

THE SPATIAL STRUCTURE OF ODOMETERS IN CERTAIN CELLULAR AUTOMATA

MARCUS PIVATO ¹ AND REEM YASSAWI ¹

E-mail address: `marcuspivoto@trentu.ca, ryassawi@trentu.ca`

¹ Department of Mathematics, Trent University, Peterborough, Ontario, Canada

ABSTRACT. Recent work has shown that many cellular automata (CA) have configurations whose orbit closures are isomorphic to odometers. We investigate the geometry of the spacetime diagrams of these ‘odometer configurations’. For boolean linear CA, we exactly determine the positions of the consecutive ‘gears’ of the odometer mechanism in the configuration. Then we characterize and explain the self-similar structure visible in the spacetime diagrams of odometer configurations for two classes of nonlinear CA: ratchet CA and Coven CA.

1. Introduction

Let \mathcal{A} be a finite alphabet. If $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton (CA), with left radius 0, then Φ can also be treated as a one-sided CA $\Phi_{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$. In [CPY07], the authors showed:

Theorem 1.1. *Let $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be a left permutative CA with left radius 0. If $\mathbf{z} \in \mathcal{A}^{\mathbb{N}}$ is $\Phi_{\mathbb{N}}$ -periodic, $\mathbf{x} = \mathbf{y}.\mathbf{z} \in \mathcal{A}^{\mathbb{Z}}$ and the Φ -orbit $\mathcal{O}_{\Phi}(\mathbf{x}) := \{\Phi^t(\mathbf{x})\}_{t=0}^{\infty}$ is infinite, then the orbit closure $(\overline{\mathcal{O}_{\Phi}(\mathbf{x})}, \Phi)$ is topologically conjugate to an odometer.*

(See §2 for definitions and notation). In this article, we discuss which odometers can be embedded in certain linear cellular automata, and the physical bounds on how these odometers are embedded. We also investigate the self similarity of the spacetime diagrams which display these odometers, in some linear and also non linear cellular automata. We start by generalising a result in [CY07] concerning which odometers can be embedded in the Ledrappier CA:

Proposition 1.2. *Suppose that the set of infinite multiplicity prime divisors of the quotient set \mathcal{Q} are $\{q_1, q_2, \dots, q_n\}$, and let $\tau : \mathcal{Z}(\mathcal{Q}) \rightarrow \mathcal{Z}(\mathcal{Q})$ be an odometer. Let $N = \prod_{j=1}^n q_j$ and let $\mathcal{A} = \mathbb{Z}/N$. Then τ embeds in any linear CA $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ with $\Phi(\mathbf{x}) = \mathbf{x} + \sum_{i=1}^r a_i \sigma^i(\mathbf{x})$, where for each $j \in \{1, \dots, n\}$, at least one of the a_i ’s does not divide q_j . Furthermore no other odometer can be embedded in Φ .*

2000 ACM Subject Classification: F.1.1.

Key words and phrases: linear cellular automata; dynamics; Coven cellular automata; odometer; self-similar; substitution.



With the conditions on \mathbf{x} in Theorem 1.1, left permutativity of Φ implied that the columns of the Φ -spacetime diagram of \mathbf{x} , $\mathbf{ST}_\Phi(\mathbf{x})$, were all periodic, and the fact that \mathbf{x} had an infinite Φ -orbit meant that there was a sequence of columns \mathbf{C}_{k_n} in $\mathbf{ST}_\Phi(\mathbf{x})$ whose periodicities $\{p_n\}$ increased to infinity. A conjugacy was then constructed between $\overline{\mathcal{O}_\Phi(\mathbf{x})}$ and the odometer with quotient set $\mathcal{Q} := \{\dots, \frac{p_3}{p_2}, \frac{p_2}{p_1}, \frac{p_1}{p}, p\}$. For many Φ , and many initial points \mathbf{x} , $\mathbf{ST}_\Phi(\mathbf{x})$ had a clear self-similar structure, reminiscent of the Sierpinski gasket - this is of course known if Φ is linear, but it also turned out to be the case for many nonlinear CA, such as some Coven CA. This fact led to interest in analysing these spacetime diagrams, and the rigidity imposed on them by the odometers that $(\overline{\mathcal{O}_\Phi(\mathbf{x})}, \Phi)$ are. One result in this direction is

Theorem 1.3. *Let $\Phi(\mathbf{x}) = \mathbf{x} + \sum_{p=1}^L \sigma^{a_p}(\mathbf{x})$ be defined on $\mathbb{Z}/2^{\mathbb{Z}}$, with $0 < a_1 < a_2 < \dots < a_L$. Suppose that $\mathbf{x}_{[0..\infty)}$ is $\Phi_{\mathbb{N}}$ -fixed, but $\mathbf{x}_{[-1..\infty)}$ is not $\Phi_{\mathbb{N}}$ -fixed. Then, letting $\mathbf{ST}_\Phi(\mathbf{x}) \in \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ be the Φ -spacetime diagram of \mathbf{x} , the sequence $\{\mathbf{C}_{k_n}\}_{n \geq 1}$ of columns, where the periodicity first jumps to 2^n , are those where $\{k_n\} = \{2^n a_1 - a_1 + 1\}_{n \geq 1}$.*

Theorem 1.3 implies that $\mathcal{O}_\Phi(\mathbf{x})$ is infinite; then Theorem 1.1 says that $(\overline{\mathcal{O}_\Phi(\mathbf{x})}, \Phi)$ is conjugate to any odometer that Theorem 1.2 allows. The columns $\{\mathbf{C}_{k_n}\}_{n \geq 1}$ can be thought of as the ‘gears’ of this odometer mechanism, and Theorem 1.3 says that the location of these gears is determined entirely by Φ , and is independent of \mathbf{x} . The odometer structures of linear CA often exhibit self-similar spacetime diagrams (see §3); Theorem 1.3 forces the ‘scaling factor’ of this self-similarity to be independent of the initial point generating the self similar diagram. Some version of this theorem may well be true for some linear Φ defined on larger alphabets. However the condition that $\Phi(\mathbf{x}) = \mathbf{x} + [\text{something}]$ cannot be relaxed. For example, if $\Phi(\mathbf{x}) = 2x + \sigma\mathbf{x}$ on $(\mathbb{Z}/3)^{\mathbb{Z}}$, and if $\mathbf{x}_{[-1..\infty)} = [1, 0, 0, 0, \dots]$, then $\Phi(\mathbf{x}_{[0..\infty)}) = \mathbf{x}_{[0..\infty)}$, the column \mathbf{C}_{-1} in $\mathbf{ST}_\Phi(\mathbf{x})$ has period 2, and \mathbf{C}_{-2} has period 6, irrespective of the choice of x_{-2} . However, the choice of x_{-2} affects the periodicity of \mathbf{C}_{-3} : if $x_{-2} = 2$, then \mathbf{C}_{-3} has period 6; otherwise it has period 18.

