

Structural stability of finite dispersion-relation preserving schemes

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Abstract

The goal of this work is to determine classes of travelling solitary wave solutions for a differential approximation of a finite difference scheme by means of a hyperbolic ansatz. It is shown that spurious solitary waves can occur in finite-difference solutions of nonlinear wave equation. The occurrence of such a spurious solitary wave, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to as a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (solitary waves in the present case) that are not solution of the original continuous equations. This paper extends our previous work about classical schemes to dispersion-relation preserving schemes.

1 Introduction: The DRP scheme

The Burgers equation:

$$u_t + cuu_x - \mu u_{xx} = 0, \quad (1)$$

c, μ being real constants, plays a crucial role in the history of wave equations. It was named after its use by Burgers [2] for studying turbulence in 1939.

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i, n denoting natural integers, a linear finite difference scheme for this equation can be written under the form:

$$\sum \alpha_{lm} u_l^m = 0 \quad (2)$$

where:

$$u_l^m = u(lh, m\tau) \quad (3)$$

$l \in \{i-1, i, i+1\}$, $m \in \{n-1, n, n+1\}$, $j = 0, \dots, n_x$, $n = 0, \dots, n_t$. The α_{lm} are real coefficients, which depend on the mesh size h , and the time step τ .

The Courant-Friedrichs-Lewy number (cfl) is defined as $\sigma = c\tau/h$.

A numerical scheme is specified by selecting appropriate values of the coefficients α_{lm} . Then, depending on them, one can obtain optimum schemes, for which the error will be minimal.

m being a strictly positive integer, the first derivative $\frac{\partial u}{\partial x}$ is approximated at the l^{th} node of the spatial mesh by:

$$\left(\frac{\partial u}{\partial x}\right)_l \simeq \sum_{k=-m}^m \gamma_k u_{i+k}^n \quad (4)$$

Following the method exposed by C. Tam and J. Webb in [5], the coefficients γ_k are determined requiring the Fourier Transform of the finite difference scheme (4) to be a close approximation of the partial derivative $\left(\frac{\partial u}{\partial x}\right)_l$.

(4) is a special case of:

$$\left(\frac{\partial u}{\partial x}\right)_l \simeq \sum_{k=-m}^m \gamma_k u(x + kh) \quad (5)$$

where x is a continuous variable, and can be recovered setting $x = lh$.

Denote by ω the phase. Applying the Fourier transform, referred to by $\hat{}$, to both sides of (5), yields:

$$j\omega \hat{u} \simeq \sum_{k=-m}^m \gamma_k e^{jk\omega h} \hat{u} \quad (6)$$

j denoting the complex square root of -1 .

Comparing the two sides of (6) enables us to identify the wavenumber $\bar{\lambda}$ of the finite difference scheme (4) and the quantity $\frac{1}{j} \sum_{k=-m}^m \gamma_k e^{jk\omega h}$, i. e.: The wavenumber of

the finite difference scheme (4) is thus:

$$\bar{\lambda} = -j \sum_{k=-m}^m \gamma_k e^{j k \omega h} \quad (7)$$

To ensure that the Fourier transform of the finite difference scheme is a good approximation of the partial derivative $(\frac{\partial u}{\partial x})_l$ over the range of waves with wavelength longer than $4h$, the a priori unknowns coefficients γ_k must be chosen so as to minimize the integrated error:

$$\begin{aligned} \mathcal{E} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\lambda h - \bar{\lambda} h|^2 d(\lambda h) \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\lambda h + j \sum_{k=-m}^m \gamma_k e^{j k \omega h} h|^2 d(\lambda h) \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\zeta + j \sum_{k=-m}^m \gamma_k \{\cos(k \zeta) + j \sin(k \zeta)\}|^2 d\zeta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \left[\zeta - \sum_{k=-m}^m \gamma_k \sin(k \zeta) \right]^2 + \left[\sum_{k=-m}^m \gamma_k \cos(k \zeta) \right]^2 \right\} d\zeta \\ &= 2 \int_0^{\frac{\pi}{2}} \left\{ \left[\zeta - \sum_{k=-m}^m \gamma_k \sin(k \zeta) \right]^2 + \left[\sum_{k=-m}^m \gamma_k \cos(k \zeta) \right]^2 \right\} d\zeta \end{aligned} \quad (8)$$

