
UNDECIDABLE PROPERTIES ON THE DYNAMICS OF REVERSIBLE ONE-DIMENSIONAL CELLULAR AUTOMATA

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ABSTRACT. Many properties of the dynamics of one-dimensional cellular automata are known to be undecidable. However, the undecidability proofs often rely on the undecidability of the nilpotency problem, and hence cannot be applied in the case the automaton is reversible. In this talk we review some recent approaches to prove dynamical properties of reversible 1D CA undecidable. Properties considered include equicontinuity (=periodicity), sensitivity, variants of mortality, one-sided expansivity and regularity. All these properties are undecidable, according to recent proofs obtained in collaboration with N.Ollinger or V.Lukkarila.

Introduction

Let S be a finite set – the state alphabet – and consider bi-infinite sequences of symbols of S , called configurations over S . The set $S^{\mathbb{Z}}$ of all configurations is endowed with the standard compact and metrizable topology obtained as the countably infinite product of the discrete topology on S . A one-dimensional cellular automaton (CA for short) over state set S is a continuous transformation $G : S^{\mathbb{Z}} \rightarrow S^{\mathbb{Z}}$ that commutes with the shift function $\sigma : S^{\mathbb{Z}} \rightarrow S^{\mathbb{Z}}$, defined by $\sigma(x)_i = x_{i+1}$ for all $x \in S^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

It follows from the classical Garden-of-Eden theorem that every injective CA transformation G is also surjective, and hence bijective. Because $S^{\mathbb{Z}}$ is compact, the inverse function G^{-1} is continuous and commutes with the shift — it is also a CA. We call an injective G a reversible cellular automaton, and G^{-1} its inverse automaton. See [6] for more details and further historical results.

To treat cellular automata algorithmically one needs a finite representation for them. This is provided by the famous Curtis-Hedlund-Lyndon theorem, which states that cellular automata are exactly the functions that can be defined by a simultaneous, parallel application of a local update rule [3]. More precisely, a non-negative integer r (called the

Key words and phrases: cellular automata.

Research supported by the Academy of Finland grant 211967.

neighborhood radius) and a local rule $f : S^{2r+1} \rightarrow S$ define the CA function G over state set S where

$$G(x)_i = f(x_{i-r}, \dots, x_{i+r})$$

for all $i \in \mathbb{Z}$. All cellular automata functions G arise in this fashion.

Many dynamical properties of cellular automata are known to be undecidable: there is no algorithm that would determine if a given CA (given in terms of its local update rule) has the property. Examples of such undecidable properties include nilpotency, equicontinuity and sensitivity (see Section 3 below). These properties are known to be undecidable among the general, not necessarily reversible cellular automata. The undecidability proofs rely mostly on nilpotency, which is a property possessed only by non-reversible CA. Until recently, no analogous undecidability results were known for reversible CA. Yet reversible CA constitute an important family of dynamical systems whose properties and long time behavior should be better understood. In this paper we report some recent undecidability results for a few dynamical properties for reversible CA. The results are obtained with N.Ollinger and V.Lukkarila. We give the results without proofs here — the proofs will appear elsewhere.

The paper is organized as follows: in Section 1 we define and discuss the dynamical properties studied in this paper. In Section 2 we briefly recall some concepts and results on Wang tiles. Section 3 reviews some basic undecidability results for non-reversible CA, and in Section 4 we give undecidability results for reversible CA.

1. Dynamical properties

This paper summarizes some undecidability results on dynamical properties of reversible one-dimensional cellular automata. Properties considered are equicontinuity (=periodicity), sensitivity, variants of mortality, one-sided expansivity and regularity. In this section we define these concepts.

First some general terms and notations: elements $i \in \mathbb{Z}$ are referred to as cells. For configuration $x \in S^{\mathbb{Z}}$ and integers $m \leq n$ we use the notation $x_{[[m,n]]}$ to denote the word $x_m x_{m+1} \dots x_n$. Configuration x is called spatially periodic if $\sigma^m(x) = x$ for some $m > 0$. Spatially periodic configurations form a dense subset of $S^{\mathbb{Z}}$. Configuration x is called temporally periodic (or just periodic) for G if $G^p(x) = x$ for some period $p > 0$, and it is called (temporally) eventually periodic for G if $G^{m+p}(x) = G^m(x)$ for some pre-period $m \geq 0$ and period $p > 0$. A spatially periodic configuration is always eventually periodic. In reversible CA eventually periodic configurations are all periodic, so in reversible CA periodic configurations are dense. A well known open problem asks whether the same holds for all surjective CA.