In §3, self-similarity of $\mathbf{ST}_\Phi(\mathbf{x})$ is defined in terms of two-dimensional substitution systems. We consider two classes of CA: the \mathbb{Z}/n -ratchet CA, and the range- R Coven CA (both nonlinear generalisations of the Ledrappier CA). We prove that for these CA, there exist points \mathbf{x} such that $\mathbf{ST}_\Phi(\mathbf{x})$ is self-similar (Propositions 3.7 and 3.9). In addition to being intrinsically interesting, the self-similarity allows us to easily characterize the spatiotemporal structure of these odometers. Finally, an Appendix contains most of the proofs.

2. Preliminaries

Cellular automata. Let \mathcal{A} be a finite alphabet, and let $\mathcal{A}^{\mathbb{Z}}$ be the space of all doubly infinite sequences with entries from \mathcal{A} . Elements $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ will sometimes be written as $\mathbf{x} = \mathbf{y}, \mathbf{z}$, where $\mathbf{y}, \mathbf{z} \in \mathcal{A}^{\mathbb{N}^+}, \mathcal{A}^{\mathbb{N}}$ respectively. A *cellular automaton* (CA) is a continuous, shift-commuting self-map $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$. It is known that every CA Φ is given by a local rule $\phi : \mathcal{A}^{[-l..r]} \rightarrow \mathcal{A}$, for some $l, r \geq 0$ (the *left and right radii* of Φ), such that for all $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$, and all $i \in \mathbb{Z}$,

$$[\Phi(\mathbf{x})]_i = \phi(x_{i-l}, x_{i-l+1}, \dots, x_{i+r}).$$

If l or r are 0, then Φ can also act on sequences from $\mathcal{A}^{\mathbb{N}}$. Φ is *left permutative* if for every $x_1, \dots, x_r \in \mathcal{A}$, the map $\phi(\bullet, x_1, \dots, x_r)$ is a permutation of \mathcal{A} , similarly for *right permutative*. Let \mathbb{Z}/p be the cyclic group of p elements. The CA Φ is *linear* if Φ is a group

endomorphism of $(\mathbb{Z}/p)^\mathbb{Z}$, where the group operation is componentwise addition. Φ can then be written as $\Phi(\mathbf{x}) = \sum_{i=0}^r a_i \sigma^i(\mathbf{x})$ where σ is the left shift map on $(\mathbb{Z}/p)^\mathbb{Z}$. If $x \in \mathcal{A}^\mathbb{Z}$, the Φ -spacetime diagram of \mathbf{x} , $\mathbf{ST}_\Phi(\mathbf{x})$, is the element in $\mathcal{A}^{\mathbb{N} \times \mathbb{Z}}$ whose k th row is $\Phi^k(\mathbf{x})$. For any integer n , we let $\mathbf{C}_n := \{(\Phi^k(\mathbf{x}))_n\}_{k \geq 0}$.

Odometers. Let $\mathcal{Q} := (q_1, q_2, \dots, q_n)$ be an ordered set (or sequence, if $n = \infty$) of integers ≥ 2 (the *quotient set*). Let $\mathcal{Z}(\mathcal{Q}) := \prod_{l=1}^n \mathbb{Z}_{q_l}$ be the Cartesian product set. $(\mathcal{Z}(\mathcal{Q}), \oplus)$ is a (compact, abelian) group where “ \oplus ” is defined as addition “with carry”: if $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, then $\mathbf{x} \oplus \mathbf{y} := (r_1, r_2, \dots, r_n)$ where $x_1 + y_1 = k_1 q_1 + r_1$, and inductively, $\sum_{l=1}^{n-1} k_l + x_n + y_n = k_n q_n + r_n$, $k_l + x_n + y_n = k_n q_n + r_n$, for $n > 1$, with $k_n \geq 0$ and $0 \leq r_n < q_n$ for each n .

$\mathbf{g} \in \mathcal{Z}(\mathcal{Q})$ is a *topological generator* for $\mathcal{Z}(\mathcal{Q})$ if $\{n\mathbf{g}\}_{n \in \mathbb{N}}$ is dense in $\mathcal{Z}(\mathcal{Q})$. For such \mathbf{g} we define the *\mathbf{g} -odometer* $\tau_{\mathbf{g}} : \mathcal{Z}(\mathcal{Q}) \rightarrow \mathcal{Z}(\mathcal{Q})$ as $\tau_{\mathbf{g}}(\mathbf{z}) = \mathbf{z} \oplus \mathbf{g}$, $\forall \mathbf{z} \in \mathcal{Z}(\mathcal{Q})$. If $\mathbf{g} = (g_1, g_2, \dots) \in \mathcal{Z}(\mathcal{Q})$ then $\{n\mathbf{g}\}_{n \in \mathbb{N}}$ is dense in $\mathcal{Z}(\mathcal{Q})$ if and only if for each n , the n -tuple (g_1, g_2, \dots, g_n) is a generator for the finite cyclic group $(\mathcal{Z}(\{q_1, q_2, \dots, q_n\}), \oplus)$. If \mathbf{g} and \mathbf{g}^* are both topological generators for $\mathcal{Z}(\mathcal{Q})$, then $(\mathcal{Z}(\mathcal{Q}), \tau_{\mathbf{g}})$ is topologically conjugate to $(\mathcal{Z}(\mathcal{Q}), \tau_{\mathbf{g}^*})$. Thus we will assume that the generator is $\mathbf{1} = (1, 0, 0, \dots)$ and write the odometer $\tau_{\mathbf{g}}$ as τ .

If p is prime, then the *multiplicity* of p in \mathcal{Q} is the sum number of times (possibly infinite) that p occurs in the prime decomposition of the elements of sequence \mathcal{Q} .

Theorem 2.1. [BS95] *$(\mathcal{Z}(\mathcal{Q}), \tau)$ and $(\mathcal{Z}(\mathcal{Q}^*), \tau)$ are topologically conjugate if and only if for every prime p , the multiplicity of p in \mathcal{Q} and \mathcal{Q}^* is equal.*

Theorem 2.1 also tells us that we can assume that all elements in \mathcal{Q} are prime. When convenient we will assume this. If $\mathcal{Q} = (p, p, \dots)$, then let $\mathcal{Z}(p) := \mathcal{Z}(\mathcal{Q})$ (this is the group of *p -adic integers*). We consider quotient sets \mathcal{Q} for whom only finitely many primes have positive multiplicity.

3. Self-similar structures in spacetime diagrams

Spacetime diagrams of linear CA often exhibit self-similar structures, as in Figure 1. This self-similarity reflects the self-similarity of Pascal’s Triangle in \mathbb{Z}/p , as described by Lucas’ theorem [Luc78], and has been intensively studied [Wil87, Tak93, vHPS01, AvHP⁺97, BvHPS03]. Self-similar structures also arise in *nonlinear* CA [e.g. see Figures 2 and 3 below], but these *cannot* be explained using Lucas’ theorem. In this section we will develop an analytic framework to understand this self-similarity as ‘compatibility’ of the CA with a suitable substitution mappings on $\mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$.

Let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a CA, with $\phi : \mathcal{A}^{\{0,1\}} \rightarrow \mathcal{A}$, and let $\mathbf{a} \in \mathcal{A}^\mathbb{Z}$. The *spacetime subshift* of Φ is the set $\mathcal{ST}(\Phi) \subseteq \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ of all spacetime diagrams of Φ . In other words, $\mathcal{ST}(\Phi)$ is the two-dimensional subshift of finite type in $\mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$, generated by the set of *admissible triominos*

$$\left\{ \begin{bmatrix} a & b \\ c \end{bmatrix} ; a, b \in \mathcal{A}; c = \phi(a, b) \right\}. \quad (3.1)$$

For example, Figures 1, 2 and 3 show spacetime diagrams, all exhibiting self-similarity. We will now explain this using the theory of substitution systems.

