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CLASSIFICATION OF DIRECTIONAL DYNAMICS FOR ADDITIVE CELLULAR AUTOMATA

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Abstract. We continue the study of cellular automata (CA) directional dynamics, *i.e.* the behavior of the joint action of CA and shift maps. This notion has been investigated for general CA in the case of expansive dynamics by Boyle and by Sablik for sensitivity and equicontinuity. In this paper we give a detailed classification for the class of additive CA providing non-trivial examples for some classes of Sablik's classification. Moreover, we extend the directional dynamics studies by considering also factor languages and attractors.

1. Introduction

Cellular automata (CA) are simple formal models for complex systems. They have been widely studied in a number of disciplines (Computer Science, Physics, Mathematics, Biology, Chemistry, *etc.*) with different purposes (simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, quasi-crystals, *etc.*). For an extensive and up-to-date bibliography, for example, see [13, 8, 17, 21, 31, 11, 23].

The huge variety of distinct dynamical behaviors is one of the main features which determined the success of CA in applications. Paradoxically, the formal (decidable) classification of such behaviors is still a major open problem in CA theory. Indeed, many classifications have been introduced over the years but none of them is decidable [14, 9, 5, 19, 16, 12, 22].

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Inspired by [28, 4], M. Sablik proposed to refine Kůrka's equicontinuity classification along "directions" different from the standard time arrow [32]. The idea is to see how "robust" a given topological behavior is when changing the way by which time samples are taken into account. In other words, Sablik studies the space-time structure of CA evolutions by classifying the dynamics of $\sigma^k \circ F^h$, where σ is the shift map and F is the global rule of a CA ($k \in \mathbb{Z}, h \in \mathbb{N}^+$, see Section 2 for the definitions). Sablik's work is concerned particularly with directions of equicontinuity and (left/right) expansivity: he provides a directional dynamics classification of CA according to such properties. Despite his classification sheds new light about the complexity of CA behavior, most of his classes are still not well understood. Moreover, it is actually unknown whether his classification is (at least partially) decidable or not.

Additive CA (ACA) are the subclass of CA whose local rule is defined by an additive function. Despite their simplicity that makes it possible a detailed algebraic analysis, ACA exhibit many of the complex features of general CA. Several important properties of ACA have been studied during the last twenty years and in some cases exact characterizations have been obtained [15, 33, 27, 26, 7, 6].

In this paper we use ACA to further illustrate the work of Sablik and we extend the directional dynamics picture by further introducing attractors and factor languages directions. We provide a very detailed directional dynamics classification of ACA and we compare our classes with Sablik's ones. Moreover, we show that our classification is completely decidable.

The paper is organized as follows. Sections 2 to 4 are devoted to the basic background on the subject of CA and ACA. In Section 5, we consider factor languages directions, in particular we show that all ACA are regular. In Section 6 we consider attractor directions. In Section 7 we provide a directional dynamics classification of ACA and compare our classes with Sablik's ones. In Section 8, we draw some conclusions and provide arguments for the decidability of our classification.

For lack of space, proofs are put in the Appendix.

2. Cellular automata

A CA consists in an infinite set of finite automata distributed over a regular lattice $\mathcal{L}$. All finite automata are identical. Each automaton assumes a *state*, chosen from a finite set A, called the *set of states* or the *alphabet*. A *configuration* is a snapshot of all the states of the automata *i.e.* a function from $\mathcal{L}$ to A. In the present paper, we consider one dimensional CA in which $\mathcal{L} = \mathbb{Z}$. A *local rule* updates the state of each automaton on the basis of its current state and the ones of a fixed set of neighboring automata individuated by the neighborhood frame $N = \{m, m+1, \ldots, a\}$, where $m, a \in \mathbb{Z}$, with $m \leq a$. The integers m, a and $r = \max\{|m|, |a|\}$ are called the *memory*, the *anticipation* and the *radius* of the CA, respectively. Formally, the local rule is a function $f : A^{a-m+1} \to A$. All automata in the lattice are updated synchronously. In other words, the local rule f induces a *global* rule $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ describing the evolution of the whole system from time t to $t+1$:

$$\forall c \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(c)_i = f(c_{i+m}, \ldots, c_{i+a}) \ . \tag{2.1}$$

We say that a CA is *one-sided* if either $m \geq 0$ or $a \leq 0$. The *shift map* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined as $\forall c \in A^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$, $\sigma(c)_i = c_{i+1}$ is one of the simplest examples of CA (it is induced by the local rule $f : A^2 \to A$ defined as $\forall x_0, x_1 \in A,\ f(x_0, x_1) = x_1$ with memory $m = 0$ and anticipation $a = 1$).

In this work we restrict our attention to the class of additive CA, *i.e.*, CA based on an additive local rule defined over the ring $\mathbf{Z}_s = \{0, 1, \ldots, s-1\}$. A function $f : \mathbf{Z}_s^{a-m+1} \to \mathbf{Z}_s$ is said to be additive if there exist $\lambda_m, \ldots, \lambda_a \in \mathbf{Z}_s$ such that it can be expressed as:

$$\forall (x_m, \ldots, x_a) \in \mathbf{Z}_s^{a-m+1}, \quad f(x_m, \ldots, x_a) = \left[\sum_{j=m}^{a} \lambda_j x_j\right]_s$$

where $[x]_s$ is the integer x taken modulo s. A CA is *additive* if its local rule is additive. In this case, Equation (2.1) becomes

$$\forall c \in \mathbf{Z}_s^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(c)_i = \left[\sum_{j=m}^{a} \lambda_j c_{i+j}\right]_s \quad .$$

A rule $f : A^{a-m+1} \to A$ is *permutive in the position* i if $\forall b_m, b_{m+1}, ..., b_{i-1}, b_{i+1}, .., b_a \in A, \forall b \in A, \exists! b_i \in A,\ f(b_m, ..., b_{i-1}, b_i, b_{i+1}, ..., b_a) = b$. The local rule of an ACA is permutive in the position i iff $gcd(s, \lambda_i) = 1$.