1.1. Equicontinuity points

Configuration x is an *equicontinuity point* for G if for all $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all configurations y

$$x_{[[-M, M]]} = y_{[[-M, M]]} \implies G^i(x)_{[[-m, m]]} = G^i(y)_{[[-m, m]]} \text{ for all } i \in \mathbb{N}.$$

The evolution of equicontinuity points can be reliably simulated up to any precision on a computer: all forthcoming states within an observation window of radius m are uniquely determined by the initial states within a window of radius M .

If G is reversible one may be interested in the evolution of x both in backwards and forwards in time. Let us define *two-way equicontinuity* of x analogously, only difference being that time i now takes also negative values: for all m there exists M such that for all $y \in S^{\mathbb{Z}}$

$$x_{[-M,M]} = y_{[-M,M]} \implies G^i(x)_{[-m,m]} = G^i(y)_{[-m,m]} \text{ for all } i \in \mathbb{Z}.$$

The two definitions, however, coincide:

Lemma 1.1. *Let G be reversible. Point $x \in S^{\mathbb{Z}}$ is an equicontinuity point for G if and only if it is a two-way equicontinuity point for G .*

Proof. Let x be a configuration that is not a two-way equicontinuity point for reversible G . Then for some window radius m holds the following: for all M there is some configuration y and time $i \in \mathbb{Z}$ such that $x_{[-M,M]} = y_{[-M,M]}$ while $G^i(x)_{[-m,m]} \neq G^i(y)_{[-m,m]}$. Periodic configurations are dense, so there are periodic x', y' such that

$$x'_{[-M,M]} = x_{[-M,M]} = y_{[-M,M]} = y'_{[-M,M]}$$

and

$$G^i(x')_{[-m,m]} = G^i(x)_{[-m,m]} \neq G^i(y)_{[-m,m]} = G^i(y')_{[-m,m]}.$$

Since x' and y' are temporally periodic, we have a positive j such that

$$G^j(x')_{[-m,m]} = G^i(x')_{[-m,m]} \neq G^i(y')_{[-m,m]} = G^j(y')_{[-m,m]}.$$

Then, either $G^j(x')_{[-m,m]} \neq G^j(x)_{[-m,m]}$ or $G^j(y')_{[-m,m]} \neq G^j(x)_{[-m,m]}$. In any case, a configuration contradicting the equicontinuity of x exists for all M , that is, x is not equicontinuous.

The other direction is trivial. ■

1.2. Equicontinuity

Cellular automaton G is called *equicontinuous* if all configurations are equicontinuity points. It easily follows from the compactness of $S^{\mathbb{Z}}$ that the radius M of the initial window can be chosen independently of configuration x , that is, G is equicontinuous iff the following holds:

$$(\forall m \in \mathbb{N})(\exists M \in \mathbb{N})(\forall x, y \in S^{\mathbb{Z}}) \\ x_{[-M,M]} = y_{[-M,M]} \implies G^i(x)_{[-m,m]} = G^i(y)_{[-m,m]} \text{ for all } i \in \mathbb{N}.$$

For reversible G one can define *two-way equicontinuity* by considering also negative time $i < 0$ (that is, by requiring all configurations to be two-way equicontinuity points) but according to Lemma 1.1 the concept obtained would be equivalent to normal equicontinuity.

A CA is called *periodic (eventually periodic)* if all configurations are periodic (eventually periodic, respectively). One can easily see that the pre-period m and period p can be then chosen independently of the configuration x , so that a CA is periodic (eventually periodic) if and only if there exists p (or m and p) such that $G^p(x) = x$ (or $G^{m+p}(x) = G^m(x)$, respectively) for all x .

It is known that one-dimensional equicontinuous cellular automata are exactly the eventually periodic CA, and among surjective CA equicontinuity is equivalent to periodicity [12, 2]. In particular, all surjective, equicontinuous CA are reversible. These facts are summarized in the following theorem:

Theorem 1.2. *The following hold [12, 2]:*

- (a) *One-dimensional CA is equicontinuous if and only if it is eventually periodic.*
- (b) *One-dimensional surjective CA is equicontinuous if and only if it is periodic.*

A cellular automaton is called *nilpotent* if there exists a positive integer i such that $G^i(S^{\mathbb{Z}})$ contains only one element. Nilpotent CA have the most trivial dynamics possible: all activity dies out within time i . It is clear that a nilpotent CA is eventually periodic, and hence equicontinuous.