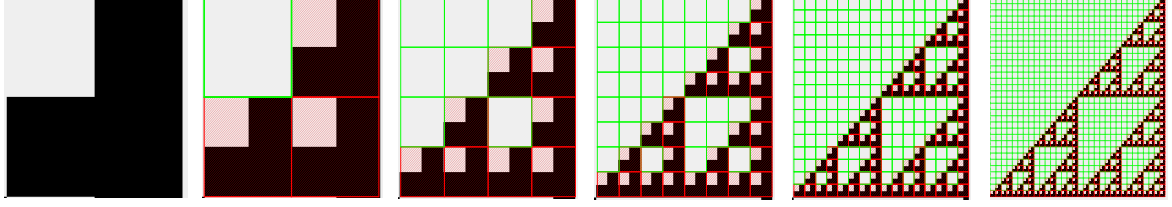


Figure 1: A self-similar spacetime diagram for the ‘Ledrappier’ CA on $\mathcal{A} = \mathbb{Z}/2$ with local rule $\phi(x_0, x_1) = x_0 + x_1$. The five images show the same spacetime diagram on larger and larger scales. Each diagram can be obtained from the previous one by applying the substitution rule described in Examples 3.1. See also Examples 3.3 and 3.4.

Substitution configurations. Let $W, H \in \mathbb{N}$ and let \mathcal{A} be a finite alphabet. A $W \times H$ substitution rule is a function $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{W \times H}$. This defines a function $\varsigma : \mathcal{A}^{\mathbb{Z} \times \mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ where

$$\varsigma \left(\begin{array}{ccc|ccc} \cdots & b_{-1}^0 & b_0^0 & b_1^0 & \cdots & \\ \cdots & b_{-1}^1 & b_0^1 & b_1^1 & \cdots & \\ \cdots & b_{-1}^2 & b_0^2 & b_1^2 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) := \begin{array}{ccc|ccc} \cdots & \varsigma(b_{-1}^0) & \varsigma(b_0^0) & \varsigma(b_1^0) & \cdots & \\ \cdots & \varsigma(b_{-1}^1) & \varsigma(b_0^1) & \varsigma(b_1^1) & \cdots & \\ \cdots & \varsigma(b_{-1}^2) & \varsigma(b_0^2) & \varsigma(b_1^2) & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \quad (3.2)$$

(the lines indicate the positions of the axes). If $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then we likewise define $\varsigma^n(a) \in \mathcal{A}^{W^n \times H^n}$ in the obvious way. The *language* of ς is the set $\mathcal{L}(\varsigma)$ of all $n \times m$ blocks (for any $n, m \in \mathbb{N}$) which occur in $\varsigma^k(a)$ for some $a \in \mathcal{A}$ and $k \in \mathbb{N}$. The ς -substitution shift $\mathbf{Sub}(\varsigma)$ is the subshift of $\mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ defined by $\mathcal{L}(\varsigma)$ [Que87, Fog02]. If Φ is a CA, we say that ς is *compatible* with Φ if $\mathbf{Sub}(\varsigma) \subseteq \mathbf{ST}(\Phi)$.

EXAMPLE 3.1: Let $\mathcal{A} = \mathbb{Z}/2$, and define $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{2 \times 2}$ by $\varsigma(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; and $\varsigma(1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then Figure 1 shows an element of $\mathbf{Sub}(\varsigma)$. This is also the spacetime diagram generated by the ‘Ledrappier’ CA with local rule $\phi(x_0, x_1) = x_0 + x_1$. This suggests that ς is compatible with Φ . \diamond

A ς -seed is a pair $[a, b] \in \mathcal{A}^2$ such that:

- (i): $[a, b] \in \mathcal{L}(\varsigma)$; (ii): For all $c \in \mathcal{A}$, $\exists n \in \mathbb{N}$ such that c occurs in $\varsigma^n[a, b]$.
- (iii): $\varsigma(a) = \begin{bmatrix} * & \cdots & * & a \\ * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \end{bmatrix}$; (iv): $\varsigma(b) = \begin{bmatrix} b & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}$;

Note that, since $\mathcal{L}(\varsigma) = \mathcal{L}(\varsigma^n)$ for any natural n , then it is always possible to find a pair $[a, b]$ satisfying (i), (iii) and (iv), by the pigeonhole principle. Define $\mathbf{A} \in \mathcal{A}^{(-\infty \dots 0] \times \mathbb{N}}$ by the property that $\mathbf{A}_{(-W^n \dots 0] \times [0 \dots H^n]} = \varsigma^n(a)$ for all $n \in \mathbb{N}$ (this definition is consistent because $\varsigma(a)_0^0 = a$). Likewise, define $\mathbf{B} \in \mathcal{A}^{[1 \dots \infty) \times \mathbb{N}}$ by the property that $\mathbf{B}_{(0 \dots W^n] \times [0 \dots H^n]} = \varsigma^n(b)$ for all $n \in \mathbb{N}$ (this definition is consistent because $\varsigma(b)_1^0 = b$). We write “ $\mathbf{A} := \varsigma^\infty(a)$ ” and “ $\mathbf{B} := \varsigma^\infty(b)$ ”. Let $\varsigma^\infty[a, b] := \overline{\mathbf{A} | \mathbf{B}}$ be the obvious element of $\mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$.

If $\mathbf{A} \in \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ and $n \in \mathbb{N}$, then the n th row of \mathbf{A} is the biinfinite sequence $[\dots a_{-1}^n, a_0^n, a_1^n, \dots]$. If $\mathbf{r} = [r_1, r_2] \in \mathcal{A}^2$, we say that \mathbf{r} occurs in \mathbf{A} if there is some $z \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a_z^n = r_1$ and $a_{z+1}^n = r_2$. Let $\mathcal{L}_2(\varsigma) := \{\mathbf{r} \in \mathcal{A}^2 ; \mathbf{r} \text{ occurs in some } \mathbf{A} \in \mathbf{Sub}(\varsigma)\}$. A configuration $\mathbf{S} \in \mathcal{A}^{\mathbb{Z} \times \mathbb{N}}$ is ς -fixed if $\varsigma(\mathbf{S}) = \mathbf{S}$.

Lemma 3.2. *Let ς be a substitution and let $\mathbf{s} \in \mathcal{A}^2$ be a ς -seed. Then*

- (a) $\mathbf{S} := \varsigma^\infty(\mathbf{s})$ is a ς -fixed configuration, and $\mathbf{Sub}(\varsigma)$ is the σ -orbit closure of \mathbf{S} .

