On Numbers Badly Approximable by q-adic Rationals

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Preface

The work and results on Diophantine Approximation and Badly Approximable Numbers collected in this thesis are based upon the four papers,

P1. J. Nilsson. On Numbers Badly Approximable by Dyadic Rationals.

P2. J. Nilsson. On Numbers Badly Approximable Via the β-shift.

P3. J. Nilsson. The Fine Structure of Dyadically Badly Approximable Numbers.

P4. J. Nilsson. The Fine Structure of q-adically Badly Approximable Numbers.

In the first two papers we consider, for the unit circle $\mathbb{S}$, the set $F^1(c,\beta) = \{x \in \mathbb{S} : \beta^n x \geq c \pmod 1 \; n \geq 0\}$. In the first paper, **P1**, we restrict ourselves to have $\beta \geq 2$ an integer and we give elementary proofs that $F^1(c,\beta)$ is a fractal set whose Hausdorff dimension depends continuously on c and is constant on intervals which form a set of Lebesgue measure 1. Hence it has a fractal graph. We completely describe the intervals where the dimension remains unchanged. The proofs uses methods from symbolic dynamics and the field of combinatorics on words. In **P2** we refine the results achieved in **P1** to hold for an arbitrary real $\beta > 1$, by using methods and results form the theory of the β-shift. The results obtained implies that we can completely describe the graph of $c \mapsto \dim_H\{x \in [0,1] : x - \frac{m}{\beta^n} < \frac{c}{\beta^n} \pmod 1 \text{ finitely often}\}$.

The two last papers concerns the far more complex set $F^2(c,q) = \{x \in \mathbb{S} : \|q^n x\| \geq c\,, n \geq 0\}$, where $\|\cdot\|$ denotes the smallest distance to an integer and for the integer $q \geq 2$. In **P3** we restrict ourselves to the binary case i.e. having $q = 2$ and in **P4** we generalise the achieved results to hold in the case for an arbitrary integer $q \geq 2$. By similar elementary methods as used in **P1** we prove that $F^2(c,q)$ is a fractal set whose Hausdorff dimension depends continuously on c, is constant on intervals which form a set of Lebesgue measure 1 and is self-similar. Hence it has a fractal graph. We completely characterise the intervals where the dimension remains unchanged. Moreover, we show that the threshold for having positive dimension of $F^2(c,q)$ is closely related to the classical Thue-Morse sequence. A consequence of our results is that we can completely describe the graph of $c \mapsto \dim_H\{x \in [0,1] : \|x - \frac{m}{q^n}\| < \frac{c}{q^n} \text{ finitely often}\}$.

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Contents

Introduction

1 Diophantine Approximation and Badly Approximable Numbers

Let (X, d) be a metric space. Given a sequence $\{x_n\} \subset X$, of maybe random numbers, and a sequence $\{l_n\}$ of positive real numbers we define the following two sets $I = \{y \in X : d(x_n, y) < l_n \text{ infinitely often}\}$, and $F = X \setminus I$. By the notion diophantine approximation we shall mean the study of the sets I and F. Let us make the following remark: if the sequence $\{x_n\}$ is dense in X then I is a non-empty and hence a residual set in the sense of Baire.

For the sequences $\{x_{n,m}\}_{n\in\mathbb{N},\, 0\leq m<n}$ and $\{l_n\}$ such that

$$x_{n,m} = \frac{m}{n} \quad \text{and} \quad l_n = \frac{1}{n^\alpha}, \tag{1.1}$$

with $\gcd(m, n) = 1$, we are in the case of the classical diophantine approximation with rational numbers. It is a well know fact that if $\alpha > 2$ then F is non-empty while it is empty when $\alpha < 2$.

Inspired by the above example, we continue in this direction and refine the definition of the set F to be the following set

$$F(\alpha) = \left\{ y \in X : d(x_{n,m}, y) < \frac{1}{n^\alpha} \text{ finitely often} \right\}. \tag{1.2}$$

An interesting question is to look at the critical exponent, α_0, such that $F(\alpha)$ is empty if $\alpha < \alpha_0$ and is non-empty when $\alpha > \alpha_0$. For this special value α_0 we say that the set $F(\alpha_0)$ is the set of *Badly Approximable Numbers*, *BAN*.

A second step in refinement of (1.2) is to introduce the dependence on an extra parameter, c,

$$F_c(\alpha) = \left\{ y \in X : d(x_{n,m}, y) < \frac{c}{n^\alpha} \text{ finitely often} \right\}.$$

In the one-dimensional case this refinement leads to the area of continued fraction, which was first systematically studied by the Dutch astronomer Huygens in the 17-th century, motivated by technical problems while constructing a model of our solar system. Briefly, the continued fraction for a positive real x is,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where the a_n's are called *partial denominators*. For brevity the continued fractions are often denoted by $[a_0, a_1, a_2, \dots]$. The following theorem gives a neat connection between the badly approximable numbers and the continued fractions, for a proof see [9].

Theorem 1.1 *An irrational x is a BAN if and only if its partial denominators are bounded.*

Yet another version, or refinement, of the F set can be introduced via a condition on the partial denominators. We set

$$F(2,N) = \{x : x = [a_0, a_1, a_2, \dots] \text{ with } \ a_j < N\}.$$

The theory of iterated function system, *IFS*-theory, gives an implicit formula for the $\dim_H F(2,N)$. The set $F_c(2)$ is finer as $F(2,N)$ counts only the maximal a_i while the c takes into account all a_i. In 1891 Hurwitz found that if $c < \frac{1}{\sqrt{5}}$ then $F_c(2)$ is empty and moreover the constant $\frac{1}{\sqrt{5}}$ is the best possible, but otherwise little is known about the set $F_c(2)$.

1.1 q-adically Badly Approximable Numbers

The subject of this thesis is badly approximable numbers under approximation of q-adic numbers. We say that a real number x is q-adic, for a fixed $q > 1$, if it is of the form $x = \frac{m}{q^n}$ for some positive integers n and m. Similar to (1.1) we let

$$x_{n,m} = \frac{m}{q^n} \quad \text{and} \quad l_n = \frac{c}{q^n},$$

for $0 < c < 1$.

We will consider two types of approximations, a one-sided model and a two-sided one. The one-sided q-adically BAN is the set

$$F^1(c,q) = \left\{ x \in \mathbb{S} : x - \frac{m}{q^n} < \frac{c}{q^n} \pmod 1 \text{ finitely often} \right\}$$

and the two-sided q-adically BAN is the set

$$F^2(c,q) = \left\{ x \in \mathbb{S} : \left\| x - \frac{m}{q^n} \right\| < \frac{c}{q^n} \text{ finitely often} \right\},$$

where $\| \cdot \|$ denotes the shortest distance to an integer.

In Part I we study the one-sided BAN. we will present two different approaches. The first, presented in Section 5, is when letting $q \geq 2$ be an integer and the second is when having $q = \beta$ for a real $\beta > 1$, given in Section 7. The main results achieved, when stated for the general β-case, is

Main Result 1.2 *For $\beta > 1$ the function $\phi_\beta : c \mapsto \dim_H F^1(c,\beta)$ is continuous, has derivative zero Lebesgue a.e. and the complementary zero-set, to where the derivative of ϕ_β is zero, has full Hausdorff dimension. Moreover we give the complete description of the intervals where the derivative of ϕ_β is zero.*

The main idea used to prove our main result is to transform the set $F^1(c,q)$ into a set of sequences. This is done by identify a real number $x \in [0,1]$ by its expansion $\boldsymbol{x}$ in base q. Hence the question now becomes a problem in symbolic dynamics. In Section 4 we introduce the concept of minimal sequences, and we present an algorithm how to obtain them. The minimal sequences are limit points of sequences of minimal sequences, which implies the continuity. Later on, the minimal sequences will play a crucial role when describing the intervals where the derivative of ϕ_β is zero in Section 5. For a minimal sequence and an integer $\beta = q \geq 2$ we can associate a transition matrix A_c to the set $F^1(c,q)$. This matrix A_c will be primitive and we can apply the classical Perron-Frobenius Theorem 2.2 to show that the intervals obtained indeed are the right one. In Section 7 we reuse most of the results presented in previous sections and reformulate them to fit into the language of the β-shift. This allows us to use all the machinery developed therein and the generalised results follows elegantly. Ideas and methods in this section are also collected from the works of Persson and Schmeling [18, 20]. We end Part I by giving some numerical presentations of the function ϕ_β.

Our work on the one-sided model is a specialisation and completes earlier works by Urbanski [23], (see Section 3 for a more detailed survey of Urbanski's work).

In Part II we study the set of two-sided q-adically BAN. The set of the two-sided BAN is a far more complex set than the set of one-sided BAN. The results we achieve are the following,

Main Result 1.3 *For an integer $q \geq 2$ the function*

$$\phi_q : c \mapsto \dim_H F^2(c, q)$$

is continuous, is partly self similar, has derivative zero Lebesgue a.e., the complementary zero-set, to where the derivative of ϕ_q is zero, has full Hausdorff dimension. Moreover we give the complete description of the intervals where the derivative of ϕ_q is zero.

As in the study of the one-sided model we transfer the problem of the two-sided approximation into a problem in symbolic dynamics. The ideas in the proofs of our main results follows mainly the same path as in the first part of the one-sided model, but where more attention has to be spent on details. In Section 11 we define the technical concept of shift-bounded sequence which will be a key tool. We define a new kind of minimal sequences in Section 12, present an algorithm to find them and show how they cluster. As in the one-sided model we have for a minimal sequence that we can associate a transition matrix A_c to the set $F^2(c, q)$. We prove that this matrix A_c will be primitive, for certain choices of the sequence $\boldsymbol{c}$, and therefore we can apply the classical Perron-Frobenius Theorem 2.2 to show that the intervals obtained indeed are the right one. The part ends with a few illustrations of the graph of ϕ_q.

Our result will extend and complete previous results given by Allouche, Cosnard, Moreira, Labarca and others, see [1, 2, 12, 14] and Section 9, for more details.

2 Prerequisites

2.1 General Symbolic Dynamics

For an integer $q \geq 2$ let

$$S^{\infty}(q) = \{0, 1, \ldots, q-1\}^{\mathbb{N}} = \{\, \boldsymbol{x} = x_1 x_2 \ldots : x_i \in \{0, 1, \ldots, q-1\}\}$$

be the space of the one-sided infinite sequences on q symbols, equipped with the product topology and the metric

$$\delta_q(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{q^i}. \tag{2.1}$$

Let similarly

$$S^{*}(q) = \{\, \boldsymbol{x} = x_1 x_2 \ldots x_m : x_i \in \{0, 1, \ldots, q-1\}\}$$

be the set of all finite sequences on q symbols. There is a natural embedding of the finite sequences into the set of infinite sequences, we can see a finite sequence as an infinite sequence ending with zeros.

The word sequence will be used both for a finite sequence as well as for an infinite one, but always based on a finite alphabet. Therefore we set

$$S(q) = S^{\infty}(q) \cup S^{*}(q),$$

that is, $S(q)$, is the set of all one-sided sequences on q symbols.

There is a correspondence between $S^{\infty}(q)$ and the real interval $[0, 1]$, by simply consider the q-nary expansion of a real number. That is, for $x \in [0, 1]$ we have

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i} \quad \text{with } x_i \in \{0, 1, \ldots, q-1\} \tag{2.2}$$

and we let $\boldsymbol{x} = x_1 x_2 x_3 \ldots$. This correspondence is one-to-one except for a countable set where it is two-to-one, but this will not cause us any trouble. We introduce here some notation that will be used.

- By a *concatenation* we mean that we append a sequence to a finite sequence, that is, the concatenation of $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{uv}$, similarly we write $\boldsymbol{uu} = \boldsymbol{u}^2$.

- We use the lexicographical order, $\leq$ and $<$, to compare sequences. To compare sequences of different length we introduce the symbol ε which has the property of being smaller than any symbol in $\{0, 1, \ldots, q-1\}$. We can extend any finite sequence by concatenating infinitely many copies of ε.

- We say that $\boldsymbol{x}$ is a *prefix* of $\boldsymbol{s}$ if there exists a sequence $\boldsymbol{u}$ such that $\boldsymbol{s} = \boldsymbol{x}\boldsymbol{u}$ and similarly we then say that $\boldsymbol{u}$ is a *suffix* of $\boldsymbol{s}$. If $\boldsymbol{u}$ is non-void then $\boldsymbol{x}$ is a proper prefix and similarly for a suffix.

- By $\boldsymbol{s}[k, n]$ we mean the subsequence $\boldsymbol{s}[k, n] = s_k s_{k+1} \ldots s_n$ of length $n - k + 1$. For a set A of sequences the notation $A[k, n]$ is the set of subsequences, $A[k, n] = \{\boldsymbol{s}[k, n] : \boldsymbol{s} \in A\}$.

- The notation $| \cdot |$ will mean the length of a sequence, that is, $|\boldsymbol{s}[k, n]| = n - k + 1$. We will also use the $| \cdot |$-notation for the cardinality of a set.

- For a sequence $\boldsymbol{x}$ we define the left-shift σ by $(\sigma(\boldsymbol{x}))_i = x_{i+1}$ and we let $\sigma^n = \sigma \circ \sigma^{n-1}$. If $\boldsymbol{x}$ is a finite sequence then $\sigma^{|\boldsymbol{x}|}(\boldsymbol{x})$ is the empty sequence.

- By the notation $\boldsymbol{x}^*$ we mean the sequence $(\boldsymbol{x}^*)_n = q - 1 - (\boldsymbol{x})_n$; the bit-wise inverse of $\boldsymbol{x}$. If $\boldsymbol{x}$ is finite then $\boldsymbol{x}^*$ can be seen as the inverse element of $\boldsymbol{x}$ in $S^*(q)$.

- The notation $\boldsymbol{x}'$ will mean the inverse when seeing $\boldsymbol{x}$ as a real number, that is, the inverse element of $\boldsymbol{x}$ in $S^\infty(q)$. If $\boldsymbol{x}$ is an infinite sequence then $\boldsymbol{x}^* = \boldsymbol{x}'$ but this equality does not hold in the finite case, as we then have to cast $\boldsymbol{x}$ to an element in $S^\infty(q)$, i.e. we have to append zeros at the end. We have $1' = (10^\infty)' = (q-2)(q-1)^\infty = (q-1)$ but $1^* = (q-2)$. We will always let $|\boldsymbol{x}| = |\boldsymbol{x}'|$.

- For a finite sequence $\boldsymbol{x}$, not ending with 0, the notation $\tilde{\boldsymbol{x}}$ means the sequence where the last symbol of $\boldsymbol{x}$ has been decreased by one.

- For a finite sequence $\boldsymbol{x}$, not ending with $(q-1)$, the notation $\hat{\boldsymbol{x}}$ means the sequence where the last symbol of $\boldsymbol{x}$ has been increased by one.

Let A be a square $\{0,1\}$ matrix with rows and columns indexed by $\{0,1,\ldots,n-1\}$. The matrix A defines a closed, shift invariant subset S_A of $S^\infty(n)$. The subset S_A defined by A is defined by selecting sequence as

$$S_A = \left\{\boldsymbol{x} \in S^\infty(n) : A_{x_i x_{i+1}} = 1 \text{ for all } i > 0\right\}.$$

The dynamical system and the restriction of the shift transformation is the *one-sided shift of finite type defined by A*. We call such a matrix A a *transition matrix*.

Example 2.1 The matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

where the rows and columns are indexed in the order $0,1,2$, defines the shift $S_A \subset S^\infty(3)$ of sequences which does not contain a repetition of a symbol, as $A_{aa} = 0$ for $a \in \{0,1,2\}$. □

The representation of a subshift via a transition matrix is not unique, two different matrices A and B may describe the same subshift. We say that a transition matrix A is *irreducible* if there for each pair of indices i,j exists an n such that $(A^n)_{ij} > 0$. Similarly, if there is an m such that $(A^m)_{ij} > 0$ for all pairs i,j we say that the matrix is *primitive*. Clearly primitivity implies irreducibility. A subshift of finite type is topologically transitive if and only if it can be represented by an irreducible transition matrix and a subshift of finite type is topologically mixing if and only if it can be represented by a primitive transition matrix.

For irreducible transition matrices we have the Perron-Frobenius theorem, (see [10, 16]).

Theorem 2.2 (Perron-Frobenius) *Suppose A is a nonnegative, square matrix. If A is irreducible there exists a real eigenvalue $\lambda > 0$ such that*

1. *λ is a simple root of the characteristic polynomial*
2. *λ has strictly positive left and right eigenvectors*
3. *the eigenvectors for λ are unique up to constant multiple*
4. *$\lambda > |\mu|$, where μ is any other eigenvalue*
5. *if $0 \le B \le A$ (entry by entry) and β is an eigenvalue for B then $|\beta| \le \lambda$ and equality occurs if and only if $B = A$.*

The special eigenvalue λ, is the Perron value of the matrix A. A positive eigenvector corresponding to λ is called a Perron eigenvector.

Note that the notion of Perron value coincides for non-negative irreducible matrices with the notion of spectral radius $\rho(A)$.

By coding each symbol in the alphabet $\{0, 1, \ldots, n-1\}$ with a finite word out of q-symbols, we can transfer a sequence of n symbols into a q-nary sequence. We may assume that all these coding words used in the translation have the same length m. Then we have that the rows and columns in a transition matrix can be indexed with finite words of length m. Hence for $\boldsymbol{u}_k = \boldsymbol{x}[k, k+m-1]$ and $\boldsymbol{v}_k = \boldsymbol{x}[k+1, k+m]$ the set

$$\{\boldsymbol{x} \in S^\infty(q) : A_{\boldsymbol{u}_k \boldsymbol{v}_k} = 1 \text{ for all } k > 0\}$$

describes a shift of finite type.

Example 2.3 For a binary sequence $\boldsymbol{x}$ consider transitions of length 2, that is, $\boldsymbol{u}_k = \boldsymbol{x}[k, k+1]$ and $\boldsymbol{v}_k = \boldsymbol{x}[k+1, k+2]$.

$$\boldsymbol{x} = x_1\ x_2\ \underbrace{x_3\ \overbrace{x_4}^{}}_{\boldsymbol{u}_3}\!\!\overbrace{\ x_5}^{\boldsymbol{v}_3}\ x_6\ x_7\ x_8\ x_9\ \ldots$$

Then the matrix

$$A = \begin{array}{c} \scriptstyle 00 \\ \scriptstyle 01 \\ \scriptstyle 10 \\ \scriptstyle 11 \end{array} \overset{\begin{array}{cccc} \scriptstyle 00 & \scriptstyle 01 & \scriptstyle 10 & \scriptstyle 11 \end{array}}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

defines the shift $S_A \subset S^\infty(2)$ of sequences which does not contain a cube of a symbol. This because the sequence 000 would give rise to $\boldsymbol{u}_k = 00$ and $\boldsymbol{v}_k = 00$. But as $A_{00,00} = 0$ we have that 000 is not an allowed subsequence of sequences in $S_A \subset S^\infty(2)$. Similarly $A_{11,11} = 0$ gives that 111 is not an allowed subsequence. □

Example 2.4 For a finite sequence $\boldsymbol{c}$ let

$$E(\boldsymbol{c}) = \{\boldsymbol{x} \in S^\infty(q) : \boldsymbol{c}' \geq \sigma^n(\boldsymbol{x}) \geq \boldsymbol{c} \text{ for all } n \geq 0\}.$$

By considering the set of prefixes we can describe the set $E(\boldsymbol{c})$ by a transition matrix A. An element $\boldsymbol{s} \in E(\boldsymbol{c})[1, |\boldsymbol{c}|]$ defines the entry $A_{\boldsymbol{s}[1,|\boldsymbol{c}|-1],\boldsymbol{s}[2,|\boldsymbol{c}|]} = 1$. Note that we can also consider $E(\boldsymbol{c})[1, k]$ for some $k > |\boldsymbol{c}|$, but we then only obtain a transition matrix of greater size. □

2.2 Dimension

Definition 2.5 *Let $s \in [0, \infty]$. The s-dimensional Hausdorff measure $H^s(Y)$ of a subset of a metric space X is defined by*

$$H^s(Y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{1=1}^{\infty} \operatorname{diam}(U_i)^s : Y \subset \bigcup_{i=1}^{\infty} U_i \,,\, \sup_i \operatorname{diam}(U_i) \leq \varepsilon \right\}.$$

The unique s_0 such that

$$H^s(Y) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \end{cases}$$

we call the Hausdorff dimension of the set Y and it will be denoted by $\dim_H Y$.

A way of estimating the Hausdorff dimension of a set is to use the connection between the Hölder exponent and the Hausdorff dimension. The following result is well known.

Proposition 2.6 *Let $X \subset \mathbb{R}^n$ and suppose that $f : X \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq C\, |x - y|^\alpha \quad (x, y \in X).$$

Then $\dim_H f(X) \leq \frac{1}{\alpha} \dim_H X$.

For a deeper discussion of dimension theory and methods used therein see Falconer's book [8] or Pesin's book [17] on dimension theory.

Let $E \subset S^{\infty}(q)$ and recall that by $E[1,n]$ we denote the set of prefixes of length n of sequences in E, that is, $E[1,n] = \{\boldsymbol{x}[1,n] : \boldsymbol{x} \in E\}$.

Definition 2.7 *We define the topological entropy h_{top} of the subshift of finite type E as the growth rate of the number of sequences allowed as the length n increases,*

$$h_{\text{top}}(E) = \lim_{n\to\infty} \frac{1}{n} \log \big|E[1,n]\big|,$$

where $|\cdot|$ denotes the cardinality of a set.

The existence of the above limit follows by simply noticing the subadditivity property of the function $n \mapsto \log|E[1,n]|$:

$$\log \big|E[1,n+m]\big| \le \log \big|E[1,n]\big| + \log \big|E[1,m]\big|.$$

The existence of the limit also implies that there exists constants k_1 and k_2 with $k_1\lambda^n \le |E[1,n]| \le k_2\lambda^n$ for all n and where $\log\lambda$ is the topological entropy of E.

Theorem 2.8 *Let $E \subset S^{\infty}(q)$ be a subshift of finite type described by the transition matrix A_E, with the spectral radius $\rho(A_E)$. Then*

1. $h_{\text{top}}(E) = \log \rho(A_E)$
2. $\dim_H E = \dfrac{\log \rho(A_E)}{\log q}$.

Theorem 2.8 gives a link between the topological entropy and the Hausdorff dimension via transition matrices for subshifts of finite type. For a proof Theorem 2.8 see Pesin's book [17] on dimension theory.

Part I

One-sided q-adically Badly Approximable Numbers

3 Introduction

In this part we are going to study a special case of diophantine approximation, one-sided approximation by real numbers of the form $\frac{m}{\beta^n}$ for the real $\beta > 1$ and an integer m. Similar to the approximation by rationals in (1.1) we set the sequences $\{x_n\}$ and $\{l_n\}$ to be

$$x_{n,m} = \frac{m}{\beta^n} \quad \text{and} \quad l_n = \frac{c}{\beta^n},$$

for $0 \leq c \leq 1$. We will turn our interest to the same type of questions as in the classical approximation case and look at the set of badly approximable numbers under these special form of $\{x_n\}$ and $\{l_n\}$. We define $F^1(c, \beta)$ to be the set

$$F^1(c,\beta) = \left\{ x \in \mathbb{S} : x - \frac{m}{\beta^n} < \frac{c}{\beta^n} \pmod 1 \text{ finitely often} \right\}. \tag{3.1}$$

As we are going to study dimensional properties of $F^1(c, \beta)$ we can restrict ourselves to the case when the condition in (3.1) is not finitely often fulfilled, but is never fulfilled. So we introduce $F(c, \beta)$ by

$$F(c,\beta) = \{x \in \mathbb{S} : \beta^n x \geq c \pmod 1 \text{ for all } n \geq 0\}. \tag{3.2}$$

Then $F^1(c, \beta)$ is the countable union of preimages of $F(c, \beta)$ under multiplication by β. Hence we have $\dim_H F^1(c, \beta) = \dim_H F(c, \beta)$. For $\beta > 1$ we define the dimension function $\phi_\beta : [0, 1] \to [0, 1]$ by $\phi_\beta(c) = \dim_H F(c, \beta)$. The main results concerning the function ϕ_β are,

Main Result 3.1 *For $\beta > 1$ the function $\phi_\beta : c \mapsto \dim_H F(c, \beta)$ is continuous, has derivative zero Lebesgue a.e. and the complementary zero-set, to where the derivative of ϕ_β is zero, has full Hausdorff dimension. Moreover we give the complete description of the intervals where the derivative of ϕ_β is zero.*

In [23], Urbanski studies the more general set

$$K_g(c) = \{x \in \mathbb{S} : g^n(x) \geq c \text{ for all } n \geq 0\},$$

where $g : \mathbb{S} \to \mathbb{S}$ is a C^2 expanding and orientation preserving map. The results given in [23] include the continuity of the map $c \mapsto \dim_H K_g(c)$, and that this map has derivative zero Lebesgue a.e., but leaves out the complete characterisation of the intervals where the dimension remains unchanged.

In Section 5 we are going to consider the case when we restrict ourselves to have $\beta = q \in \mathbb{N}$ and strictly larger than 1. We prove there, in a simpler way than presented by Urbanski [23], that the map ϕ_q is continuous, has derivative zero Lebesgue a.e. and we characterise completely the intervals where the dimension remains unchanged. The main tool in proving these results is the classical Perron-Frobenius Theorem 2.2 for irreducible transition matrices.

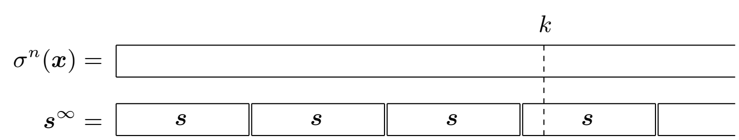
In Section 7 we reprove and refine the results obtained for the map ϕ_β in Section 5, that is, we prove that the results achieved for $\beta = q$ also can be obtained for an arbitrary real $\beta > 1$. This refinement heavily relies on result on the β-shift.

4 Minimal Sequences

This section is devoted to study certain types of sequences, which will serve as main tools in proving our results in the forthcoming sections. We start with a simple, but fundamental lemma.

Lemma 4.1 *Let* $\boldsymbol{s} \in S^*(q)$. *Then any sequence* $\boldsymbol{x} \in S^\infty(q)$ *fulfils* $\sigma^n(\boldsymbol{x}) \geq \boldsymbol{s}$ *for all* $n \geq 0$ *if and only if* $\sigma^n(\boldsymbol{x}) \geq \boldsymbol{s}^\infty$ *for all* $n \geq 0$.

Proof: It is clear that if $\sigma^n(\boldsymbol{x}) \geq \boldsymbol{s}^\infty$ for $n \geq 0$ then $\sigma^n(\boldsymbol{x}) \geq \boldsymbol{s}$ for $n \geq 0$ since $\boldsymbol{s}^\infty \geq \boldsymbol{s}$. Conversely, assume there is an n such that $\sigma^n(\boldsymbol{x}) < \boldsymbol{s}^\infty$. Let k be the first position where $\sigma^n(\boldsymbol{x})$ differs from $\boldsymbol{s}^\infty$.