Finally, remark that if $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is additive, then for all $k \in \mathbb{Z}, h \in \mathbb{N}^+$ the automaton $(\mathbf{Z}_s^{\mathbb{Z}}, \sigma^k \circ F^h)$ is also additive.

3. Dynamical properties of topological dynamical systems and CA

A topological dynamical system is a pair (X, g) where X is a compact topological space and g is a continuous mapping from X to itself. When A is equipped with the discrete topology and $A^{\mathbb{Z}}$ with the induced product topology, for any CA F, the pair $(A^{\mathbb{Z}}, F)$ is a topological dynamical system. The study of the dynamical behavior of CA is interesting and captured the attention of researchers in the last decades. We now illustrate several properties which are widely recognized as fundamental in the characterization of the behavior of dynamical system.

Dynamical and set theoretical properties for topological dynamical systems. Let (X, g) be a topological dynamical system. It is *injective* (resp., *surjective*, *open*) iff g is injective (resp., surjective, open). It is *sensitive to the initial conditions* (or simply *sensitive*) if $\exists \varepsilon > 0$ such that $\forall x \in X, \forall \delta > 0, \exists y \in X$ such that $d(y, x) < \delta$ and $d(g^n(y), g^n(x)) > \varepsilon$ for some $n \in \mathbb{N}$. It is *positively expansive* if $\exists \varepsilon > 0$ such that $\forall x, y \in X, x \neq y, d(g^n(y), g^n(x)) \geq \varepsilon$ for some $n \in \mathbb{N}$. If X is a perfect set, any positively expansive dynamical system is also sensitive. When g is a homeomorphism it cannot be positively expansive. In this case the notion of *expansivity* can be considered. It is obtained by replacing $n \in \mathbb{N}$ with $n \in \mathbb{Z}$ in the definition of positive expansivity. Both sensitivity and expansivity are referred to as elements of unstability for the system. We now recall two notions which represent conditions of stability for topological dynamical systems. An element $x \in X$ is an *equicontinuity point* for g if $\forall \varepsilon > 0, \exists, \delta > 0$ such that $\forall y \in X, d(y, x) < \delta$ implies that $\forall n \in \mathbb{N},\ d(g^n(y), g^n(x)) < \varepsilon$. The dynamical systems (X, g) is said to be *equicontinuous* iff every point in X is an equicontinuity point. It is *almost equicontinuous* if the set of equicontinuity points is residual (*i.e.*, it can be obtained by an infinite intersection of dense open subsets).

The system (X, g) is *(topologically) transitive* if for any pair of non-empty open sets $U, V \subseteq X, \exists n \in \mathbb{N}$ such that $g^n(U) \cap V \neq \emptyset$. It is *(topologically) mixing* if for any pair of non-empty open sets $U, V \subseteq A^{\mathbb{Z}}, \exists n \in \mathbb{N}$ such that $\forall t \geq n, g^t(U) \cap V \neq \emptyset$. Trivially, a mixing dynamical system is transitive.

A morphism between two dynamical systems (X, g) and (Y, h) is a continuous map $\phi : X \to Y$ such that $h \circ \phi = \phi \circ g$. If ϕ is surjective, (Y, h) is a *factor* of (X, g). If ϕ is a homeomorphism, the two systems are said to be *(topologically) conjugated.* The conjugacy preserves most of the properties seen so far. In the sequel we recall some notions useful to understand the long term behavior of dynamical systems. For a given (X, g), a subset $V \subseteq X$ is said to be *invariant* if $g(V) \subseteq V$.The *omega limit* of a closed invariant subset $V \subseteq X$ is defined as

$$\omega(V) = \cap_{n>0}\overline{\cup_{m>n} g^m(V)}$$

The *limit set* of (X, g) is $\omega(X)$. A dynamical system is called *stable* if it reaches its limit set in a fine amount of time, i.e., if there exists some $n > 0$ such that $\forall m > n$, $g^m(X) = g^n(X)$. A set $Y \subseteq X$ is an attractor if there exists a nonempty open set V such that $F(\overline{V}) \subseteq V$ and $Y = \omega(V)$. In totally disconnected spaces, attractors are omega limit sets of clopen invariant sets. A set $Y \subseteq X$ is a *minimal* attractor if it is an attractor and no proper subset of Y is an attractor. A *quasi-attractor* is a countable intersection of attractors which is not an attractor.

Topology on CA configurations and related properties. In order to study the dynamical properties of CA, $A^{\mathbb{Z}}$ is usually equipped with the Cantor metric d defined as

$$\forall c, c' \in A^{\mathbb{Z}},\ d(c, c') = 2^{-n},\ \text{where } n = \min\left\{i \geq 0\ :\ c_i \neq c'_i \text{ or } c_{-i} \neq c'_{-i}\right\}\ .$$

The topology induced by d coincides with the product topology defined above. In this case, $A^{\mathbb{Z}}$ is a Cantor space, *i.e.*, it is compact, perfect and totally disconnected.