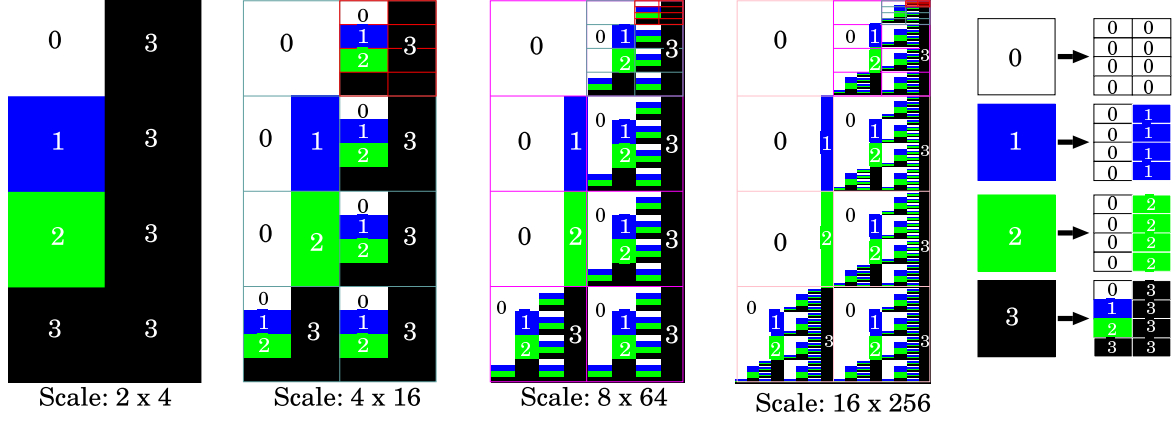


Figure 2: Self-similarity in the $\mathbb{Z}/4$ -ratchet CA. The left four images are the same spacetime diagram, shown on larger and larger scales. The numerical labels show how each spacetime diagram can be obtained from the previous one by applying the substitution mapping illustrated on the far right.

(b) *There exists $n \in \mathbb{N}$ so that $\mathcal{L}_2(\varsigma)$ is the set of all 2-words which occur in $\varsigma^n(\mathbf{s})$.*

Proof. See Appendix. □

EXAMPLE 3.3: Let $\mathcal{A} = \mathbb{Z}/2$, and define $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{2 \times 2}$ as in Example 3.1 Then $[1, 0]$ is a ς -seed. ◇

We say that ϕ commutes with ς if, for every $[a, b] \in \mathcal{L}_2(\varsigma)$ with $c = \phi(a, b)$,

$$\varsigma \begin{bmatrix} a & b \\ c \end{bmatrix} := \begin{bmatrix} \varsigma(a) & \varsigma(b) \\ \varsigma(c) \end{bmatrix} \text{ is a fragment of a spacetime diagram of } \Phi.$$

EXAMPLE 3.4: Let $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{2 \times 2}$ be as in Examples 3.1 and 3.3. The Ledrappier CA (with local rule $\phi(x_0, x_1) = x_0 + x_1$) commutes with ς , as shown by the following computations

$$\varsigma \begin{bmatrix} 0 & 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (0) & \\ 0 & 0 & & \end{bmatrix} \quad \varsigma \begin{bmatrix} 0 & 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & (0) & \\ 1 & 1 & & \end{bmatrix} \quad \varsigma \begin{bmatrix} 1 & 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & (0) & \\ 1 & 1 & & \end{bmatrix} \quad \varsigma \begin{bmatrix} 1 & 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & (0) & \\ 0 & 0 & & \end{bmatrix}.$$

Observe that the image of each Φ -admissible triomino is a fragment of a Φ -spacetime diagram (to illustrate this, we have completed these diagram by adding one entry in the bottom right box, in parentheses). ◇

Proposition 3.5. *Let ς be a substitution with seeds. Let Φ be a CA. The following are equivalent:*

- (a) ς is compatible with Φ .
- (b) *There is some ς -seed $\mathbf{s} \in \mathcal{A}^2$ such that $\varsigma^\infty(\mathbf{s}) \in \mathcal{ST}(\Phi)$.*
- (c) *For every ς -seed $\mathbf{s} \in \mathcal{A}^2$, we have $\varsigma^\infty(\mathbf{s}) \in \mathcal{ST}(\Phi)$.*
- (d) ς commutes with Φ .

Proof. See Appendix. □

A substitution ς is *aperiodic* if there is no $\mathbf{z} \in \mathbb{Z} \times \mathbb{N}$ such that $\text{Sub}(\varsigma) \subseteq \text{Fix}[\sigma^{\mathbf{z}}]$.

Corollary 3.6. *Let $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{W \times H}$ be an aperiodic substitution compatible with Φ . Let $\mathbf{A} \in \mathcal{ST}(\Phi)$ be a ς -fixed configuration [which exists by Prop.3.5(b)], whose first row is $\mathbf{x} = \mathbf{a}\mathbf{b}$, where $\mathbf{a} \in \mathcal{A}^{(-\infty \dots 0]}$ and where $\mathbf{b} \in \mathcal{A}^{[1 \dots \infty)}$ is Φ -periodic [as in Thm.1.1]. Then $\overline{\mathcal{O}_\Phi(\mathbf{x})}$ has a p -adic odometer as a factor, for at least one prime factor p of H .*

Proof. See Appendix. □

Let $n \in \mathbb{N}$, and let $\mathcal{A} := \mathbb{Z}/n$. The \mathbb{Z}/n -ratchet CA is the left-permutative CA $\Psi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ with right-sided local rule $\psi : \mathcal{A}^{\{0,1\}} \rightarrow \mathcal{A}$ defined

$$\psi(a, b) := \begin{cases} a & \text{if } b \neq n'; \\ a + 1 & \text{if } b = n'; \end{cases} \quad \text{where } n' := n - 1.$$

For example, the $\mathbb{Z}/2$ -ratchet CA is just the Ledrappier CA shown in Figure 1. Figure 2 shows a spacetime diagram of $\mathbb{Z}/4$ -ratchet CA. This diagram is visibly self-similar, and some of the columns are strongly reminiscent of the 4-adic odometer, as explained by the next result.

Proposition 3.7. *Let $n \in \mathbb{N}$, let $\mathcal{A} := \mathbb{Z}/n$, and let $n' := n - 1$. Let $\Psi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be the \mathbb{Z}/n -ratchet CA. Then*

(a) Ψ is compatible with the substitution $\varsigma : \mathcal{A} \rightarrow \mathcal{A}^{2 \times n}$ defined by

$$\varsigma(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}; \quad \varsigma(1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}; \quad \varsigma(2) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ \vdots & \vdots \\ 0 & 2 \end{bmatrix}; \quad \dots \quad \varsigma(n-2) = \begin{bmatrix} 0 & n-2 \\ 0 & n-2 \\ \vdots & \vdots \\ 0 & n-2 \end{bmatrix}; \quad \varsigma(n') = \begin{bmatrix} 0 & n' \\ 1 & n' \\ \vdots & \vdots \\ n' & n' \end{bmatrix};$$

(b) Let $\mathbf{a} = [\dots 0, 0, 0, 0, n', 0, 0, 0, 0 \dots]$. Then $\mathbf{ST}_\Psi(\mathbf{a})$ is a ς -fixed point.

(c) $(\overline{\mathcal{O}_\Psi(\mathbf{a})}, \Psi)$ is conjugate to a $\mathcal{Z}(n)$ -odometer (where $\mathcal{Z}(n)$ is the n -adic integers).