1.3. Sensitivity

A CA is *sensitive*, if there is an observation window radius $m \in \mathbb{N}$ such that for every initial configuration x and every $M \in \mathbb{N}$ there exists a configuration y and time $i \in \mathbb{N}$ such that $x_{[-M, M]} = y_{[-M, M]}$ but $G^i(x)_{[-m, m]} \neq G^i(y)_{[-m, m]}$. In other words, every configuration x can be modified arbitrarily far away in space in such a way that the modification eventually propagates inside the observation window. Note that the definition states that in sensitive CA no configuration is equicontinuous, and moreover, the observation window size m that contradicts the equicontinuity at point x can be chosen independently of x . It turns out that the second condition is automatically satisfied: If a one-dimensional CA has no equicontinuity points then arbitrarily distant modifications can be made to any configuration in such a way that the change propagates into the observation window whose radius m equals the neighborhood radius r of the CA [12]. So we have the following:

Theorem 1.3. *A one-dimensional CA is sensitive if and only if it has no equicontinuity points [12].*

A CA is called *almost equicontinuous* if it has some equicontinuity points. Almost equicontinuity and sensitivity are complementary properties.

Two-way sensitivity of a reversible CA is defined analogously to sensitivity, only difference being that the time i when $G^i(x)_{[-m, m]} \neq G^i(y)_{[-m, m]}$ may be negative. It is easy to see, analogously to Theorem 1.3, that a reversible one-dimensional CA is two-way sensitive if and only if it has no two-way equicontinuity points. It follows then from Lemma 1.1 that sensitivity and two-way sensitivity are equivalent concepts among reversible CA.

1.4. Expansivity

CA G is *positively expansive* if any difference in configurations eventually propagates inside a fixed observation window: there exists $m \in \mathbb{N}$ such that

$$x \neq y \implies G^i(x)_{[-m, m]} \neq G^i(y)_{[-m, m]} \text{ for some } i \in \mathbb{N}.$$

A reversible CA is *expansive* if time i is allowed to have a negative value: there exists $m \in \mathbb{N}$ such that

$$x \neq y \implies G^i(x)_{[-m, m]} \neq G^i(y)_{[-m, m]} \text{ for some } i \in \mathbb{Z}.$$

According to our naming convention above, expansivity should be termed two-way expansivity, but we rather decided to follow the historical terminology. Unlike for the dynamical properties discussed so far, the concepts of expansivity and positive expansivity are not equivalent. For example, the shift function σ is expansive but not positively expansive. In fact, it is easy to see that a reversible CA can never be positively expansive.

Expansivity and positive expansivity of one-dimensional CA have quite natural left- and right-sided variants. Let us call a CA *positively left-expansive* if there exists $m \in \mathbb{N}$ such that

$$x_{[[0,\infty)} \neq y_{[[0,\infty)} \implies G^i(x)_{[[-m,m]]} \neq G^i(y)_{[[-m,m]]} \text{ for some } i \in \mathbb{N}.$$

Analogously, in *positively right-expansive* CA differences propagate to the right. A CA is clearly positively expansive iff it is both positively left- and positively right-expansive. The shift σ is an example of a positively left expansive CA. For reversible CA we analogously define *left-* and *right-expansive* CA by allowing time i to obtain also negative values: G is left-expansive if for some $m \in \mathbb{N}$ holds

$$x_{[[0,\infty)} \neq y_{[[0,\infty)} \implies G^i(x)_{[[-m,m]]} \neq G^i(y)_{[[-m,m]]} \text{ for some } i \in \mathbb{Z}.$$

We see in Theorem 4.4 below that it is undecidable whether a given reversible CA is left-expansive.

1.5. Equicontinuity classification of CA

In [12] Kurka proposed a classification of one-dimensional cellular automata into four classes based on their degree of sensitivity to initial conditions. The classes are as follows:

- (K1) Equicontinuous CA.
- (K2) Almost equicontinuous CA that are not equicontinuous.
- (K3) Sensitive CA that are not positively expansive.
- (K4) Positively expansive CA.