For any configuration $c \in A^{\mathbb{Z}}$ and any pair $i, j \in \mathbb{Z}$, with $i \leq j$, denote by $c_{[i,j]}$ the word $c_i \cdots c_j \in A^{j-i+1}$, *i.e.*, the portion of the configuration $c \in A^{\mathbb{Z}}$ inside the interval $[i, j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $C_i(u) = \{c \in A^{\mathbb{Z}} : c_{[i,i+k-1]} = u\}$. Cylinders are clopen (*i.e.* closed and open) sets *w.r.t.* the Cantor metric.

In the case of CA, it is possible to study other forms of expansivity. For any $n \in \mathbb{Z}$, let $c_{[n,\infty)}$ (resp., $c_{(-\infty,n]}$) denote the portion of a configuration c inside the infinite integer interval $[n, \infty)$ (resp., $(-\infty, n]$). A CA $(A^{\mathbb{Z}}, F)$ is *right* (resp., *left*) *expansive* if there exists a constant $\varepsilon > 0$ such that for any pair of configurations $c, c' \in A^{\mathbb{Z}}$ with $c_{[0,\infty)} \neq c'_{[0,\infty)}$ (resp., $c_{(-\infty,0]} \neq c'_{(-\infty,0]}$) we have $d(F^n(c), F^n(c')) \geq \varepsilon$ for some $n \in \mathbb{N}$. Remark that a CA is positively expansive iff it is both left and right expansive. A simple class of left (resp., right) expansive CA is the one of automata whose local rule is permutive in its leftmost (resp., rightmost) position.

Subshifts and column subshifts. A *subshift* on the alphabet A is a pair (S, σ) where S is a closed σ-invariant subset of the *full shift* $A^{\mathbb{N}}$ (or $A^{\mathbb{Z}}$). From now on, for the sake of simplicity, when it is clear from the context, we identify a subshift (S, σ) with the set S. For $w = w_1 \cdots w_n \in A^n$ and $y \in A^{\mathbb{N}}$, $w \prec y$ means that w is a proper factor of $y \in A^{\mathbb{N}}$, *i.e.*, there exists $i \in \mathbb{N}$ such that $y_{[i,i+n-1]} = w$. Let $\mathcal{F} \subseteq A^*$ and $S_{\mathcal{F}} = \left\{y \in A^{\mathbb{N}} : \forall w \prec y,\ w \notin \mathcal{F}\right\}$. $S_{\mathcal{F}}$ is a subshift, and $\mathcal{F}$ is its set of *forbidden patterns.* A subshift S is said to be a *subshift of finite type* (SFT) if $S = S_{\mathcal{F}}$ for some finite set $\mathcal{F}$. The language of a subshift S is $L_S = \left\{w \in A^* : \exists y \in A^{\mathbb{N}}, w \prec y\right\}$. A subshift is *sofic* if it is a factor of some SFT. We refer to [24] for more on subshifts. Let S_1 and S_2 be two subshifts. A function $\varphi : S_1 \to S_2$ is said to be a *block map* if it is continuous and σ-commuting, i.e., $\varphi \circ \sigma = \sigma \circ \varphi$. In particular, CA are block maps from the sushift $A^{\mathbb{Z}}$ to itself. The *column subshift* of width $k > 0$ of a given CA $(A^{\mathbb{Z}}, F)$, is the subshift $(\Sigma_k(F), \sigma)$ on the $B = A^k$ where

$$\Sigma_k(F) = \left\{ y \in B^{\mathbb{N}} : \exists c \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, y_i = F^i(c)_{[1,k]} \right\} .$$

Remark that, for a given CA F, the set $\Sigma_k(F)$ does not change when replacing the interval $[1, k]$ involved in the previous definition with any other interval of width k . A language $L \subseteq A^*$ is *bounded periodic* if there exist two integers $l \geq 0$ and $n > 0$ such that for every $u \in L$ and $i \geq l$ we have $u_i = u_{i+n}$. A CA is said to be *bounded periodic* (resp., *regular*) if for any $k > 0$ the the language of the column subshift $(\Sigma_k(F), \sigma)$ is bounded periodic (resp., regular).

Directional dynamics of CA. The directional dynamics of CA concerns the study of the joint action of CA with the shift map. More precisely, for a given CA F and for any rational $k/h \in \mathbb{Q}$, the focus is the dynamical behavior of the CA $\sigma^k F^h$. A CA F is said to be equicontinuous (resp., almost equicontinuous, resp. left expansive, resp., right expansive, resp., positively expansive, resp., expansive) along the direction k/h, $k \in \mathbb{Z}$, $h \in \mathbb{N}^+$, if the CA $\sigma^k F^h$ is equicontinuous (resp., almost equicontinuous, resp. left expansive, resp., right expansive, resp., positively expansive, resp., expansive). Note that all the above properties are preserved along directions, i.e., if $\sigma^k F^h$ has property $\mathcal{P}$ then $\forall n > 0, (\sigma^k F^h)^n$ has property $\mathcal{P}$.

3.1. Classifications of CA

This section reviews three important classifications of CA based on the complexity of their column subshift languages, the degree of stability/unstability of their behavior, and the existence of attractors, respectively. All these classifications have been defined and compared in [19].

Theorem 3.1. [19] *Every CA $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes:*

L1 $(A^{\mathbb{Z}}, F)$ *is bounded periodic.*
L2 $(A^{\mathbb{Z}}, F)$ *is regular not bounded periodic.*
L3 $(A^{\mathbb{Z}}, F)$ *is not regular.*

Theorem 3.2. [19] *Every CA $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes:*

E1 $(A^{\mathbb{Z}}, F)$ *is equicontinuous;*
E2 $(A^{\mathbb{Z}}, F)$ *is almost equicontinuous but not equicontinuous;*
E3 $(A^{\mathbb{Z}}, F)$ *is sensitive but not positively expansive;*
E4 $(A^{\mathbb{Z}}, F)$ *is positively expansive.*

Factor languages of equicontinuous and positively expansive CA have been studied in deep. Here we just recall some results that will be useful later.