Proof. (a,b) Note that $\mathcal{L}_2(\varsigma) = \mathcal{A}^2$, and that ψ commutes with ς (this can be checked by direct computation, similar to Example 3.4). Also, $[n', 0]$ is a seed for ς , so Proposition 3.5(c) says that the ς -fixed array $\mathbf{S} := \varsigma^\infty([1, 0])$ is in $\mathcal{ST}(\Psi)$. But the zeroth row of \mathbf{S} is \mathbf{a} ; hence $\mathbf{S} = \mathbf{A}$.

(c) Suppose $\mathbf{A} = [A_s^t]_{s \in \mathbb{Z}, t \in \mathbb{N}}$. For all $k \in \mathbb{N}$, let $\mathbf{C}_k := [A_{-2^k}^t]_{t \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ be the -2^k th column of \mathbf{A} . Then \mathbf{C}_k is the sequence of the k th digit in the standard ‘ n -ary number’ representation of the n -ary odometer. That is,

$$\mathbf{C}_k = \left[\underbrace{0, \dots, 0}_{n^k}, \underbrace{1, \dots, 1}_{n^k}, \underbrace{2, \dots, 2}_{n^k}, \dots, \underbrace{n', \dots, n'}_{n^k}, \underbrace{0, \dots, 0}_{n^k}, \underbrace{1, \dots, 1}_{n^k}, \dots \right]$$

This yields an obvious surjection $\Gamma : \overline{\mathcal{O}_\Psi(\mathbf{a})} \rightarrow \mathcal{Z}(n)$. Also, Γ is injective, because the information in the columns $\{\mathbf{C}_k\}_{k=0}^\infty$ is sufficient to reconstruct all the other columns in the Ψ -spacetime diagram \mathbf{A} . □

If $\mathcal{A} := \mathbb{Z}/2$ and $R \in \mathbb{N}$, then the range R Coven CA is the CA $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ with local rule $\phi(x_0, x_1, \dots, x_R) = x_0 + x_1 x_2 \cdots x_R$; these were introduced in [CH79]. For example, the range 1 Coven CA is just the Ledrappier CA with local rule $\phi(x_0, x_1) = x_0 + x_1$ [Figure 1], while the range 2 Coven CA has local rule $\phi(x_0, x_1, x_2) = x_0 + x_1 x_2$ [Figure 3]. Nonlinear (i.e. $R \geq 2$) Coven CA exhibit self-similar spacetime diagrams which *cannot* be explained simply by compatibility with a substitution map. Also, we remark that the configuration in Figure 3 is *not* automatic (see [vH03] for an introduction to automatic configurations). Instead, these diagrams are self-similar because they can be ‘recoded’ as the diagrams of

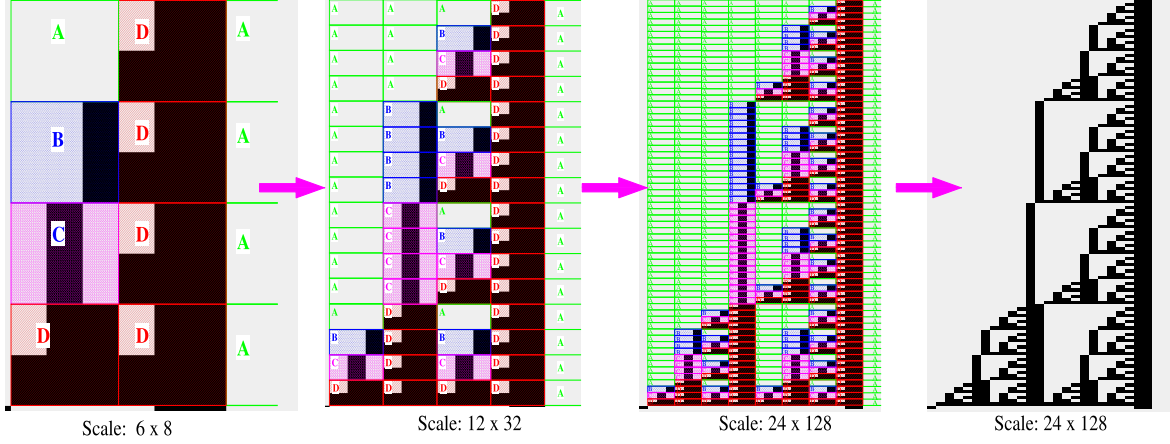


Figure 3: Self-similarity in the Coven CA with local rule $\phi(x_0, x_1, x_2) = x_0 + x_1x_2$. The left three images are the same spacetime diagram, shown on larger and larger scales. The alphabetic labels show how this spacetime diagram can be obtained from Figure 2 via the function Ξ described in Example 3.8.

ratchet CA, which are self-similar by Proposition 3.7. We will explain this in Proposition 3.9, but first we illustrate with an example.

EXAMPLE 3.8: Let $\mathcal{A} := \mathbb{Z}/2$ and let $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be the range 2 Coven CA with local rule $\phi(x_0, x_1, x_2) = x_0 + x_1x_2$. Let $\mathbf{a} := [0, 0, 0]$, $\mathbf{b} := [0, 0, 1]$, $\mathbf{c} := [0, 1, 0]$ and $\mathbf{d} := [0, 1, 1]$. Let $\mathcal{B} := \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\} \subset \mathcal{A}^3$, and let $\mathfrak{B} \subset \mathcal{A}^{\mathbb{Z}}$ be the set of all sequences obtained by concatenating words from \mathcal{B} , such that a word boundary lies at zero. Clearly, \mathfrak{B} is σ^3 -invariant, and it can be checked by direct computation that $\Phi^2(\mathfrak{B}) \subseteq \mathfrak{B}$. Let $\Xi : (\mathbb{Z}/4)^{\mathbb{Z}} \rightarrow \mathfrak{B}$ be the bijection with local rule given by $\xi(0) := \mathbf{a}$, $\xi(1) := \mathbf{b}$, $\xi(2) := \mathbf{c}$, and $\xi(3) := \mathbf{d}$.

Let $\Psi : (\mathbb{Z}/4)^{\mathbb{Z}} \rightarrow (\mathbb{Z}/4)^{\mathbb{Z}}$ be the $\mathbb{Z}/4$ -ratchet CA. Direct computation shows that $\Xi \circ \sigma = \sigma^3 \circ \Xi$ and $\Xi \circ \Phi^2 = \Psi \circ \Xi$. In other words, Ξ is a dynamical isomorphism from $((\mathbb{Z}/4)^{\mathbb{Z}}, \Psi, \sigma)$ to $(\mathfrak{B}, \Phi^2, \sigma^3)$.

The first row of Figure 3 is $[\dots, 0, 0, 0, 0, 1, 1, 0, 0, 0, \dots]$, which equals $[\dots, \mathbf{a}, \mathbf{a}, \mathbf{d}, \mathbf{a}, \mathbf{a}, \dots]$ (an element of \mathfrak{B}), which is the Ξ -image of $[\dots, 0, 0, 3, 0, 0, \dots]$, which is the first row of Figure 2. Thus, Ξ maps the spacetime diagram of Figure 2 into that of Figure 3. Proposition 3.7(d) implies that Figure 3 is conjugate to a dyadic odometer. \diamond

Example 3.8 generalizes as follows.