We can write $k = m|\boldsymbol{s}| + r$ for some positive integers m, r with $r < |\boldsymbol{s}|$. But then we must have $\sigma^{n+m|\boldsymbol{s}|}(\boldsymbol{x}) < \boldsymbol{s}$. □

We are interested in looking at sequences $\boldsymbol{s}$ where we have a maximal difference between $\boldsymbol{s}$ and $\boldsymbol{s}^\infty$. Therefore we introduce the notion of minimal prefixes and minimal sequences.

Definition 4.2 *For a sequence $\boldsymbol{s} \in S(q)$ we define the integer $n_{\boldsymbol{s}}$, which may be infinite, by*

$$n_{\boldsymbol{s}} = \inf \left\{ n \in \mathbb{N} : \boldsymbol{s}[1,n]^\infty \geq \boldsymbol{s} \right\}.$$

We say that $\boldsymbol{s}[1, n_{\boldsymbol{s}}]$ is a minimal prefix of $\boldsymbol{s}$ and if $\boldsymbol{s}[1, n_{\boldsymbol{s}}] = \boldsymbol{s}$ or $n_{\boldsymbol{s}}$ undefined we say that $\boldsymbol{s}$ is a minimal sequence or simply just minimal.

Example 4.3 The sequence $\boldsymbol{s}_1 = 001(01)^\infty$ lacks a finite minimal prefix, hence it is an infinite minimal sequence, while $\boldsymbol{s}_2 = 01(001)^\infty$ has the finite minimal prefix 01. □

Lemma 4.4 *A finite sequence $\boldsymbol{s}$ is minimal if and only if $\sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for all $0 < n < |\boldsymbol{s}|$. An infinite sequence $\boldsymbol{s}$ is minimal if and only if $\sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for all $n > 0$.*

Proof: For the finite case, assume $\boldsymbol{s}$ is a finite minimal sequence. For $0 < n < |\boldsymbol{s}|$ there is a maximal N such that $\boldsymbol{s} = \boldsymbol{s}[1,n]^N \boldsymbol{d}$ for some sequence $\boldsymbol{d}$ with $\boldsymbol{s}[1,n] < \boldsymbol{d}$. This clearly implies $\sigma^n(\boldsymbol{s}) > \boldsymbol{s}$.

Conversely, assume for contradiction that there is an $0 < n < |\boldsymbol{s}|$ such that $\boldsymbol{s}[1,n]^\infty > \boldsymbol{s}$. Then there is a smallest N and a largest K such that $\boldsymbol{s}[1,n]^N > \boldsymbol{s}[1,n]^K \boldsymbol{d} = \boldsymbol{s}$, where the sequence $\boldsymbol{d}$ fulfils $\boldsymbol{d} < \boldsymbol{s}[1,n]$. But then $\sigma^{Kn}(\boldsymbol{s}) < \boldsymbol{s}$, a contradiction.

For the infinite case, let $\boldsymbol{s}$ fulfil $\sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for all $n > 0$. Assume that $\boldsymbol{s}$ has a finite minimal prefix. There is a smallest N such that $\boldsymbol{s}[1, n_{\boldsymbol{s}}]^N > \boldsymbol{s}$, since otherwise we would have $\boldsymbol{s}[1, n_{\boldsymbol{s}}]^\infty = \boldsymbol{s}$. Hence, for some sequence $\boldsymbol{d}$ we have $\boldsymbol{s} = \boldsymbol{s}[1, n_{\boldsymbol{s}}]^{N-1} \boldsymbol{d}$ with $\boldsymbol{d} < \boldsymbol{s}[1, n_{\boldsymbol{s}}]$. By shifting $n_{\boldsymbol{s}}$ times we have $\sigma^{n_{\boldsymbol{s}}}(\boldsymbol{s}) < \boldsymbol{s}$, a contradiction.

Conversely, if we for some n have $\sigma^n(\boldsymbol{s}) \leq \boldsymbol{s}$ then we have $\boldsymbol{s} \leq \boldsymbol{s}[1,n]^\infty$, and it follows that $\boldsymbol{s}$ has a finite minimal prefix. □

Lemma 4.5 *Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be a prefix and a suffix respectively of the finite minimal prefix $\boldsymbol{s}[1, n_{\boldsymbol{s}}]$ such that $|\boldsymbol{u}| = |\boldsymbol{w}| \leq n_{\boldsymbol{s}}/2$. Then $\boldsymbol{u} < \boldsymbol{w}$.*

Proof: We have for some sequence $\boldsymbol{v}$, which may be empty, $\boldsymbol{s}[1,n_{\boldsymbol{s}}] = \boldsymbol{uvw}$. Assume for contradiction that $\boldsymbol{u} \geq \boldsymbol{w}$. If the assumed inequality is strict, that is, $\boldsymbol{u} > \boldsymbol{w}$ then we would have,

$$\boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty = (\boldsymbol{uvw})^\infty < \boldsymbol{uvu} < (\boldsymbol{uv})^\infty,$$

which contradicts the definition of $n_{\boldsymbol{s}}$, as this gives a shorter prefix bounding $\boldsymbol{s}$. Therefore we shall from now-on assume that $\boldsymbol{u} = \boldsymbol{w}$.

First consider the case when $\boldsymbol{u}$ and $\boldsymbol{v}$ are of the same length. If $\boldsymbol{u} \geq \boldsymbol{v}$ then clearly $\boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty \leq \boldsymbol{u}^\infty$ which again gives a contradiction to the minimality of $n_{\boldsymbol{s}}$. Similarly if $\boldsymbol{u} < \boldsymbol{v}$ we would have $\boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty < (\boldsymbol{uv})^\infty$.

For the case when $|\boldsymbol{u}| < |\boldsymbol{v}|$ we make the implicit definition of the sequence $\boldsymbol{v}_1$ by factoring out all the $\boldsymbol{u}$-prefixes, that is, $\boldsymbol{v} = \boldsymbol{u}^{n_1}\boldsymbol{v}_1$. Similarly we define the sequence $\boldsymbol{u}_1$ by $\boldsymbol{u} = \boldsymbol{v}_1^{m_1}\boldsymbol{u}_1$. These prefix arguments can now be recursively repeated, that is, $\boldsymbol{u}_1$ might be a prefix of $\boldsymbol{v}_1$, and thereby we have to continue to define new sequences. By doing so we have defined a process by taking shorter and shorter prefixes. As we are dealing with sequences of finite length, and in each step of the process we factor out a sequence of positive length the process must end after a finite number of steps.

Assume that the process ends after an even number of steps. We then have, by retracing backwards, for some sequences $\Delta_{\boldsymbol{u}}$ and $\Delta_{\boldsymbol{v}}$,

$$\boldsymbol{u} = \boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{u}}\,\boldsymbol{u}_k, \qquad \boldsymbol{v} = \boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{v}}\,\boldsymbol{v}_k.$$

Using this factorisation we see that if $\boldsymbol{u}_k \leq \boldsymbol{v}_k$ we would have

$$\begin{aligned}\boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty &= \big((\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{u}}\,\boldsymbol{u}_k)(\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{v}}\,\boldsymbol{v}_k)(\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{u}}\,\boldsymbol{u}_k)\big)^\infty \\ &\leq (\boldsymbol{v}_k)^\infty,\end{aligned}$$

since $\boldsymbol{u}_k$ is not a prefix of $\boldsymbol{v}_k$ and then $\boldsymbol{u}_k\boldsymbol{z} \leq \boldsymbol{v}_k$ for any sequence $\boldsymbol{z}$. If we would have $\boldsymbol{u}_k > \boldsymbol{v}_k$ then notice that in the process the $\boldsymbol{u}_k$'s are separated by at least $\boldsymbol{v}_k^{m_k}$. This gives

$$\begin{aligned}\boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty &= \big((\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{u}}\,\boldsymbol{u}_k)(\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{v}}\,\boldsymbol{v}_k)(\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k\,\Delta_{\boldsymbol{u}}\,\boldsymbol{u}_k)\big)^\infty \\ &\leq (\boldsymbol{v}_k^{m_k}\boldsymbol{u}_k)^\infty,\end{aligned}$$

which concludes the case of even steps. The case when the process ends after an odd number of steps is treated in the same way. The case when $|\boldsymbol{u}| > |\boldsymbol{v}|$ follows a similar pattern. □

Lemma 4.6 *A minimal prefix is a minimal sequence, i.e.* $\boldsymbol{s}[1,m]^\infty < \boldsymbol{s}[1,n_{\boldsymbol{s}}]$ *for* $m < n_{\boldsymbol{s}}$.

Proof: The statement is clear if $n_{\boldsymbol{s}} = \infty$. Hence we only have to deal with the finite case. Let $k_{\boldsymbol{s}} = n_{\boldsymbol{s}[1,n_{\boldsymbol{s}}]}$, that is, $k_{\boldsymbol{s}}$ is the smallest integer such that $\boldsymbol{s}[1,k_{\boldsymbol{s}}]^\infty \geq \boldsymbol{s}[1,n_{\boldsymbol{s}}]$. Assume for contradiction that $k_{\boldsymbol{s}} < n_{\boldsymbol{s}}$. As $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$ is of finite length there is a smallest positive integer N such that $\boldsymbol{s}[1,k_{\boldsymbol{s}}]^N \geq \boldsymbol{s}[1,n_{\boldsymbol{s}}]$. The case of equality can be out-ruled, since otherwise $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$ would be a repetition of $\boldsymbol{s}[1,k_{\boldsymbol{s}}]$, which contradicts the definition of $n_{\boldsymbol{s}}$. Hence we have

$$\boldsymbol{s}[1,k_{\boldsymbol{s}}]^N > \boldsymbol{s}[1,n_{\boldsymbol{s}}]. \tag{4.1}$$

Again by the definition of $n_{\boldsymbol{s}}$ it follows that N must be chosen so that $Nk_{\boldsymbol{s}}$ is strictly larger than $n_{\boldsymbol{s}}$, since otherwise we would have found a shorter bounding prefix of $\boldsymbol{s}$. On the other hand, as any minimal prefix ends with a non-zero symbol, it follows that this is also a sufficient choice of N. Hence we have

$$k_{\boldsymbol{s}}(N-1) < n_{\boldsymbol{s}} < k_{\boldsymbol{s}}N.$$

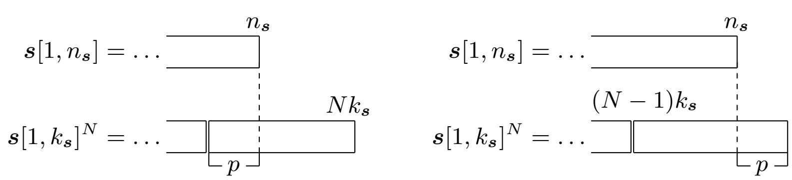
As $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$ is the minimal prefix of $\boldsymbol{s}$ we have $\boldsymbol{s}[1,k_{\boldsymbol{s}}]^\infty < \boldsymbol{s}[1,n_{\boldsymbol{s}}]^\infty$. This implies the inequality, by considering prefixes,

$$\left(\boldsymbol{s}[1,k_{\boldsymbol{s}}]^N\right)[1,n_{\boldsymbol{s}}] \leq \boldsymbol{s}[1,n_{\boldsymbol{s}}]. \tag{4.2}$$

Combining (4.1) and (4.2) shows that $\left(\boldsymbol{s}[1,k_{\boldsymbol{s}}]^N\right)[1,n_{\boldsymbol{s}}]$ must be equal to $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$. Now define the positive integer p by

$$p = \min\left\{Nk_{\boldsymbol{s}} - n_{\boldsymbol{s}},\ n_{\boldsymbol{s}} - (N-1)k_{\boldsymbol{s}}\right\}.$$

This number p is the length of the smallest of the parts, of the last repetition of $\boldsymbol{s}[1,k_{\boldsymbol{s}}]$, that either extends beyond or overlaps $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$.



Denote by $\boldsymbol{u} = \boldsymbol{s}[1,p]$ and $\boldsymbol{w} = \boldsymbol{s}[k_{\boldsymbol{s}} - p + 1, k_{\boldsymbol{s}}]$. We can then for some sequence $\boldsymbol{v}$ write $\boldsymbol{s}[1,k_{\boldsymbol{s}}] = \boldsymbol{uvw}$. By the equality in the first positions it follows that then $\boldsymbol{s}[1,n_{\boldsymbol{s}}]$ must be of the form, depending on the choice in the definition of p,

$$\boldsymbol{s}[1,n_{\boldsymbol{s}}] = \begin{cases} (\boldsymbol{uvw})^{N-1}\boldsymbol{u} & \text{if } n_{\boldsymbol{s}} - k_{\boldsymbol{s}}(N-1) < Nk_{\boldsymbol{s}} - n_{\boldsymbol{s}}, \\ (\boldsymbol{uvw})^{N-1}\boldsymbol{uv} & \text{otherwise.} \end{cases}$$

But this is impossible as Lemma 4.5 would imply $\boldsymbol{u} < \boldsymbol{u}$, or $\boldsymbol{uv} < \boldsymbol{uv}$, a contradiction. □

Corollary 4.7 *Let $\boldsymbol{s}$ be a finite minimal sequence. Then all sequences $\boldsymbol{s} \leq \boldsymbol{x} \leq \boldsymbol{s}^\infty$ have the same minimal prefix, i.e. $\boldsymbol{s} = \boldsymbol{x}[1,n_{\boldsymbol{x}}]$.*

Proof: If we assume $n_{\boldsymbol{x}} < n_{\boldsymbol{s}}$ then we have by Lemma 4.6 that $\boldsymbol{x} \leq \boldsymbol{x}[1,n_{\boldsymbol{x}}]^\infty = \boldsymbol{s}[1,n_{\boldsymbol{x}}]^\infty < \boldsymbol{s}[1,n_{\boldsymbol{s}}] = \boldsymbol{s}$, a contradiction. Similarly, if $n_{\boldsymbol{x}} > n_{\boldsymbol{s}}$ then again by Lemma 4.6 we have that $\boldsymbol{s}^\infty = \boldsymbol{x}[1,n_{\boldsymbol{s}}]^\infty < \boldsymbol{x}[1,n_{\boldsymbol{x}}] \leq \boldsymbol{x}$, again a contradiction. □

We introduce here two sequences $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$, which will be crutial to us in proving our theorems. For a real number $\beta > 1$ we let $u = \lceil \beta \rceil - 1$. Then for a sequence $\boldsymbol{s}$ with finite minimal prefix we define

$$\begin{aligned} \boldsymbol{a}_k = \boldsymbol{a}_k(\boldsymbol{s},\beta) &= \boldsymbol{s}[1,n_{\boldsymbol{s}}-1]\big((\boldsymbol{s})_{n_{\boldsymbol{s}}} - 1\big)\, u^k, \\ \boldsymbol{b}_k = \boldsymbol{b}_k(\boldsymbol{s},\beta) &= \boldsymbol{s}[1,n_{\boldsymbol{s}}]^k u. \end{aligned} \tag{4.3}$$

Let us give an example of $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ sequences.

Example 4.8 The binary sequence $\boldsymbol{s} = 0011$ is a minimal sequence and we have $\boldsymbol{a}_k(\boldsymbol{s},2) = 0010(1)^k$ and $\boldsymbol{b}_k(\boldsymbol{s},2) = (0011)^k 1$. The ternary sequence $\boldsymbol{u} = 0011$ is also a minimal sequence. Hence $\boldsymbol{a}_k(\boldsymbol{u},3) = 0010(2)^k$ and $\boldsymbol{b}_k(\boldsymbol{u},3) = (0011)^k 2$. Similarly, the ternary sequence $\boldsymbol{v} = 1122$ is a minimal sequence, and therefore $\boldsymbol{a}_k(\boldsymbol{v},3) = 1121(2)^k$ and $\boldsymbol{b}_k(\boldsymbol{v},3) = (1122)^k 2$. □

It is clear from its definition that $\boldsymbol{b}_k$ is a minimal sequence for all $k \geq 0$. Moreover we have

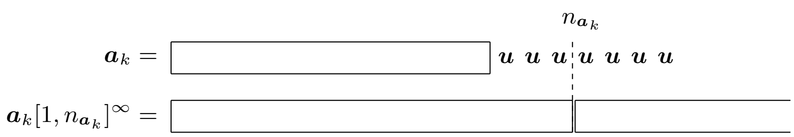
Lemma 4.9 *For $\beta > 1$ the sequence $\boldsymbol{a}_k(\boldsymbol{s},\beta)$ is minimal for all k larger than some N.*

Proof: Assume for contradiction that $n_{\boldsymbol{a}_k}$ is less than $n_{\boldsymbol{s}} - 1 + k = |\boldsymbol{a}_k|$, that is, $\boldsymbol{a}_k$ is not minimal. If $n_{\boldsymbol{a}_k}$ would be equal to $n_{\boldsymbol{s}}$ it would contradict the minimality of $\boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]$.

If we assume that $n_{\boldsymbol{a}_k} > n_{\boldsymbol{s}}$ then, for the symbol $u = \lceil \beta \rceil - 1$, we would have that

$$\boldsymbol{a}_k = \boldsymbol{s}[1, n_{\boldsymbol{s}} - 1]\big((\boldsymbol{s})_{n_{\boldsymbol{s}}} - 1\big)\, u^k > \boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]^\infty$$

as the sequences $\boldsymbol{a}_k$ and $\boldsymbol{s}$ are equal in the first $n_{\boldsymbol{a}_k}$ positions, a contradiction to the assumed minimality of $\boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]$, since a minimal sequence cannot start and end with the same symbol.



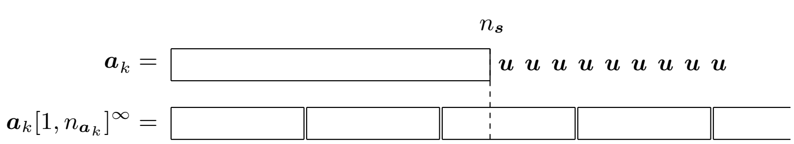
Finally, if $n_{\boldsymbol{a}_k} < n_{\boldsymbol{s}}$ then we have

$$\boldsymbol{a}_k = \boldsymbol{s}[1, n_{\boldsymbol{s}} - 1]\big((\boldsymbol{s})_{n_{\boldsymbol{s}}} - 1\big)\, \boldsymbol{u}^k \leq \boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]^\infty = \boldsymbol{s}[1, n_{\boldsymbol{a}_k}]^\infty. \qquad (4.4)$$

But by Lemma 4.6 we have the inequality

$$\boldsymbol{s} > \boldsymbol{s}[1, n_{\boldsymbol{a}_k}]^\infty \qquad (4.5)$$

Combining (4.4) and (4.5) we see that when decreasing the symbol at position $n_{\boldsymbol{s}}$ the inequality is reversed.



This implies that $\boldsymbol{a}_k$ is equal to $\boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]^\infty$ in the first $n_{\boldsymbol{s}}$ positions, that is,

$$\boldsymbol{a}_k[1, n_{\boldsymbol{s}}] = \big(\boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]^\infty\big)[1, n_{\boldsymbol{s}}].$$

If now N is chosen large enough then this implies that $\boldsymbol{a}_k[1, n_{\boldsymbol{a}_k}]$ must be larger than a block of u's of length $n_{\boldsymbol{a}_k}$, which is not possible. $\square$

Remark: Lemma 4.9 shows that a minimal sequence is a limit point of minimal sequences.

We end the section by a result on the set of infinite minimal sequences. Let $IM(q)$ be the set of all infinite minimal sequences on a q letter alphabet. By Lemma 4.4 we can characterise these sequences via the left-shift, that is,

$$IM(q) = \{\boldsymbol{x} \in S^{\infty}(q) : \sigma^n(\boldsymbol{x}) > \boldsymbol{x} \text{ for all } n \geq 1\} \tag{4.6}$$

Lemma 4.10 *The set $IM(q)$ has Lebesgue measure zero.*

Proof: Let λ be the Lebesgue measure. As λ is invariant under $x \mapsto qx$ on the unit circle λ-almost every x has a dense orbit. Hence as $IM(q)$ is a set of x's with bounded orbit it must have Lebesgue measure 0. □

5 Main Results via Perron-Frobenius

Throughout this section let $q \geq 2$ be a fixed integer. From the expansion (2.2) of a real number $x \in [0, 1]$ into an integer base q we see that we can redefine the set $F(c, \beta)$, defined in (3.2), into a set of sequences of q symbols. We have

$$F(\boldsymbol{c}, q) = \{\boldsymbol{x} \in S^{\infty}(q) : \sigma^n(\boldsymbol{x}) \geq \boldsymbol{c} \text{ for all } n \geq 0\},$$

where clearly $\boldsymbol{c}$ is the sequence from the q-nary expansion of the real number $c \in [0, 1]$. We define the intervals $I(\boldsymbol{c}, q)$ via the concept of minimal prefixes, that is, we let

$$I(\boldsymbol{c}, q) = \left\{\boldsymbol{x} \in S^{\infty}(q) : \boldsymbol{c}[1, n_{\boldsymbol{c}}] \leq \boldsymbol{x} \leq \boldsymbol{c}[1, n_{\boldsymbol{c}}]^{\infty}\right\}$$

The next theorem gives that the definition of $I(\boldsymbol{c}, q)$ is independent of the choice of the representative $\boldsymbol{c}$.

Theorem 5.1 *For any $\boldsymbol{d} \in I(\boldsymbol{c}, q)$ we have $I(\boldsymbol{d}, q) = I(\boldsymbol{c}, q)$.*

Proof: We may assume that $\boldsymbol{c}$ is a finite minimal sequence. Let $\boldsymbol{d} \in I(\boldsymbol{c}, q)$. Then Corollary 4.7 gives that $\boldsymbol{d}[1, n_{\boldsymbol{d}}] = \boldsymbol{c}$, and therefore we get $I(\boldsymbol{d}, q) = I(\boldsymbol{c}, q)$. □

We can now state a first result of the map $\phi_q(\boldsymbol{c}) = \dim_H F(\boldsymbol{c}, q)$.

Theorem 5.2 *The derivative of ϕ_q is zero Lebesgue a.e.*

Proof: The sequences which give rise to one-point intervals $I(\boldsymbol{c}, q) = \{\boldsymbol{c}\}$ are precisely the sequences $\boldsymbol{c} \in IM(q)$. As $IM(q)$ has Lebesgue measure 0, by Lemma 4.10 we must have that the complementary set, the set formed by the intervals, has full Lebesgue measure. □

Lemma 5.3 *For any $\boldsymbol{c} \in S(q)$ the dynamical system $\sigma : F(\boldsymbol{c}, q) \to F(\boldsymbol{c}, q)$ is topologically mixing.*

Proof: Let $U \subset F(\boldsymbol{c}, q)$ be the cylinder-set defined by $[u_1 u_2 \dots u_k]$ and let similarly $V \subset F(\boldsymbol{c}, q)$ be the cylinder-set defined by $[v_1 v_2 \dots v_j]$. By choosing the number N sufficiently large the cylinder-set $[u_1 u_2 \dots u_k\, (q-1)^N v_1 v_2 \dots v_j]$ is a subset of U. This shows that the intersection $\sigma^n(U) \cap V$ is non-empty for all $n \geq N$. □

For the next corollary recall from Example 2.4 how to associate a subshift with a transition matrix.

Corollary 5.4 *Let $\boldsymbol{c} \in S(q)$ be such that $F(\boldsymbol{c}, q)$ is a subshift of finite type. Then the transition matrix $A_{\boldsymbol{c}}$ corresponding to $F(\boldsymbol{c}, q)$ is primitive.*

Theorem 5.5 *The interval $I(\boldsymbol{c}, q)$ is the largest interval I on which $\phi_q(\boldsymbol{d}) = \phi_q(\boldsymbol{c})$ for $\boldsymbol{d} \in I$.*

Proof: We may assume that $\boldsymbol{c}$ is a finite minimal sequence. From Lemma 4.1 we have that $F(\boldsymbol{c}, q) = F(\boldsymbol{c}^\infty, q)$, and hence $\dim_H F(\boldsymbol{c}, q) = \dim_H F(\boldsymbol{c}^\infty, q)$.

From Lemma 4.9 the sequences $\{\boldsymbol{a}_k(\boldsymbol{c}, q)\}$ can be assumed minimal. Let $A_{\boldsymbol{a}_k}$ be a transition matrix corresponding to $F(\boldsymbol{a}_k, q)$ and let $A_{\boldsymbol{c}}$ be a transition matrix corresponding to $F(\boldsymbol{c}, q)$. We can scale them to be of the same size if needed. Since $F(\boldsymbol{a}_k, q) \setminus F(\boldsymbol{c}, q)$ is non-void, we have $(A_{\boldsymbol{a}_k})_{ij} \geq (A_{\boldsymbol{c}})_{ij}$, entry by entry, where there is one pair of indices r, s such that the inequality is strict. As $A_{\boldsymbol{a}_k}$ is irreducible it follows from the Perron-Frobenius Theorem 2.2 that $\phi_q(\boldsymbol{a}_k) > \phi_q(\boldsymbol{c})$.

That we cannot go beyond the right endpoint of $I(\boldsymbol{c}, q)$ follows by the same arguments when considering the sequences $\{\boldsymbol{b}_k(\boldsymbol{c}, q)\}$. □

Example 5.6 Let $c = 0.25$ and $q = 2$. Then c corresponds to the sequence $\boldsymbol{c} = 01$, which is a minimal sequence, and we have that a transition matrix connected to $F(\boldsymbol{c}, q)$ is

$$A_{\boldsymbol{c}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

The spectral radius of the transition matrix $A_{\boldsymbol{c}}$ gives

$$\dim_H F(\boldsymbol{c}, q) = \dim_H F(01, 2) = \frac{\log \frac{1+\sqrt{5}}{2}}{\log 2} \approx 0.69424.$$

Lemma 4.1 gives that $F(01, q) = F\big((01)^\infty, q\big)$, i.e., $I(\boldsymbol{c}, q) = [01, (01)^\infty]$, which correspond to the real interval $[\frac{1}{4}, \frac{1}{3}]$. □

The drawback by finding the dimension of $F(\boldsymbol{c}, q)$ via a transition matrix is that the size of the transition matrix is $q^{|\boldsymbol{c}|-1} \times q^{|\boldsymbol{c}|-1}$, which clearly grows out of hand for long sequences in a large alphabet.