The conditions that \mathcal{E} is a minimum are:

$$\frac{\partial \mathcal{E}}{\partial \gamma_i} = 0, \quad i = -m, \dots, m \quad (9)$$

i. e.:

$$\int_0^{\frac{\pi}{2}} \left\{ -\zeta \sin(i \zeta) + \sum_{k=-m}^m \gamma_k \cos((k-i) \zeta) \right\} d\zeta = 0 \quad (10)$$

Changing i into $-i$, and k into $-k$ in the summation yields:

$$\int_0^{\frac{\pi}{2}} \left\{ \zeta \sin(i \zeta) + \sum_{k=-m}^m \gamma_{-k} \cos((-k+i) \zeta) \right\} d\zeta = 0 \quad (11)$$

i. e.:

$$\int_0^{\frac{\pi}{2}} \left\{ \zeta \sin(i \zeta) + \sum_{k=-m}^m \gamma_{-k} \cos((k-i) \zeta) \right\} d\zeta = 0 \quad (12)$$

Thus:

$$\int_0^{\frac{\pi}{2}} \sum_{k=-m}^m \{\gamma_{-k} + \gamma_k\} \cos((k-i)\zeta) d\zeta = 0 \quad (13)$$

which yields:

$$\frac{\pi}{2} \{\gamma_{-i} + \gamma_i\} + \sum_{k \neq i, k=-m}^m \left\{ \frac{\gamma_{-k} + \gamma_k}{k-i} \right\} \sin\left((k-i)\frac{\pi}{2}\right) = 0 \quad (14)$$

which can be considered as a linear system of $2m+1$ equations, the unknowns of which are the $\gamma_{-i} + \gamma_i$, $i = -m, \dots, m$. The determinant of this system is not equal to zero, while it is the case of its second member: the Cramer formulae give then, for $i = -m, \dots, m$:

$$\gamma_{-i} + \gamma_i = 0 \quad (15)$$

or:

$$\gamma_{-i} = -\gamma_i \quad (16)$$

For $i = 0$, one of course obtains:

$$\gamma_0 = 0 \quad (17)$$

All this ensures:

$$\sum_{k=-m}^m \gamma_k = 0 \quad (18)$$

The values of the γ_k coefficients are obtained by substituting relations (16) into (10):

$$\sum_{k=-m}^m \gamma_k = 0 \quad (19)$$

m being a strictly positive integer, a $2m+1$ -points *DRP* scheme ([5]) is thus given by:

$$-u_i^{n+1} + u_i^n + \frac{\tau}{h} \sum_{k=-m}^m \gamma_k u_{i+k}^n = 0 \quad (20)$$

where the γ_k , $k \in \{-m, m\}$ are the coefficients of the considered scheme, and satisfy the relations (16).

Considering again the u_i^m terms as functions of the mesh size h and time step τ , expanding them at a given order by means of their Taylor series expansion, and neglecting the $o(\tau^p)$ and $o(h^q)$ terms, for given values of the integers p, q , leads to the following differential approximation (see [6]):

$$-u_i^{n+1} + u_i^n + \frac{\tau}{h} \sum_{k=-m}^m \gamma_k \mathcal{F}_{i+k}^n \left(u, \frac{\partial^r u}{\partial x^r}, \frac{\partial^s u}{\partial t^s}, h, \tau \right) = 0 \quad (21)$$

where \mathcal{F}_{i+k}^n denotes the function of $u, \frac{\partial^r u}{\partial x^r}, \frac{\partial^s u}{\partial t^s}, h, \tau$ obtained by means of the above Taylor expansion, r, s being integers.

For sake of simplicity, a non-dimensional form of Eq. (21) will be used:

$$-\tilde{u}_i^{n+1} + \tilde{u}_i^n + \sum_{k=-m}^m \gamma_k \mathcal{F}_{i+k}^n \left(\tilde{u}, \frac{\partial^r \tilde{u}}{\partial x^r}, \frac{\partial^s \tilde{u}}{\partial t^s} \right) = 0 \quad (22)$$

Depending on this differential approximation (22), solutions, as solitary waves, may arise.

The paper is organized as follows. DRP schemes are analyzed in section 2. The general method is exposed in Section 3. Classical DRP schemes are studied in section 4, where it is shown that out of the two studied schemes, only one leads to solitary waves. A related class of travelling wave solutions of equation (21) is thus presented, by using a hyperbolic ansatz.