Theorem 3.3. [19] **L1** = **E1**.

Theorem 3.4. [19, 29] *Let $(A^{\mathbb{Z}}, F)$ be a positively expansive CA with memory and anticipation $m < 0 < a$. Then, it is conjugated to $(\Sigma_{a-m+1}(F), \sigma)$ which is a mixing SFT.*

In particular, Nasu proved that $(\Sigma_{a-m+1}(F), \sigma)$ is conjugated to a full shift [29].

Theorem 3.5. [3] *Let $(A^{\mathbb{N}}, F)$ be a positively expansive CA with anticipation $a > 0$. Then, it is conjugated to $(\Sigma_a(F), \sigma)$ which is a mixing SFT.*

Theorem 3.6. [19] *Every CA $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes.*

A1 *There exist two disjoint attractors.*
A2 *There exists a unique minimal quasi-attractor.*
A3 *There exists a unique minimal attractor different from $\omega(A^{\mathbb{Z}})$.*
A4 *There exists a unique attractor $\omega(A^{\mathbb{Z}}) \neq A^{\mathbb{Z}}$.*
A5 *There exists a unique attractor $\omega(A^{\mathbb{Z}}) = A^{\mathbb{Z}}$.*

We report some results concerning attractors of CA. They will be useful in the sequel.

Theorem 3.7. [20] *An equicontinuous CA has either a pair of disjoint attractors or a unique attractor which is a singleton.*

If an equicontinuous CA is surjective then it must have two disjoint attractors. CA with a unique attractor which is a singleton are called *nilpotent*.

Theorem 3.8. [19] *A transitive CA has a unique attractor.*

Theorem 3.9. [1] *Let $(A^{\mathbb{Z}}, F)$ be a surjective CA with memory m and anticipation a. If either $m > 0$ or $a < 0$, then F is mixing.*

Since transitive CA are surjective and mixing CA are transitive (see [20], for example), from Theorem 3.7 and Theorem 3.9 it follows that surjective CA with either memory $m > 0$ or anticipation $a < 0$ have a unique attractor.

A recent classification concerns the directional dynamics of a CA F. In order to illustrate it, we introduce the following notation.

Definition 3.10. The *equicontinuous, almost equicontinuous, expansive* and *left-or-right expansive* direction sets of a CA $(A^{\mathbb{Z}}, F)$ are defined as follows

- $\mathfrak{E}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ is equicontinuous}\}$.
- $\mathfrak{A}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ is almost equicontinuous}\}$.
- $\mathfrak{X}_F^- = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ is left expansive}\}$.
- $\mathfrak{X}_F^+ = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ is right expansive}\}$.
- $\mathfrak{X}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ is expansive}\}$.

Note that the sets $\mathfrak{E}_F, \mathfrak{A}_F, \mathfrak{X}_F^-$ and $\mathfrak{X}_F^+$ are convex (in $\mathbb{Q}$ or in $\mathbb{R}$). Moreover, note that the set of positively expansive directions is $\mathfrak{X}_F^+ \cap \mathfrak{X}_F^-$.

Theorem 3.11. [32] *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory m and anticipation a.*

- *If $|\mathfrak{E}_F| > 1$ then $\mathfrak{E}_F = \mathbb{Q}$ and $(A^{\mathbb{Z}}, F)$ is nilpotent.*
- *If $\mathfrak{E}_F \neq \emptyset$ and $\mathfrak{E}_F \neq \mathbb{Q}$ then $\exists! \alpha \in [-a, -m], \mathfrak{E}_F = \{\alpha\}$ and $\mathfrak{X}_F^- = (-\infty, \alpha)$, $\mathfrak{X}_F^+ = (\alpha, \infty)$. In particular, $(A^{\mathbb{Z}}, F)$ is injective.*

Theorem 3.12. [32] *Every $(A^{\mathbb{Z}}, F)$ CA with memory m and anticipation a falls exactly in one of the following classes:*

C1. $\mathfrak{E}_F = \mathfrak{A}_F = \mathbb{Q}$ *and* $\mathfrak{X}_F^- = \mathfrak{X}_F^+ = \emptyset$. *This happens iff $(A^{\mathbb{Z}}, F)$ is nilpotent.*
C2. *There exists $\alpha \in [-a, -m]$, $\mathfrak{E}_F = \mathfrak{A}_F = \{\alpha\}$. Moreover, if $(A^{\mathbb{Z}}, F)$ is surjective, $\mathfrak{X}_F^- = (-\infty, \alpha)$ and $\mathfrak{X}_F^+ = (\alpha, \infty)$.*
C3. *There exists $\alpha \in [-a, -m]$, $\mathfrak{E}_F = \emptyset, \mathfrak{A}_F = \{\alpha\}$.*

C4. *There exist* $\alpha_1 < \alpha_2$ *such that* $(\alpha_1, \alpha_2) \subseteq \mathfrak{A}_F \subseteq [\alpha_1, \alpha_2] \subseteq [-a, -m]$ *and* $\mathfrak{C}_F = \mathfrak{X}_F^- = \mathfrak{X}_F^+ = \emptyset$.
C5. $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset$. *This implies* $\mathfrak{C}_F = \mathfrak{A}_F = \emptyset$.
C6. $\mathfrak{C}_F = \mathfrak{A}_F = \emptyset$ *and* $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$.

3.2. Main properties of ACA

The dynamical behavior of ACA has been extensively studied. We briefly report the main results which characterize the most important dynamical and set theoretical properties for ACA.