Proposition 3.9. *Let $\mathcal{A} := \mathbb{Z}/2$ and let $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be the range R Coven CA. Let $\mathcal{C} := \mathbb{Z}/2^R$ and let $\Psi : \mathcal{C}^{\mathbb{Z}} \rightarrow \mathcal{C}^{\mathbb{Z}}$ be the \mathcal{C} -ratchet CA.*

(a) *There is a (Φ^2, σ^R) -invariant subset $\mathfrak{B} \subset \mathcal{A}^{\mathbb{Z}}$ such that (\mathfrak{B}, Φ^2) is isomorphic to (\mathcal{C}, Ψ) .*

(b) *The point $\mathbf{x} := [\dots, 0, 0, 0, 0, \overbrace{1, \dots, 1}^R, 0, 0, 0, \dots]$ is in \mathfrak{B} , and $(\overline{\mathcal{O}_{\Phi}(\mathbf{x})}, \Phi^2)$ is conjugate to $(\overline{\mathcal{O}_{\Phi}(\mathbf{a})}, \Psi)$, where $\mathbf{a} \in \mathcal{C}^{\mathbb{Z}}$ is as in Proposition 3.7(b,c). Hence $(\overline{\mathcal{O}_{\Phi}(\mathbf{x})}, \Phi)$ is isomorphic to a dyadic odometer.*

Proof. (a) Let $\mathcal{B} := \{\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{2^R-1}\} \subset \mathcal{A}^{R+1}$, where $\mathbf{b}_0 := [0, \dots, 0, 0]$, $\mathbf{b}_1 := [0, \dots, 0, 1]$, $\mathbf{b}_2 := [0, \dots, 1, 0], \dots, \mathbf{b}_{2^R-1} := [0, 1, \dots, 1, 1]$. Let $\mathfrak{B} := \mathcal{B}^{\mathbb{Z}}$, as a subset of $\mathcal{A}^{\mathbb{Z}}$. Clearly \mathfrak{B} is σ^R -invariant. The function $\xi : \mathcal{C} \ni n \mapsto \mathbf{b}_n \in \mathcal{B}$ yields a bijection

$\Xi : \mathcal{C}^{\mathbb{Z}} \rightarrow \mathfrak{B}$, and direct computation shows that $\Xi \circ \sigma = \sigma^{R+1} \circ \Xi$ and $\Xi \circ \Phi^2 = \Psi \circ \Xi$, so Ξ is a dynamical isomorphism from $(\mathcal{C}^{\mathbb{Z}}, \Psi, \sigma)$ to $(\mathfrak{B}, \Phi^2, \sigma^{R+1})$. It follows that \mathfrak{B} is also Φ^2 -invariant.

(b) $\mathbf{x} = \Xi(\mathbf{a})$ where \mathbf{a} is as in Proposition 3.7(b); thus Proposition 3.7(c) states that $(\overline{\mathcal{O}_{\Psi}(\mathbf{a})}, \Psi)$ is conjugate to the 2^{R+1} -adic (hence dyadic) odometer. Thus, part (a) implies that $(\overline{\mathcal{O}_{\Phi^2}(\mathbf{x})}, \Phi^2)$ is also isomorphic to a dyadic odometer. Thus $(\overline{\mathcal{O}_{\Phi}(\mathbf{x})}, \Phi)$ is also isomorphic to a dyadic odometer. \square

Acknowledgements. Initial work was done while the authors were on research leave at Wesleyan University, and was partially supported by the Van Vleck Fund. We thank Ethan Coven for generous hospitality and many stimulating discussions. This work was also supported by NSERC Canada.

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Appendix: Proofs

Proof of Theorem 1.2 : The Chinese Remainder Theorem implies that any non-trivial CA Φ with rule $\Phi(\mathbf{x}) = \mathbf{x} + \sum_{i=1}^r a_i \sigma^i(\mathbf{x})$ on $(\mathbb{Z}/N)^{\mathbb{Z}}$ is topologically conjugate to $\prod_{j=1}^n \Phi_{q_j}$, where Φ_{q_j} acts on $(\mathbb{Z}/q_j)^{\mathbb{Z}}$ by $\Phi_{q_j}(\mathbf{x}) := \mathbf{x} + \sum_{i=1}^r a_i \sigma^i(\mathbf{x})$. The conditions on the a_i 's guarantee that each Φ_{q_j} is non-trivial. Note that $\prod_{j=1}^n (\mathcal{Z}(q_j), \tau)$ is topologically conjugate to $(\mathcal{Z}(Q), \tau)$. We will show that $\prod_{j=1}^n (\mathcal{Z}(q_j), \tau)$ can be embedded in $\prod_{j=1}^n \Phi_{q_j}$.

Case 1. Suppose first that the multiplicity of q in \mathcal{Q} is infinite for each p in $\{q_1, q_2, \dots, q_n\}$. Find $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ such that $\mathbf{x}_{[0..\infty)}$ is Φ -fixed and such that $\mathcal{O}_{\Phi}(\mathbf{x}) := \{\Phi^t(\mathbf{x}) ; t \in \mathbb{N}\}$ is infinite. This can be done, since Φ_{q_j} is conjugate to a full one sided shift, which has fixed points — thus one can find some Φ_{q_j} -fixed $\mathbf{x}_{[0..\infty)}$; further if x_{-1} is chosen so that $\mathbf{x}_{[-1..\infty)}$ is not fixed, then the proof of Theorem 1.3 shows that \mathbf{x} has an infinite Φ_{q_j} orbit. Using Theorem 4 in [CPY07], Φ_{q_j} embeds $(\mathcal{Z}(q_j), \tau)$.

Case 2. Suppose that $\mathcal{P} = \mathcal{P}_f \cup \mathcal{P}_i$ where p in \mathcal{P}_f have finite multiplicity in \mathcal{Q} and p in \mathcal{P}_i have infinite multiplicity in \mathcal{Q} . Let $P := \prod_{p \in \mathcal{P}_f} p$. As in the Corollary to Theorem 1 in [CPY07], find $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ such that $\mathbf{x}_{[0..\infty)}$ is Φ_{q_1} -periodic with least period P , and such that $\mathcal{O}_{\Phi}(\mathbf{x})$ is infinite. Then Φ_{q_1} embeds $(\mathcal{Z}(P, q_1, q_1, \dots), \tau)$, and, by Case 1, Φ_{q_j} embeds $(\mathcal{Z}(q_j), \tau)$ for $1 < j \leq n$. The embedding result follows. That no other odometer can be embedded in these linear CA is proved similarly to the result for $\Phi(x) = x + \sigma(x)$ in [CY07]. \square

Lemma 3.10. *Let $\mathcal{A} := \mathbb{Z}/2$, and suppose that $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ and $\mathbf{x} = (x_i)_{i=-\infty}^{\infty} \in \mathcal{A}^{\mathbb{Z}}$ satisfy the conditions of Theorem 1.3. Then for each $j \geq 1$ and each $k \geq 0$, we have*

$$\sum_{p=1}^L x_{2^k(a_p - a_1) + a_1 - 1 + j} = 0.$$

Thus $\Phi^{2^k}(\mathbf{x})|_{-(2^k a_1 - a_1 + 1) + j} = x_{-(2^k a_1 - a_1 + 1) + j}$ for each $k \geq 0$ and $j \geq 1$.