2 Analysis of DRP schemes

Consider u_i^n as a function of the time step τ , and expand it at the second order by means of its Taylor series:

$$u_i^{n+1} = u(i h, (n+1) \tau) = u(i h, n \tau) + \tau u_t(i h, n \tau) + \frac{\tau^2}{2} u_{tt}(i h, n \tau) + o(\tau^2) \quad (23)$$

It ensures:

$$\frac{u_i^{n+1} - u_i^n}{\tau} = u_t(i h, n \tau) + \frac{\tau}{2} u_{tt}(i h, n \tau) + o(\tau) \quad (24)$$

In the same way, for $k \in \{-m, m\}$, consider u_{i+k}^n as a function of the mesh size h , and expand it at the fourth order by means of its Taylor series expansion:

$$\begin{aligned} u_{i+k}^n &= u((i+k)h, n\tau) \\ &= u(h, n\tau) + kh u_x(ih, n\tau) \\ &\quad + \frac{k^2 h^2}{2} u_{xx}(ih, n\tau) + \frac{k^4 h^4}{4!} u_{xxxx}(ih, n\tau) + o(h^4) \end{aligned} \quad (25)$$

Equation (21) can thus be written as:

$$\begin{aligned} & -u_t(ih, n\tau) - \frac{\tau}{2} u_{tt}(ih, n\tau) + o(\tau) \\ & + \frac{\tau}{h} \sum_{k=-m}^m \gamma_k \left\{ u(ih, n\tau) + kh u_x(ih, n\tau) + \frac{k^2 h^2}{2} u_{xx}(ih, n\tau) + \frac{k^4 h^4}{4!} u_{xxxx}(ih, n\tau) + o(h^4) \right\} = 0 \end{aligned} \quad (26)$$

i. e., at $x = ih$ and $t = n\tau$:

$$-u_t - \frac{\tau}{2} u_{tt} + o(\tau) + \frac{\tau}{h} \sum_{k=-m}^m \gamma_k \left\{ u + kh u_x + \frac{k^2 h^2}{2} + u_{xx} \frac{k^4 h^4}{4!} u_{xxxx} + o(h^4) \right\} = 0 \quad (27)$$

(19) ensures then:

$$-u_t - \frac{\tau}{2} u_{tt} + o(\tau) + \frac{\tau}{h} \sum_{k=-m}^m k \gamma_k \{h u_x + o(h^4)\} = 0 \quad (28)$$

The related first differential approximation of the Burgers equation (1) is thus obtained neglecting the $o(\tau)$ and $o(h^2)$ terms, yielding:

$$-u_t - \frac{\tau}{2} u_{tt} + \tau \sum_{k=-m}^m k \gamma_k u_x = 0 \quad (29)$$

For sake of simplicity, this latter equation can be adimensionalized in the following way:
set:

$$\begin{cases} u &= U_0 \tilde{u} \\ t &= \tau_0 \tilde{t} \\ x &= h_0 \tilde{x} \end{cases} \quad (30)$$

where:

$$U_0 = \frac{h_0}{\tau_0} \quad (31)$$

In the following, Re_h will denotes the mesh Reynolds number, defined as:

$$Re_h = \frac{U_0 h}{\mu} \quad (32)$$

For $j \in \mathbb{N}$, the change of variables (30) leads to:

$$\begin{cases} u_t &= \frac{U_0}{\tau_0} \tilde{u}_{\tilde{t}} \\ u_{tj} &= \frac{U_0}{\tau_0^j} \tilde{u}_{\tilde{t}j} \\ u_{xj} &= \frac{U_0}{h_0^j} \tilde{u}_{\tilde{x}j} \end{cases} \quad (33)$$

(29) becomes:

$$-\frac{U_0}{\tau_0} \tilde{u}_{\tilde{t}} - \frac{\tau}{2} \frac{U_0}{\tau_0^2} \tilde{u}_{\tilde{t}\tilde{t}} + 2\tau \sum_{k=1}^m k \gamma_k \frac{U_0}{h_0} \tilde{u}_{\tilde{x}} = 0 \quad (34)$$

Multiplying (34) by $\frac{\tau_0}{U_0}$ yields:

$$-\tilde{u}_{\tilde{t}} - \frac{\tau}{2\tau_0} \tilde{u}_{\tilde{t}\tilde{t}} + 2 \frac{\tau \tau_0}{h_0} \sum_{k=1}^m k \gamma_k h \tilde{u}_{\tilde{x}} = 0 \quad (35)$$