Theorem 3.13. [15, 25, 27, 7, 6] *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA with local rule* $f(x_m, \ldots, x_a) = \left[\sum_{j=-m}^{a} \lambda_j x_j\right]_s$ *and with* $s = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_l^{n_l}$ *where* $p_1, .., p_l$ *are primes. Then,*

- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is surjective iff* $\gcd(s, \lambda_{-m}, ..., \lambda_a) = 1$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is injective iff* $\forall p_i, \exists! \lambda_j, p_i \nmid \lambda_j$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is equicontinuous iff* $\forall p_i, p_i \mid \gcd(\lambda_{-m}, ..., \lambda_{-1}, \lambda_1, ..., \lambda_a)$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is sensitive iff* $\exists p_i, p_i \nmid \gcd(\lambda_{-m}, ..., \lambda_{-1}, \lambda_1, ..., \lambda_a)$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is transitive iff it is mixing iff* $\gcd(s, \lambda_{-m}, ... \lambda_{-1}, \lambda_1, ..., \lambda_a) = 1$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is pos. expansive iff* $\gcd(s, \lambda_{-m}, ..., \lambda_{-1}) = \gcd(s, \lambda_1, ..., \lambda_a) = 1$
- $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is expansive iff* $\gcd(s, \lambda_{-m}, ..., \lambda_{-1}, \lambda_1, ..., \lambda_a) = 1$

Remark that, as immediate consequence of Theorem 3.13, **E2** $= \emptyset$ for ACA. Moreover, all the characterizations are given in terms of coefficients of the local rule and hence they are decidable.

In the sequel, we are going to recall two fundamental tools. The former states that any ACA has a canonical decomposition into simple basic ACA. The latter tells us that in order to study ACA one can focus on the surjective ones. This is possible since ACA are stable and the class of possible dynamics on their limit sets is equivalent to the class of dynamics of surjective ACA.

Theorem 3.14. [10] *Consider an ACA* $(\mathbf{Z}_{pq}^{\mathbb{Z}}, F)$ *with* $\gcd(p, q) = 1$. *Then* $(\mathbf{Z}_{pq}^{\mathbb{Z}}, F)$ *is conjugated to the ACA* $(\mathbf{Z}_p^{\mathbb{Z}} \times \mathbf{Z}_q^{\mathbb{Z}}, [F]_p \times [F]_q)$.

On the basis of this theorem, if $s = p_1^{n_1} \cdots p_l^{n_l}$ is the prime factor decomposition of s, an ACA on $\mathbf{Z}_s$ is conjugated to the product of ACA on $\mathbf{Z}_{p_i^{n_i}}$. So all the properties which are preserved under product and under topological conjugacy are lifted from ACA on $\mathbf{Z}_{p^k}$ to $\mathbf{Z}_s$.

Theorem 3.15. [26] *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA. Then* $\forall h \geq \lfloor \log_2 s \rfloor$, $F^h(\mathbf{Z}_s^{\mathbb{Z}}) = F^{\lfloor \log_2 s \rfloor}(\mathbf{Z}_s^{\mathbb{Z}}) = \omega(\mathbf{Z}_s^{\mathbb{Z}})$. *Moreover,* $(F^h(\mathbf{Z}_s^{\mathbb{Z}}), F)$ *is conjugated to some surjective ACA* $(\mathbf{Z}_{s*}^{\mathbb{Z}}, F*)$.

Remark that the conjugacy map involved in the proof of Theorem 3.15 preserves factor languages complexities, i.e., for $k > 0$, the column factor of width k of $(F^h(\mathbf{Z}_s^{\mathbb{Z}}), F)$ is a SFT if and only if the column factor of width k of $(\mathbf{Z}_{s*}^{\mathbb{Z}}, F*)$ is SFT. This property will be useful in the sequel.

4. Surjective ACA

Thanks to Theorem 3.14 and Theorem 3.15, most of the properties of general ACA can be deduced from *undecomposable ACA*, namely surjective ACA over $\mathbf{Z}_{p^k}$ for some prime number p. In this section we restrict our attention to the class of surjective ACA and, in particular, we classify the possible dynamics of undecomposable surjective ACA. The results contained in this section will be useful later to understand the directional dynamics of general ACA.

One useful property of undecomposable ACA is that there always exist powers of the *undecomposable maps* which are permutive in both their leftmost and rightmost positions.

Lemma 4.1. *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with* p *prime whose local rule has memory* m *and anticipation* a. *Then there exists* $i \in [m, a]$ *such that* $\gcd(\lambda_i, p) = 1$.

Lemma 4.2. [10] *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with* p *prime. Set*

$$L = \min\{j : \gcd(\lambda_j, p) = 1\} \quad \textit{and} \quad R = \max\{j : \gcd(\lambda_j, p) = 1\}.$$

Then there exists $h \geq 1$ *such that the local rule* f^h *associated to* F^h *has the form*

$$f^h(x_{hm}, ..., x_{ha}) = \left[\Sigma_{i=hL}^{hR} \mu_i x_i\right]_{p^k} \ \textit{with} \ \gcd(\mu_{hL}, p) = \gcd(\mu_{hR}, p) = 1.$$

Recall that the condition $\gcd(\mu_{hL}, p) = \gcd(\mu_{hR}, p) = 1$ implies permutivity in hL and hR. The following proposition characterizes the possible dynamics of undecomposable CA.

Proposition 4.3. *Consider a surjective ACA* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *with* p *prime. Then, exactly one of the following cases occurs:*

1. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is equicontinuous.*
2. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is positively expansive.*
3. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is either left or right expansive.*

Remark 4.4. By the same property used in the previous proof, one can show that right/left expansive ACA on $\mathbf{Z}_{p_i^{n_i}}$ are mixing.