Proof. We prove this by induction on k . Since $\mathbf{x}_{[0..\infty)}$ is fixed by Φ , we have $x_k + \sum_{p=1}^L x_{k+a_p} = x_k$, for each $k \geq 0$, so that $\sum_{p=1}^L x_{a_p - 1 + j} = 0$ is true for each $j \geq 1$. Let $\Delta_i := a_{i+1} - a_i$, for $1 \leq i \leq L-1$, and assume that for each $j \geq 1$,

$$\sum_{p=1}^L x_{2^k(a_p - a_1) + a_1 - 1 + j} = \sum_{p=1}^L x_{a_1 - 1 + j + 2^k(\sum_{i=1}^{p-1} \Delta_i)} = 0. \quad (3.3)$$

Given a positive j , let $\star_p := 2^{k+1}(a_p - a_1) + a_1 - 1 + j$ for $1 \leq p \leq L$, and define $\star_{11} := \star_1$ and $\star_{1p} := \star_1 + 2^k \left(\sum_{i=1}^{p-1} \Delta_i \right)$ for $p = 2, \dots, L$. By Equation 3.3, $\sum_{p=1}^L x_{\star_{1p}} = 0$.

For $2 \leq q \leq L$, define $\star_{q1} := \star_{1q}$, and $\star_{qp} := \star_{q1} + 2^k \left(\sum_{i=1}^{p-1} \Delta_i \right)$. Using Equation 3.3

and $j_q := j + 2^k \left(\sum_{i=1}^{q-1} \Delta_i \right)$, we have $\sum_{p=1}^L x_{\star_{qp}} = 0$.

Consider the matrix $\{x_{\star_{pq}}\}_{p,q=1}^L$. We have the following two claims:

CLAIM 1: $\star_{pq} = \star_{qp}$ for $1 \leq p, q \leq L$.

Proof. If $q = 1$ or $p = 1$ this is true by definition. If $2 \leq p, q \leq L$, $*_{pq} = *_{1p} + 2^k \sum_{i=1}^{q-1} \Delta_i = *_{1p} + 2^k \sum_{i=1}^{p-1} \Delta_i + 2^k \sum_{i=1}^{q-1} \Delta_i = *_{1q} + 2^k \sum_{i=1}^{p-1} \Delta_i = *_{qp}$. \diamond Claim 1

CLAIM 2: $*_{pp} = *_{1p}$.

Proof. $*_{pp} = *_{1p} + 2^k \left(\sum_{i=1}^{p-1} \Delta_i \right) = *_{1p} + 2^{k+1} \left(\sum_{i=1}^{p-1} \Delta_i \right) = a_1 - 1 + j + 2^{k+1}(a_p - a_1) = *_{1p}$. \diamond Claim 2

These last two claims tell us that the matrix $\{x_{*_{pq}}\}_{p,q=1}^L$ is a symmetric 0, 1-matrix each of whose rows sum to zero. Thus $0 = \sum_{p=1}^L \sum_{q=1}^L x_{*_{pq}} = \sum_{p=1}^L x_{*_{pp}} = \sum_{p=1}^L x_{*_{1p}}$, which shows that $\sum_{p=1}^L x_{2^{k+1}(a_p - a_1) + a_1 - 1 + j} = 0$. \square

Lemma 3.11. *Let $\mathcal{A} := \mathbb{Z}/2$, and suppose that $\Phi : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ and $\mathbf{x} = (x_i)_{i=-\infty}^{\infty} \in \mathcal{A}^{\mathbb{Z}}$ satisfy the conditions of Theorem 1.3. Then for each $k \geq 0$, $\Phi^{2^k}(\mathbf{x})|_{-(2^k a_1 - a_1 + 1)} \neq x_{-(2^k a_1 - a_1 + 1)}$.*

Proof. We prove this by induction on k . If \mathbf{x} satisfies the conditions of Theorem 1.3, then $\Phi(\mathbf{x})|_{-1} \neq x_{-1}$. Next, assume that $\Phi^{2^k}(\mathbf{x})|_{-(2^k a_1 - a_1 + 1)} \neq x_{-(2^k a_1 - a_1 + 1)}$. Thus

$$x_{-(2^k a_1 - a_1 + 1)} + x_{a_1 - 1} + \sum_{p=2}^L x_{a_1 - 1 + 2^k(a_p - a_1)} \neq x_{-(2^k a_1 - a_1 + 1)}, \quad (3.4)$$

What follows is essentially the same as that of the previous lemma, except that this time we have a symmetric, 0-1 matrix all of whose rows sum to 0, *except* the first, which sums to one. We claim that

$$x_{a_1 - 1} + \sum_{p=2}^L x_{a_1 - 1 + 2^{k+1}(a_p - a_1)} = 1. \quad (3.5)$$

Let $*_{1p} := a_1 - 1 + 2^k(a_p - a_1)$, for $1 \leq p \leq L$. For $2 \leq p \leq L$, let $*_{p1} := *_{1p} = a_1 - 1 + 2^k(a_p - a_1)$, and $*_{pq} := a_1 - 1 + 2^k(a_p - a_1) + 2^k(a_q - a_1)$. Let $j_p := 2^k(a_p - a_1)$.

Lemma 3.10 tells us that $x_{a_1 - 1 + j} + \sum_{p=2}^L x_{2^k(a_p - a_1) + a_1 - 1 + j} = 0$, so $\sum_{q=1}^L x_{*_{pq}} = 0$; Equation

3.4 implies that $\sum_{q=1}^L x_{*_{1q}} = 1$.

As in Lemma 3.10, we have a symmetric 0-1 matrix $[x_{*_{pq}}]_{p,q=1}^L$ whose diagonal terms are the summands in Equation 3.5, and all of whose rows sum to zero, save the first row. The result follows. \square

Proof of Theorem 1.3. Lemma 3.11 tells us that $k_n \geq 2^n a_1 - a_1 + 1$, and Lemma 3.10 tells us that $k_n = 2^n a_1 - a_1 + 1$. \square

Proof of Lemma 3.2. **(a)** is a standard argument [Fog02, §1.2.6]. For **(b)** note that $\mathbf{s} \in \mathcal{L}_2(\varsigma)$. For any $n \in \mathbb{N}$, let $\mathcal{R}_n := \{\mathbf{r} \in \mathcal{A}^2; \mathbf{r} \text{ occurs in } \varsigma^n(\mathbf{s})\}$. Then $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \dots \subseteq \mathcal{L}_2(\varsigma)$. Let $\mathcal{R}_\infty = \bigcup_{n=1}^{\infty} \mathcal{R}_n$. Then $\mathcal{R}_\infty = \mathcal{L}_2(\varsigma)$, because \mathbf{s} is a ς -seed. But $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \dots \subseteq \mathcal{R}_\infty$ are finite sets, so there exists $n \in \mathbb{N}$ such that $\mathcal{R}_\infty = \mathcal{R}_n$. \square

Proof of Proposition 3.5. “(a) \Rightarrow (c)” If \mathbf{s} is a ζ -seed, then $\zeta^\infty(\mathbf{s}) \in \mathfrak{Sub}(\zeta)$. If $\mathfrak{Sub}(\zeta) \subseteq \mathcal{ST}(\Phi)$, then $\zeta^\infty(\mathbf{s}) \in \mathcal{ST}(\Phi)$.