For $h = h_0$, due to $\sigma = \frac{U_0 \tau}{h}$, Eq. (35) becomes:

$$-\tilde{u}_{\tilde{t}} - \frac{\tau}{2\tau_0} \tilde{u}_{\tilde{t}\tilde{t}} + 2 \frac{\tau h_0}{\mu Re_h} \sum_{k=1}^m k \gamma_k \tilde{u}_{\tilde{x}} = 0 \quad (36)$$

which simplifies in:

$$-\tilde{u}_{\tilde{t}} - \frac{\sigma}{2} \tilde{u}_{\tilde{t}\tilde{t}} + \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k \tilde{u}_{\tilde{x}} = 0 \quad (37)$$

3 Solitary waves

Approximated solutions of the Burgers equation (1) by means of the difference scheme (20) strongly depend on the values of the time and space steps. For specific values of τ and h , equation (37) can, for instance, exhibit travelling wave solutions which can represent great disturbances of the searched solution.

We presently aim at determining the conditions, depending on the values of the parameters τ and h , which give birth to travelling wave solutions of (37).

Following Feng [3] and our previous work [4], in which travelling wave solutions of the *CBKDV* equation were exhibited as combinations of bell-profile waves and kink-profile waves, we aim at determining travelling wave solutions of (37) (see [7],[8], [9], [10], [11], [12], [13], [14], [15]).

Following [3], we assume that equation (37) has travelling wave solutions of the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \tilde{u}(\xi), \quad \xi = \tilde{x} - v \tilde{t} \quad (38)$$

where v is the wave velocity. Substituting (38) into equation (22) leads to:

$$\tilde{\mathcal{F}}(\tilde{u}, \tilde{u}^{(r)}, (-v)^s \tilde{u}^{(s)}) = 0, \quad (39)$$

Performing an integration of (39) with respect to ξ leads to an equation of the form:

$$\tilde{\mathcal{F}}_{\xi}^{\mathcal{P}}(\tilde{u}, \tilde{u}^{(r)}, (-v)^s \tilde{u}^{(s)}) = C, \quad (40)$$

where C is an arbitrary integration constant, which will be the starting point for the determination of solitary waves solutions.

It is important to note that, contrary to other works, the integration constant is not taken equal to zero, which would lead to a loss of solutions.

4 Travelling Solitary Waves

4.1 Hyperbolic Ansatz

The discussion in the preceding section provides us useful information when we construct travelling solitary wave solutions for equation (39). Based on these results, in this section, a class of travelling wave solutions of the equivalent equation (29) is searched as a combination of bell-profile waves and kink-profile waves of the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \sum_{i=1}^n (U_i \tanh^i [C_i(\tilde{x} - v\tilde{t})] + V_i \operatorname{sech}^i [C_i(\tilde{x} - v\tilde{t} + x_0)]) + V_0 \quad (41)$$

where the U_i 's, V_i 's, C_i 's, ($i = 1, \dots, n$), V_0 and v are constants to be determined. In the following, c is taken equal to 1.

4.2 Theoretical analysis

Substitution of (52) into equation (40) leads to an equation of the form

$$\sum_{i,j,k} A_i \tanh^i(C_i \xi) \operatorname{sech}^j(C_i \xi) \sinh^k(C_i \xi) = C \quad (42)$$

the A_i being real constants.

The difficulty for solving equation (42) lies in finding the values of the constants U_i , V_i , C_i , V_0 and v by solving the over-determined algebraic equations. Following [3], after balancing the higher-order derivative term and the leading nonlinear term, we deduce $n = 1$.

Then, following [4] we replace $\operatorname{sech}(C_1 \xi)$ by $\frac{2}{e^{C_1 \xi} + e^{-C_1 \xi}}$, $\sinh(C_1 \xi)$ by $\frac{e^{C_1 \xi} - e^{-C_1 \xi}}{2}$, $\tanh(C_1 \xi)$ by $\frac{e^{C_1 \xi} - e^{-C_1 \xi}}{e^{C_1 \xi} + e^{-C_1 \xi}}$, and multiply both sides by $(1 + e^{2\xi C_1})^2$, so that equation (42) can be rewritten in the following form:

$$\sum_{k=0}^4 P_k(U_1, V_1, C_1, v, V_0) e^{k C_1 \xi} = 0, \quad (43)$$

where the P_k ($k = 0, \dots, 4$), are polynomials of U_1, V_1, C_1, V_0 and v . Depending whether (42) admits or no consistent solutions, spurious solitary waves solutions may, or not, appear.