Lemma 5.7 *Let* $\boldsymbol{c} \in S(q)$. *Then* $\dim_H F(\boldsymbol{c}, q) = 0$ *if and only if* $\boldsymbol{c} \geq (q-2)(q-1)^\infty$

Proof: Let $\boldsymbol{c}_n = (q-2)(q-1)^n$ for $n \geq 0$ and put $\boldsymbol{c}_\infty = (q-2)(q-1)^\infty$. Then the set

$$Q_n = \big\{(q-2)(q-1)^n, (q-1)^{n+1}\big\}^{\mathbb{N}}$$

is a subset of $F(\boldsymbol{c}_n, q)$. Let θ be the map $\big((q-2)(q-1)^n, (q-1)^{n+1}\big) \mapsto (0, 1)$. This gives, by Proposition 2.6,

$$\begin{aligned} \dim_H F(\boldsymbol{c}_n, q) &\geq \dim_F Q \\ &= \frac{1}{n+1} \frac{\log 2}{\log q} \dim_H S^\infty(2) \\ &= \frac{1}{n+1} \frac{\log 2}{\log q} > 0. \end{aligned}$$

Hence, $\dim_H F(\boldsymbol{c}, q) > 0$ for $\boldsymbol{c} < \boldsymbol{c}_\infty$. Conversely, we have

$$F(\boldsymbol{c}_\infty, q) = \{(q-1)^\infty\} \cup \bigcup_{n=0}^{\infty} \{(q-1)^n \boldsymbol{c}_\infty\}.$$

It is clear that $F(\boldsymbol{c}_\infty, q)$ is a countable set and therefore $\dim_H F(\boldsymbol{c}, q) = 0$ for $\boldsymbol{c} \geq \boldsymbol{c}_\infty$, which concludes the proof. □

Corollary 5.8 *For fixed $q \geq 2$ we have $\dim_H F(c,q) = 0$ if and only if $c \geq 1 - \frac{1}{q}$.*

Next let us turn to the question of the continuity of the map ϕ_q. Recall that by $|\cdot|$ we mean the cardinality of a set. We need the following estimating lemma.

Lemma 5.9 *Let $\boldsymbol{c} \in S(q)$ be a finite minimal sequence and such that $\boldsymbol{c} \neq q-1$. Put $\boldsymbol{a}_k = \boldsymbol{a}_k(\boldsymbol{c}, q)$ and $\boldsymbol{b}_k = \boldsymbol{b}_k(\boldsymbol{c}, q)$.*

1. *There is a constant C such that $|F(\boldsymbol{a}_k, q)[1,k]| \leq C|F(\boldsymbol{c}, q)[1,k]|$ for all $k \geq 1$.*
2. *Given k large enough, there is a constant C such that for all $n \geq 1$ we have*

$$\big|F(\boldsymbol{b}_k, q)[1,n]\big| \geq C\big|F(\boldsymbol{c}, q)[1,n]\big| \left(1 - \frac{1}{k}\right)^n.$$

Proof: (1). A sequence $\boldsymbol{d} \in F(\boldsymbol{a}_k, q)[1,k] \backslash F(\boldsymbol{c}, q)[1,k]$, is a sequence that must start with a prefix of a sequence in $F(\boldsymbol{c}, q)[1,k]$, but at some point it must have a subsequence which is smaller than $\boldsymbol{c}$, that is, there exists an n such that $\sigma^n(\boldsymbol{d}) < \boldsymbol{c}$. For this n we must also have $\sigma^n(\boldsymbol{d}) \geq \boldsymbol{a}_k$. Hence we have

$$\begin{aligned}
\big|F(\boldsymbol{a}_k, q)[1,k]\big| &\leq \sum_{i=1}^{k} \big|F(\boldsymbol{c}, q)[1,i]\big| \\
&\leq k_1 \sum_{i=1}^{k} \lambda_{\boldsymbol{c}}^i \\
&\leq k_2 \lambda_{\boldsymbol{c}}^k \\
&\leq C\big|F(\boldsymbol{c}, q)[1,k]\big|
\end{aligned}$$

for some constants k_1 and k_2 and where $\log(\lambda_{\boldsymbol{c}})$ is the topological entropy of $F(\boldsymbol{c}, q)$.

(2). Similarly, a sequences $\boldsymbol{d} \in F(\boldsymbol{c}, q)[1,n] \setminus F(\boldsymbol{b}_k, q)[1,n]$ must contain, at least once, the pattern $\boldsymbol{u} = \boldsymbol{c}^k v$, where $c_1 \leq v < q-1$. To see

this, the sequence $\boldsymbol{b}_k$ is strictly larger than $\boldsymbol{u}$ and hence no sequence in $F(\boldsymbol{b}_k, q)[1, n]$ can contain the subsequence $\boldsymbol{u}$. Conversely, as $\boldsymbol{u}$ is larger than $\boldsymbol{c}^\infty[1, |\boldsymbol{u}|]$ we have that $\boldsymbol{u}$ is an allowed pattern in sequences in $F(\boldsymbol{c}, q)[1, n]$ The number of sequences of length n that contains this pattern precisely p times, is bounded by

$$q\binom{n - p|\boldsymbol{u}|}{p}\left|F(\boldsymbol{b}_k, q)\,[1, n - p|\boldsymbol{u}|]\,\right|,$$

which we obtain by looking at the number of places the pattern $\boldsymbol{u}$ can be placed in. We have by summing up for a k large enough

$$\begin{aligned}
C\big|F(\boldsymbol{c},q)[1, n]\big| \le & \\
& \le \lambda_{\boldsymbol{b}_k}^n + \binom{n-|\boldsymbol{u}|}{1}\lambda_{\boldsymbol{b}_k}^{n-|\boldsymbol{u}|} + \binom{n-2|\boldsymbol{u}|}{2}\lambda_{\boldsymbol{b}_k}^{n-2|\boldsymbol{u}|} + \dots \\
& \le \lambda_{\boldsymbol{b}_k}^n \left(1 + \binom{n}{1}\frac{1}{\lambda_{\boldsymbol{b}_k}^{|\boldsymbol{u}|}} + \binom{n}{2}\frac{1}{\lambda_{\boldsymbol{b}_k}^{2|\boldsymbol{u}|}} + \dots\right) \\
& \le \lambda_{\boldsymbol{b}_k}^n \left(1 + \frac{1}{\lambda_{\boldsymbol{b}_k}^{|\boldsymbol{u}|}}\right)^n \\
& \le \lambda_{\boldsymbol{b}_k}^n \left(1 + \frac{1}{k}\right)^n,
\end{aligned}$$

as by Lemma 5.7 we have $\lambda_{\boldsymbol{b}_k} > 1$, which gives the desired result and concludes the lemma. □

We can now prove the main result on continuity.

Theorem 5.10 *The map ϕ_q is continuous.*

Proof: By Theorem 2.8 we just have to show that the entropy of $F(\boldsymbol{c}, q)$ depends continuously on $\boldsymbol{c}$. It is clear that for any sequence $\boldsymbol{c}$ the estimate $|F(\boldsymbol{c}, q)[1, rn]| \le |F(\boldsymbol{c}, q)[1, n]|^r$ holds. Let $\boldsymbol{a}_k = \boldsymbol{a}_k(\boldsymbol{c}, q)$ and

$\boldsymbol{b}_k = \boldsymbol{b}_k(\boldsymbol{c}, q)$. Hence by Lemma 5.9 it follows that

$$\begin{aligned} h_{\text{top}}\big(F(\boldsymbol{c},q)\big) &\leq \lim_{k\to\infty} h_{\text{top}}\big(F(\boldsymbol{a}_k,q)\big) \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\big|F(\boldsymbol{a}_k,q)[1,n]\big| \\ &= \lim_{k\to\infty}\lim_{r\to\infty}\frac{1}{rk}\log\big|F(\boldsymbol{a}_k,q)[1,rk]\big| \\ &\leq \lim_{k\to\infty}\lim_{p\to\infty}\frac{1}{rk}\log\big|F(\boldsymbol{a}_k,q)[1,k]\big|^r \\ &\leq \lim_{k\to\infty}\frac{1}{k}\log C\big|F(\boldsymbol{c},q)[1,k]\big| \\ &= h_{\text{top}}\big(F(\boldsymbol{c},q)\big), \end{aligned}$$

which shows the left-continuity of the entropy in the left endpoint of the interval $I(\boldsymbol{c}, q)$. The right-continuity follows trivially as the entropy is constant in a neighbourhood to the right of this point. In the same way the left-continuity in the right endpoint of $I(\boldsymbol{c}, q)$ is also clear. Again by Lemma 5.9 we have

$$\begin{aligned} h_{\text{top}}\big(F(\boldsymbol{c},q)\big) &\geq \lim_{k\to\infty} h_{\text{top}}\big(F(\boldsymbol{b}_k,q)\big) \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\big|F(\boldsymbol{b}_k,q)[1,n]\big| \\ &\geq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\left(C\big|F(\boldsymbol{c},q)[1,n]\big|\left(1-\frac{1}{k}\right)^n\right) \\ &= h_{\text{top}}\big(F(\boldsymbol{c},q)\big) + \lim_{k\to\infty}\log\left(1-\frac{1}{k}\right) \\ &= h_{\text{top}}\big(F(\boldsymbol{c},q)\big), \end{aligned}$$

and the right-continuity in the right endpoints follows and concludes the theorem. □

We end this section by some remarks on the recursive embedding of $F(\boldsymbol{c}, q)$ into $F(\boldsymbol{d}, q + 1)$.

Definition 5.11 *Let be R the re-alphabetisation function such that for $\boldsymbol{x} \in S(q)$ we have $R(\boldsymbol{x}) \in S(q+1)$ and $R(\boldsymbol{x})_i = x_i + 1$.*

Theorem 5.12 *Let $\boldsymbol{c} \in S(q)$. Then*

$$R(F(\boldsymbol{c}, q)) = F(R(\boldsymbol{c}), q+1).$$

Proof: It is clear that $R(F(\boldsymbol{c}, q)) \subset F(R(\boldsymbol{c}), q+1)$. For the reversed inclusion, let $\boldsymbol{x} \in F(R(\boldsymbol{c}), q+1)$. If $\boldsymbol{x}$ would contain a 0 at position n then $\sigma^n(\boldsymbol{x}) < R(\boldsymbol{c})$ which would contradict $\boldsymbol{x} \in F(R(\boldsymbol{c}), q+1)$. Hence $R^{-1}(\boldsymbol{x})$ is well defined and $R^{-1}(\boldsymbol{x}) \in S^\infty(q)$. If there exists an m such that $\sigma^m(R^{-1}(\boldsymbol{x})) < \boldsymbol{c}$ then we would have $R(\sigma^m(R^{-1}(\boldsymbol{x}))) = \sigma^m(\boldsymbol{x}) < R(\boldsymbol{c})$, a contradiction to our assumption. Hence $R^{-1}(\boldsymbol{x}) \in F(\boldsymbol{c}, q)$ and therefore $R(F(\boldsymbol{c}, q)) \supset F(R(\boldsymbol{c}), q+1)$. □

Corollary 5.13 *Let $\boldsymbol{c} \in S(q)$. Then*

$$\dim_H F(\boldsymbol{c}, q) = \frac{\log(q+1)}{\log q} \dim_H F(R(\boldsymbol{c}), q+1).$$

Proof: Apply the metric defined in (2.1) and Proposition 2.6 to the result of Theorem 5.11. □

6 The β-shift

The field of β-shift originated in the late fifties by Rényi [19] who introduced the representation of a real number with an arbitrary base $\beta > 1$. One of the most studied problems in this field is the link between expansions to base β and ergodic properties of the corresponding β-shift.

More precisely, the definition of the β-expansion, where $[\cdot]$ means the integral part, is the following;

Definition 6.1 *The expansion of a number $x \in [0,1]$ in base β, or β-expansion, is a sequence $\boldsymbol{x}$ of integers out of $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$ such that*

$$x_n = [\beta T_\beta^{n-1}(x)], \qquad n \geq 1,$$

where $T_\beta : [0,1] \to [0,1)$ is the transformation $T_\beta(x) = \beta x \pmod 1$.

Definition 6.2 *We denote by* $d(x,\beta)$ *the* β*-expansion of* x *in base* β

Definition 6.3 *The closure of the set of all* β*-expansions of* $x \in [0,1]$ *is called the* β*-shift,* S_β*.*

The expansion of 1 in base β turns out to be crucial for characterising the β-shift. Parry [15] proved that S_β is totally determined by the expansion of 1.

Theorem 6.4 (Parry) *If* $d(1,\beta)$ *is not finite (i.e. it will not terminate with zeros only), then* $\boldsymbol{s} \in S^\infty(\lceil\beta\rceil - 1)$ *belongs to* S_β *if and only if*

$$\sigma^n(\boldsymbol{s}) < d(1,\beta) \qquad \text{for all } n \geq 1.$$

If $d(1,\beta) = \boldsymbol{i} = i_1 i_2 \dots i_M 0^\infty$ *then* $\boldsymbol{s}$ *belongs to* S_β *if and only if*

$$\sigma^n(\boldsymbol{s}) < \big(i_1 i_2 \dots i_{M-1}(i_M - 1)\big)^\infty \qquad \text{for all } n \geq 1. \tag{6.1}$$

Moreover Parry proved the following theorem.

Theorem 6.5 (Parry) *A sequence* $\boldsymbol{s}$ *is an expansion of* 1 *for some* β *if and only if*

$$\sigma^n(\boldsymbol{s}) < \boldsymbol{s} \qquad \text{for all } n \geq 1$$

and then β *is unique. Moreover the map* $\Xi : \beta \mapsto d(1,\beta)$ *is monotone increasing.*

7 Main Results via the β-shift

Recall the definition of the set $F(c,\beta)$ from (3.2) for any $\beta > 1$,

$$F(c,\beta) = \{x \in \mathbb{S} : \beta^n x \geq c \ (\text{mod } 1) \text{ for all } n \geq 0\}.$$

We can clearly choose to study the symmetrically identical set

$$F'(c,\beta) = \{x \in \mathbb{S} : \beta^n x \leq 1 - c \ (\text{mod } 1) \text{ for all } n \geq 0\}$$

By using the β-expansion of real numbers we may turn into dealing with a set of sequences. That is, we let

$$F_\beta(c) = \{\boldsymbol{x} \in S_\beta : \sigma^n(\boldsymbol{x}) < d(1-c,\beta) \text{ for all } n \geq 0\}. \tag{7.1}$$

Note that we may change the inequality in (7.1) to be strict, as this only removes the periodic sequences, which are countable and hence does not affect the dimension. We emphasise also that $\dim_H F(c,\beta) = \dim_H F_\beta(c)$.

From Lemma 4.1 and Lemma 4.4 we have the following two corollaries by, simply consider inverse sequences.

Corollary 7.1 *Let $\boldsymbol{s} \in S^*(\lceil\beta\rceil - 1)$ be a finite sequence of length n not ending with a 0. Then $\sigma^n(\boldsymbol{x}^*) \leq (\tilde{\boldsymbol{s}})^*$ for all $n \geq 0$ if and only if $\sigma^n(\boldsymbol{x}^*) \leq ((\tilde{\boldsymbol{s}})^*)^\infty$ for all $n \geq 0$.*

Corollary 7.2 *A finite sequence $\boldsymbol{s} \in S^*(\lceil\beta\rceil - 1)$ is minimal if and only if $\sigma^n((\tilde{\boldsymbol{s}})^*) < (\tilde{\boldsymbol{s}})^*$ for all $0 < n < |\boldsymbol{s}|$. An infinite sequence $\boldsymbol{s} \in S^\infty(\lceil\beta\rceil - 1)$ is minimal if and only if $\sigma^n(\boldsymbol{s}^*) < \boldsymbol{s}^*$ for all $n > 0$.*

For $c \in [0,1]$ let $\boldsymbol{s} = d(1-c,\beta)$. If $\boldsymbol{s}$ is finite let $\boldsymbol{u}$ be the unique minimal prefix of $(\tilde{\boldsymbol{s}})^*$ and if $\boldsymbol{s}$ is infinite let $\boldsymbol{u}$ be the minimal prefix of $\boldsymbol{s}^*$. If $\boldsymbol{u}$ is finite define

$$m(c,\beta) = 1 - \sum_{i=1}^{|\boldsymbol{u}|} \frac{((\tilde{\boldsymbol{u}})^*)_i}{\beta^i} \qquad \text{and} \qquad M(c,\beta) = 1 - \sum_{i=1}^{\infty} \frac{((\boldsymbol{u}^\infty)^*)_i}{\beta^i}.$$

In the case when $\boldsymbol{u}$ is an infinite sequence we have that $m(c,\beta)$ and $M(c,\beta)$ coincides. That is,

$$m(c,\beta) = M(c,\beta) = 1 - \sum_{i=1}^{\infty} \frac{(\boldsymbol{u}^*)_i}{\beta^i}.$$

Theorem 7.3 *For any $c \in [0,1]$ the real numbers $m(c,\beta)$ and $M(c,\beta)$ are unique and such that $F_\beta(c) = F_\beta(m(c,\beta))$ and $d(1-m(c,\beta),\beta) = d(1,B(c,\beta))$ with a unique $B(c,\beta) \leq \beta$.*

Proof: The theorem follows from Parry's Theorem 6.5, Corollary 7.1 and Corollary 7.2. □

The statement of Theorem 7.3 says that if given the reals $c \in (0,1)$ and $\beta > 1$ there is a $c_0 = m(c,\beta)$ such that $F_\beta(c) = F_\beta(c_0)$. Moreover this special c_0 is such that the β-expansion of $1 - c_0$, the sequence $\boldsymbol{s}$, is such that $\sigma^n(\boldsymbol{s}) < \boldsymbol{s}$, that is, it is the expansion of 1 for some $\beta_0 = B(c, \beta)$. This means that $F_{\beta_0}(c_0)$ is the β-shift S_{β_0}.

Definition 7.4 *We shall from now on use the notation $B(c, \beta)$ for the unique β-value connected to c as defined in Theorem 7.3.*

Theorem 7.5 *For $c \in [0, 1]$ we have*

$$\dim_H F_\beta(c) = \frac{\log B(c, \beta)}{\log \beta}.$$

Proof: Given $c \in [0, 1]$ there exists, by Theorem 7.3, an $m(c, \beta)$ such that $F_\beta(c) = F_\beta(m(c, \beta))$ and $d(1 - m(c, \beta), \beta) = d(1, B(c, \beta))$. Hence, by (7.1) and Parry's Theorem 6.4, $F_\beta(m(c, \beta)) = S_{B(c,\beta)}$ is a β-shift contained in S_β. This gives

$$\dim_H F_\beta(c) = \dim_H F_\beta(m(c, \beta)) = \frac{\log B(c, \beta)}{\log \beta} \dim_H S_\beta,$$

which completes the proof. □

We define the intervals $I_\beta(c)$ via the functions m and M. That is, we let

$$I_\beta(c) = [m(c, \beta), M(c, \beta)]$$

From Corollary 4.7 we have that the definition of $I_\beta(c)$ is independent of the choice of the representative $\boldsymbol{c}$. To summarise, (compare to Theorem 5.1),

Theorem 7.6 *For any $d \in I_\beta(c)$ we have $I_\beta(d) = I_\beta(c)$.*

Theorem 7.7 *The interval $I_\beta(c)$ is the largest interval I such that $F_\beta(d) = F_\beta(c)$ for all $d \in I$.*

Proof: From Corollary 7.1 it follows that $F_\beta(d) = F_\beta(c)$ for all $d \in I_\beta(c)$. To see that we can not extend the interval further follows from that minimal sequences are limit points of minimal sequences and from Theorem 7.5. □

Example 7.8 Let $\beta = 2.0$ and let $c = 0.25$. Then $d(1-0.25, 2) = 110^\infty = d(1, \frac{1}{2}(1+\sqrt{5}))$, since $\frac{1}{2}(1+\sqrt{5})$ is the largest root of the equation $1 = \frac{1}{x} + \frac{1}{x^2}$. Hence

$$\dim_H F_\beta(c) = \dim_H F_2(0.25) = \frac{\log \frac{1+\sqrt{5}}{2}}{\log 2} \approx 0.69424.$$

Moreover we have $m(0.25, 2) = \frac{1}{4}$ and $M(0.25, 2) = \frac{1}{3}$ from the sequence $(10)^\infty$. (Compare this to Example 5.6). □

Example 7.9 Let $\beta = 1.7$ and let $c = 0.25$. Then $d(1-0.25, 1.7) = 100101000\ldots$ with the inverted minimal prefix 101. This gives

$$m(0.25, 1.7) = 1 - \frac{1}{1.7} - \frac{1}{1.7^3} \approx 0.20822$$

and and the sequence $(100)^\infty$ gives $M(0.25, 1.7) \approx 0.26144$. To find the value $B(0.25, 1, 7) \approx 1.46557$ we have to solve the equation $1 = \frac{1}{x} - \frac{1}{x^3}$. Hence

$$\dim_H F_\beta(c) = \dim_H F_{1.7}(0.25) \approx \frac{\log 1.46557}{\log 1.7} \approx 0.72036.$$

□

Example 7.10 Again let $c = 0.25$. To find the β satisfying $d(1 - 0.25, \beta) = 200(20)^\infty$ we have to solve the equation $0.75 = \frac{2}{\beta} + \frac{2}{\beta^4} + \frac{2}{\beta^6} + \frac{2}{\beta^8} + \ldots$, which gives $\beta \approx 2.77690$. We clearly have $m(c, \beta) = M(c, \beta) = 0.25$. The equation

$$1 = \frac{2}{x} + \frac{2}{x^4} + \frac{2}{x^6} + \frac{2}{x^8} + \ldots,$$

gives $B(c, \beta) \approx 2.16286$. Hence

$$\dim_H F_\beta(c) \approx \dim_H F_{2.77690}(0.25) \approx \frac{\log 2.16286}{\log 2.77690} \approx 0.75532.$$

□

Remark: From Example 7.8, Example 7.9 and Example 7.10 we see that the function $\beta \mapsto \dim F_\beta(c)$ is not monotone for fixed c, (see also Figure 4).

Theorem 7.11 *For fixed $\beta > 1$ the function $c \mapsto \dim_H F_\beta(c)$ is continuous.*

Proof: Let c be such that $d(1-c,\beta) = d(1, B(c,\beta))$. Then by the existence of the sequences $\{\boldsymbol{a}_k(c,\beta)\}$ and $\{\boldsymbol{b}_k(c,\beta)\}$ from (4.3) there exists a sequence or real numbers $\{c_n\}$ converging to c and such that $d(1-c_n,\beta) = d(1, B(c_n,\beta))$. The continuity now follows from Theorem 7.5 and the continuity of the β-expansion. □

Proposition 7.12 *For fixed $\beta > 1$ we have $\dim_H F_\beta(c) = 0$ if and only if $c \geq 1 - \frac{1}{\beta}$.*

Proof: There is an N such that $(10^n)^\infty \in S_\beta$ for $n > N$. Let r_k be the largest root of the equation $1 = \frac{1}{x} + \frac{1}{x^{n+k}}$. Then

$$\dim_H F_\beta\left(1 - \frac{1}{\beta} - \frac{1}{\beta^{n+k}}\right) = \frac{\log r_k}{\log \beta} \searrow 0$$

as r_k decreases monotonically to 1 when k tends to infinity. □

Now let us turn to the points which correspond to those sequences with infinite minimal prefix. It is clear that these points fulfils $m(x,\beta) = M(x,\beta)$, that is, it is those points falling between the intervals $I_\beta(c)$. By Lemma 4.4 we can define the set of infinite expansions of one $IEO(c,\beta)$ in the interval $[B(c,\beta),\beta]$ by

$$IEO(c,\beta) = \{\boldsymbol{x} = d(1,\beta_0) \in S_\beta : B(c,\beta) \leq \beta_0 \leq \beta \text{ and } \boldsymbol{x} \text{ is infinite}\}.$$

Theorem 7.13 *The function $c \mapsto \dim_H F_\beta(c)$ has derivative zero Lebesgue a.e.*

Proof: Let λ_β be the Lebesgue measure on S_β. If $\boldsymbol{x} \in IEO(c,\beta)$ then $\boldsymbol{x} = d(1,\beta_0)$ for some $\beta_0 \leq \beta$. Hence

$$IEO(c,\beta) \subset \bigcup_{\beta_0 < \beta} S_{\beta_0} \cup \{d(1,\beta)\}.$$

But as $\lambda(S_{\beta_0}) = 0$ for $\beta_0 < \beta$ and since we may consider the union above as countable we have $\lambda(IEO(c,\beta)) = 0$ □

Finally we show that when considering the dimension of the set $F_\beta(c)$ we can restrict ourselves to consider the set $IEO(c,\beta)$ of expansions of 1. Again we see the importance of the expansion of 1 when studying the β-shift.

Theorem 7.14 *For* $c \in [0,1]$ *we have* $\dim_H IEO(c,\beta) = \dim_H F_\beta(c)$.

Proof: It is clear that $IEO(c,\beta) \subset F_\beta(c)$ and hence $\dim_H IEO(c,\beta) \leq \dim_H F_\beta(c)$. For the reversed inequality, put $\boldsymbol{s} = d(1, B(c,\beta))$ or let $\boldsymbol{u}$ be the minimal prefix of $\boldsymbol{s}^*$ if $\boldsymbol{s}$ is infinite otherwise let $\boldsymbol{u}$ be the minimal prefix of $(\tilde{\boldsymbol{s}})^*$. We may assume that $\boldsymbol{u}$ is finite. From (4.3) there is a sequence $\{\boldsymbol{b}_k(\boldsymbol{u},\beta)\}$ of minimal sequences decreasing to $\boldsymbol{u}^\infty$. Then also $\boldsymbol{v}_k = \boldsymbol{b}_k\big(\boldsymbol{b}_k(\boldsymbol{u},\beta),\beta\big)$ decreases to $\boldsymbol{u}^\infty$. Put

$$c_k = 1 - \sum_{i=1}^{|\boldsymbol{v}_k|} \frac{((\tilde{\boldsymbol{v}}_k)^*)_i}{\beta^i}$$

and define the set

$$N_k = \left\{ (\tilde{\boldsymbol{b}}_k(\boldsymbol{u},\beta))^* \, \boldsymbol{w} : \boldsymbol{w} \in [(\tilde{\boldsymbol{v}}_k)^*] \cap F_\beta(c_k) \right\},$$

where $[\cdot]$ denotes the cylinder-set. Then we have $N_k \subset IEO(c,\beta)$ and $\dim_H N_k = \dim_H F_\beta(c_k)$. By choosing k sufficiently large we have $\dim_H N_k$ arbitrarily close to $\dim_H F_\beta(c)$. $\square$

By combining Theorem 7.13 and Theorem 7.14 we have that the set $IEO(0,\beta)$ is a set with Lebesgue measure zero and Hausdorff dimension one.

8 Numerics

By characterising the dimension of $F(c,\beta)$, when β is an integer, via the spectral radius of a primitive transition matrix the problem of numerically calculate an approximative value of ϕ_q reduces to calculate the eigenvalues of the transition matrix.

The graph of ϕ_2, (see figure 1) was calculate in Maple 9.5 by considering minimal sequence of length at-most 8, which gives transition

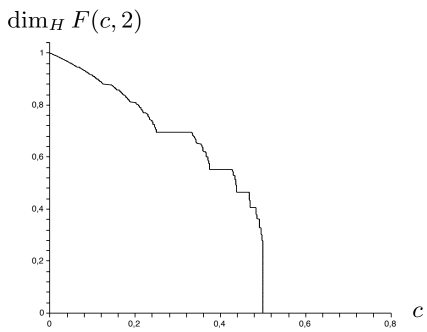


Figure 1: The graph of $c \mapsto \dim_H F(c, 2)$.

