$  for any theta-characteristic  $\kappa'$ . By the deformation theory of determinant and Pfaffian divisors (see e.g. [L], [LS]) the point  $E_\kappa \in \mathcal{M}_4$  is a smooth point of the divisor  $\Delta_{\kappa'} \subset \mathcal{M}_4$  if and only if the following two conditions hold

- (1)  $h^0(\Lambda^2 E_\kappa \otimes \kappa') = 2$ ,
- (2) the natural linear form

$$\Phi_{\kappa'} : T_{E_\kappa} \mathcal{M}_4 = H^1(\text{End}_0(E_\kappa)) \longrightarrow \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa')^*$$

is non-zero.

Moreover if these two conditions holds, then  $T_{E_\kappa} \Delta_{\kappa'} = \ker \Phi_{\kappa'}$ . The map  $\Phi_{\kappa'}$  is built up as follows: the exceptional isomorphism of Lie algebras  $\mathfrak{sl}_4 \cong \mathfrak{so}_6$  induces a natural vector bundle isomorphism

$$(8) \quad \text{End}_0(E_\kappa) \xrightarrow{\sim} \Lambda^2(\Lambda^2 E_\kappa).$$

Then  $\Phi_{\kappa'}$  is the dual of the linear map given by the wedge product of global sections

$$\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \longrightarrow H^0(\Lambda^2(\Lambda^2 E_\kappa) \otimes K) = H^0(\text{End}_0(E_\kappa) \otimes K).$$

**Proposition 2.9.** *The 0-dimensional scheme  $\mathcal{B}_4$  is reduced.*

*Proof.* Since  $E_\kappa$  is a smooth point of  $\mathcal{M}_4$  and  $\dim T_{E_\kappa} \mathcal{M}_4 = 15$ , it is sufficient to show that for any theta-characteristic  $\kappa' \neq \kappa$  the divisor  $\Delta_{\kappa'}$  is smooth at  $E_\kappa$  and that the 15 hyperplanes  $\ker \Phi_{\kappa'} \subset T_{E_\kappa} \mathcal{M}_4$  are linearly independent: using the isomorphism (5) we obtain that for  $\kappa' \neq \kappa$

$$h^0(\Lambda^2 E_\kappa \otimes \kappa') = \sharp S(\kappa) \cap S(\kappa') = 2$$

and using the isomorphism (8) we obtain that

$$\text{End}_0(E_\kappa) = \bigoplus_{\alpha \in J[2] \setminus \{0\}} \alpha.$$

On the other hand one easily sees that if  $\gamma, \delta \in J[2]$  are the two 2-torsion points in the intersection  $S(\kappa) \cap S(\kappa')$ , then  $\kappa' = \kappa\gamma\delta$ , hence  $\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \cong H^0(K\gamma\delta)$ . This implies that the linear form

$$\Phi_{\kappa'} : \bigoplus_{\alpha \in J[2] \setminus \{0\}} H^1(\alpha) \longrightarrow H^0(K\gamma\delta)^* = H^1(\beta)$$

is projection onto the direct summand  $H^1(\beta)$ , where  $\beta = \kappa^{-1}\kappa' \in J[2]$ . This description of the linear forms  $\Phi_{\kappa'}$  clearly shows that they are non-zero and linearly independent.  $\square$

This completes the proof of Theorem 1.1.

## 3. PROOF OF COROLLARY 1.2

Since by Theorem 1.1  $\mathcal{B}_4$  is a reduced 0-dimensional scheme of length 16, the degree of the theta map  $\theta$  is given by the formula

$$\deg \theta + 16 = c_{15},$$

where  $\frac{c_{15}}{15!}$  is the leading coefficient of the Hilbert polynomial

$$P(n) = \chi(\mathcal{M}_4, \mathcal{L}^n) = \frac{c_{15}}{15!}n^{15} + \text{lower degree terms.}$$