“(c) \Rightarrow (b)” is immediate. For “(b) \Rightarrow (a)”, let $\mathbf{S} = \zeta^\infty(\mathbf{s})$, and suppose $\mathbf{S} \in \mathcal{ST}(\Phi)$. Then the σ -orbit closure of \mathbf{S} is contained in $\mathcal{ST}(\Phi)$, because $\mathcal{ST}(\Phi)$ is closed and σ -invariant. Thus, Lemma 3.2(a) says $\mathfrak{Sub}(\zeta) \subseteq \mathcal{ST}(\Phi)$; ie. ζ is compatible with ϕ .

“(d) \Rightarrow (c)” Let $\mathbf{s} = [a, b]$ be a ζ -seed and let $\mathbf{S} := \zeta^\infty(\mathbf{s}) = \overline{\mathbf{A}|\mathbf{B}}$, where $\mathbf{A} = \zeta^\infty(a)$ and $\mathbf{B} := \zeta^\infty(b)$.

CLAIM 1: For all $n \in \mathbb{N}$, let $\mathbf{A}_n := \zeta^n(a)$ and $\mathbf{B}_n := \zeta^n(b)$. Then $\overline{\mathbf{A}_n|\mathbf{B}_n}$ is $\mathcal{ST}(\Phi)$ -admissible.

Proof. Case (n=1): By definition, $[a, b] \in \mathcal{L}_2(\zeta)$, and by hypothesis, ϕ commutes with ζ , so $\overline{\mathbf{A}_1|\mathbf{B}_1} = \zeta[a, b]$ is $\mathcal{ST}(\Phi)$ -admissible.

Induction: Suppose $\overline{\mathbf{A}_n|\mathbf{B}_n}$ is $\mathcal{ST}(\Phi)$ -admissible. Let $\begin{bmatrix} x & y \\ z & \end{bmatrix}$ be a triomino appearing somewhere in $\overline{\mathbf{A}_{n+1}|\mathbf{B}_{n+1}}$; we must show that $\begin{bmatrix} x & y \\ z & \end{bmatrix}$ is $\mathcal{ST}(\Phi)$ -admissible [ie. that $z = \phi(x, y)$]. Now, $\overline{\mathbf{A}_{n+1}|\mathbf{B}_{n+1}} = \zeta(\overline{\mathbf{A}_n|\mathbf{B}_n})$, so there is some triomino $\begin{bmatrix} u & v \\ w & \end{bmatrix}$ in $\overline{\mathbf{A}_n|\mathbf{B}_n}$ such that $\begin{bmatrix} x & y \\ z & \end{bmatrix}$ appears inside $\zeta\begin{bmatrix} u & v \\ w & \end{bmatrix}$. By definition, $[u, v] \in \mathcal{L}_2(\zeta)$, and by hypothesis, ϕ commutes with ζ . Hence $\zeta\begin{bmatrix} u & v \\ w & \end{bmatrix}$ is a $\mathcal{ST}(\Phi)$ -admissible fragment, which in particular means that $\begin{bmatrix} x & y \\ z & \end{bmatrix}$ is $\mathcal{ST}(\Phi)$ -admissible.

This works for any $\begin{bmatrix} x & y \\ z & \end{bmatrix}$ in $\overline{\mathbf{A}_{n+1}|\mathbf{B}_{n+1}}$. Thus, $\overline{\mathbf{A}_{n+1}|\mathbf{B}_{n+1}}$ is $\mathcal{ST}(\Phi)$ -admissible, because $\mathcal{ST}(\Phi)$ is the SFT generated by the set of triominos in eqn.(3.1). \diamond claim 1
It follows that $\mathbf{S} = \overline{\mathbf{A}|\mathbf{B}}$ is $\mathcal{ST}(\Phi)$ -admissible.

“(b) \Rightarrow (d)” Suppose $\mathbf{S} = \zeta^\infty(\mathbf{s})$ for some ζ -seed $\mathbf{s} \in \mathcal{A}^2$. If $\mathbf{S} \in \mathcal{ST}(\Phi)$, then ζ commutes with Φ on all $[u, v]$ which occur in \mathbf{S} . But Lemma 3.2(b) says this is all of $\mathcal{L}_2(\zeta)$. \square

Proof of Corollary 3.6. \mathbf{x} has infinite Φ -orbit because otherwise, \mathbf{A} would be fixed under some vertical shift, contradicting the aperiodicity of ζ . Thus, Theorem 1.1 says $\mathfrak{D} := (\overline{\mathcal{O}_\Phi(\mathbf{x})}, \Phi)$ is isomorphic to some odometer. We must show that \mathfrak{D} has a p -adic odometer as a factor, for some prime p dividing H .

For all $k \in \mathbb{N}$, let $\mathbf{A}_k := \mathbf{A}_{(-W^k \dots 0] \times \mathbb{N}} = [\Phi^t(\mathbf{x})_{(-W^k \dots 0]}]_{t=0}^\infty$ (i.e. the first W^k ‘columns’ in the spacetime diagram of \mathbf{x}). Each \mathbf{A}_k is vertically periodic (because \mathfrak{D} is an odometer); let T_k be its minimal period. Thus, $T_0 \leq T_1 \leq T_2 \leq \dots$.

CLAIM 2: (a) T_k divides $H^k T_0$. (b) $\lim_{k \rightarrow \infty} T_k = \infty$.

Proof. (a) $\zeta(\mathbf{A}) = \mathbf{A}$, so $\zeta(\mathbf{A}_{k-1}) = \mathbf{A}_k$, so \mathbf{A}_k is vertically (HT_{k-1}) -periodic, so its least period T_k must divide HT_{k-1} . By induction, this means T_k divides $H^k T_0$.

(b) By contradiction, suppose the sequence $\{T_k\}_{k=1}^\infty$ was bounded. Then there would be some k such that $T_k = T_{k+1} = T_{k+2} = \dots$, and then \mathbf{x} would be Φ^{T_k} -periodic, contradicting the fact that \mathbf{x} has infinite Φ -orbit. \diamond claim 2

For any N , Claim 1(b) yields some $k \geq N$ such that $T_k \geq H^N T_0$. Let $d := \gcd(T_k, T_0)$, and let $T'_k = T_k/d$ and $T'_0 := T_0/d$; then $T'_k \geq H^N T'_0 \geq H^N$. But Claim 1(a) says that T_k divides $H^k T_0$, which means T'_k divides $H^k T'_0$, which means T'_k divides H^k (because T'_k is coprime to T'_0). Thus, all prime factors of T_k are prime factors of H . But $T'_k \geq H^N$, so T_k must be divisible by p^N for at least one prime factor p of H . It follows that p^N divides T_k , which means that \mathfrak{D} contains a factor of minimal period p^N . But N can be made arbitrarily large, so \mathfrak{D} must have a p -adic odometer as a factor. \square