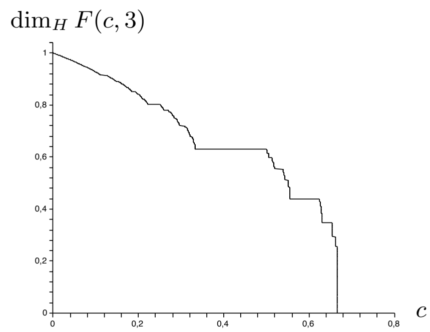


Figure 2: The graph of $c \mapsto \dim_H F(c, 3)$.

matrices of size 128×128. A finer subdivision of the interval [0,1] would require harder calculation as the runtime complexity of the computation is exponential in the length of the minimal sequences.

Characterising the dimension of $F(c, \beta)$ via the expansion of 1 in a β-shift gives a neat way of calculating an approximative picture of the graph of $\phi_\beta : c \mapsto \dim_H F(c, \beta)$. If we let n be the number of points we wish to evaluate the graph in then the procedure becomes

$\texttt{For}\ j := n\ \texttt{downto}\ 0\ \texttt{do}$

$$b := 1 + (\beta - 1)\frac{j}{n}$$

$$\texttt{plot}\left(1 - \sum_{k=1}^{\infty} \frac{d(1,b)_k}{\beta^k}, \frac{\log b}{\log \beta}\right)$$

$\texttt{endfor}$

In Figure 2 for the graph of $c \mapsto \dim_H F(c, \beta)$ for $\beta = 3$ and Figure 3 for the graph of $c \mapsto \dim_H F(c, \beta)$ for $\beta = 2.7$ calculated in $n = 1000$ points. This method is far less time consuming than the method described above, where eigenvalues of large transition matrices had to be calculated.

Note that Corollary 5.13 gives that the graph of $c \mapsto \dim_H F(c, 2)$ is contained in the graph of $c \mapsto \dim_H F(c, 3)$.

$\dim_H F(c, 2.7)$

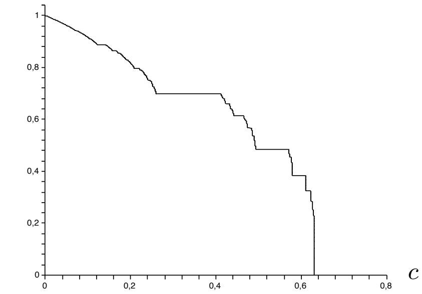


Figure 3: The graph of $c \mapsto \dim_H F(c, 2.7)$.

$\dim_H F(0.25, \beta)$

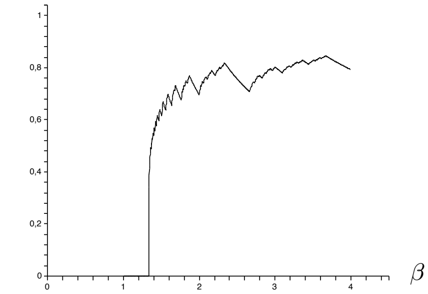


Figure 4: The graph of $\beta \mapsto \dim_H F(0.25, \beta)$.

Part II

Two-sided q-adically Badly Approximable Numbers

9 Introduction

In this part we are going to study a special case of diophantine approximation, two-sided approximation by real numbers of the form $\frac{m}{q^n}$ for the integer $q \geq 2$ and an integer m. Similar to the approximation by rationals in (1.1) we set the sequences $\{x_n\}$ and $\{l_n\}$ to be

$$x_{n,m} = \frac{m}{q^n} \quad \text{and} \quad l_n = \frac{c}{q^n},$$

for $0 < c < 1$. We will turn our interest to the same type of questions as in the classical approximation case and look at the set of badly approximable numbers under these special form of $\{x_n\}$ and $\{l_n\}$. We define $F^2(c, q)$ to be the set

$$F^2(c, q) = \left\{ x \in \mathbb{S} : \left\| x - \frac{m}{q^n} \right\| < \frac{c}{q^n} \text{ finitely often} \right\}, \qquad (9.1)$$

where the norm $\|\cdot\|$ denotes the shortest distance to an integer. As we are going to study dimensional properties of $F^2(c, q)$ we can restrict ourselves to the case when the condition in (9.1) is infinitely often fulfilled but is never fulfilled. So we introduce $F(c, q)$ by

$$F(c, q) = \{x \in \mathbb{S} : \|q^n x\| \geq c \text{ for all } n \geq 0\} \qquad (9.2)$$

Then $F^2(c, q)$ is the countable union of preimages of $F(c, \beta)$ under multiplication by q. Hence we have $\dim_H F^2(c, q) = \dim_H F(c, q)$. From the expansion (2.2) of a real number $x \in [0, 1]$ into an integer base q we see that we can redefine the set $F(c, q)$ into a set of sequences from a q letter alphabet. We have

$$F(\boldsymbol{c}, q) = \left\{ x \in S^\infty(q) : \boldsymbol{c}' \geq \sigma^n(\boldsymbol{x}) \geq \boldsymbol{c} \text{ for all } n \geq 0. \right\} \qquad (9.3)$$

We define the dimension function $\phi_q : [0, 1] \to [0, 1]$ by

$$\phi_q(c) = \dim_H F(c, q).$$

The main results concerning the function ϕ_q are,

Main Result 9.1 *For $q \geq 2$ the function ϕ_q is continuous, is partly self similar, has derivative zero Lebesgue a.e., the complementary zero-set, to where the derivative of ϕ_q is zero, has full Hausdorff dimension and we give the complete characterisation of the intervals where the derivative of ϕ_q is zero.*

In [1, 2], (see also [4]) Allouche and Cosnard consider iterations of unimodal functions. (A continuous function f is said to be unimodal if for $a \in (0,1)$, $f(1) = 0$ and $f(a) = 1$, it is strictly increasing on $[0,a)$ and strictly decreasing on $(a,1]$). They give the result that the existence of unimodal functions is connected to elements in the set of binary sequences $\Gamma(2)$, where

$$\Gamma(q) = \{\boldsymbol{x} \in S^{\infty}(q) : \boldsymbol{x}' \leq \sigma^n(\boldsymbol{x}) \leq \boldsymbol{x} \text{ for all } n \geq 0\}. \tag{9.4}$$

Allouche and Cosnard presents some properties of the set $\Gamma(2)$. They show that it is a self similar set and therefore a fractal set. In Corollary 16.24 we show that the dimensional structure of $\Gamma(2)$ is the same as the dimensional structure of $F(\boldsymbol{c},2)$. Furthermore in [1, 2], Allouche and Cosnard consider also the more general set $\Gamma_{\boldsymbol{a}}(2)$, where

$$\Gamma_{\boldsymbol{a}}(q) = \{\boldsymbol{x} \in S^{\infty}(q) : \boldsymbol{a}' \leq \sigma^n(\boldsymbol{x}) \leq \boldsymbol{a} \text{ for all } n \geq 0\}. \tag{9.5}$$

One of the main results achieved by Allouche and Cosnard on $\Gamma_{\boldsymbol{a}}(2)$ is to present the threshold sequence $\boldsymbol{t}_2$ such that $\Gamma_{\boldsymbol{a}}(2)$ is countable if and only if $\boldsymbol{a} < \boldsymbol{t}_2$. In [14], Moreira improves this result and shows that $\dim_H \Gamma_{\boldsymbol{a}}(2) = 0$ if and only if $\boldsymbol{a} \leq \boldsymbol{t}_2$.

Moreira also turn his interest to how the dimension of sets like $\Gamma_{\boldsymbol{a}}(2)$ depends on the parameter $\boldsymbol{a}$. In [12], Labarca and Moreira show that for $(\boldsymbol{a},\boldsymbol{b}) \in S^{\infty}(2) \times S^{\infty}(2)$ the map

$$(\boldsymbol{a},\boldsymbol{b}) \mapsto \dim_H\{\boldsymbol{x} \in S^{\infty}(2) : \boldsymbol{a} \geq \sigma^n(\boldsymbol{x}) \geq \boldsymbol{b} \text{ for all } n \geq 0\}$$

is continuous in both $\boldsymbol{a}$ and $\boldsymbol{b}$. In Section 14 we present in more detail some technical results by Allouche and Cosnard that we will make use of.

It is clear that the threshold sequence $\boldsymbol{t}_2$ given by Allouche, Cosnard and Moreira also applies to our set $F(\boldsymbol{c},2)$, that is, $F(\boldsymbol{c},2)$ is countable if and only if the binary sequence $\boldsymbol{c}$ fulfils $\boldsymbol{c} > \boldsymbol{t}_2'$ and $\dim_H F(\boldsymbol{c}, 2) = 0$ if and only if $\boldsymbol{c} \geq \boldsymbol{t}_2'$. We will generalise this result and give the threshold sequence $\boldsymbol{t}_q$ for set $F(\boldsymbol{c}, q)$ for any integer $q \geq 2$, (see Theorem 16.9).

We will give a new proof that the map ϕ_2 is continuous, which is a special case of the result that ϕ_q is continuous. We prove that ϕ_q has derivative zero Lebesgue a.e. and show how to completely describe the intervals where the dimension remains unchanged.

The main idea is that we start by considering the binary case, $q = 2$, which we then show has a partly direct translation to the case $q = 3$. These two case then serve as the base in a two step induction when extending the result to hold for the case with $q + 2$.

10 Fundamental Properties

From the definition of $F(\boldsymbol{c}, q)$ and by symmetry it is clear that we have the equaivalence

$$\boldsymbol{x} \in F(\boldsymbol{c}, q) \quad \text{if and only if} \quad \boldsymbol{x}' \in F(\boldsymbol{c}, q).$$

Lemma 10.1 *Let $\boldsymbol{c} \in S^*(q)$. Then $F(\boldsymbol{c}, q) = F(\boldsymbol{c}^\infty, q)$.*

Proof: The lemma is a direct consequence from Lemma 4.1. ☐

Lemma 10.2 *Let $\boldsymbol{c} \in S(q)$ be of the form $\boldsymbol{c} = \tilde{\boldsymbol{u}}\,(\boldsymbol{u}^*)^k \boldsymbol{u}'\boldsymbol{v}$ for some $k \geq 0$ and a finite sequence $\boldsymbol{u}$. If $\boldsymbol{x} \in F(\boldsymbol{c}, q)$ contains the subsequence $\tilde{\boldsymbol{u}}$, (or symmetrically $\boldsymbol{u}'$), then $\boldsymbol{x}$ must be of the form*

$$\boldsymbol{w}\,\tilde{\boldsymbol{u}}\,(\boldsymbol{u}^*)^{k_1}\,\boldsymbol{u}'\,\boldsymbol{u}^{k_2}\,\tilde{\boldsymbol{u}}\,(\boldsymbol{u}^*)^{k_3}\,\boldsymbol{u}'\,\boldsymbol{u}^{k_4}\,\tilde{\boldsymbol{u}}\ldots, \tag{10.1}$$

with $0 \leq k_i \leq k$ and where the sequence $\boldsymbol{w}$ does not contain the subsequence $\tilde{\boldsymbol{u}}$.

Proof: Let n be the smallest integer such that $\sigma^n(\boldsymbol{x}) = \tilde{\boldsymbol{u}}\ldots$. Let $\sigma^n(\boldsymbol{x}) = \tilde{\boldsymbol{u}}\,\boldsymbol{a}_1\,\boldsymbol{a}_2\ldots$, with $|\boldsymbol{a_i}| = |\boldsymbol{u}|$. Let m be the smallest integer such that $\boldsymbol{a}_m \neq \boldsymbol{u}^*$. From the inequality $\sigma^n(\boldsymbol{x}) = \tilde{\boldsymbol{u}}\,\boldsymbol{a}_1\,\boldsymbol{a}_2\ldots \geq \tilde{\boldsymbol{u}}\,(\boldsymbol{u}^*)^k\boldsymbol{u}'\boldsymbol{v}$ we have that $1 \leq m \leq k+1$.

$\sigma^n(\boldsymbol{x}) =$	$\tilde{\boldsymbol{u}}$	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$
$\boldsymbol{c} =$	$\tilde{\boldsymbol{u}}$	$\boldsymbol{u}^*$	$\boldsymbol{u}^*$	$\boldsymbol{u}^*$	$\boldsymbol{u}'$	$\boldsymbol{v}$

This implies that $\boldsymbol{a}_m \geq \boldsymbol{u}'$. By shifting the sequence $\boldsymbol{x}$ additionally $m|\boldsymbol{u}|$ times we obtain $\boldsymbol{u}' \geq \boldsymbol{a}_m$.

$\boldsymbol{c}' =$	$\boldsymbol{u}'$	$\boldsymbol{u}$	$\boldsymbol{u}$	$\boldsymbol{u}$	$\tilde{\boldsymbol{u}}$	$\boldsymbol{v}'$
$\sigma^{n+m\|\boldsymbol{u}\|}(\boldsymbol{x}) =$	$\boldsymbol{a}_m$	$\boldsymbol{a}_{m+1}$	$\boldsymbol{a}_{m+2}$	$\boldsymbol{a}_{m+3}$	$\boldsymbol{a}_{m+4}$	$\boldsymbol{a}_{m+5}$

Hence $\boldsymbol{u}' = \boldsymbol{a}_m$. The result now follows by symmetry. □

For the special case when $k = 0$ in Lemma 10.2 we have the following corollary, which also was given by Allouche in [1].

Corollary 10.3 *Let $\boldsymbol{c} \in S^*(q)$ be of the form $\boldsymbol{c} = \boldsymbol{u}\,\boldsymbol{u}^*$. If $\boldsymbol{x} \in F(\boldsymbol{c}, q)$ contains the subsequence $\boldsymbol{u}$, (or symmetrically $\boldsymbol{u}^*$), then $\boldsymbol{x}$ must be of the form*

$$\boldsymbol{w}(\boldsymbol{u}\,\boldsymbol{u}^*)^\infty$$

for some sequence $\boldsymbol{w}$ not containing the subsequence $\boldsymbol{u}$.

11 Shift-Bounded Sequences

Definition 11.1 *A finite sequence $\boldsymbol{s}$ fulfilling $\boldsymbol{s}' > \sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for $0 < n < |\boldsymbol{s}|$ is said to be a finite shift-bounded sequence. Similarly, an infinite sequence $\boldsymbol{s}$ fulfilling $\boldsymbol{s}' > \sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for all $n > 0$ is said to be an infinite shift-bounded sequence. We also say that a sequence $\boldsymbol{s} \neq 0$ and $|\boldsymbol{s}| = 1$ is shift-bounded.*

Our definition of shift-bounded sequences coincides with and extends the definition of *admissible* sequences considered by Komornik and Loreti in [11] and by Allouche and Cosnard in [3]. From the Definition 11.1 we have directly the following lemma, which states some very useful properties of suffixes and prefixes of a shift-bounded sequence.

Proposition 11.2 *Let $\boldsymbol{s}$ be a finite shift-bounded sequence and let α and $\boldsymbol{\gamma}$ be a prefix and a suffix respectively of $\boldsymbol{s}$ such that $|\boldsymbol{\alpha}| = |\gamma| < |\boldsymbol{c}|$. Then $\boldsymbol{\alpha}^* \geq \boldsymbol{\gamma} > \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^* > \tilde{\gamma} \geq \boldsymbol{\alpha}$.*

Lemma 11.3 *Let $\boldsymbol{s}$ be a shift-bounded sequence. If $\boldsymbol{a}\,\boldsymbol{a}^*$ is a prefix of $\boldsymbol{s}$ then $\boldsymbol{s} = \boldsymbol{a}\,\boldsymbol{a}^*$.*

Proof: Assume $\boldsymbol{s} = \boldsymbol{a}\,\boldsymbol{a}^*\boldsymbol{w}$, where $\boldsymbol{w}$ is non-empty. Let N be the maximal integer such that $\boldsymbol{s} = (\boldsymbol{a}\,\boldsymbol{a}^*)^N\boldsymbol{u}$ for some sequence $\boldsymbol{u}$. This number N exists, since otherwise $\boldsymbol{s}$ would be periodic and hence not shift-bounded. If $\boldsymbol{u}$ does not have $\boldsymbol{a}$ as a prefix we have $\boldsymbol{a}^*\boldsymbol{a}\,\boldsymbol{u}' > \sigma^{2|\boldsymbol{a}|N-|\boldsymbol{a}|}(\boldsymbol{s}) = \boldsymbol{a}^*\boldsymbol{u}$. But also $\sigma^{2|\boldsymbol{a}|N}(\boldsymbol{s}) = \boldsymbol{u} > \boldsymbol{a}$. Hence $\boldsymbol{a}\,\boldsymbol{u}' > \boldsymbol{u} > \boldsymbol{a}$, a contradiction as $\boldsymbol{a}$ is not a prefix of $\boldsymbol{u}$. For the second case, if $\boldsymbol{u}$ has $\boldsymbol{a}$ as prefix we can write $\boldsymbol{s} = (\boldsymbol{a}\,\boldsymbol{a}^*)^N\boldsymbol{a}\,\boldsymbol{v}$ for some sequence $\boldsymbol{v}$ not having $\boldsymbol{a}^*$ as prefix. This gives $\boldsymbol{a}^*\boldsymbol{a}\,\boldsymbol{a}^*\boldsymbol{v}' > \sigma^{2|\boldsymbol{a}|N-|\boldsymbol{a}|}(\boldsymbol{s}) = \boldsymbol{a}^*\boldsymbol{a}\,\boldsymbol{v}$ and $\sigma^{2|\boldsymbol{a}|N}(\boldsymbol{s}) = \boldsymbol{a}\,\boldsymbol{v} > \boldsymbol{a}\,\boldsymbol{a}^*$, that is, $\boldsymbol{a}^*\boldsymbol{v}' > \boldsymbol{v} > \boldsymbol{a}^*$, a contradiction. □

Corollary 11.4 *Let $\boldsymbol{s}$ be a shift-bounded sequence and let $\boldsymbol{s} = \boldsymbol{abe}$ with $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $|\boldsymbol{e}| > 0$. Then $\boldsymbol{b} < \boldsymbol{a}^*$.*

Lemma 11.5 *Let $\boldsymbol{s} \in S(q)$ be a shift-bounded sequence. Then $\boldsymbol{s}$ is a sequence from the alphabet $\{s_1, s_1 + 1, s_1 + 2, \ldots, q - s_1 - 1\}$.*

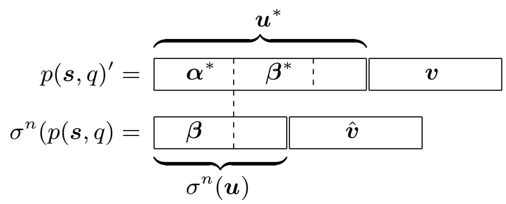
Proof: Assume $s_n < s_1$. Then $\sigma^{n-1}(\boldsymbol{s}) = s_n s_{n+1} s_{n+2} \ldots < s_1 s_2 s_3 \ldots = \boldsymbol{s}$, contradicting $\boldsymbol{s}$ being shift-bounded. Similarly, if $q - s_n - 1 < s_n$ then $\boldsymbol{s}' = (q - s_1 - 1)(q - s_2 - 1)(q - s_3 - 1)\ldots < s_n s_{n+1} s_{n+2} \ldots = \sigma^{n-1}(\boldsymbol{s})$, again a contradiction. □

Definition 11.6 *For a finite shift-bounded sequence $\boldsymbol{s} = \boldsymbol{uvu}^* < 1$ in $S^*(q)$, where $\boldsymbol{u}$ is the longest possible, we define the prefix-suffix reduction function $p : S^*(q) \to S^*(q)$ by $p(\boldsymbol{s}, q) = \widehat{\boldsymbol{uv}}$.*

The shift-boundedness of $\boldsymbol{s}$ in the definition gives that $p(\boldsymbol{s}, q)$ is well defined, that $\frac{1}{2}|\boldsymbol{s}| \leq |p(\boldsymbol{s}, q)| \leq |\boldsymbol{s}|$ and $\boldsymbol{s} < p(\boldsymbol{s}, q)$.

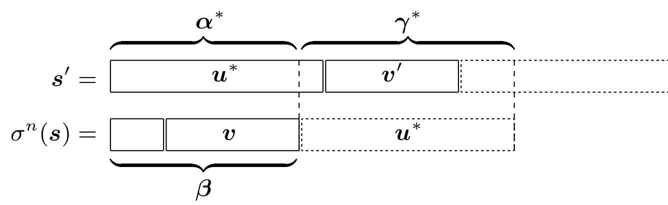
Lemma 11.7 *Let $\boldsymbol{s} \in S(q)$ be a finite shift-bounded sequence such that $|\boldsymbol{s}| > 1$. Then $p(\boldsymbol{s}, q)$ is shift-bounded.*

Proof: Let $\boldsymbol{s} = \boldsymbol{u}\boldsymbol{v}\boldsymbol{u}^*$ where $p(\boldsymbol{s}, q) = \widehat{\boldsymbol{u}\boldsymbol{v}}$. The inequality $\sigma^n(p(\boldsymbol{s}, q)) > p(\boldsymbol{s}, q)$ for $0 < n < |p(\boldsymbol{s}, q)|$ follows from the definition of p and that $\boldsymbol{s}$ is shift-bounded. For the upper bounding inequality for shift-boundedness we consider first the case when $0 < n < |\boldsymbol{u}|$. Let $\boldsymbol{\alpha} = p(\boldsymbol{s}, q)[1, n]$ and $\boldsymbol{\beta} = p(\boldsymbol{s}, q)[n+1, 2n]$.



Then as $\boldsymbol{s}$ is shift-bounded we have by Corollary 11.4 that $\boldsymbol{\alpha}^* > \boldsymbol{\beta}$ and therefore $p(\boldsymbol{s}, q)' > \sigma^n(p(\boldsymbol{s}, q))$.

For $|\boldsymbol{u}| \leq n < |p(\boldsymbol{s}, q)|$ assume first that $\boldsymbol{u}$ is non-void. Then let $\boldsymbol{\alpha} = \boldsymbol{s}[1, |\boldsymbol{u}\boldsymbol{v}| - n]$, $\beta = \boldsymbol{s}[n+1, |\boldsymbol{u}\boldsymbol{v}|]$ and $\boldsymbol{\gamma} = \boldsymbol{s}[|\boldsymbol{u}\boldsymbol{v}| - n + 1, |\boldsymbol{u}\boldsymbol{v}| - n + |\boldsymbol{u}|]$.



By the definition of $\boldsymbol{u}$ we have $\boldsymbol{\alpha}^*\boldsymbol{\gamma}^* > \boldsymbol{\beta}\boldsymbol{u}^*$. But as $\boldsymbol{u}^* \geq \boldsymbol{\gamma}^*$ we must have $\boldsymbol{\alpha}^* > \boldsymbol{\beta}$ and hence $p(\boldsymbol{s}, q)' > \sigma^n(p(\boldsymbol{s}, q))$. If $\boldsymbol{u}$ is void let $\boldsymbol{\alpha} = \boldsymbol{s}[1, |\boldsymbol{s}| - n]$ and $\boldsymbol{\gamma} = \boldsymbol{s}[n+1, |\boldsymbol{s}|]$. Then since $\boldsymbol{s}$ is shift-bounded we have $\boldsymbol{\alpha}^* \geq \boldsymbol{\gamma}$, but since $\boldsymbol{u}$ is empty we must have $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}$. Hence $\boldsymbol{\alpha}^* \geq \hat{\boldsymbol{\gamma}}$, which implies $p(\boldsymbol{s}, q)' > \sigma^n(p(\boldsymbol{s}, q))$. □

Lemma 11.8 *Let $\boldsymbol{s} \in S^*(q)$ be a finite shift-bounded sequence starting with the symbol s_1. Then there is an n such that $p^n(\boldsymbol{s}, q) = s_1(q - s_1 - 1)$.*

Proof: Since $\frac{1}{2}|\boldsymbol{s}| \leq |p(\boldsymbol{s}, q)|$ and $p^n(\boldsymbol{s}, q)$ is shift-bounded and decreasing in length under n there is a first m such that $|p^m(\boldsymbol{s}, q)| = 2$. We have $p^m(\boldsymbol{s}, q) = s_1 r$ for some $r \in \{s_1 + 1, s_1 + 2, \ldots, q - s_1 - 1\}$. This gives $p^{m+q-s_1-r-1}(\boldsymbol{s}, q) = s_1(q - s_1 - 1)$. □

Definition 11.9 *For a finite sequence $\boldsymbol{s}$ of length n not ending with a 0, we define the map $f : S^*(q) \to S^*(q)$ by $f(\boldsymbol{s}, q) = \tilde{\boldsymbol{s}}\,\boldsymbol{s}'$. We define* *the function* $d : S^*(q) \to S^*(q)$ *as the function taking* $\boldsymbol{s}$ *to its limit point under self-composition of* f,

$$d(\boldsymbol{s}, q) = \lim_{k\to\infty} f^k(\boldsymbol{s}, q).$$

The function f could equally have been defined on the rational numbers. By a straight forward calculation we have

Theorem 11.10 *Let* $\frac{a}{q^n} \in \mathbb{Q}^+ \setminus \{0\}$*. Then the limit*

$$\lim_{k\to\infty} f^k(\frac{a}{q^n}) = \frac{a}{q^n} \prod_{i=0}^{\infty} \left(1 - \frac{1}{q^{2^i n}}\right)$$

is a well defined real number and moreover it is a transcendental number.

The second part of the theorem is a direct consequence of the following theorem by Mahler [13],

Theorem 11.11 (Mahler) *Let* $0 < |a| < 1$ *be an algebraic number. Then the product* $\prod_{i=0}^{\infty}(1 - a^{2^i})$ *is transcendental.*

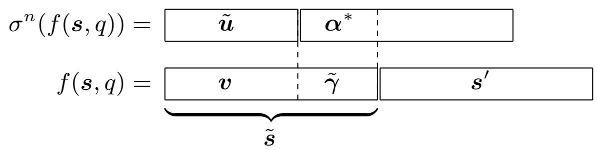
In [1, 2, 3] Allouche and Cosnard define the function φ on periodic sequences by $\varphi((\boldsymbol{x}0)^\infty) = (\boldsymbol{x}1\boldsymbol{x}^*0)^\infty$. Our function f is φ restricted to finite sequences. As we are going to consider properties of finite sequences we prefer f rather than Allouche and Cosnard's φ. The sequence $d(\boldsymbol{s}, q)$ coincides with Allouche and Cosnard's notion of t-mirror sequences, where the t is the length of $\boldsymbol{s}$.

Lemma 11.12 (Allouche, Cosnard) *Let* Γ *be the set defined in (9.4) and let* $\boldsymbol{x} = (\boldsymbol{a}0)^\infty$, *where* $(\boldsymbol{a}0)$ *is the shortest period. Then* $\boldsymbol{x} \in \Gamma$ *if and only if* $\varphi(\boldsymbol{x}) \in \Gamma$ *and moreover* $\boldsymbol{x} \in \Gamma$ *if and only if* $\lim_{n\to\infty} \varphi^n(\boldsymbol{x}) \in \Gamma$.