Each 1D CA belongs to exactly one of these classes. First three classes are based on the number of equicontinuity points (none, some but not all, and all configurations are equicontinuity points, respectively). No reversible CA is in class (K4), so when classifying reversible CA it makes sense to replace (K3) and (K4) by

- (K3') Sensitive CA that are not expansive.
- (K4') Expansive CA.

1.6. Mortality

Mortality of a dynamical system refers to the property that the evolution leads from every initial configuration into an "accepting" configuration. In cellular automata theory, acceptance can be defined in various ways which leads to different variants of mortality. We consider two variants.

Let us specify a set $F \subseteq S$ of accepting states. In our first variant acceptance happens when some cell enters an accepting state, and in the second variant acceptance happens when a fixed, predefined cell enters an accepting state. This leads to the following definitions: G is *globally mortal* with respect to set $F \subseteq S$ if for every $x \in S^{\mathbb{Z}}$ there are $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ such that $G^i(x)_j \in F$. It is *locally mortal* with respect to set F if for every $x \in S^{\mathbb{Z}}$ there is $i \in \mathbb{N}$ such that $G^i(x)_0 \in F$. Every locally mortal CA is also globally mortal, while the converse is not true.

1.7. Column subshifts and the language classification of CA

In [12] also another classification of CA was proposed based on the complexity of their column subshifts. A column subshift is the set of all possible temporal sequences of views to the CA configurations through a fixed finite window. More precisely, let G be a one-dimensional CA and let k be a positive integer, the width of the observation window. The column subshift $\Sigma_k(G)$ of width k associated to G is

$$\Sigma_k(G) = \{y \in (S^k)^{\mathbb{N}} \mid \exists x \in S^{\mathbb{Z}} : y_i = G^i(x)_{[[1,k]]} \text{ for all } i \in \mathbb{N}\}.$$

It is a one-sided subshift over the alphabet S^k . Term *trace subshift* was used in [4] for the column subshift $\Sigma_1(G)$ of width one.

The column language $L_k(G)$ of width k is defined as the set of finite subwords of sequences in $\Sigma_k(G)$. It is a language over the alphabet S^k . Language $L_k(G)$ is always context-sensitive [14].

As in several sections above, if G is reversible then it makes sense to allow also negative time. The two-way column subshift of width k associated to G is

$$\{y \in (S^k)^{\mathbb{Z}} \mid \exists x \in S^{\mathbb{Z}} : y_i = G^i(x)_{[[1,k]]} \text{ for all } i \in \mathbb{Z}\}.$$

Among reversible (and even surjective) CA this does not, however, bring any new information on the dynamics because the language of this two-sided subshift is the same $L_k(G)$ extracted from the one-sided case.

Cellular automaton G is called *regular* if $\Sigma_k(G)$ is a sofic shift for every $k > 0$. This is equivalent to $L_k(G)$ being a regular language. Many questions on the dynamics of a given CA become decidable when restricted to regular CA [10]. This fact is largely based on the following result:

Theorem 1.4. *If G is regular then the column subshift $\Sigma_k(G)$ of width k can be effectively constructed, for every $k > 0$ [10].*

CA G is equicontinuous if and only if it is eventually periodic with some pre-period m and period p (see Theorem 1.2). This is equivalent to the column subshift $\Sigma_k(G)$ being bounded periodic: For all $y \in \Sigma_k(G)$ we have $y_i = y_{i+p}$ whenever $i \geq m$. It is easy to see that such subshifts are always sofic (even of finite type), so all equicontinuous CA are regular.

The language classification of CA proposed in [12] has three classes:

- (L1) Equicontinuous CA (the column subshifts are all bounded periodic).
- (L2) Regular but not equicontinuous CA (the column subshifts are sofic but not bounded periodic).
- (L3) Non-regular CA (a column subshift is not sofic).

2. Wang tiles

Wang tiles and the tiling problem have been used in many undecidability proofs concerning cellular automata. While the tiling problem relates naturally to two-dimensional CA, also limiting behavior of one-dimensional CA can be treated by interpreting the space-time diagrams as plane tilings. This leads naturally to the definition of determinism in Wang tiles.