In order to compute the polynomial  $P$  we write

$$(9) \quad P(X) = \sum_{k=0}^{15} \alpha_k Q_k(X), \quad \text{with} \quad Q_k(X) = \frac{1}{k!}(X+7)(X+6)\cdots(X+8-k)$$

and  $Q_0(X) = 1$ . Note that  $\deg Q_k = k$  and that  $c_{15} = \alpha_{15}$ . The canonical bundle of  $\mathcal{M}_4$  equals  $\mathcal{L}^{-8}$ . By the Grauert-Riemenschneider vanishing theorem we obtain that  $h^i(\mathcal{M}_4, \mathcal{L}^n) = 0$  for any  $i \geq 1$  and  $n \geq -7$ . Hence  $P(n) = h^0(\mathcal{M}_4, \mathcal{L}^n)$  for  $n \geq -7$ . Moreover  $P(n) = 0$  for  $n = -7, -6, \dots, -1$  and  $P(0) = 1$ . The values  $P(n)$  for  $n = 1, 2, \dots, 8$  can be computed by the Verlinde formula and with the use of MAPLE. They are given in the following table.

$n$	1	2	3	4	5	6	7	8
$P(n)$	16	140	896	4680	21024	83628	300080	984539

Using the expression (9) of  $P$  one straightforwardly deduces the coefficients  $\alpha_k$  by increasing induction on  $k$ :  $\alpha_k = 0$  for  $k = 0, 1, \dots, 6$  and the values  $\alpha_k$  for  $k = 7, \dots, 15$  are given in the following table.

$k$	7	8	9	10	11	12	13	14	15
$\alpha_k$	1	8	32	96	214	328	324	184	46

Hence  $\deg \theta = \alpha_{15} - 16 = 30$ .

## REFERENCES

- [B1] A. Beauville: Vector bundles and theta functions on curves of genus 2 and 3, Amer. J. of Math. 128 (2006), 607-618
- [B2] A. Beauville: Orthogonal bundles on curves and theta functions, Ann. Inst. Fourier 56 (2006), 1405-1418
- [BV] S. Brivio, A. Verra: The Brill-Noether curve of a stable vector bundle on a genus two curve, in "Algebraic cycles and motives" (Leiden 2004), to appear
- [Bu] D. Butler: Normal generation of vector bundles over a curve, J. Differential Geom. 39 (1994), 1-34
- [He] G. Hein: Raynaud's vector bundles and base points of the generalized Theta divisor, Math. Zeitschrift 257 (2007), 597-611
- [Hi] G. Hitching: Rank four symplectic bundles without theta divisor over a curve of genus two, math.AG/0604637, to appear in International J. of Math.
- [L] Y. Laszlo: Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables sur une courbe, Duke Math. J. 64 (1991), 333-347
- [LS] Y. Laszlo, C. Sorger: The line bundles of the moduli of parabolic  $G$ -bundles over curves and their sections, Ann. Scient. Ec. Norm. Sup. 30 (1997), 499-525
- [NR] M.S. Narasimhan, S. Ramanan: Moduli of vector bundles on a compact Riemann surface, Ann. of Math. (2) 89 (1969), 14-51
- [O] A. Ortega: On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic, J. Algebraic Geom. 14 (2005), 327-356
- [P] C. Pauly: On the base locus of the linear system of generalised theta functions, arXiv:0707.3364, to appear in Mathematical Research Letters
- [R] M. Raynaud: Sections de fibrés vectoriels sur une courbe, Bull. Soc. Math. France, 110 (1982), 103-125

- [S] O. Serman: Espaces de modules de fibrés orthogonaux sur une courbe algébrique, Thèse de Doctorat (2007), Université de Nice-Sophia-Antipolis
- [Se] J.-P. Serre: Revêtements à ramification impaire et thêta-caractéristiques, C. R. Acad. Sci. Paris Math. 311 (1990), 547-552

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONTPELLIER II - CASE COURRIER 051, PLACE  
EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE  
*E-mail address:* `pauly@math.univ-montp2.fr`