In the same spirit as Lemma 11.12 we have

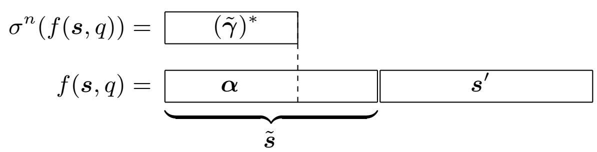
Lemma 11.13 *Let* $\boldsymbol{s} \in S^*(q)$ *be a finite sequence. Then* $\boldsymbol{s}$ *is shift-bounded if and only if* $f(\boldsymbol{s}, q)$ *is shift-bounded.*

Proof: Assume that $\boldsymbol{s}$ is shift-bounded. For $0 < n < |\boldsymbol{s}|$ let $\boldsymbol{\alpha}^* = f(\boldsymbol{s},q)[|\boldsymbol{s}|+1, |\boldsymbol{s}|+n]$, $\boldsymbol{v} = f(\boldsymbol{s},q)[1, |\boldsymbol{s}|-n]$, $\tilde{\boldsymbol{u}} = f(\boldsymbol{s},q)[n+1, |\boldsymbol{s}|]$ and $\tilde{\boldsymbol{\gamma}} = f(\boldsymbol{s},q)[|\boldsymbol{s}|-n+1, |\boldsymbol{s}|]$.



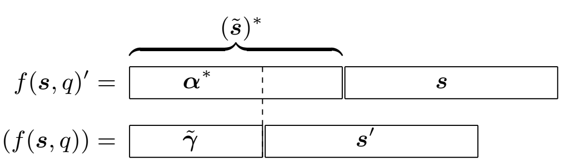
The shift-boundedness of $\boldsymbol{s}$ gives $\tilde{\boldsymbol{u}} \geq \boldsymbol{v}$ and $\boldsymbol{\alpha}^* > \tilde{\boldsymbol{\gamma}}$. This implies that the inequality $\sigma^n(f(\boldsymbol{s},q)) > f(\boldsymbol{s},q)$ holds. For the case $n = |\boldsymbol{s}|$ we have that $\sigma^n(f(\boldsymbol{s},q)) = \boldsymbol{s}' > \tilde{\boldsymbol{s}}\boldsymbol{s}' = f(\boldsymbol{s},q)$.

For $|\boldsymbol{s}| < n < 2|\boldsymbol{s}|$ let $\boldsymbol{\alpha} = f(\boldsymbol{s},q)[1, 2|\boldsymbol{s}|-n]$, $(\tilde{\boldsymbol{\gamma}})^* = f(\boldsymbol{s},q)[n+1, 2|\boldsymbol{s}|]$.



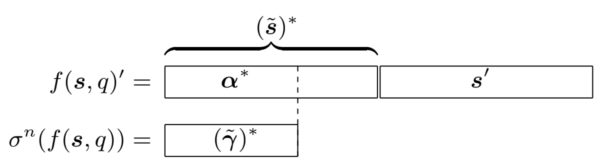
Hence $(\tilde{\boldsymbol{\gamma}})^* > \boldsymbol{\alpha}$ as $\boldsymbol{s}$ is shift-bounded and therefore $\sigma^n(f(\boldsymbol{s},q)) > f(\boldsymbol{s},q)$.

For $0 < n < |\boldsymbol{s}|$ let $\boldsymbol{\alpha}^* = \big(f(\boldsymbol{s},q)[1, |\boldsymbol{s}|-n]\big)'$ and $\tilde{\boldsymbol{\gamma}} = f(\boldsymbol{s},q)[n+1, |\boldsymbol{s}|]$.



From Proposition 11.2 the shift-boundedness of $\boldsymbol{s}$ gives $\boldsymbol{\alpha}^* > \tilde{\boldsymbol{\gamma}}$ and therefore $f(\boldsymbol{s},q) > \sigma^n(f(\boldsymbol{s},q))$. For $n = |\boldsymbol{s}|$ we have $f(\boldsymbol{s},q)' = \boldsymbol{s}'\boldsymbol{s} > \boldsymbol{s}' = \sigma^n(f(\boldsymbol{s},q))$.

For $|\boldsymbol{s}| < n < 2|\boldsymbol{s}|$ let $\boldsymbol{\alpha}^* = \big(f(\boldsymbol{s},q)[1, 2|\boldsymbol{s}|-n]\big)'$, $(\tilde{\boldsymbol{\gamma}})^* = f(\boldsymbol{s},q)[n+1, 2|\boldsymbol{s}|]$.



Again, as $\boldsymbol{s}$ is shift-bounded we have $\boldsymbol{\alpha}^* \geq (\tilde{\gamma})^*$, which implies $f(\boldsymbol{s}, q) > \sigma^n(f(\boldsymbol{s}, q))$.

Now conversely, assume that $f(\boldsymbol{s}, q)$ is shift-bounded. Since $f(\boldsymbol{s}, q) = \tilde{\boldsymbol{s}}\boldsymbol{s}' = \tilde{\boldsymbol{s}}(\tilde{\boldsymbol{s}})$ we have by Lemma 11.7 that $p(f(\boldsymbol{s}, q), q) = \boldsymbol{s}$ and that $\boldsymbol{s}$ is shift-bounded, which ends the proof. □

Corollary 11.14 *If $\boldsymbol{s} \in S^*(q)$ is a finite shift-bounded sequence then $d(\boldsymbol{s}, q)$ is an infinite shift-bounded sequence.*

Proof: From Lemma 11.13 we have that

$$d(\boldsymbol{s}, q)' \geq \sigma^n\big(d(\boldsymbol{s}, q)\big) \geq d(\boldsymbol{s}, q) \tag{11.1}$$

for all $n > 0$. The sequence $d(\boldsymbol{s}, q)$ corresponds to the q-nary expansion of a real number, x and Mahler's Theorem 11.11 gives that this number x is transcendental. If we for some n would have equality in one of the inequalities then $d(\boldsymbol{s}, q)$ must be periodic and therefore x must be rational, a contradiction. □

Lemma 11.15 *Let $\boldsymbol{s} \in S^*(q)$ be a finite shift-bounded sequence such that there exists no sequence $\boldsymbol{u} \in S^*(q)$ such that $f(\boldsymbol{u}, q) = \boldsymbol{s}$. Then the sequences $\{\boldsymbol{u}_k = f^k(\boldsymbol{s}, q) : k \geq 0\}$ are the only shift-bounded sequences in the interval $\big(d(\boldsymbol{s}, q), \boldsymbol{s}^\infty\big]$.*

Proof: Let $\boldsymbol{v} \in S(q)$ be a finite or infinite shift-bounded sequence in the interval $\big(d(\boldsymbol{s}, q), \boldsymbol{s}\big]$. Then there is a $k \geq 0$ such that

$$\boldsymbol{u}_{k+1} < \boldsymbol{v} \leq \boldsymbol{u}_k. \tag{11.2}$$

Hence $\boldsymbol{u}_k[1, |\boldsymbol{u}_k| - 1] = \boldsymbol{v}[1, |\boldsymbol{u}_k| - 1]$. If $\boldsymbol{u}_k = \boldsymbol{v}[1, |\boldsymbol{u}_k|]$ then by (11.2) we must have $\boldsymbol{u}_k = \boldsymbol{v}$. For the case $\boldsymbol{u}_{k+1}[1, |\boldsymbol{u}_k|] = \boldsymbol{v}[1, |\boldsymbol{u}_k|]$, we have by Lemma 11.3 and (11.2) that $\boldsymbol{u}_{k+1}$ can not be a prefix of $\boldsymbol{v}$. Hence there is first a position $|\boldsymbol{u}_k| < i \leq |\boldsymbol{u}_{k+1}|$ where $\boldsymbol{u}_{k+1}$ and $\boldsymbol{v}$ differ. But then $\sigma^{|\boldsymbol{u}_k|}(\boldsymbol{v}) > \boldsymbol{v}'$, contradicting $\boldsymbol{v}$ being shift-bounded.

If $\boldsymbol{v}$ is a shift-bounded sequence in the interval $(\boldsymbol{s}\,, \boldsymbol{s}^\infty)$ then we must have $\boldsymbol{v} = \boldsymbol{s}^k\boldsymbol{b}$ where $\boldsymbol{b} > \boldsymbol{s}$. But then $\boldsymbol{v} > \boldsymbol{s}^\infty$, a contradiction. □

Lemma 11.16 *Let $\boldsymbol{c} \in S^*(q)$ be a finite sequence. Then*

$$\dim_H F(f^n(\boldsymbol{c}, q), q) = \dim_H F(\boldsymbol{c}, q)$$

for all $n \geq 0$.

Proof: Let $\boldsymbol{x} \in F(f(\boldsymbol{c}), q) \setminus F(\boldsymbol{c}, q)$. Then x must at least once contain the pattern $\tilde{\boldsymbol{c}}$. But as $f(\boldsymbol{c}) = \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{c}})^*$ Corollary 10.3 gives that $\boldsymbol{x}$ must end with $(\tilde{\boldsymbol{c}}(\tilde{\boldsymbol{c}})^*)^\infty$. Hence $F(f(\boldsymbol{c}), q)$ contains only countably more elements than $F(\boldsymbol{c}, q)$. To prove that we may extend to f^2 replace $\boldsymbol{c}$ by $\boldsymbol{c}_1 = f(\boldsymbol{c})$ and repeat the argumentation. □

We end the section with a remark on the infinite shift-bounded sequences. Let $ISB(q)$ be the set of all infinite shift-bounded sequences on the alphabet $\{0, 1, \dots, q-1\}$, that is,

$$ISB(q) = \{\boldsymbol{x} \in S^\infty(q) : \boldsymbol{x}' > \sigma^n(\boldsymbol{x}) > \boldsymbol{x} \text{ for all } n > 0\}. \tag{11.3}$$

Lemma 11.17 *The set $ISB(q)$ has Lebesgue measure* 0.

Proof: We have $ISB(q) \subset IM(q)$, so the result follows directly from Lemma 4.10. □

12 Minimal Sequences

Definition 12.1 *For a finite sequence $\boldsymbol{s} \in S^*(q)$ not ending with a 0, we define the function $e : S^*(q) \to S^\infty(q)$ by $e(\boldsymbol{s}, q) = \tilde{\boldsymbol{s}}\,(\boldsymbol{s}^*)^\infty$.*

The motivation for the definition of the function e is that e gives sequences which are the extremal case, $k = \infty$, in Lemma 10.2.

Definition 12.2 *Let $\boldsymbol{e}_{q,i} = e(f^i([\frac{q}{2}], q), q)$ for $i \geq 0$ and we let $\boldsymbol{e}^0_{q,0} = 0^\infty$ and $\boldsymbol{e}^r_{q,0} = e(r, q)$ for $r \in \{1, 2, \dots, [\frac{q}{2}]\}$.*

A sequence $\boldsymbol{s} \in S(q)$, not containing only zeros, finite or infinite, is an $e_{q,i}$-sequence if $\boldsymbol{e}_{q,i-1} \leq \boldsymbol{s} < \boldsymbol{e}_{q,i}$ or an $e_{q,0}$-sequence if $\boldsymbol{s} < \boldsymbol{e}_{q,0}$. If $\boldsymbol{e}^{r-1}_{q,0} \leq \boldsymbol{s} < \boldsymbol{e}^r_{q,0}$ then $\boldsymbol{s}$ is said to be an $e^r_{q,0}$-sequence.

Note that $\boldsymbol{e}_{q,i}$ grows monotonically to $d([\frac{q}{2}], q)$ as i tends to infinity and moreover, any $\boldsymbol{e}^r_{q,0}$-sequence is an $e_{q,0}$-sequence and $\boldsymbol{e}_{q,0} = \boldsymbol{e}^k_{q,0}$ for $k = [\frac{q}{2}]$. Note also that the sequence $\boldsymbol{e}_{q,i}$ are $e_{q,i+1}$-sequences.

Example 12.3 For $q = 2, 3, 4, 5$ we have for small values of i

$$\begin{array}{lcllcl}
\boldsymbol{e}^0_{2,0} & = & 0^\infty & \boldsymbol{e}^0_{3,0} & = & 0^\infty \\
\boldsymbol{e}^1_{2,0} & = & 0^\infty & \boldsymbol{e}^1_{3,0} & = & 0(1)^\infty \\
\boldsymbol{e}_{2,0} & = & 0^\infty & \boldsymbol{e}_{3,0} & = & 0(1)^\infty \\
\boldsymbol{e}_{2,1} & = & 00(10)^\infty & \boldsymbol{e}_{3,1} & = & 01(20)^\infty \\
\boldsymbol{e}_{2,2} & = & 0010(1100)^\infty & \boldsymbol{e}_{3,2} & = & 0120(2101)^\infty \\
\boldsymbol{e}_{2,3} & = & 00101100(11010010)^\infty & \boldsymbol{e}_{3,3} & = & 01202101(21020120)^\infty
\end{array}$$

$$\begin{array}{lcllcl}
\boldsymbol{e}^0_{4,0} & = & 0^\infty & \boldsymbol{e}^0_{5,0} & = & 0^\infty \\
\boldsymbol{e}^1_{4,0} & = & 0(2)^\infty & \boldsymbol{e}^1_{5,0} & = & 0(3)^\infty \\
\boldsymbol{e}^2_{4,0} & = & 1^\infty & \boldsymbol{e}^2_{5,0} & = & 1(2)^\infty \\
\boldsymbol{e}_{4,0} & = & 1^\infty & \boldsymbol{e}_{5,0} & = & 1(2)^\infty \\
\boldsymbol{e}_{4,1} & = & 11(21)^\infty & \boldsymbol{e}_{5,1} & = & 12(31)^\infty \\
\boldsymbol{e}_{4,2} & = & 1121(2211)^\infty & \boldsymbol{e}_{5,2} & = & 1231(3212)^\infty \\
\boldsymbol{e}_{4,3} & = & 11212211(22121121)^\infty & \boldsymbol{e}_{5,3} & = & 12313212(32131231)^\infty
\end{array}$$

Note that $\boldsymbol{e}_{4,i}$ is just a re-alphabetisation of $\boldsymbol{e}_{2,i}$ and that $\boldsymbol{e}_{5,i}$ is a re-alphabetisation of $\boldsymbol{e}_{3,i}$. □

Definition 12.4 *For a sequence $\boldsymbol{s} \in S(q)$ with $|\boldsymbol{s}| > n$ we define the map g by $g_{n,k}(\boldsymbol{s}) = \boldsymbol{s}[1, n-1]k$, where $k \in \{0, 1, \dots, q-1\}$.*

Definition 12.5 *Let $\boldsymbol{s}$ be an $e_{q,i}$-sequence. We define the integer $m_{\boldsymbol{s}}$ by*

$$m_{\boldsymbol{s}} = \min_{1 \le k < q} \inf \left\{ n \ge 2^i : e(g_{n,k}(\boldsymbol{s}), q) \le \boldsymbol{s} \le g_{n,k}(\boldsymbol{s})^\infty \right\}$$

and we denote by $k_{\boldsymbol{s}}$ the unique k for which the minimum is obtained.

If $m_{\boldsymbol{s}}$ is undefined we set $m_{\boldsymbol{s}} = \infty$. We say that $g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})$ is an $e_{q,i}$-minimal prefix of $\boldsymbol{s}$. A sequence $\boldsymbol{s}$ is an $e_{q,i}$-minimal sequence if $g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s}) = \boldsymbol{s}[1, m_{\boldsymbol{s}}] = \boldsymbol{s}$ or if $m_{\boldsymbol{s}} = \infty$.

Example 12.6 The binary sequence 001 is an $e_{2,1}$-minimal sequence and the sequence 001011 is an $e_{2,2}$-minimal sequence. The non-zero sequences $\boldsymbol{e}_{q,i}$ are infinite $e_{q,i+1}$-minimal sequences. The ternary sequence 0111 has the $e_{3,1}$-minimal prefix 012. □

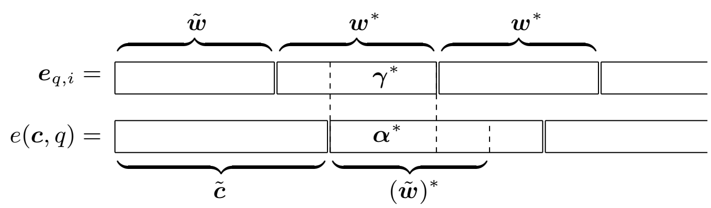
Lemma 12.7 *Let $\boldsymbol{s}$ be an $e_{q,i}$-sequence with a finite $e_{q,i}$-minimal prefix. Then either $s_{m_{\boldsymbol{s}}} = k_{\boldsymbol{s}}$ or $s_{m_{\boldsymbol{s}}} = k_{\boldsymbol{s}} - 1$.*

Proof: Assume that $s_{m_{\boldsymbol{s}}} < k_{\boldsymbol{s}} - 1$. Then $e(g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s}), q) > \boldsymbol{s}$, contradicting the definition of $k_{\boldsymbol{s}}$. Similarly, if $s_{m_{\boldsymbol{s}}} > k_{\boldsymbol{s}}$ then $\boldsymbol{s} > g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})^{\infty}$, again a contradiction. □

Lemma 12.8 *An $e_{q,i}$-minimal prefix is an $e_{q,i}$-sequence.*

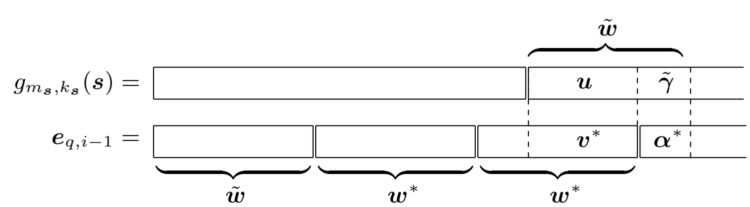
Proof: Let $\boldsymbol{s}$ be an $e_{q,i}$-sequence. The lemma is clear if the $e_{q,i}$-minimal prefix of $\boldsymbol{s}$ is an infinite sequence. Hence we assume that $\boldsymbol{s}$ has the finite $e_{q,i}$-minimal prefix $\boldsymbol{c}$, that is, $\boldsymbol{c} = g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})$.

Assume for contradiction that $\boldsymbol{c} > \boldsymbol{e}_{q,i}$. We must have that $\tilde{\boldsymbol{c}}$ is a prefix of $\boldsymbol{e}_{q,i}$. Let $\boldsymbol{w} = f^i([\frac{q}{2}], q)$, $\boldsymbol{\alpha}^* = e(\boldsymbol{c}, q)[|\boldsymbol{c}| + 1, k \cdot 2^i]$ and $\boldsymbol{\gamma}^* = \boldsymbol{e}_{q,i}[|\boldsymbol{c}|+1, k \cdot 2^i]$, where k is the smallest integer such that $k \cdot 2^i \geq |\boldsymbol{c}|$.



If $k > 1$ and $k \cdot 2^i > |\boldsymbol{c}|$ then as $\boldsymbol{w}$ is shift-bounded we have $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}^*$. Therefore $\boldsymbol{s} \geq e(\boldsymbol{c}, q) > \boldsymbol{e}_{q,i}$, contradicting $\boldsymbol{s}$ being an $e_{q,i}$-sequence. If $k = 1$ then we would have $e(\boldsymbol{c}, q) = \boldsymbol{e}_{q,i}$, again contradicting that $\boldsymbol{s}$ is an $e_{q,i}$-sequence. For $k > 1$ and $k \cdot 2^i = |\boldsymbol{c}|$ then we reach a contradiction as $\boldsymbol{w}^* < (\tilde{\boldsymbol{w}})^*$, which conclude the case.

For the case if $\boldsymbol{c}$ is to small, assume for contradiction that $\boldsymbol{c} < \boldsymbol{e}_{q,i-1}$. We must then have that $\boldsymbol{c}$ is a prefix of $\boldsymbol{e}_{q,i-1}$. This because Lemma 12.7 gives that $k_{\boldsymbol{s}}$ must be $s_{m_{\boldsymbol{s}}}$ or $s_{m_{\boldsymbol{s}}} + 1$. If $s_{m_{\boldsymbol{s}}} - 1 = k_{\boldsymbol{s}}$ then $\boldsymbol{s} < \boldsymbol{e}_{q,i-1}$, so $\boldsymbol{s}$ would not be an $e_{q,i}$-sequence. Let $\boldsymbol{w} = f^{i-1}([\frac{q}{2}], q)$. Note that $|\boldsymbol{w}| = 2^{i-1}$. Furthermore let $\boldsymbol{v}^* = \boldsymbol{e}_{q,i-1}[|\boldsymbol{c}| + 1, k \cdot 2^{i-1}]$, $\boldsymbol{u} = g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})^{\infty}[|\boldsymbol{c}| + 1, k \cdot 2^{i-1}]$, $\boldsymbol{\alpha}^* = \boldsymbol{e}_{q,i}[k \cdot 2^{i-1}, |\boldsymbol{c}| + 2^{i-1}]$ and $\tilde{\boldsymbol{\gamma}} = g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})^{\infty}[k \cdot 2^{i-1}, |\boldsymbol{c}| + 2^{i-1}]$ where k is the smallest integer such that $k \cdot 2^{i-1} \geq |\boldsymbol{c}|$.



If $k \cdot 2^{i-1} > |\boldsymbol{c}|$ then as $\boldsymbol{w}$ is shift-bounded we have $\boldsymbol{v}^*\boldsymbol{\alpha}^* > \boldsymbol{u}\tilde{\boldsymbol{\gamma}} = \tilde{\boldsymbol{w}}$ and therefore

$$\boldsymbol{s} \leq g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})^\infty < \boldsymbol{e}_{q,i-1},$$

a contradiction. If $k \cdot 2^{i-1} = |\boldsymbol{c}|$ the result follows as $\tilde{\boldsymbol{w}} \leq \boldsymbol{w}^*$, which concludes the proof. □

Example 12.9 If the condition of having $n \geq 2^i$ in the definition of $e_{q,i}$-minimal prefixes is dropped then for any $e_{q,i}$-sequence $\boldsymbol{s}$ with $i \geq 1$ we would have $m_{\boldsymbol{s}} = 1$. For illustration, the $e_{2,2}$-minimal sequence 001011 is bounded by $e(1,2) = 0^\infty \leq 001011 \leq 1^\infty$. □

Lemma 12.10 *The $e_{q,0}$-minimal prefix of an $e^r_{q,0}$-sequence is again an $e^r_{q,0}$-sequence.*

Proof: Let $\boldsymbol{s}$ be an $e^r_{q,0}$-sequence. The lemma is clear if the $e_{q,0}$-minimal prefix of $\boldsymbol{s}$ is an infinite sequence. Therefore we only have to consider the case when $\boldsymbol{s}$ has the finite $e_{q,0}$-minimal prefix $\boldsymbol{c}$, that is, $\boldsymbol{c} = g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})$.

Assume for contradiction that $\boldsymbol{c} > \boldsymbol{e}^r_{q,0}$. First consider the case when $|\boldsymbol{c}| = 1$. Then we must have $\boldsymbol{c} = r$ and therefore

$$\boldsymbol{s} \geq e(\boldsymbol{c}, q) = (r-1)(q-r)^\infty = \boldsymbol{e}^r_{q,0}.$$

a contradiction to $\boldsymbol{s}$ being an $e^r_{q,0}$-sequence. Similarly, if $|\boldsymbol{c}| \geq 2$ we must have that $\tilde{\boldsymbol{c}}$ is a prefix of $\boldsymbol{e}^r_{q,0}$, that is, $\boldsymbol{c} = (r-1)(q-r-1)^N(q-r)$ for some N. But then

$$\begin{aligned} \boldsymbol{s} &\geq e(\boldsymbol{c}, q) \\ &= (r-1)(q-r-1)^N(q-r-1)\left((q-r)r^{N+1}\right)^\infty \\ &> (r-1)(q-r-1)^\infty \\ &= \boldsymbol{e}^r_{q,0}, \end{aligned}$$

again a contradiction to $\boldsymbol{s}$ being an $e^r_{q,0}$-sequence.

Now, assume that $\boldsymbol{c} < \boldsymbol{e}^{r-1}_{q,0}$. We must then have that $\boldsymbol{c}$ is a prefix of the sequence $\boldsymbol{e}^{r-1}_{q,0}$. This because Lemma 12.7 gives that $k_{\boldsymbol{s}}$ must be $s_{m_{\boldsymbol{s}}}$ or $s_{m_{\boldsymbol{s}}} + 1$. If $s_{m_{\boldsymbol{s}}} - 1 = k_{\boldsymbol{s}}$ then $\boldsymbol{s} < \boldsymbol{e}^{r-1}_{q,0}$, so $\boldsymbol{s}$ would not be an

$e^r_{q,0}$-sequence. This gives that $\boldsymbol{c} = (r-2)(q-r-2)^N$ for some N. Then we get

$$\begin{aligned} \boldsymbol{s} &\leq g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{c})^\infty \\ &= \big((r-2)(q-r-2)^N\big)^\infty \\ &< (r-2)(q-r-2)^\infty \\ &= \boldsymbol{e}^{r-1}_{q,0}, \end{aligned}$$

again a contradiction to that $\boldsymbol{s}$ is an $e^r_{q,0}$-sequence. □

Lemma 12.11 *Let $\boldsymbol{s}$ be an $e_{q,i}$-sequence. Then $\boldsymbol{s}' > \sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for all $0 < n < 2^i$.*

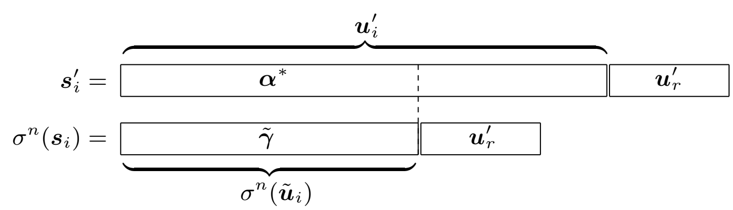
Proof: The result is clear for any $e_{q,0}$- and $e_{q,1}$-sequence. Hence we only have to consider the case with $i > 1$. Let $\boldsymbol{u}_k = f^k([\frac{q}{2}], q)$ and put

$$\boldsymbol{s}_i := \tilde{\boldsymbol{u}}_i\, \boldsymbol{u}'_{i-2} = \tilde{\boldsymbol{u}}_{i-2}\, \boldsymbol{u}^*_{i-2}\, \boldsymbol{u}'_{i-2}\, \tilde{\boldsymbol{u}}_{i-2}\, \boldsymbol{u}'_{i-2}.$$

Note that $|\boldsymbol{s}_i| = 5 \cdot 2^{i-2}$. We have that $\boldsymbol{s}_i$ is a prefix of all $e_{q,i}$-sequences and moreover $\boldsymbol{s}_i$ is a prefix of $f^{i+1}([\frac{q}{2}], q)$. To prove the lemma it is enough to show that $\boldsymbol{s}'_i > \sigma^n(\boldsymbol{s}_i) > \boldsymbol{s}_i$ holds for $0 < n < 2^i$ as $\boldsymbol{s}_{\boldsymbol{i}}$ is a prefix of any $e_{q,i}$-sequence. Since $f^{i+1}([\frac{q}{2}], q)$ is a shift-bounded sequence we have that

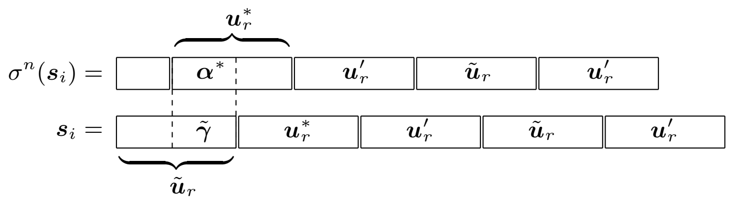
$$\boldsymbol{s}'_i[1, |\boldsymbol{s}_i| - n] \geq \sigma^n(\boldsymbol{s}_i) \geq \boldsymbol{s}_i[1, |\boldsymbol{s}_i| - n] \tag{12.1}$$

holds for $0 < n < |\boldsymbol{s}_i|$. Hence we have to show that these (12.1) shift-inequalities are strict for $1 < n < 2^i$. For the upper bounding inequality let $\boldsymbol{\alpha} = \boldsymbol{s}_i[|\boldsymbol{u}_i| + 1, |\boldsymbol{u}_i| + n]$ and $\tilde{\boldsymbol{\gamma}} = \boldsymbol{s}_i[|\boldsymbol{u}_i| - n + 1, |\boldsymbol{u}_i|]$.