A Wang tile is an oriented unit square tile with labeled edges. We denote by $N(t)$, $E(t)$, $S(t)$ and $W(t)$ the label of the north, east, south and west edge of tile t , respectively.

A tile set T is a finite set of such tiles. A valid tiling by T is an assignment $c \in T^{\mathbb{Z}^2}$ of tiles on two-dimensional lattice in such a way that abutting edges of adjacent tiles are everywhere identical, that is, for every $(i, j) \in \mathbb{Z}^2$ we have $N(c(i, j)) = S(c(i, j + 1))$ and $E(c(i, j)) = W(c(i + 1, j))$.

The tiling problem is the algorithmic question of determining whether a given set T admits at least one valid tiling. This problem was proved undecidable by R.Berger in [13]. This is the starting point in many reductions to prove undecidability results for cellular automata.

To deal with one-dimensional CA we add the following constraint on tile sets: tile set T is *NW-deterministic* if the colors of the north and the west edges determine each tile uniquely, that is, for all $t, u \in T$

$$N(t) = N(u) \wedge W(t) = W(u) \implies t = u.$$

We define analogously NE- SW- and SE-determinism. Tile set T is called *two-way deterministic* if it is both NW- and SE-deterministic (i.e. deterministic in two opposite directions), and T is *four-way deterministic* if determinism holds in all four directions.

If c is a valid tiling admitted by a NW-deterministic tile set T then any southwest-to-northeast diagonal of c uniquely determines the next diagonal below. A local rule using just two tiles of the previous diagonal gives each tile. Valid tilings become space-time diagrams of a one-dimensional CA if we interpret the diagonals as configurations of the CA. With a similar interpretation a two-way deterministic tile set yields a reversible CA. This method is better explained below in the proofs of Theorems 3.1 and 4.1.

The following result was proved in [15]. A weaker version dealing with NW-deterministic tile sets only was proved earlier in [5]. The result provides a basis for some of the undecidability results reported here:

Theorem 2.1. *It is undecidable whether a given 4-way deterministic tile set admits a valid tiling [15].*

3. Non-reversible CA

In this section we review some undecidability results concerning general, not necessarily reversible CA. Using the undecidability of the tiling problem among NW-deterministic tile sets (Theorem 2.1) we easily obtain the following:

Theorem 3.1. *It is undecidable whether a given one-dimensional CA is nilpotent [5]. The question is undecidable even among CA over the binary state set $S = \{0, 1\}$ [1].*

Proof. For a given NW-deterministic set T of Wang tiles we construct a one-dimensional CA over state set $S = T \cup \{q\}$ where $q \notin T$ is a new state. The local update rule only uses the state s_1 of the cell and state s_2 of its right neighbor. If $s_1, s_2 \in T$ and there exists a tile $t \in T$ such that

$$W(t) = E(s_1) \text{ and } N(t) = S(s_2)$$

then the new state of the cell is t . (Note that this t is unique due to NW-determinism.) In all other cases the new state is q . It is clear that the configuration ${}^\omega q^\omega = \dots qqqq \dots$ is a fixed point of this CA. Any southwest-to-northeast diagonal of a valid tiling is a configuration that never evolves to this fixed point, so if T admits a valid tiling then the CA is not nilpotent.

Conversely, if a valid tiling does not exist then, by a compactness argument, the $n \times n$ square can not be properly tiled for some n . This means that inside every segment of length n state q will appear within the first $2n$ time steps. As state q spreads the configuration becomes ${}^\omega q^\omega$ in at most $3n$ time steps. Hence the CA is nilpotent. The first result now follows from the undecidability of the tiling problem among NW-deterministic tile sets.

See [1] for the technique to reduce the state set into the binary alphabet. ■

Corollary 3.2. *It is undecidable whether a given 1D CA is equicontinuous [1]. It is undecidable whether it is sensitive [1]. These two properties are recursively inseparable. This is true even among CA over the binary state set $S = \{0, 1\}$.*

Proof. Let G be an arbitrary 1D CA of radius r . Then clearly $G \circ \sigma^{r+1}$ is nilpotent (and hence equicontinuous) if G is nilpotent, and it is sensitive if G is not nilpotent. ■

Among equicontinuity classification, the decidability status of positive expansivity remains a challenging open problem. Concerning the mortality of non-reversible CA, we have the following easy corollary:

Corollary 3.3. *Local mortality and global mortality are undecidable among one-dimensional CA.*