The following theorem classifies the directional dynamics of undecomposable surjective ACA: any undecomposable ACA either contains exactly one equicontinuous direction (and it is injective) or contains a positively expansive direction (and it is not injective).

Theorem 4.5. *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with* p *prime. Then exactly one of the following cases can occur*

1. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is injective. Then,*
$$\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset, |\mathfrak{E}_F| = 1 \ \textit{and} \ \mathfrak{X}_F = \mathfrak{X}_F^- \cup \mathfrak{X}_F^+ = \mathbb{Q} \setminus \mathfrak{E}_F.$$
2. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is not injective. Then,*
$$\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset \ \textit{and} \ \mathfrak{E}_F = \mathfrak{X}_F = \emptyset.$$

Remark 4.6. Since the openness property is preserved in every direction and it is preserved also under product, by Theorem 3.14 and Theorem 4.5 it follows that any surjective ACA is open. For proofs of this property in a more general setting see, for example, [33] and [18].

5. Directional dynamics of ACA according to regularity

In this section we show that all ACA are regular. This fact implies that the dynamics of ACA is regular in all rational directions.

Theorem 5.1. [30] *A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is a SFT if and only if $\sigma : \Sigma \to \Sigma$ is open.*

Lemma 5.2. *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift. Then the following conditions are equivalent:*

1. *(Σ, σ) is open*
2. *$\forall n > 0, (\Sigma, \sigma^n)$ is open.*
3. *$\exists n > 1$ such that (Σ, σ^n) is open.*

A proof of Lemma 5.2 in a more general setting can be found in [2].

Lemma 5.3. *Let $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F)$ be a right (left) expansive ACA with p prime. Then for all sufficiently large k, $(\Sigma_k(F), \sigma)$ is a SFT.*

Note that the condition that for all sufficiently large $k > 0, \Sigma_k(F)$ is a SFT is sufficient to conclude that $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F)$ is regular.

Theorem 5.4. *Any ACA is regular.*

Actually, since the conjugacy of Theorem 3.15, preserves factor languages, we can obtain the following more strong property.

Corollary 5.5. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA. Then for all sufficiently large $k > 0, \Sigma_k(F)$ is a SFT.*

Question 1. Is there any ACA having a strictly sofic column factor $\Sigma_k(F)$?

6. Directional dynamics of ACA according to attractors

In this section we study the class of attractors of ACA according to rational directions. In [26], Manzini and Margara show that any ACA can have either a unique attractor or a pair of disjoint attractors. Here we show some properties of *disjoint attractor* directions of ACA. We will need the two following results.

Lemma 6.1. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be a surjective ACA and let $s = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_l^{n_l}$ be the prime factor decomposition of s. Then the following conditions are equivalent:*

1. *$(\mathbf{Z}_s^{\mathbb{Z}}, F)$ has two disjoint attractors,*
2. *$(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is not mixing,*
3. *$(\mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}}, [F]_{p_i^{n_i}})$ is equicontinuous for some $p_i^{n_i}$.*

We can easily characterize the class of attractors of ACA from the class of attractors of surjective undecomposable ACA.

Theorem 6.2. [26] *Any ACA has either a unique attractor or a pair of disjoint attractors.*

We can now study the set of *disjoint attractor directions* of ACA.

Definition 6.3. Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA. The *disjoint attractors* direction set of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is

$$\mathfrak{D}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{ has two disjoint attractors}\}.$$

The following proposition shows some properties of the set $\mathfrak{D}_F$. In particular, we have that $\mathfrak{D}_F$ is finite and that between two disjoint attractors directions $\alpha_1, \alpha_2 \in \mathfrak{D}_F$ there cannot exist left/right expansive directions.

Proposition 6.4. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA with memory m and anticipation a. Then the following conditions hold.*

1. *If $|\mathfrak{E}_F| > 1$ then $\mathfrak{D}_F = \emptyset$.*
2. *If $\mathfrak{E}_F = \{\alpha\}$ then $\mathfrak{D}_F = \{\alpha\}$.*
3. *If $|\mathfrak{D}_F| > 1$ then $\mathfrak{E}_F = \emptyset$.*
4. *$\mathfrak{D}_F \subset [-a, -m]$ is finite.*
5. *If $\mathfrak{D}_F = \{\alpha_1, ..., \alpha_n\}$ then $\forall \alpha_i \leq \alpha_j, [\alpha_i, \alpha_j] \not\subset \mathfrak{X}_F^- \cup \mathfrak{X}_F^+$.*

To conclude we enumerate some classes of ACA for which $\mathfrak{D}_F$ is easy to characterize.

Corollary 6.5. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA.*

- *If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is nilpotent then $\mathfrak{D}_F = \emptyset$.*
- *If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is equicontinuous and not nilpotent then $\mathfrak{D}_F = \{0\}$.*
- *If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is positively expansive then $\mathfrak{D}_F = \emptyset$.*
- *If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is expansive then $\mathfrak{D}_F \neq \emptyset$*

In the case of ACA, the presence of a direction with two disjoint attractors is tightly linked to the presence of some form of equicontinuity. Indeed, such an ACA is either equicontinuous (not nilpotent) or it is the product of an ACA having an equicontinuous direction with some other ACA (see Lemma 6.1). It is not known if the same holds for general CA.

7. Directional dynamics of ACA

In this section we classify the directional dynamics of ACA according to equicontinuous, left/right expansive, expansive and disjoint attractor directions. We do not consider explicitly factor languages directions since, by Theorem 5.4, for ACA all language directions are regular, and, by Theorem 3.3, directions which have bounded periodic languages coincide exactly with equicontinuous directions. To have a more clear picture we introduce explicitly the class of strictly sensitive nonexpansive directions.