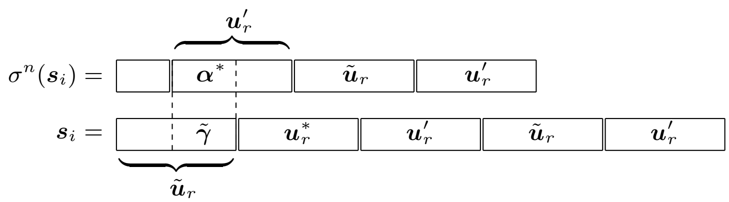
As $\boldsymbol{u}_i$ is shift-bounded we have $\boldsymbol{\alpha}^* > \tilde{\boldsymbol{\gamma}}$ and therefore $\boldsymbol{s}'_i[1, \boldsymbol{s}_i - n] > \sigma^n(\boldsymbol{s}_i)$.

To prove the lower inequality of (12.1) we consider first the case when $0 < n < |\boldsymbol{u}_r|$. Let $\boldsymbol{\alpha}^* = \boldsymbol{s}_i[|\boldsymbol{u}_r|+1, |\boldsymbol{u}_r|+n]$ and $\tilde{\boldsymbol{\gamma}} = \boldsymbol{s}_i[|\boldsymbol{u}_r|-n+1, |\boldsymbol{u}_r|]$.



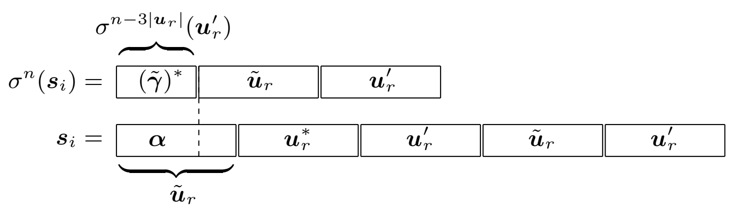
As $\boldsymbol{u}_r$ is shift-bounded we have $\boldsymbol{\alpha}^* > \tilde{\boldsymbol{\gamma}}$ and therefore $\sigma^n(\boldsymbol{s}_i) > \boldsymbol{s}_i$. The case $n = |\boldsymbol{u}_r|$ is clear as $\tilde{\boldsymbol{u}}_r < \boldsymbol{u}_r^*$.

For $|\boldsymbol{u}_r| < n < 2|\boldsymbol{u}_r|$ let $\boldsymbol{\alpha}^* = \boldsymbol{s}_i[2|\boldsymbol{u}_r|+1, |\boldsymbol{u}_r|+n]$ and $\tilde{\boldsymbol{\gamma}} = \boldsymbol{s}_i[|\boldsymbol{u}_r|-n+1, |\boldsymbol{u}_r|]$.



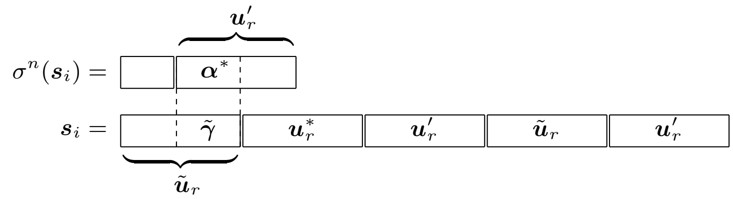
Again by the shift-boundedness of $\boldsymbol{u}_r$ we have $\boldsymbol{\alpha}^* > \tilde{\boldsymbol{\gamma}}$ and therefore $\sigma^n(\boldsymbol{s}_i) > \boldsymbol{s}_i$. The case $n = 2|\boldsymbol{u}_r|$ follows as $\tilde{\boldsymbol{u}}_r < \boldsymbol{u}_r'$.

For $2|\boldsymbol{u}_r| < n < 3|\boldsymbol{u}_r|$ let $\boldsymbol{\alpha} = \boldsymbol{s}_i[1, 3|\boldsymbol{u}_r| - n]$ and $(\tilde{\gamma})^* = \boldsymbol{s}_i[n+1, 3|\boldsymbol{u}_r|]$.



The shift-boundedness of $\boldsymbol{u}_r$ gives again $\boldsymbol{\alpha} < (\tilde{\gamma})^*$ and therefore $\sigma^n(\boldsymbol{s}_i) > \boldsymbol{s}_i$. The case $n = 3|\boldsymbol{u}_r|$ is clear as $\boldsymbol{u}_r^* < \boldsymbol{u}_r'$.

For $3|\boldsymbol{u}_r| < n < 4|\boldsymbol{u}_r|$ let $\boldsymbol{\alpha} = \boldsymbol{s}_i[4|\boldsymbol{u}_r| + 1, |\boldsymbol{u}_r| + n]$ and $\tilde{\boldsymbol{\gamma}} = \boldsymbol{s}_i[4|\boldsymbol{u}_r| - n + 1, |\boldsymbol{u}_r|]$.



As $\boldsymbol{u}_r$ is shift-bounded we have $\boldsymbol{\alpha} < (\tilde{\gamma})^*$ and therefore $\sigma^n(\boldsymbol{s}_i) > \boldsymbol{s}_i$. The case $n = 4|\boldsymbol{u}_r|$ is as before clear as $\tilde{\boldsymbol{u}}_r < \boldsymbol{u}'_r$, concluding the proof of the lower inequality of (12.1). □

Lemma 12.12 *An $e_{q,i}$-minimal prefix is an $e_{q,i}$-minimal sequence.*

Proof: It is clear that the statement holds in the infinite case. Let $\boldsymbol{c}$ be the finite $e_{q,i}$-minimal prefix of the sequence $\boldsymbol{s}$, i.e. $\boldsymbol{c} = g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s})$. We have to show that the $e_{q,i}$-minimal prefix of $\boldsymbol{c}$ is $\boldsymbol{c}$ itself, that is, $\boldsymbol{c} = g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})$.

If $m_{\boldsymbol{c}} = m_{\boldsymbol{s}}$ then Lemma 12.7 gives that $k_{\boldsymbol{s}} = k_{\boldsymbol{c}}$ or that $k_{\boldsymbol{s}} = k_{\boldsymbol{c}} - 1$. From the definition of $e_{q,i}$-minimal prefixes we have

$$e(g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(\boldsymbol{s}), q) \leq \boldsymbol{s} \leq \boldsymbol{c} \leq g_{m_{\boldsymbol{s}},k_{\boldsymbol{s}}}(s)^{\infty}.$$

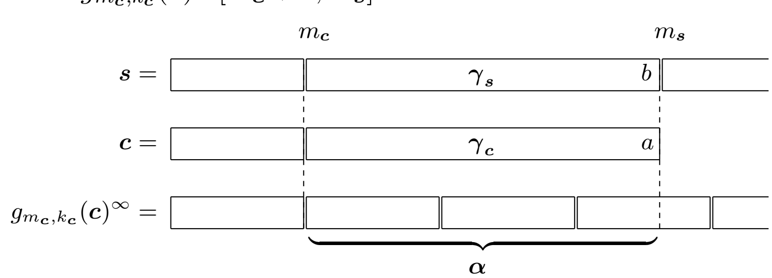
Hence by definition we must have $k_{\boldsymbol{s}} = k_{\boldsymbol{c}}$.

If $m_{\boldsymbol{c}} < m_{\boldsymbol{s}}$ and $c_{m_{\boldsymbol{c}}} = k_{\boldsymbol{c}} - 1$ then by definition of an $e_{q,i}$-minimal prefix we have

$$e(g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c}), q) < \boldsymbol{c}[1, m_{\boldsymbol{c}}] \leq \boldsymbol{s} < g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})^{\infty},$$

contradicting $\boldsymbol{c}$ being the $e_{q,i}$-minimal prefix of $\boldsymbol{s}$.

For the case $c_{m_{\boldsymbol{c}}} = k_{\boldsymbol{c}} =: a$ and $m_{\boldsymbol{c}} < m_{\boldsymbol{s}}$ we consider two cases. First, if $s_{m_{\boldsymbol{s}}} = k_{\boldsymbol{s}} =: b$ then let $\boldsymbol{\gamma}_s = \boldsymbol{s}[m_{\boldsymbol{c}} + 1, m_{\boldsymbol{s}}]$, $\boldsymbol{\gamma}_c = \boldsymbol{c}[m_{\boldsymbol{c}} + 1, m_{\boldsymbol{s}}]$ and $\boldsymbol{\alpha} = g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})^{\infty}[m_{\boldsymbol{c}} + 1, m_{\boldsymbol{s}}]$.



It is clear that $\boldsymbol{\gamma}_s < \boldsymbol{\gamma}_c$. As $\boldsymbol{c}$ is the $e_{q,i}$-minimal prefix of $\boldsymbol{s}$ we have $\boldsymbol{\alpha} \leq \boldsymbol{\gamma}_{\boldsymbol{s}}$ and as $g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})$ is the $e_{q,i}$-minimal prefix of $\boldsymbol{c}$ we have $\boldsymbol{\alpha} \geq \boldsymbol{\gamma}_c$. Hence $\boldsymbol{\alpha} \leq \boldsymbol{\gamma}_s < \boldsymbol{\gamma}_c \leq \boldsymbol{\alpha}$, a contradiction.

Secondly, if $s_{m_{\boldsymbol{c}}} = k_{\boldsymbol{s}}$ then by Lemma 12.11 $m_{\boldsymbol{s}}$ coincides with the integer $n_{\boldsymbol{s}}$ defined by $n_{\boldsymbol{s}} = \inf\{n \geq 1 : \boldsymbol{s}[1,n]^\infty \geq \boldsymbol{s}\}$ in Definition 4.2. From Lemma 12.11 we have $n_{\boldsymbol{s}} \geq 2^i$. Lemma 4.4 now implies that $m_{\boldsymbol{c}} = m_{\boldsymbol{s}}$, concluding the proof. □

Lemma 12.13 *An $e_{q,i}$-minimal sequence is shift-bounded.*

Proof: Let $\boldsymbol{s}$ be an $e_{q,i}$-minimal sequence. From Lemma 4.4 and Lemma 12.11 we have that $n_{\boldsymbol{s}} \geq 2^i$. But as $\boldsymbol{s}$ is an $e_{q,i}$-minimal sequence we have also that $\boldsymbol{s} > g_{n,s_n}(\boldsymbol{s})^\infty$ for $2^i \leq n < |\boldsymbol{s}|$. Hence $\boldsymbol{s} > \boldsymbol{s}[1,n]^\infty$ for $0 < n < |\boldsymbol{s}|$, which by Lemma 4.4 implies $\sigma^n(\boldsymbol{s}) > \boldsymbol{s}$ for $0 < n < |\boldsymbol{s}|$.

For the upper bounding inequality in the definition of shift-boundedness we have by Lemma 12.11 that $\boldsymbol{s}' > \sigma^n(\boldsymbol{s})$ for $0 < n < 2^i$. Moreover, by the $e_{q,i}$-minimality of $\boldsymbol{s}$ we have that $e(g_{n,s_n+1}(\boldsymbol{s}), q) > \boldsymbol{s}$ for $2^i \leq n < |\boldsymbol{s}|$. For $2^i \leq n < |\boldsymbol{s}|$ let $\boldsymbol{a} = \boldsymbol{s}[1,n]$. Then $e(g_{n,s_n+1}(\boldsymbol{s}), q) = \boldsymbol{a}((\hat{\boldsymbol{a}})^*)^\infty$ and $\boldsymbol{s} = \boldsymbol{ab}$ for some sequence $\boldsymbol{b}$ such that $(\hat{\boldsymbol{a}})^* > \boldsymbol{b}$. This implies $\boldsymbol{s}' \geq \boldsymbol{a}^* > (\hat{\boldsymbol{a}})^* > \boldsymbol{b} = \sigma^n(\boldsymbol{s})$. □

Example 12.14 A shift-bounded $e_{q,i}$-sequence does not have to be an $e_{q,i}$-minimal sequence. The binary $e_{2,1}$-sequence 000111 is shift-bounded but not an $e_{2,1}$-minimal sequence, it has the $e_{2,1}$-minimal prefix 001. □

Lemma 12.15 *Let $\boldsymbol{s}$ be a finite $e_{q,i}$-minimal sequence and let $\boldsymbol{s} = \boldsymbol{uvu}^*$ where $p(\boldsymbol{s}, q) = \widehat{\boldsymbol{uv}}$. Put $\boldsymbol{a}_k = \tilde{\boldsymbol{s}}(\boldsymbol{s}^*)^k\boldsymbol{u}^*$ for $k > 0$. Then the $\boldsymbol{a}_k$'s are $e_{q,i}$-minimal and $\boldsymbol{a}_k \nearrow e(\boldsymbol{s}, q)$ when k tends to infinity.*

Proof: We have to show that the inequalities

$$e(g_{n,c}(\boldsymbol{a}_k), q) > \boldsymbol{a}_k, \tag{12.2}$$

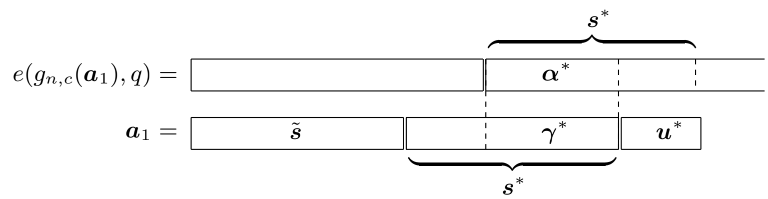
where $c = (\boldsymbol{a}_k)_n + 1$, and

$$\boldsymbol{a}_k > g_{n,c}(\boldsymbol{a}_k)^\infty, \tag{12.3}$$

where $c = (\boldsymbol{a}_k)_n$, holds for $2^i \leq n < |\boldsymbol{a}_k|$. The $e_{q,i}$-minimality of $\boldsymbol{s}$ gives directly that (12.2) and (12.3) holds for $2^i \leq n < |\boldsymbol{s}|$. For $n = r|\boldsymbol{s}|$ with

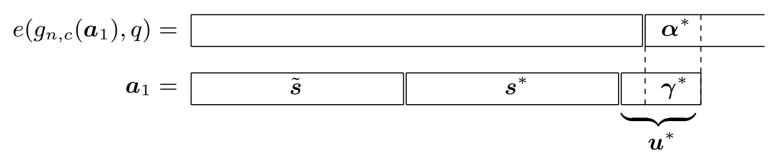
$1 < r < k$ we have that (12.2) holds as $(\tilde{\boldsymbol{s}})^* > \boldsymbol{s}^* \geq \boldsymbol{u}^*$ and that (12.3) holds because $\boldsymbol{s}^* > \tilde{\boldsymbol{s}}$. For the remaning cases let us first turn to the inequality (12.2).

For $j|\boldsymbol{s}| < n < (j+1)|\boldsymbol{s}|$ with $0 < j < k$ let $c = (\boldsymbol{a}_k)_n + 1$, $\boldsymbol{\alpha}^* = e(g_{n,c}(\boldsymbol{a}_k, q)[n+1, j|\boldsymbol{s}|]$ and $\boldsymbol{\gamma}^* = \boldsymbol{a}_k[n+1, j|\boldsymbol{s}|]$.



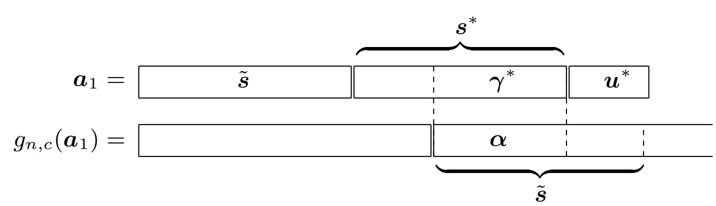
As $\boldsymbol{s}$ is shift-bounded we have $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}^*$ and therefore (12.2) holds.

For $k|\boldsymbol{s}| < n < |\boldsymbol{a}_k|$ let $c = (\boldsymbol{a}_k)_n + 1$, $\boldsymbol{\alpha}^* = e(g_{n,c}(\boldsymbol{a}_k, q)[n+1, |\boldsymbol{a}_k|]$ and $\boldsymbol{\gamma}^* = \boldsymbol{a}_k[n+1, |\boldsymbol{a}_k|]$.



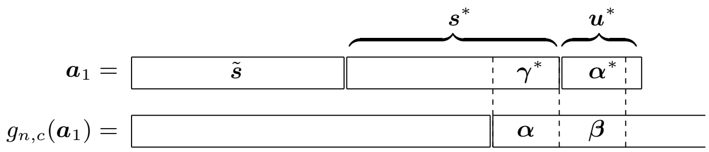
The shift-boundedness of $\boldsymbol{s}$ and the definition of $\boldsymbol{u}$ gives $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}^*$ and hence (12.2) holds.

Now let us turn to the inequality (12.3). Let $0 < j < k$. Then for $j|\boldsymbol{s}| < n < (j+1)|\boldsymbol{s}| - |\boldsymbol{u}|$ let $c = (\boldsymbol{a}_k)_n$, $\boldsymbol{\alpha} = g_{n,c}(\boldsymbol{a}_k)^\infty[n+1, (j+1)|\boldsymbol{s}|]$ and $\boldsymbol{\gamma} = \boldsymbol{a}_k[n+1, (j+1)|\boldsymbol{s}|]$.



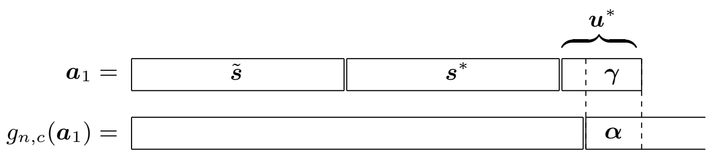
By the shift-boundedness of $\boldsymbol{s}$ and the definition of $\boldsymbol{u}$ we must have $\boldsymbol{\gamma}^* > \boldsymbol{\alpha}$, which implies (12.3).

For $(j+1)|\boldsymbol{s}| - |\boldsymbol{u}| \leq n < (j+1)|\boldsymbol{s}|$ we let $c = (\boldsymbol{a}_k)_n$, $\boldsymbol{\alpha} = g_{n,c}(\boldsymbol{a}_k)^\infty[n, (j+1)|\boldsymbol{s}|]$, $\boldsymbol{\gamma} = \boldsymbol{a}_k[n+1, (j+1)|\boldsymbol{s}|]$ and $\boldsymbol{\beta} = g_{n,c}(\boldsymbol{a}_k)^\infty[(j+1)|\boldsymbol{s}|+1, 2(j+1)|\boldsymbol{s}| - n]$.



Again, as $\boldsymbol{s}$ is shift-bounded we have $\boldsymbol{\gamma}^*\boldsymbol{\alpha}^* > \boldsymbol{\alpha\beta}$ since $\boldsymbol{\gamma}^* \geq \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^* > \boldsymbol{\beta}$, which gives (12.3).

For $k|\boldsymbol{s}| < n < |\boldsymbol{a}_k|$ let $c = (\boldsymbol{a}_k)_n$, $\boldsymbol{\gamma} = \boldsymbol{a}_k[n+1, |\boldsymbol{a}_k|]$ and $\boldsymbol{\alpha} = g_{n,c}(\boldsymbol{a}_k)^\infty[n+1, |\boldsymbol{a}_k|]$.



Then as $\boldsymbol{s}$ is shift-bounded and having $\boldsymbol{u}^*$ as a suffix we must have $\boldsymbol{\gamma} > \boldsymbol{\alpha}$, which again gives (12.3) and completing the proof. $\square$

Lemma 12.16 *Let $\boldsymbol{s}$ be a finite $e_{q,i}$-minimal sequence. Define $\boldsymbol{b}_k = \boldsymbol{s}^k p(\boldsymbol{s}, q)$ for $k > 0$. Then the $\boldsymbol{b}_k$'s are $e_{q,i}$-minimal and $\boldsymbol{b}_k \searrow \boldsymbol{s}^\infty$ when k tends to infinity.*

Proof: We have to show that the inequalities

$$e(g_{n,c}(\boldsymbol{b}_k), q) > \boldsymbol{b}_k, \tag{12.4}$$

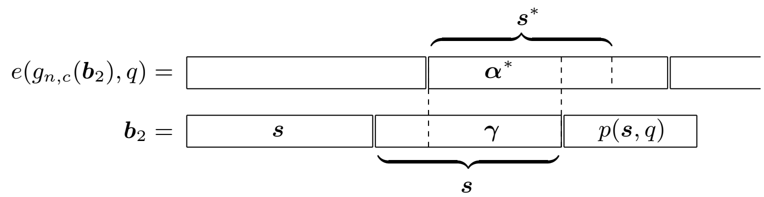
where $c = (\boldsymbol{b}_k)_n + 1$, and

$$\boldsymbol{b}_k > g_{n,c}(\boldsymbol{b}_k)^\infty, \tag{12.5}$$

where $c = (\boldsymbol{b}_k)_n$, holds for $2^i \leq n < |\boldsymbol{b}_k|$. The $e_{q,i}$-minimality of $\boldsymbol{s}$ gives that (12.4) and (12.5) holds for $2^i \leq n < |\boldsymbol{s}|$. Moreover the definition of the function p implies (12.5) holds for all n. Hence we only have to deal with (12.4).

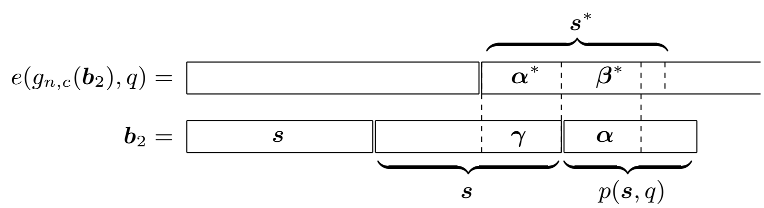
For $n = r|\boldsymbol{s}|$ with $1 < r < k$ we have that (12.4) holds as $(\hat{\boldsymbol{s}})^* > \boldsymbol{s} \geq p(\boldsymbol{s}, q)$. Note that we only have to consider the case when $\hat{\boldsymbol{s}}$ is defined.

For $j|\boldsymbol{s}| < n < j|\boldsymbol{s}| + |p(\boldsymbol{s}, q)|$ where $0 < j < k$ let $c = (\boldsymbol{b}_k)_n + 1$, $\boldsymbol{\alpha}^* = e(g_{n,c}(\boldsymbol{b}_k), q)[n + 1, (j + 1)|\boldsymbol{s}|]$ and $\boldsymbol{\gamma} = \boldsymbol{b}_k[n + 1, (j + 1)|\boldsymbol{s}|]$.



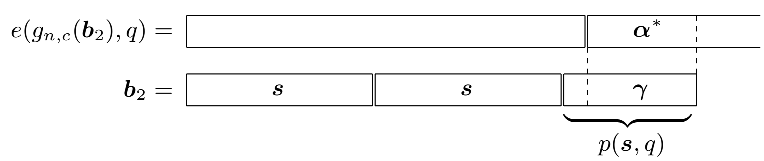
From the shift-boundedness of $\boldsymbol{s}$ and the definition of $p(\boldsymbol{s}, q)$ we have that $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}$, which implies (12.4).

For $j|\boldsymbol{s}| + |p(\boldsymbol{s}, q)| \leq n < (j+1)|\boldsymbol{s}|$ where $0 < j < k$ let $c = (\boldsymbol{b}_k)_n + 1$, $\boldsymbol{\alpha}^* = e(g_{n,c}(\boldsymbol{b}_k), q)[n + 1, (j + 1)|\boldsymbol{s}|]$, $\boldsymbol{\gamma} = \boldsymbol{b}_k[n + 1, (j + 1)|\boldsymbol{s}|]$ and $\boldsymbol{\beta}^* = e(g_{n,c}(\boldsymbol{b}_k), q)[(j + 1)|\boldsymbol{s}| + 1, (j + 1)|\boldsymbol{s}| - n]$.



Again, as $\boldsymbol{s}$ is shift-bounded we have $\boldsymbol{\alpha}^*\boldsymbol{\beta}^* > \boldsymbol{\gamma}\boldsymbol{\alpha}$ since $\boldsymbol{\alpha}^* \geq \boldsymbol{\gamma}$ and $\boldsymbol{\beta}^* > \boldsymbol{\alpha}$, which gives (12.4).

For $k|\boldsymbol{s}| < n < |\boldsymbol{b}_k|$ let $c = (\boldsymbol{b}_k)_n + 1$, $\boldsymbol{\alpha}^* = e(g_{n,c}(\boldsymbol{b}_k), q)[n + 1, |\boldsymbol{b}_k|]$ and $\boldsymbol{\gamma} = \boldsymbol{b}_k[n + 1, |\boldsymbol{b}_k|]$.