Proof. Let $F = \{q\}$ where q is the external non-tile state in the proof of Theorem 3.1. The CA constructed in the proof is locally (and globally) mortal with respect to F if and only if the tile set does not admit a valid tiling. ■

Finally, it was shown in [10] that it is undecidable to determine if a given one-dimensional CA is regular. In fact, nilpotency and non-regularity are recursively inseparable properties. In particular, this means that the classes (L1) and (L3) in the language classification of Section 1.7 are recursively inseparable from each other:

Corollary 3.4. *It is undecidable whether a given one-dimensional CA is regular [10]. More precisely, nilpotency and non-regularity are recursively inseparable.*

Proof. Suppose there exists an algorithm to separate nilpotent CA from non-regular CA. Then we can decide nilpotency of a given G as follows: The separating algorithm either tells that G is definitely not nilpotent or that G is definitely regular. In the first case we know the non-nilpotency of G directly, and in the second case we can effectively construct the trace subshift $\Sigma_1(G)$ of G using Theorem 1.4. Note that G is nilpotent if and only if $\Sigma_1(G)$ is eventually constant, that is, there is state $s \in S$ and number N such that $y_i = s$ for all $y \in \Sigma_1(G)$ and $i > N$. This condition can be easily verified from the finite automaton representation of the trace subshift. ■

4. The case of reversible CA

4.1. Mortality

Analogously to Theorem 3.1, a direct application of the undecidability of the tiling problem among two-way deterministic tile sets was used in [9] to show that global mortality is undecidable among reversible CA:

Theorem 4.1. *It is undecidable whether a given reversible 1D CA is globally mortal. This holds even if the CA is known to be expansive [9].*

Proof. For a given two-way deterministic tile set T we start by adding to it tiles until we have a tile set $S = T \cup F$ that is two-way deterministic and complete in the sense that for every horizontal color h and vertical color v there is a unique tile $t \in S$ with $N(t) = h$ and $W(t) = v$, as well as a unique tile $t' \in S$ with $S(t) = h$ and $E(t) = v$. The added tile set F can be effectively constructed by arbitrarily pairing the missing horizontal/vertical color pairs among NW- and SE-edges in T .

One can next construct a one-dimensional CA over the state set S analogously to the proof of Theorem 3.1: Let s_1 and s_2 be the states of a cell and its right neighbor, respectively. Then the new state of the cell is the unique tile $t \in S$ such that

$$W(t) = E(s_1) \text{ and } N(t) = S(s_2).$$

It follows from two-way determinism of S that the CA is reversible.

The CA is globally mortal with respect to the added tiles F if and only if T does not admit a tiling. Indeed, if T admits a tiling then a diagonal of a valid tiling is a configuration whose space-time diagram only contains elements of T . So the CA is not globally F -mortal.

Conversely, if T does not admit a tiling then there exists a number n such that T does not admit a tiling of an $n \times n$ square. Then every segment of length n will contain a state in F within $2n$ time steps, so the CA is globally mortal.

The first part of the theorem now follows from the undecidability of the tiling problem among two-way deterministic tile sets.

For the second part we observe that global mortality is invariant under composing the CA with the shift: G is globally mortal if and only if $G \circ \sigma^{r+1}$ globally mortal. But $G \circ \sigma^{r+1}$ is always expansive where r denotes the neighborhood radius. ■

A layer of binary signals was introduced in [9] to obtain the following result.

Theorem 4.2. *It is undecidable whether a given reversible 1D CA is locally mortal. This holds even if the CA is known to be left-expansive [9].*

Proof. We reduce global immortality to local immortality. Let G be an expansive one-dimensional CA, with state set $S = T \cup F$ where $T \cap F = \emptyset$. Let us construct a new CA G' with state set $S \times \{0, 1\}^2$. Each state has two binary signals associated with it. The S -states evolve according to G . Normally (if the underlying S -state belongs to T) the two signals travel left and right, respectively. An exception happens in every cell whose state belongs to F : If the incoming right and left moving signals have binary values a and b , respectively, the outgoing right and left moving signals will have binary values $1 + b \pmod{2}$ and $a + b \pmod{2}$, respectively. Let F' (the new acceptance set) consist of all states that contain a signal with binary value 1.