4.3 Numerical scheme analysis

Equation (39) is presently given by:

$$-v \tilde{u}'(\xi) - \frac{v^2 \sigma}{2} \tilde{u}''(\xi) + \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k \tilde{u}'(\xi) = 0 \quad (44)$$

Performing an integration of (44) with respect to ξ yields:

$$-v \tilde{u}(\xi) - \frac{v^2 \sigma}{2} \tilde{u}'(\xi) + \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k \tilde{u}(\xi) = C \quad (45)$$

i. e.:

$$\left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \tilde{u}(\xi) - \frac{v^2 \sigma}{2} \tilde{u}'(\xi) = C \quad (46)$$

where C is an arbitrary integration constant.

Substitution of (52) for $n = 1$ into equation (46) leads to:

$$\left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \{U_1 \tanh[C_1 \xi] + V_1 \operatorname{sech}[C_1 \xi] + V_0\} - \frac{v^2 \sigma}{2} \left\{ U_1 \operatorname{sech}^2[C_1 \xi] - V_1 \frac{\sinh[C_1 \xi]}{\cosh^2[C_1 \xi]} \right\} = C \quad (47)$$

i. e.:

$$\left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \left\{ U_1 \frac{e^{C_1 \xi} - e^{-C_1 \xi}}{e^{C_1 \xi} + e^{-C_1 \xi}} + \frac{2V_1}{e^{C_1 \xi} + e^{-C_1 \xi}} + V_0 \right\} - \frac{v^2 \sigma}{2} \left\{ U_1 \left(\frac{2}{e^{C_1 \xi} + e^{-C_1 \xi}} \right)^2 - 2V_1 \frac{e^{C_1 \xi} - e^{-C_1 \xi}}{(e^{C_1 \xi} + e^{-C_1 \xi})^2} \right\} = C \quad (48)$$

Multiplying both sides by $(1 + e^{2C_1 \xi})^2$ yields:

$$\begin{aligned} & \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \left\{ U_1 (e^{4C_1 \xi} - 1) + 2V_1 (e^{3C_1 \xi} + e^{C_1 \xi}) + V_0 (1 + e^{2C_1 \xi})^2 \right\} \\ & - \frac{v^2 C_1 \sigma C_1}{2} \left\{ 4U_1 - 2V_1 (e^{3C_1 \xi} - 1) \right\} = C \end{aligned} \quad (49)$$

which is a fourth-order equation in $e^{C_1 \xi}$. This equation being satisfied for any real value of ξ , one therefore deduces that the coefficients of $e^{k C_1 \xi}$, $k = 0, \dots, 4$ must be equal to zero, i.e.:

$$\left\{ \begin{array}{l} 2 \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \{-U_1 + V_0\} - \frac{v^2 C_1 \sigma}{2} \{4U_1 + 2V_1\} = C \\ \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} 2V_1 = 0 \\ 2 \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} V_0 = 0 \\ 2 \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} V_1 + v^2 C_1 \sigma V_1 = 0 \\ \left\{ \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k - v \right\} \{U_1 + V_0\} = 0 \end{array} \right. \quad (50)$$

$v = \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k$, $V_1 \neq 0$ leads to the trivial nul solution. Therefore, V_1 is necessarily equal to zero, which implies:

$$\left\{ \begin{array}{l} v = \frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k \\ U_1 = -\frac{C}{2C_1 v^2 \sigma} \\ V_0 \in \mathbb{R}, C_1 \in \mathbb{R} \end{array} \right. \quad (51)$$

All *DRP* schemes admit thus kink-profile travelling solitary waves solutions, given by:

$$\tilde{u}(\tilde{x}, \tilde{t}) = -\frac{C}{2C_1 \left(\frac{2\sigma}{\mu Re_h} \sum_{k=1}^m k \gamma_k \right)^2} \tanh [C_1(\tilde{x} - v\tilde{t})] + V_0 \quad (52)$$

5 Conclusions

The analysis of the nonlinear equivalent differential equation for finite-differenced *DRP* schemes for the Burgers equation has been carried out. We show that all *DRP* schemes admit spurious travelling solitary waves solutions, which make them, as regards this point, structurally instable.

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