Since $p(\boldsymbol{s}, q)$ is shift-bounded we have that $\boldsymbol{\alpha}^* \geq \boldsymbol{\gamma}$, which again gives (12.4). □

Lemma 12.17 *Let* $\boldsymbol{c} \in S^*(q)$ *be such that* $[\boldsymbol{c}] \cap F(\boldsymbol{c}, q) \neq \emptyset$ *and let* $\boldsymbol{u} \in S^*(q)$ *be such that* $[\boldsymbol{u}] \cap F(\boldsymbol{c}, q) \neq \emptyset$. *Then there exists* $1 \leq k \leq |\boldsymbol{u}|$ *such that* $[\boldsymbol{u}[1, k]\boldsymbol{c}] \cap F(\boldsymbol{c}, q) \neq \emptyset$

Proof: Assume there exists a smallest k such that $\boldsymbol{u}[k+1, |\boldsymbol{u}|] = \boldsymbol{c}[1, |\boldsymbol{u}| - k+1]$. If we for some $n < k$ would have $\sigma^n(\boldsymbol{u}) = \boldsymbol{c}[1, n-k]$ then we would have a contradiction to the minimality of k. Hence $\sigma^n(\boldsymbol{u}) > \boldsymbol{c}[1, n-k]$ for $n < k$. For any continuation $\boldsymbol{v}$ of $\boldsymbol{u}$ such that $\boldsymbol{uv} \in F(\boldsymbol{c}, q)$ we have

$$\boldsymbol{c}' > \sigma^n\big(\boldsymbol{u}\ \boldsymbol{v}[1, |\boldsymbol{c}| - |\boldsymbol{u}| + k - 1]\big) \geq \sigma^n(\boldsymbol{u}\ \boldsymbol{c}[k+1, |\boldsymbol{c}|]),$$

as $\boldsymbol{c}$ is lexicographically smallest sequence in $F(\boldsymbol{c}, q)$. If $\boldsymbol{c}$ does not overlap $\boldsymbol{u}$ then clearly we must have both $\sigma^n(\boldsymbol{u}) > \boldsymbol{c}$ and $\boldsymbol{c}' > \sigma^n(\boldsymbol{uc})$ for $0 < n < |\boldsymbol{u}|$. □

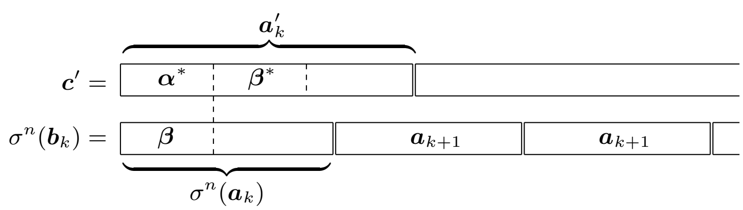
Lemma 12.18 *Let $\boldsymbol{c}$ be a finite $e_{q,0}$- or $e_{q,1}$-minimal sequence starting with the symbol c_1. Put $\boldsymbol{z} = c_1\,(q - c_1 - 1)$. Then there exists a finite sequence $\boldsymbol{w}$ such that $\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^\infty \in F(\boldsymbol{c}, q)$.*

Proof: Let $\boldsymbol{a}_k = p^k(\boldsymbol{c}, q)$ for $0 \leq k \leq N$ where N is such that $\boldsymbol{a}_N = \boldsymbol{z}$, which exists by Lemma 11.8. Now let

$$\boldsymbol{b}_k = \boldsymbol{a}_k\,(\boldsymbol{a}_{k+1})^\infty = \boldsymbol{uvu}^*(\widehat{\boldsymbol{uv}})^\infty.$$

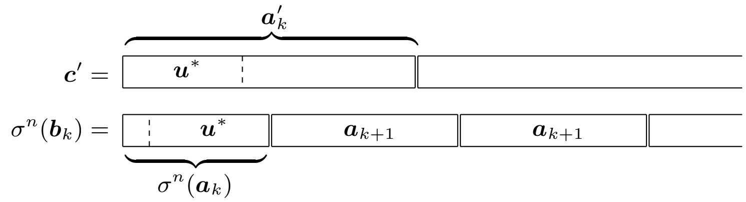
We claim that $\boldsymbol{c}' > \sigma^n(\boldsymbol{b}_k) > \boldsymbol{c}$ for $0 \leq n < |\boldsymbol{a}_k|$. To prove the claim it is enough to prove that it holds for $0 \leq n \leq |\boldsymbol{a}_k|$ as $\boldsymbol{c}' > \sigma^n(\boldsymbol{a}_r^\infty) > \boldsymbol{c}$ for $n \geq 0$ and $0 \leq r \leq N$. The lower inequality, $\sigma^n(\boldsymbol{b}_k) > \boldsymbol{c}$, follows direct from the definition of p. For the upper inequality we start by notice that when $n = 0$ the result follows trivially as $(\boldsymbol{b}_k)_1 < (\boldsymbol{c}')_1$.

For $0 < n < \frac{1}{2}|\boldsymbol{a}_k|$ let $\boldsymbol{\alpha} = \boldsymbol{b}_k[1, n]$ and $\boldsymbol{\beta} = \boldsymbol{b}_k[n+1, 2n]$.

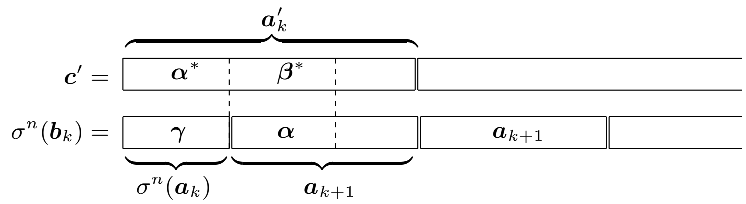


As $\boldsymbol{c}$ is shift-bounded $\boldsymbol{\alpha}^* > \boldsymbol{\beta}$ and therefore $\boldsymbol{c}' > \sigma^n(\boldsymbol{b}_k)$.

For $n = \frac{1}{2}|\boldsymbol{a}_k|$, if $\boldsymbol{v}$ is void and since $|\boldsymbol{a}_k| \geq 2$ the $e_{q,i}$-minimality of $\boldsymbol{c}$ gives $\boldsymbol{c}' > \boldsymbol{u}^*(\tilde{\boldsymbol{u}})^\infty = \sigma^n(\boldsymbol{b}_k)$. (Note that this argument can only be used for $i = 0$ or $i = 1$). If both $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-void then the result follows by the definition of $\boldsymbol{a}_{k+1}$ via p.



For $\frac{1}{2}|\boldsymbol{a}_k| < n < |\boldsymbol{a}_k|$ let $\boldsymbol{\alpha}^* = \boldsymbol{c}'[1, |\boldsymbol{a}_k| - n]$, $\boldsymbol{\beta}^* = \boldsymbol{c}'[|\boldsymbol{a}_k| - n + 1, 2|\boldsymbol{a}_k| - 2n]$ and $\boldsymbol{\gamma} = \boldsymbol{b}_k[n+1, |\boldsymbol{a}_k|]$.



We have $\boldsymbol{\alpha}^* \geq \boldsymbol{\gamma}$ and $\boldsymbol{\beta}^* > \boldsymbol{\alpha}$. If $\boldsymbol{u}$ is void we have directly $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}$. Therefore $\boldsymbol{c}' > \sigma^n(\boldsymbol{b}_k)$, which proves the claim. Put $\boldsymbol{w} = \boldsymbol{a}_1^{n_1} \boldsymbol{a}_2^{n_2} \dots \boldsymbol{a}_N$ with $n_k = [\frac{|\boldsymbol{c}|}{|\boldsymbol{a}_k|}] + 1$. By repeated use of the just proved claim we have $\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^\infty \in F(\boldsymbol{c}, q)$. $\square$

Example 12.19 Lemma 12.18 can not be generalised to hold for some $i \geq 2$ in any base $q \geq 2$. Consider the case $i \geq 2$, $q = 2$ and assume that $\boldsymbol{c}$ is a finite minimal $e_{2,i}$-minimal sequence. Then $\boldsymbol{c}$ must have a prefix $\boldsymbol{p}$ of the form $01(10)^k 11$ for some $k > 0$. If we let $\boldsymbol{u} = 01$ then we have that $\boldsymbol{p} = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^k \boldsymbol{u}'$. Lemma 10.2 gives that we can never find a sequence $\boldsymbol{w}$ such that $\boldsymbol{c}\boldsymbol{w}(01)^\infty$ is a sequence in $F(\boldsymbol{c}, 2)$.

Similarly, for $i \geq 2$, $q = 3$ we can assume that $\boldsymbol{c}$ is a finite minimal $e_{3,i}$-minimal sequence. Then $\boldsymbol{c}$ must have a prefix $\boldsymbol{p}$ of the form $02(20)^k 21$ for some $k > 0$. If we let $\boldsymbol{u} = 02$ then we have that $\boldsymbol{p} = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^k \boldsymbol{u}'$. Lemma 10.2 gives that we can never find a sequence $\boldsymbol{w}$ such that $\boldsymbol{c}\boldsymbol{w}(02)^\infty$ is a sequence in $F(\boldsymbol{c}, 3)$.

In Section 16 we give Theorem 16.5 by which we can generalise this example to any $q \geq 2$. $\square$

Proposition 12.20 *Let $\boldsymbol{c}$ be a finite shift-bounded $e_{q,i}$-sequence for $i \geq 1$. Then there exists an $n > 0$ such that $p^n(\boldsymbol{c}, q) = f^i([\frac{q}{2}], q)$.*

Proof: Let us use the notation $\boldsymbol{w}_k = f^k([\frac{q}{2}], q)$ for $k \geq 0$. Then $|\boldsymbol{w}_k| = 2^k$. Assume for contradiction that there is an n such that

$$p^n(\boldsymbol{c}, q) < \boldsymbol{w}_i < p^{n+1}(\boldsymbol{c}, q). \tag{12.6}$$

We claim that the above assumption gives the following chain of inequalities

$$\boldsymbol{e}_{q,i-1} < p^n(\boldsymbol{c}, q) < \boldsymbol{e}_{q,i} < p^{n+1}(\boldsymbol{c}, q). \tag{12.7}$$

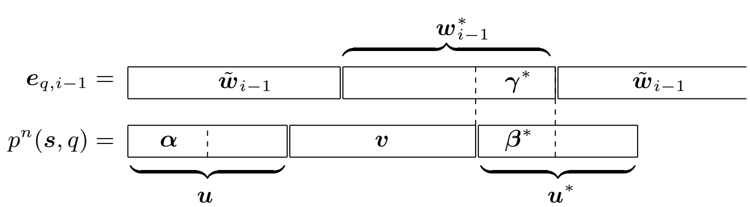
The left-most inequality of (12.7) is clear as $\boldsymbol{c}$ is an $e_{q,i}$-sequence and therefore $\boldsymbol{e}_{q,i-1} < \boldsymbol{c} < p^n(\boldsymbol{c}, q)$. The right-most inequality of (12.7) is given by our assumption $\boldsymbol{e}_{q,i} < \boldsymbol{w}_i < p^{n+1}(\boldsymbol{c}, q)$. For the middle inequality of the claim (12.7), assume that $p^n(\boldsymbol{c}, q) > \boldsymbol{e}_{q,i}$. Then $p^n(\boldsymbol{c}, q) = \tilde{\boldsymbol{w}}_i \boldsymbol{s}$ for some non-empty sequence $\boldsymbol{s}$ with $|\boldsymbol{s}| \leq |\boldsymbol{w}_i|$ and $\boldsymbol{s} > \boldsymbol{w}_i^*$. (If $\boldsymbol{s}$ would be empty then $p^n(\boldsymbol{c}, q) < \boldsymbol{e}_{q,i-1}$ since $\boldsymbol{e}_{q,i-1}[1, 2^i] = \boldsymbol{e}_{q,i}[1, 2^i]$). Thus

$$p^n(\boldsymbol{c}, q)' = \boldsymbol{w}_i^* \boldsymbol{s}' < \sigma^{|\boldsymbol{w}_i|}(p^n(\boldsymbol{c}, q)) = \boldsymbol{s},$$

contradicting the shift-boundedness of $p^n(\boldsymbol{c}, q)$.

The assumption (12.6) also gives $|p^{n+1}(\boldsymbol{c}, q)| < |\boldsymbol{w}|_i$, since otherwise $\boldsymbol{w}_i$ would be a prefix of $p^n(\boldsymbol{c}, q)$, and from the equality $\boldsymbol{e}_{q,i-1}[1, 2^i] = \boldsymbol{e}_{q,i}[1, 2^i]$ and (12.7) we also have $|\boldsymbol{w}|_i < |p^n(\boldsymbol{c}, q)|$.

Let $p^n(\boldsymbol{c}, q) = \boldsymbol{u}\boldsymbol{v}\boldsymbol{u}^*$ where $p^{n+1}(\boldsymbol{c}, q) = \widehat{\boldsymbol{u}\boldsymbol{v}}$. Put $\boldsymbol{\alpha} = p^n(\boldsymbol{c}, q)[1, 2^i - |\boldsymbol{u}\boldsymbol{v}|]$, $\boldsymbol{\beta}^* = p^n(\boldsymbol{c}, q)[|\boldsymbol{u}\boldsymbol{v}| + 1, 2^i]$ and $\boldsymbol{\gamma}^* = \boldsymbol{e}_{i-1}[|\boldsymbol{u}\boldsymbol{v}| + 1, 2^i]$.



By the definition of p we have $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and as $\boldsymbol{w}_i$ is shift-bounded we have $\boldsymbol{\alpha}^* > \boldsymbol{\gamma}^*$, that is, $\boldsymbol{\gamma} \neq \boldsymbol{\beta}$, a contradiction. □

13 The Set $\mathcal{A}$

Let $\boldsymbol{u} \in S^*(q)$ be a finite sequence not ending with 0 and let $\mathcal{A}(\boldsymbol{u})$ be the set of infinite sequences on the alphabet $\{\tilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{u}^*, \boldsymbol{u}'\}$ defined by the transition matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

where the rows and columns are ordered in the order $\tilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{u}^*, \boldsymbol{u}'$. The elements of $\mathcal{A}(\boldsymbol{u})$ are sequences similar to the suffix given in (10.1) but where the upper bounding k has been removed. Note that the transition matrix A is primitive and has the spectral radius $\rho(A) = 2$. By Proposition 2.6 we have

Lemma 13.1 *Let $\boldsymbol{u} \in S^*(q)$ be a finite sequence not ending with a 0. Then*

$$\dim_H \mathcal{A}(\boldsymbol{u}) = \frac{1}{|\boldsymbol{u}|}\frac{\log 2}{\log q}.$$

For the special case when having $\boldsymbol{u} = f^i(1,2)$ we have $\mathcal{A}(f^i(1,2)) = S^\infty(2)$ for $i = 0$ and for $i \geq 0$ we have the nested inclusions

$$\mathcal{A}(f^{i+1}(1,2)) \subset \mathcal{A}(f^i(1,2)). \tag{13.1}$$

Definition 13.2 *Let $\boldsymbol{u} \in S^*(q)$ be a finite sequence not ending with* 0 *and let*

$$\mu_{\boldsymbol{u}} : \mathcal{A}(\boldsymbol{u}) \to S^\infty(2)$$

be the map $(\tilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{u}^*, \boldsymbol{u}') \mapsto (0,1,0,1)$. Let $\mu_{\boldsymbol{u}}^{-1}$ map the first 0 in each block of zeros to $\tilde{\boldsymbol{u}}$ otherwise 0 is mapped to $\boldsymbol{u}^*$ and let the first 1 in each block of ones be mapped to $\boldsymbol{u}'$ otherwise 1 mapped to $\boldsymbol{u}$.

Note that μ could equally have been defined as a function between sets of finite sequences, that is, $\mu_{\boldsymbol{u}} : \{\boldsymbol{x}[1, n|\boldsymbol{u}|] : \boldsymbol{x} \in \mathcal{A}(\boldsymbol{u})\} \to \{\boldsymbol{x}[1,n] : \boldsymbol{x} \in S^\infty(2)\}$.

A function T similar to our $\mu_{\boldsymbol{u}}$ is defined by Allouche in [1]. The function T is there used to show that the set Γ, defined in (9.4) is self similar.

The function $\mu_{\boldsymbol{u}} : \mathcal{A}(\boldsymbol{u}) \to S^\infty(2)$ is not bijective as for $U_1 = [\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u})$ and $U_2 = [\boldsymbol{u}'] \cap \mathcal{A}(\boldsymbol{u})$ we have $\mu_{\boldsymbol{u}}^{-1}(S^\infty(2)) = U_1 \cup U_2$, where the right-hand-side clearly is a proper subset of $\mathcal{A}(\boldsymbol{u})$. The violation of the bijectivity is however only in the first positions, so by shifting these out we have

$$\sigma^{|\boldsymbol{u}|}\left(\mu_{\boldsymbol{u}}^{-1}(S^{\infty}(2))\right) = \mathcal{A}(\boldsymbol{u}).$$

If we restrict μ to map sequences from $[\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u})$ into the cylinder-set $[0]$ we obtain a bijection as the sequences causing a collisions due to the definition of the inverse of μ have been removed.

Example 13.3 Let $\boldsymbol{u} = 01$. Then $\boldsymbol{c}_1 = 0011$ and $\boldsymbol{c}_2 = 1011$ are prefixes of sequences in $\mathcal{A}(\boldsymbol{u})$. We have $\mu_{\boldsymbol{u}}(0011) = 01$ and $\mu_{\boldsymbol{u}}(1011) = 01$, but $\mu_{\boldsymbol{u}}^{-1}(01) = 0011$. □

Lemma 13.4 *Let* $\boldsymbol{u} \in S^*(q)$ *be a finite sequence not ending with* 0 *such that* $\boldsymbol{u} \leq \boldsymbol{u}'$. *Put* $U = [\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u})$ *and* $V = [0]$, *(or* $U = ([\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u}))[1, |\boldsymbol{u}|n]$ *and* $V = ([0])[1, n]$ *in the case of finite sequeces). Then* $\mu_{\boldsymbol{u}} : U \to V$ *is bijective and order-preserving.*

Proof: The bijectivity is clear from the just above reasoning of the definition of the inverse of μ. For the order preservation let $\boldsymbol{c}_1 < \boldsymbol{c}_2$ be two sequences in U and let $\boldsymbol{s}_1 = \mu(\boldsymbol{c}_1)$ and $\boldsymbol{s}_2 = \mu(\boldsymbol{c}_2)$. Assume for contradiction that $\boldsymbol{s}_1 > \boldsymbol{s}_2$. There is a smallest n such that $(\boldsymbol{s}_1)_n = 1$ and $(\boldsymbol{s}_2)_n = 0$. Let $\boldsymbol{w}_1 = \boldsymbol{c}_1[n|\boldsymbol{u}| + 1, (n+1)|\boldsymbol{u}|]$ and $\boldsymbol{w}_2 = \boldsymbol{c}_2[n|\boldsymbol{u}| + 1, (n+1)|\boldsymbol{u}|]$. That is, $\boldsymbol{w}_1$ is the subsequence in $\boldsymbol{c}_1$ mapped into $(\boldsymbol{s}_1)_n$ by $\mu_{\boldsymbol{u}}$, and similarly for $\boldsymbol{w}_2$. If $(\boldsymbol{s}_1)_{n-1} = 1$ then $\boldsymbol{w}_1 = \boldsymbol{u}$ and $\boldsymbol{w}_2 = \tilde{\boldsymbol{u}}$, contradicting $\boldsymbol{c}_1 < \boldsymbol{c}_2$. If $(\boldsymbol{s}_1)_{n-1} = 0$ then $\boldsymbol{w}_1 = \boldsymbol{u}'$ and $\boldsymbol{w}_2 = \boldsymbol{u}^*$, again contradicting $\boldsymbol{c}_1 < \boldsymbol{c}_2$. Finally, if $n = 1$ then $\boldsymbol{w}_1 = \boldsymbol{u}'$ and $\boldsymbol{w}_2 = \tilde{\boldsymbol{u}}$, then similarly this would imply $\boldsymbol{c}_1 < \boldsymbol{c}_2$. □

14 Dyadic Approximation

In this section we will turn our interest to the special case when having $q = 2$, the binary case. A central role will be played by the classical Thue-Morse sequence, see [4, 21, 22].

Definition 14.1 (Thue-Morse sequence) *The sequence* $\boldsymbol{t}$ *recursively defined by* $t_1 = 0$ *and* $t_{2n+1} = t_{n+1}$, $t_{2n+2} = t_{n+1}^*$, *is called the Thue-Morse sequence.*

The first entries of the Thue- Morse sequence $\boldsymbol{t}$ and its inverse are easily seen to be

$$\begin{aligned}\boldsymbol{t} &= 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\ldots \\ \boldsymbol{t}' &= 1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\ldots\end{aligned}$$

In [1, 2], Allouche and Cosnard observed that we may obtain the Thue-Morse sequence via the limit under iteration of the function f.

Theorem 14.2 (Allouche, Cosnard) *Let $\boldsymbol{t}$ be the Thue-Morse sequence. Then*

$$\sigma(\boldsymbol{t}') = d(1,2) = \lim_{n\to\infty} f^n(1,2)$$

Moreover the sequence $\sigma(\boldsymbol{t}')$ is shift-bounded.

From Theorem 11.10 we now have the following corollary, also observed by Dekking in [6],

Theorem 14.3 (Dekking, Mahler) *Let $\boldsymbol{t}$ be the Thue-Morse sequence. Then the Thue-Morse constant $\tau = \sum_{i=1}^{\infty} \frac{t_i}{2^{i+1}} = 0.41245403\ldots$ is transcendental.*

Theorem 14.4 (Allouche, Cosnard) *Let $\boldsymbol{t}$ be Thue-Morse sequence. Then $F(\boldsymbol{c},2)$ is countable if and only if $\boldsymbol{c} > \sigma(\boldsymbol{t}')$.*

We can say more than what stated in Theorem 14.4 about what set $F(\boldsymbol{c},2)$ looks like for $\boldsymbol{c} > \sigma(\boldsymbol{t}')$. For $\boldsymbol{c} > (01)^\infty$ we have $F(\boldsymbol{c},2) = \emptyset$ and $F(\boldsymbol{c},2) = \{(01)^\infty, (10)^\infty\}$ whenever $\boldsymbol{c} \in [01, (01)^\infty]$. By the technique of Lemma 11.16 we see that $F(\boldsymbol{c},2)$ is infinite, but countable, when $\sigma(\boldsymbol{t}') < \boldsymbol{c} < 01$.

The Thue-Morse sequence $\boldsymbol{t}$ also appears when looking at limit points in sets of the form $\{\|\xi\alpha^n\|\}$, for an algebraic number α and a real ξ. For more of this see Dubickas [7].

In [14], Moreira improved Theorem 14.4 and showed that the sequence $\sigma(\boldsymbol{t}')$ also is the threshold for the dimension of $F(\boldsymbol{c},2)$.

Theorem 14.5 (Moreira) *Let $\boldsymbol{t}$ be the Thue-Morse sequence. Then we have $\dim_H F(\boldsymbol{c},2) = 0$ if and only if $\boldsymbol{c} \geq \sigma(\boldsymbol{t}')$.*

We will generalise the result by Allouche, Cosnard and Moriera and show that the Thue-Morse sequence $\boldsymbol{t}$ is connected to the threshold for the dimension of $F(\boldsymbol{c}, q)$ for any $q \geq 2$, see Corollary 15.10 and Theorem 16.9.

Lemma 14.6 *For $\boldsymbol{u} = f^{i-1}(1,2)$ where $i \geq 1$ let $U = [\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u})$. If $\boldsymbol{c}$ is an infinite shift-bounded $e_{2,i}$-sequence then $\boldsymbol{c} \in U$. If $\boldsymbol{c}$ is a finite shift-bounded $e_{2,i}$-sequence then $\boldsymbol{c}$ is a prefix of a sequences in U and $|\boldsymbol{c}| = k \cdot 2^{i-1}$ for some $k \geq 3$.*

Proof: There is a maximal N and a sequence $\boldsymbol{v}$ such that $\boldsymbol{c} = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^N \boldsymbol{v}$ with $\boldsymbol{v} > \boldsymbol{u}^*$ as an $e_{2,i}$-sequence must start with $\tilde{\boldsymbol{u}}(\boldsymbol{u}^*)$. By shifting $n = (1+N)|\boldsymbol{u}|$ times we obtain from $\boldsymbol{c}' = \boldsymbol{u}'\boldsymbol{u}^N\boldsymbol{v}' > \sigma^n(\boldsymbol{c}) = \boldsymbol{v}'$ that $\boldsymbol{u}'$ must be a prefix of $\boldsymbol{v}$. Hence Lemma 10.2 gives that $\boldsymbol{c} \in U$ if $\boldsymbol{c}$ is infinite or that $\boldsymbol{c}$ is the prefix of a sequence in U if $\boldsymbol{c}$ is finite. Moreover, since $\boldsymbol{u}'$ is a prefix of $\boldsymbol{v}$ we have that $|\boldsymbol{c}| \geq 3|\boldsymbol{u}|$.

For the length of $\boldsymbol{c}$ in the finite case we have to show that $\boldsymbol{u}$ and $\boldsymbol{u}'$ are the only allowed suffixes of $\boldsymbol{c}$ of length $|\boldsymbol{u}|$ and moreover we may not find $\boldsymbol{u}$ or $\boldsymbol{u}'$ by cutting an ending $\tilde{\boldsymbol{u}}$, $\boldsymbol{u}$, $\boldsymbol{u}^*$ or $\boldsymbol{u}'$ off.

The sequence $\boldsymbol{c}$ cannot end with $\tilde{\boldsymbol{u}}$ or $\boldsymbol{u}^*$ as it then would end with a zero, contradicting $\boldsymbol{c}$ being shift-bounded.

If $\boldsymbol{c}$ ends with a prefix $\boldsymbol{v}$ of $\tilde{\boldsymbol{u}}$ then there is an n such that $\sigma^n(\boldsymbol{c}) = \boldsymbol{v} \leq \tilde{\boldsymbol{u}} < \boldsymbol{c}$, contradicting the shift-boundedness of $\boldsymbol{c}$. The same procedure holds for a proper prefix of $\boldsymbol{u}$.

If $\boldsymbol{c}$ ends with a proper prefix $\boldsymbol{v}$ of $\boldsymbol{u}^*$ then $\boldsymbol{c}$ must end with $\tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^m\boldsymbol{v}$ for some $0 \leq m$, as $\boldsymbol{c}$ is prefix of a sequence in U. If $\boldsymbol{c} = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^m\boldsymbol{v}$ then it would not be an $e_{2,i}$-sequence. Hence $\boldsymbol{c}$ must end with

$$\tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^r\boldsymbol{u}'(\boldsymbol{u})^s\tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^m\boldsymbol{v}.$$

But then for $n = (2+r+s)|\boldsymbol{u}|$ we have $\sigma^n(\boldsymbol{c}) = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^m\boldsymbol{v} < \boldsymbol{c}$, contradicting the shift-boundedness of $\boldsymbol{c}$. □

Corollary 14.7 *An $e_{2,i}$-minimal sequence for $i \geq 1$ is of length $k \cdot 2^{i-1}$ for some $k \geq 3$.*

Lemma 14.8 *For $\boldsymbol{u} = f^{i-1}(1,2)$ where $i \geq 1$ let $U = [\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u})$ and $V = [0]$, (or $U = ([\tilde{\boldsymbol{u}}] \cap \mathcal{A}(\boldsymbol{u}))[1, |\boldsymbol{u}|n]$ and $V = ([0])[1, n]$ in the case of* *finite sequences). Then $\mu_{\boldsymbol{u}} : U \to V$ is a bijection between $e_{2,i}$-minimal sequences and $e_{2,1}$-minimal sequences.*

Proof: It is clear that an $e_{2,1}$-minimal sequence is a prefix of a sequence in V and by Lemma 14.6 an $e_{2,i}$-minimal sequence is a prefix of a sequences in U.

Let $\boldsymbol{c}$ be a prefix of a sequence in U such that $|\boldsymbol{c}| = k|\boldsymbol{u}|$, for some $k \geq 3$, and let $\boldsymbol{s} = \mu_{\boldsymbol{u}}(\boldsymbol{c})$. Since $g_{n|\boldsymbol{u}|,1}(\boldsymbol{c})$ ends with either $\boldsymbol{u}$ or $\boldsymbol{u}'$ and begins with $\tilde{\boldsymbol{u}}$ it follows that $g_{n|\boldsymbol{u}|,1}(\boldsymbol{c})^\infty$ is an element in U. Similarly we have that $e(g_{n|\boldsymbol{u}|,1}(\boldsymbol{c}), 2)[1, n|\boldsymbol{u}|]$ ends with either $\tilde{\boldsymbol{u}}$ or $\boldsymbol{u}^*$ and since $(g_{n|\boldsymbol{u}|,1}(\boldsymbol{c}))^*$ begins with $\boldsymbol{u}'$ we have that also $e(g_{n|\boldsymbol{u}|,1}(\boldsymbol{c}), 2)$ is an element of U.