It is easy to see that G' is left-expansive, and it is locally mortal w.r.t F' if and only if G is globally mortal w.r.t F . The result now follows from Theorem 4.1. ■

If one executes the constructions in the proofs of Theorems 4.1 and 4.2 starting with an aperiodic, two-way deterministic tile set T then the final CA G' is left-expansive but not regular. Aperiodic two-way deterministic tile sets must exist due to the undecidability of the tiling problem — and an aperiodic four-way deterministic example was even explicitly constructed in [8] — so we have the following corollary:

Corollary 4.3. *There are left-expansive reversible CA that are not regular.*

By adding yet another layer of signals we can reduce local mortality among left expansive CA into the problem of determining if a given CA is left-expansive:

Theorem 4.4. *It is undecidable whether a given reversible 1D CA is left-expansive [9].*

Proof. Let G be a left-expansive CA over the state set $S = T \cup F$ where $T \cap F = \emptyset$. Construct a new CA G' with two additional binary signals in each cell. The first signal shifts left while the second one does not move. In addition, in each cell where the underlying S -state belongs to F the two signals are swapped. It is easy to see that G' is left-expansive if and only if G is locally mortal with respect to F . The result now follows from Theorem 4.2. ■

It is not known whether expansivity of reversible CA is decidable. Note that if locally mortality were undecidable among expansive CA then the construction we used in Theorem 4.4 would show that expansivity is undecidable. Global mortality is known to be undecidable among expansive CA (Theorem 4.1), but it remains a challenge to reduce global mortality to local mortality while preserving expansivity. For left-expansive CA this was successfully done in the proof of Theorem 4.2.

4.2. Equicontinuity

Using a reduction from the mortality problem of reversible Turing machine it is shown in [7] that it is undecidable if a given one-dimensional CA is periodic:

Theorem 4.5. *It is undecidable whether a given reversible 1D CA is periodic [7].*

Using Theorem 1.2(b) we directly obtain the following:

Corollary 4.6. *It is undecidable if a given reversible one-dimensional CA is equicontinuous [7].*

By observing that periodicity of a sofic shift is decidable, one sees analogously to Corollary 3.4 that the classes (L1) and (L3) in the language classification of Section 1.7 are recursively inseparable even among reversible CA.

Corollary 4.7. *It is undecidable if a given reversible one-dimensional CA is regular. More precisely, periodicity and non-regularity are recursively inseparable among reversible one-dimensional CA [16].*

Proof. Suppose there exists an algorithm to separate periodic from non-regular reversible CA. Then we can decide periodicity of given G as follows: The separating algorithm either tells that G is definitely not periodic or that G is definitely regular. In the first case we know the non-periodicity of G directly, and in the second case we can effectively construct the trace $\Sigma_1(G)$ of G using Theorem 1.4. It is an easy matter to effectively determine if $\Sigma_1(G)$ is periodic, given the finite automaton representation of its language. ■

Finally, we mention without proof that recently V.Lukkarila reduced the halting problem of reversible Turing machines to show that sensitivity is undecidable among reversible CA:

Theorem 4.8. *It is undecidable whether a given reversible 1D CA is sensitive [16].*

5. Conclusions and open problems

We have discussed recent undecidability results from [9, 7, 16]. These show that many dynamical properties that were previously known to be undecidable for general one-dimensional CA remain undecidable with the additional constraint of reversibility.

Many challenging and interesting decidability problems remain open. Most notably, it is not known whether it is decidable if a given one-dimensional reversible CA is expansive. Another, closely related open question is a conjecture of Nasu [11] that all expansive one-dimensional cellular automata are conjugate to subshifts of finite type, or equivalently, that the column subshift $\Sigma_{r+1}(G)$ of width $r + 1$ is of finite type when G is expansive and r is its neighborhood radius. It would be even interesting to know if all expansive CA are regular, a weaker statement than Nasu's conjecture. Note that Corollary 4.3 states that left-expansive CA are not necessarily regular.

One approach to the expansivity problem is to try to reduce global mortality to local mortality of expansive CA, as suggested in Section 4.1. It would seem, however, that for this to work a counter example to Nasu's conjecture would be needed, in an analogy to Corollary 4.3.

We also have not yet touched the decidability status of any mixing property such as transitivity.

Acknowledgement

The author wishes to thank N.Ollinger and V.Lukkarila for discussions and fruitful collaboration.

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