Definition 7.1. The *strictly sensitive* direction sets of the ACA $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is defined by $\mathfrak{S}_F = \mathbb{Q} \setminus (\mathfrak{E}_F \cup \mathfrak{X}_F^- \cup \mathfrak{X}_F^+ \cup \mathfrak{X}_F)$.

We consider separately the directional dynamics of non surjective, strictly surjective and injective ACA. Note that, since there are no almost equicontinuous ACA, classes **C3** and **C4** of Theorem 3.12 are empty for ACA. By Theorem 4.5, it follows that surjective ACA always have left and right expansive directions. In particular, it is not difficult to see that for any surjective ACA of memory m and anticipation a it happens that $(-\infty, -a) \subseteq \mathfrak{X}_F^-$ and $(-m, \infty) \subseteq \mathfrak{X}_F^+$. This implies that surjective ACA can only belong to classes **C2**, **C5**, **C6**. In particular, injective ACA are contained in class **C2** $\cup$ **C6** and strictly surjective ACA are contained in **C5** $\cup$ **C6**. Obviously, in the strictly surjective case there are not expansive directions which arise uniquely in the injective case. For injective ACA it happens also that $\mathfrak{D}_F \neq \emptyset$ and that expansive directions are always the complement in $\mathbb{Q}$ of $\mathfrak{D}_F$.

Theorem 7.2. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an injective ACA with memory m and anticipation a. Then $\mathfrak{X}_F = \mathbb{Q} \setminus \mathfrak{D}_F$. Moreover, exactly one of the following cases can occur:*

1. $\mathfrak{E}_F \neq \emptyset$. *Then* $\mathfrak{D}_F = \mathfrak{E}_F = \{\alpha\} \subset [-a, -m]$, $\mathfrak{X}^+{}_F = (\alpha, \infty)$, $\mathfrak{X}^-{}_F = (-\infty, \alpha)$.
2. $\mathfrak{E}_F = \emptyset$. *Then* $\mathfrak{D}_F = \{\alpha_1, .., \alpha_n\} \subset [-a, -m]$, *with* $\alpha_1 < .. < \alpha_n, n > 1$ *and* $\mathfrak{X}^-{}_F = (-\infty, \alpha_1), \mathfrak{X}^+{}_F = (\alpha_n, \infty)$.

Strictly surjective ACA trivially cannot contain equicontinuous directions but they can have disjoint attractors directions.

Theorem 7.3. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be a surjective but non injective ACA with memory* m *and anticipation* a. *Then* $\mathfrak{E}_F = \emptyset$. *Moreover, exactly one of the following cases occurs.*

1. $\mathfrak{D}_F = \emptyset$ *and* $\mathfrak{X}^-{}_F \cap \mathfrak{X}^+{}_F = \emptyset$. *Then* $\exists \alpha_1, \alpha_2 \in [-a, -m], \alpha_1 < \alpha_2, \mathfrak{X}^-{}_F = (-\infty, \alpha_1)$, $\mathfrak{X}^+{}_F = (\alpha_2, \infty)$, $\mathfrak{S}_F = [\alpha_1, \alpha_2]$.
2. $\mathfrak{D}_F = \emptyset$ *and* $\mathfrak{X}^-{}_F \cap \mathfrak{X}^+{}_F \neq \emptyset$. *Then* $\exists \alpha_1, \alpha_2 \in [-a, -m]$, $\alpha_2 \leq \alpha_1$, $\mathfrak{X}^-{}_F = (-\infty, \alpha_1)$, $\mathfrak{X}^+{}_F = (\alpha_2, \infty)$, $\mathfrak{S}_F = \emptyset$.
3. $\mathfrak{D}_F \neq \emptyset$. *Then* $\exists -a \leq \alpha_1 \leq \beta_1 \leq .. \leq \beta_n \leq \alpha_2 \leq -m, \mathfrak{D}_F = \{\beta_1, .., \beta_n\}$, $\mathfrak{X}^-{}_F = (-\infty, \alpha_1), \mathfrak{X}^+{}_F = (\alpha_2, \infty), \mathfrak{S}_F = [\alpha_1, \alpha_2]$.

For any non surjective CA trivially $\mathfrak{X}^-{}_F = \mathfrak{X}^+{}_F = \mathfrak{X}_F = \emptyset$.

Theorem 7.4. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be a non surjective ACA. Then exactly one of the following cases can occur.*

1. $\mathfrak{E}_F = \mathbb{Q}$ *and* $\mathfrak{D}_F = \mathfrak{S}_F = \emptyset$.
2. $\mathfrak{E}_F = \mathfrak{D}_F = \{\alpha\} \subseteq [-a, -m]$ *and* $\mathfrak{S}_F = \mathbb{Q} \setminus \{\alpha\}$.
3. $\mathfrak{S}_F = \mathbb{Q}, \mathfrak{E}_F = \emptyset$ *(with either* $\mathfrak{D}_F = \emptyset$ *or* $\mathfrak{D}_F \neq \emptyset$*)*.

As requested by one of the referee, in the next theorem we summarise all our results and we express them in terms of the coefficients of the local rule. We beg the reader pardon for its unreadable form.

Theorem 7.5. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA with local rule* $f(x_m, \dots, x_a) = \left[\sum_{j=m}^{a} \lambda_j x_j\right]_s$ *and with* $s = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_l^{n_l}$ *where* $p_1, .., p_l$ *are primes. Then,*

1.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1. *of Theorem 7.2 iff*
$$\exists! \lambda_j, \forall p_i, p_i \nmid \lambda_j$$
1.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2. *of Theorem 7.2 iff*
$$\forall p_i, \exists! \lambda_j, p_i \nmid \lambda_j \text{ and } \nexists! \lambda_j, \forall p_i, p_i \nmid \lambda_j$$
2.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1. *of Theorem 7.3 iff*
$$\forall p_i, \exists \lambda_{j'} \neq \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''} \text{ and } \nexists k \in [m, a], \forall p_i, \exists \lambda_{j'} < k \leq \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''}$$
2.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2. *of Theorem 7.3 iff*
$$\exists k \in [m, a], \forall p_i, \exists \lambda_{j'} < k \leq \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''}$$
2.3 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 3. *of Theorem 7.3 iff*
$$\forall p_i, \exists \lambda_j, p_i \nmid \lambda_j \text{ and } \exists p_i, \exists! \lambda_j, p_i \nmid \lambda_j \text{ and } \exists p_{i'}, \exists \lambda_{j'} \neq \lambda_{j''}, p_{i'} \nmid \lambda_{j'}, p_{i'} \nmid \lambda_{j''}$$
3.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1. *of Theorem 7.4 iff it is nilpotent iff*
$$\forall p_i, \forall \lambda_j, p_i \mid \lambda_j$$
3.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2. *of Theorem 7.4 iff*
$$\gcd(s, \lambda_m, ..., \lambda_a) \neq 1 \text{ and } \exists p_i, \exists \lambda_j, p_i \nmid \lambda_j \text{ and}$$
$$\exists k \in [m, a], \forall p_i, p_i \mid \gcd(\lambda_m, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_a)$$
3.3 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 3. *of Theorem 7.4 iff*

$$\gcd(s, \lambda_m, ..., \lambda_a) \neq 1 \ \textit{and} \ \exists p_i, \exists \lambda_j, p_i \nmid \lambda_j \ \textit{and}$$
$$\nexists k \in [m, a], \forall p_i, p_i \mid \gcd(\lambda_m, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_a)$$

8. Conclusions

In this paper we have completely characterized the directional dynamics of ACA, not only *w.r.t.* equicontinuity or expansivity (as in the Sablik's approach) but also *w.r.t.* attractors and factor languages. Figures 1 to 3 summarize all the possibles scenarios.

Looking at the pictures, one immediately sees that the algebraic nature of ACA has greatly reduced the number and complexity of the possible dynamics. For example, we have proved that the factor languages of any ACA are regular along any direction. Of course, this is not true for the general case but it would be very interesting to investigate which is the largest class of CA with such a property.

The directional classification proposed by Sablik [32] sheds some light on how information propagates space-time diagrams of CA. For instance, there is no exchange of information between zones delimited by two directions of equicontinuity (almost equicontinuity) and the rest of phase space. In this paper, we showed that this is also the case for CA having directions with two disjoint attractors (see Figure 2 right or Figure 3). Remark that in the case of ACA, directions with two disjoint attractors are always tightly linked to the presence of equicontinuity (see Lemma 6.1). We wonder if this happens also in the general case where the situation is much more complicated, since one must take into account also almost equicontinuity and other types of attractors.

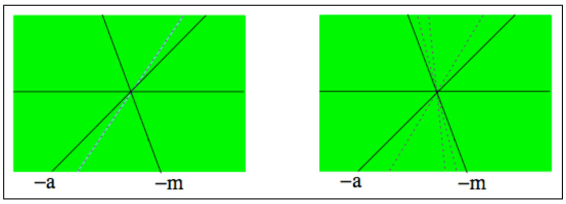

Figure 1: Directional dynamics for injective ACA. Green area depicts expansive directions. Lightblue line is a direction of equicontinuity. Magenta dotted lines are directions presenting two disjoint attractors.

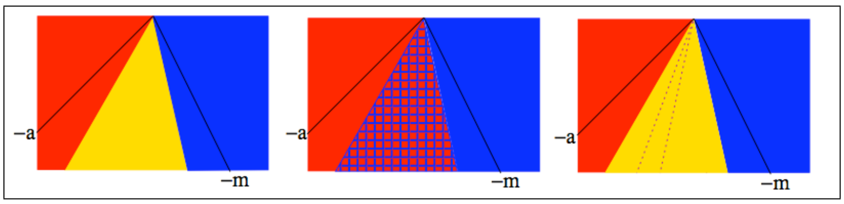

Figure 2: Directional dynamics for surjective ACA. Red (resp., blue) area depicts left (resp., right) expansive directions. Gold area indicates directions presenting sensitivity. Red/blue checkered area gives the positively expansive directions. Magenta dotted lines are directions presenting two disjoint attractors.

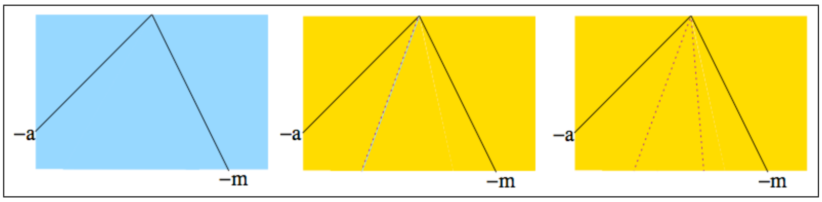

Figure 3: Directional dynamics for non-surjective ACA. Lightblue (resp., gold) area indicates equicontinuity (resp. sensitivity) directions. Magenta dotted lines are directions presenting two disjoint attractors.

To conclude we remark that, from Theorem 7.5, it follows that our classification is completely decidable.

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