Lemma 14.6 gives that we only have to check for minimality of $\boldsymbol{c}$ in prefixes of length $k|\boldsymbol{u}|$ for $k \geq 3$. Assume there is an $n \geq 2|\boldsymbol{u}| = 2^i$ such that

$$e(g_{n|\boldsymbol{u}|,1}(\boldsymbol{c}), 2) \leq \boldsymbol{c} \leq g_{n|\boldsymbol{u}|,1}(\boldsymbol{c})^\infty$$

does not hold. Then the order-preservation of μ gives that

$$e(g_{n,1}(\boldsymbol{s}), 2) \leq \boldsymbol{s} \leq g_{n,1}(\boldsymbol{s})^\infty$$

cannot hold either. □

Theorem 14.9 *For $\boldsymbol{u} = f^{i-1}(1,2)$ for some $i \geq 1$ put $U = \mathcal{A}(\boldsymbol{u})$ and $V = S^\infty(2)$, (or put $U = \mathcal{A}(\boldsymbol{u})[1, |\boldsymbol{u}|n]$ and $V = S^\infty(2)[1,n]$ for the finite case). Let $\mu_{\boldsymbol{u}} : U \to V$ and let $\boldsymbol{c}$ be an $e_{2,i}$-sequence such that $\mu_{\boldsymbol{u}}(c)$ is well defined. Then*

$$\dim_H F(\boldsymbol{c}, 2) = \frac{1}{2^{i-1}} \dim_H F\big(\mu_{\boldsymbol{u}}(\boldsymbol{c}), 2\big).$$

Proof: Let $S = ([\tilde{\boldsymbol{u}}] \cup [\boldsymbol{u}']) \cap F(\boldsymbol{c}, 2)$. By the order-preservation of μ and Lemma 10.2 we have $\mu_{\boldsymbol{u}}(S) = F(\mu_{\boldsymbol{u}}(\boldsymbol{c}), 2)$. Moreover $\mu_{\boldsymbol{u}}^{-1}(\mu_{\boldsymbol{u}}(S)) \subset F(\boldsymbol{c}, 2)$. Hence $\dim_H F(\mu_{\boldsymbol{u}}(\boldsymbol{c}), 2) \leq 2^{i-1} \dim_H F(\boldsymbol{c}, 2)$.

For the reversed inequality, let $\boldsymbol{x} \in F(\boldsymbol{c}, 2)$. If $\boldsymbol{x}$ does not contain 00 nor 11 then $\boldsymbol{x}$ is either of the sequences $(01)^\infty$ or $(10)^\infty$. If $\boldsymbol{x}$ does contain two consecutive zeros or ones then Lemma 10.2 gives that $\boldsymbol{x}$

ends with a sequence which is an element in $\mathcal{A}(f^1(1,2))$. Therefore, and by the use of the nested inclusion (13.1), we have that

$$\bigcup_{n=1}^{\infty}\left\{\boldsymbol{v}[1,n]\boldsymbol{w} : \boldsymbol{v}\in F(f^i(1,2),2),\ \boldsymbol{w}\in \bigcup_{k=|\boldsymbol{u}|}^{2|\boldsymbol{u}|-1}\sigma^k\Big(\mu_{\boldsymbol{u}}^{-1}\big(F(\mu_{\boldsymbol{u}}(\boldsymbol{c}),2)\big)\Big)\right\},$$

contains $F(\boldsymbol{c},2)$, which implies the desired inequality. □

Corollary 14.10 *Let $i \geq 0$. Then*

$$\dim_H F(\boldsymbol{e}_{2,i},2) = \frac{1}{2^i}.$$

By Corollary 14.7 we directly obtain Moreira's Theorem 14.5.

15 Triadic Approximation

The Thue-Morse sequence $\boldsymbol{t}$ has the property,

Theorem 15.1 (Thue) *The Thue-Morse sequence $\boldsymbol{t}$ does not contain a subsequence of the form $a\boldsymbol{s}a\boldsymbol{s}a$, where $a\in\{0,1\}$ and $\boldsymbol{s}\in S^*(2)$. That is, $\boldsymbol{t}$ is said to be overlap-free.*

From $\boldsymbol{t}$ Thue constructed the sequence $\boldsymbol{v}$ as follows: for $n \geq 1$ let v_n be the number of 1's between the n'th and $(n+1)$'st occurrence of 0 in the Thue-Morse sequences $\boldsymbol{t}$. Hence

$$\boldsymbol{v} = 2\,1\,0\,2\,0\,1\,2\,1\,0\,1\,2\ldots \tag{15.1}$$

From Theorem 15.1 it is clear that $\boldsymbol{v}$ is a sequence on 3 symbols. Thue used Theorem 15.1 to prove the following result

Theorem 15.2 (Thue) *The sequence $\boldsymbol{v}$ is square free, i.e. it does not contain $\boldsymbol{s}^2$ for any sequence $\boldsymbol{s}$.*

For a proof of Theorem 15.1 and Theorem 15.2 we refer to [21, 22] or see the work of Berstel [5] and references therein.

Definition 15.3 *Let $T : S(3) \to S(2)$ be the map defined by $(0, 1, 2) \mapsto (0, 01, 011)$.*

The map T is the transformation used by Thue to obtain the sequence $\boldsymbol{v}$ from the Thue-Morse sequence $\boldsymbol{t}$, that is, $T(\boldsymbol{v}) = \boldsymbol{t}$.

Lemma 15.4 *Let $i \geq 1$. Then $f^i(1,2) = T(f^{i-1}(1,3))$.*

Proof: The proof is made by induction. As the basis steps we have

$$\begin{aligned} T(f^0(1,3)) &= T(1) = 01 = f^1(1,2) \\ T(f^1(1,3)) &= T(02) = 0011 = f^2(1,2) \\ T(f^2(1,3)) &= T(0121) = 00101101 = f^3(1,2) \end{aligned}$$

Assume that the statement is true for any $i < k$ with $k \geq 3$. For the induction step, $k = i$, let $\boldsymbol{u} = f^{k-3}(1,3)$ and $\boldsymbol{w} = f^{k-2}(1,2)$. Then by the assumption we have $T(\boldsymbol{u}) = \boldsymbol{w}$ and

$$T(f(\boldsymbol{u},3)) = T\big(\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{u}})^*\big) = \tilde{\boldsymbol{w}}(\tilde{\boldsymbol{w}})^* = f(\boldsymbol{w},2).$$

Since $\boldsymbol{u}$ ends with a non-zero symbol, as it is shift-bounded, the assumption gives that $T(\tilde{\boldsymbol{u}}) = \boldsymbol{w}[1, |\boldsymbol{w}| - 1]$ and therefore

$$T\big(\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{u}})^*\big) = \boldsymbol{w}[1, |\boldsymbol{w}| - 1]0(\tilde{\boldsymbol{w}})^* = \tilde{\boldsymbol{w}}(\tilde{\boldsymbol{w}})^*.$$

This implies that we must have $T((\tilde{\boldsymbol{u}})^*) = 0(\tilde{\boldsymbol{w}})^*$, but then also $T(\boldsymbol{u}^*) = 0\ \boldsymbol{w}^*[1, |\boldsymbol{w}| - 1]$. Thus, we have

$$\begin{aligned} T(f^{k-1}(1,3)) &= T(f^2(\boldsymbol{u},3)) \\ &= T\big(\tilde{\boldsymbol{u}}\,\boldsymbol{u}^*(\tilde{\boldsymbol{u}})^*\boldsymbol{u}\big) \\ &= T\big(\tilde{\boldsymbol{u}}\big)\,T\big(\boldsymbol{u}^*\big)T\big((\tilde{\boldsymbol{u}})^*\big)T\big(\boldsymbol{u}\big) \\ &= \boldsymbol{w}[1, |\boldsymbol{w}| - 1]\,0\ \boldsymbol{w}^*[1, |\boldsymbol{w}| - 1]\,0(\tilde{\boldsymbol{w}})^*\,\boldsymbol{w} \\ &= \tilde{\boldsymbol{w}}\,\boldsymbol{w}^*(\tilde{\boldsymbol{w}})^*\boldsymbol{w} \\ &= f^2(\boldsymbol{w},2) \\ &= f^k(1,2), \end{aligned}$$

which completes the proof. □

Corollary 15.5 *Let $i \geq 1$. Then $T(\boldsymbol{e}_{3,i-1}) = \boldsymbol{e}_{2,i}$.*

Proof: Let $\boldsymbol{u} = f^{i-1}(1,3)$ and $\boldsymbol{w} = f^i(1,2)$. Then by Lemma 15.4 we have that $T(\boldsymbol{u}) = \boldsymbol{w}$. By the same arguments as in the proof of Lemma 15.4 we have $T(\tilde{\boldsymbol{u}}) = \boldsymbol{w}[1, |\boldsymbol{w}|-1]$ and $T(\boldsymbol{u}^*) = 0\ \boldsymbol{w}^*[1, |\boldsymbol{w}|-1]$. This gives

$$\begin{aligned} T(\boldsymbol{e}_{3,i-1}) &= T\big(\tilde{\boldsymbol{u}}\,(\boldsymbol{u}^*)^\infty\big) \\ &= T(\tilde{\boldsymbol{u}})\,T(\boldsymbol{u}^*)^\infty \\ &= \boldsymbol{w}[1, |\boldsymbol{w}|-1]\,\big(0\ \boldsymbol{w}^*[1, |\boldsymbol{w}|-1]\big)^\infty \\ &= \boldsymbol{w}[1, |\boldsymbol{w}|-1]\,0\big(\boldsymbol{w}^*[1, |\boldsymbol{w}|-1]\,0\big)^\infty \\ &= \tilde{\boldsymbol{w}}(\boldsymbol{w}^*)^\infty \\ &= \boldsymbol{e}_{2,i}, \end{aligned}$$

which is the desired result. □

As a second corollary we have that we may obtain Thue's sequence $\boldsymbol{v}$ directly from iterations of f and do not have to go via the Thue-Morse sequence $\boldsymbol{t}$, (compare the corollary to Allouche's and Cosnard's Theorem 14.2).

Corollary 15.6 *Let $\boldsymbol{v}$ be Thue's sequence from (15.1). Then*

$$\boldsymbol{v}' = d(1,3) = \lim_{n\to\infty} f^n(1,3).$$

Moreover the sequence $\boldsymbol{v}'$ is shift-bounded.

Theorem 15.7 *Let $\boldsymbol{c} \in S(3)$ be an $e_{3,i}$-sequence for some $i \geq 1$. Then*

$$T(F(\boldsymbol{c},3)) = F(T(\boldsymbol{c}),2) \cap [0].$$

Proof: Let $\boldsymbol{x} \in F(\boldsymbol{c},3)$ and assume there is an n such that $\sigma^n(T(\boldsymbol{x})) < T(\boldsymbol{c})$. Then $\sigma^n(T(\boldsymbol{x}))$ begins with a zero and hence $T^{-1}(\sigma^n(T(\boldsymbol{x})))$ is well defined and is equal to $\sigma^m(\boldsymbol{x})$ for some m, and thus $\sigma^m(\boldsymbol{x}) < \boldsymbol{c} = T^{-1}(T(\boldsymbol{c}))$. But this contradicts our assumption $\boldsymbol{x} \in F(\boldsymbol{c},3)$. For the upper bounding inequality assume there is an n such that $T(\boldsymbol{c})' < \sigma^n(T(\boldsymbol{x}))$. Then we also have $0T(\boldsymbol{c})' < 0\sigma^n(T(\boldsymbol{x}))$, which implies

$$\boldsymbol{c}' \leq T^{-1}(0T(\boldsymbol{c})') < T^{-1}\big(0\sigma^n(T(\boldsymbol{x}))\big) = \sigma^m(\boldsymbol{x})$$

for some m. This because we may assume that if $\boldsymbol{c}$ is finite it does not end with a zero. We have reached a contradiction to our assumption $\boldsymbol{x} \in F(\boldsymbol{c}, 3)$. Thus we must have $T(\boldsymbol{x}) \in F(T(\boldsymbol{c}), 2) \cap [0]$, and hence

$$T(F(\boldsymbol{c}, 3)) \subset F(T(\boldsymbol{c}, 2) \cap [0].$$

For the reversed inclusion, note that $T(e_{0,3}) = 00(10)^\infty = e_{2,1}$ and therefore no sequences in $F(T(\boldsymbol{c}), 2) \cap [0]$ contain more than 2 consecutive zeros nor ones, which gives that $T^{-1}(\boldsymbol{u})$ is well-defined for all $\boldsymbol{u} \in F(T(\boldsymbol{c}), 2) \cap [0]$. Let $\boldsymbol{u} \in F(T(\boldsymbol{c}), 2) \cap [0]$. Assume there is an n such that $\sigma^n(T^{-1}(\boldsymbol{u})) < \boldsymbol{c}$. But then we must have $\sigma^m(\boldsymbol{u}) < T(\boldsymbol{c})$ for some m, a contradiction. The upper inequality, $\boldsymbol{c}' \geq \sigma^n(T^{-1}(\boldsymbol{u}))$, follows in the same way. Hence $T(F(\boldsymbol{c}, 3)) \supset F(T(\boldsymbol{c}), 2) \cap [0]$, which completes the proof. □

Lemma 15.8 *Let $\boldsymbol{c}$ be an $e_{3,i}$-sequence for some $i \geq 1$. Then*

$$\dim_H F(\boldsymbol{c}, 3) = \frac{2 \log 2}{\log 3} \dim_H F(T(\boldsymbol{c}), 2).$$

Proof: An element $\boldsymbol{x} \in F(\boldsymbol{c}, 3)$, where $\boldsymbol{c}$ is an $e_{3,i}$-sequences for some $i \geq 1$, is of the form

$$0\,1^{k_1} 2\,1^{k_2} 0\,1^{k_3} 2\,1^{k_4} \dots,$$

or shifts thereof. This gives that T expands the length of $\boldsymbol{x}$ by a factor two. Combining this expansion factor with the change of alphabet we obtain the Hölder condition

$$C_1 \ \delta_3(\boldsymbol{x}, \boldsymbol{u})^{\frac{2 \log 2}{\log 3}} \leq \delta_2(T(\boldsymbol{x}), T(\boldsymbol{u})) \leq C_2 \ \delta_3(\boldsymbol{x}, \boldsymbol{u})^{\frac{2 \log 2}{\log 3}}$$

for some constants C_1 and C_2 and for the metric $\delta_q(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^\infty |x_i - u_i| q^{-i}$. The statement now follows from Proposition 2.6 and Theorem 15.7. □

Corollary 15.9 *Let $i \geq 0$. Then*

$$\dim_H F(\boldsymbol{e}_{3,i}, 3) = \frac{1}{2^i} \frac{\log 2}{\log 3}.$$

Corollary 15.10 *Let $\boldsymbol{v}$ be Thue's sequences from (15.1) and $\boldsymbol{c} \in S(3)$. Then $\dim_H F(\boldsymbol{c}, 3) = 0$ if and only if $\boldsymbol{c} \geq \boldsymbol{v}'$, and $F(\boldsymbol{c}, 3)$ is countable if and only if $\boldsymbol{c} > \boldsymbol{v}'$.*

Proof: Use Allouche's and Cosnard's Theorem14.4, Moreira's Theorem 14.5, Theorem 15.7 and Lemma 15.8. □

Lemma 15.11 *Let $\boldsymbol{s}$ be a finite shift-bounded $e_{3,i}$-sequences for some $i \geq 1$. Then $\boldsymbol{s}$ contains the same number of 0's as 2's.*

Proof: An $e_{3,i}$-sequences must be of the form

$$\boldsymbol{s} = 0\,1^{k_1}2\,1^{k_2}0\,1^{k_3}2\,1^{k_4}\ldots.$$

If $\boldsymbol{s}$ ends with 01^k then by shifting this block up to the front we obtain $\sigma^n(\boldsymbol{s}) = 01^k < 01^{k_1}2\ldots$, a contradiction to $\boldsymbol{s}$ being shift-bounded. □

Corollary 15.12 *Let $\boldsymbol{s}$ be a finite shift-bounded $e_{3,i}$-sequences for some $i \geq 1$. Then $|T(\boldsymbol{s})| = 2|\boldsymbol{s}|$.*

Lemma 15.13 *The map T is a bijection between the set of $e_{3,i}$-minimal sequences and $e_{2,i+1}$-minimal sequences for $i \geq 1$.*

Proof: Let $\boldsymbol{x} \in S^*(2)$ be an $e_{2,i+1}$-minimal sequence and let $\boldsymbol{c} = T^{-1}(\boldsymbol{x})$. Assume there is an $2^i \leq n < |\boldsymbol{c}|$ and an $r \in \{1, 2\}$ such that

$$e(g_{n,r}(\boldsymbol{c}), 3) \leq \boldsymbol{c} \leq g_{n,r}(\boldsymbol{c})^\infty.$$

The order-preservation of T gives $e(g_{m,1}(\boldsymbol{x}), 2) \leq \boldsymbol{x} \leq g_{m,1}(\boldsymbol{x})^\infty$ for some $m < |\boldsymbol{x}|$. Corollary 15.12 gives that $2^{i+1} \leq m$, that is, a contradiction to the minimality of $\boldsymbol{x}$.

Conversely let $\boldsymbol{c} \in S^*(3)$ be an $e_{3,i}$-minimal sequence and let $\boldsymbol{x} = T(\boldsymbol{c})$. Assume $m_{\boldsymbol{x}} < |\boldsymbol{x}|$. That is, $m_{\boldsymbol{x}}$ is the smallest $m \geq 2^{i+1}$ such that

$$e(g_{m,1}(\boldsymbol{x}), 2) \leq \boldsymbol{x} \leq g_{m,1}(\boldsymbol{x})^\infty.$$

Since minimality implies shift-boundedness it follows from that $g_{m_{\boldsymbol{x}},1}(\boldsymbol{x})$ does not contain three consecutive ones, as it begins with precisely two zeros, that $T^{-1}(g_{m_{\boldsymbol{x}},1}(\boldsymbol{x})^\infty)$ and $T^{-1}(e(g_{m_{\boldsymbol{x}},1}(\boldsymbol{x}), 2))$ are well defined.

This gives $e(g_{n,r}(\boldsymbol{c}), 3) \leq \boldsymbol{c} \leq g_{n,r}(\boldsymbol{c})^\infty$ for some $n < |\boldsymbol{c}|$. As $g_{m_{\boldsymbol{x}},1}(\boldsymbol{x})$ begins with precisely two zeros it cannot, by being shift-bounded, have 001 as a suffix. Hence $T^{-1}(g_{m_{\boldsymbol{x}},1}(\boldsymbol{x}))$ contains equally many 0's as 2's, which implies $2^i \leq n$ and we have therefore reached a contradiction to the minimality of $\boldsymbol{c}$. □

Let $\boldsymbol{u}_i = f^i(1, 2)$. Then by Lemma 14.8 we have that $\mu_{\boldsymbol{u}_i}$ is a bijection between the $e_{2,i}$- and $e_{2,1}$-minimal sequences. Let $\mu_{\boldsymbol{u}_i,\boldsymbol{u}_j} = \mu_{\boldsymbol{u}_j}^{-1} \circ \mu_{\boldsymbol{u}_i}$ is a bijection between the $e_{2,i}$- and $e_{2,j}$-minimal sequences.

Corollary 15.14 *Let $\mu := \mu_{\boldsymbol{u}_i,\boldsymbol{u}_2}$ be the bijection between $e_{i,2}$- and $e_{2,2}$-minimal sequences as described above. Then for an $e_{3,i}$-minimal sequence $\boldsymbol{c}$ with $i \geq 1$ we have*

$$\dim_H F(\boldsymbol{c}, 3) = \frac{1}{2^{i-1}} \dim_H F\big(T^{-1}(\mu(T(\boldsymbol{c}))), 3\big).$$

In particular, the sequence $T^{-1}(\mu(T(\boldsymbol{c})))$ is an $e_{3,1}$-minimal sequence.

Proof: By applying Theorem 15.7, Theorem 14.9 and Lemma 15.13 we get

$$\begin{aligned}
\dim_H F(\boldsymbol{c}, 3) &= \frac{2\log 2}{\log 3} \dim_H F(T(\boldsymbol{c}), 2) \\
&= \frac{1}{2^{i-1}} \frac{2\log 2}{\log 3} \dim_H F(\mu(T(\boldsymbol{c})), 2) \\
&= \frac{\log 3}{2\log 2} \frac{1}{2^{i-1}} \frac{2\log 2}{\log 3} \dim_H T^{-1}(F(\mu(T(\boldsymbol{c})), 2) \cap [0]) \\
&= \frac{1}{2^{i-1}} \dim_H F(T^{-1}(\mu(T(\boldsymbol{c}))), 3),
\end{aligned}$$

which proves the desired result. □

16 q-adic Approximation

Similar to Definition 5.11 we define the re-alphabetisation function, but with a slight modification of the image set.

Definition 16.1 *Let R be the re-alphabetisation function such that for $\boldsymbol{x} \in S(q)$ we have $R(\boldsymbol{x}) \in S(q+2)$ and $R(\boldsymbol{x})_i = x_i + 1$.*

Example 16.2 Let $\boldsymbol{x}$ be the binary sequence 01 then $R(\boldsymbol{x}) = 12 \in S^*(4)$. Similarly, for the ternary sequence $\boldsymbol{u} = 012^\infty$ we have $R(\boldsymbol{u}) = 123^\infty \in S^\infty(5)$ and $R^2(\boldsymbol{u}) = 234^\infty \in S^\infty(7)$. □

Note that we have $R(\boldsymbol{e}_{q,i}) = \boldsymbol{e}_{q+2,i}$ for $i \geq 0$, so there is just two types of $e_{q,i}$-sequences, those obtained for q odd and those obtained for q even. Note also that from Lemma 11.5 we have that R^{-1} is well defined for any shift-bounded sequence $\boldsymbol{c} \in S(q+2)$ which starts with a non-zero symbol.

Lemma 16.3 *For sequences $\boldsymbol{u}, \boldsymbol{v} \in S(q)$ the re-alphabetisation function R fulfils the Hölder condition*

$$\delta_{q+2}\big(R(\boldsymbol{u}), R(\boldsymbol{v})\big) = \delta_q(\boldsymbol{u}, \boldsymbol{v})^{\frac{\log q}{\log(q+2)}}.$$

Proof: This is a direct consequence form the definition of the distance δ_q form (2.1). □

Lemma 16.4 *If $\boldsymbol{s}$ is an $e_{q,i}$-sequence for some $i \geq 1$ then $\boldsymbol{s}$ is an $e_{q,i}$-minimal sequence if and only if $R(\boldsymbol{s})$ is an $e_{q+2,i}$-minimal sequence.*

If $\boldsymbol{s}$ is an $e^r_{q,0}$-sequence for some $r \in \{1, 2, \ldots [\frac{q}{2}] - 1\}$ then $\boldsymbol{s}$ is an $e^r_{q,0}$-minimal sequence if and only if $R(\boldsymbol{s})$ is an $e^{r+1}_{q+2,0}$-minimal sequence.

Proof: The Lemma follows directly from the order-preservation of R and Lemma 11.5. □

Theorem 16.5 *Let $\boldsymbol{c} \in S(q)$. Then*

$$R(F(\boldsymbol{c}, q)) = F(R(\boldsymbol{c}), q+2).$$

Proof: It is clear that $R(F(\boldsymbol{c}, q)) \subset F(R(\boldsymbol{c}), q+2)$. For the reversed inclusion, let $\boldsymbol{x} \in F(R(\boldsymbol{c}), q+2)$. If $\boldsymbol{x}$ would contain a 0 at position n then $\sigma^n(\boldsymbol{x}) < R(\boldsymbol{c})$ which would contradict $\boldsymbol{x} \in F(R(\boldsymbol{c}), q+2)$. Similarly we have that $\boldsymbol{x}$ does not contain the symbol $q+1$. Hence $R^{-1}(\boldsymbol{x}$ is well defined and $R^{-1}(\boldsymbol{x}) \in S^\infty(q)$. If there exists an m such that $\sigma^m(R^{-1}(\boldsymbol{x})) <$

$\boldsymbol{c}$ then we would have $R(\sigma^n(R^{-1}(\boldsymbol{x}))) = \sigma^m(\boldsymbol{x}) < R(\boldsymbol{c})$, a contradiction to our assumption. The upper inequality follows similarly. Hence $R^{-1}(\boldsymbol{x}) \in F(\boldsymbol{c}, q)$ and therefore we get $R(F(\boldsymbol{c}, q)) \supset F(R(\boldsymbol{c}), q+2)$. $\square$

Example 16.6 *(Generalisation of Example 12.19).* If q is even, $i \geq 2$ and $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequence then by applying the re-alphabetisation function R to the prefix $\boldsymbol{p} = 01(10)^k 11$ we have by Theorem 16.5 that there is no sequence $\boldsymbol{w}$ such that $\boldsymbol{cwz}^\infty \in F(\boldsymbol{c}, q)$ for $\boldsymbol{z} = c_1\,(q - c_1 - 1)$. The procedure when q is even and $i \geq 2$ follows the same pattern. $\square$

Corollary 16.7 *Let $\boldsymbol{c} \in S(q)$. Then*

$$\dim_H F(\boldsymbol{c}, q) = \frac{\log(q+2)}{\log q} \dim_H F(R(\boldsymbol{c}), q+2).$$

Proof: Apply Lemma 16.3 and Proposition 2.6 to the result of Theorem 16.5. $\square$

We can now generalise Corollary 14.10 and Corollary 15.9.

Theorem 16.8 *If $i \geq 0$ then*

$$\dim_H F(\boldsymbol{e}_{q,i}, q) = \frac{1}{2^i} \frac{\log 2}{\log q}.$$

Proof: First the case when q is even. By Corollary 16.7 and Corollary 14.10 we have

$$\dim_H F(\boldsymbol{e}_{q,i}, q) = \frac{\log 2}{\log q} \dim_H F(\boldsymbol{e}_{2,i}, 2) = \frac{1}{2^i} \frac{\log 2}{\log q}.$$

Similarly, if q is odd then Corollary 16.7 and Corollary 15.9 gives

$$\begin{aligned}
\dim_H F(\boldsymbol{e}_{q,i}, q) &= \frac{\log 3}{\log q} \dim_H F(\boldsymbol{e}_{3,i}, 3) \\
&= \frac{\log 3}{\log q} \frac{2 \log 2}{\log 3} \dim_H F(\boldsymbol{e}_{2,i+1}, 2) \\
&= \frac{\log 2}{\log q} \frac{2}{2^{i+1}},
\end{aligned}$$

which proves the theorem. □

We can now formulate the generalisation of Allouche's and Cosnard's Theorem 14.4 and Moreira's Theorem 14.5 to any $q \geq 2$.

Theorem 16.9 *Let $n = [\frac{q}{2}] - 1$ and $\boldsymbol{c} \in S(q)$.*

- *If q is even let $\boldsymbol{t}$ be the Thue-Morse sequence. Then $F(\boldsymbol{c}, q)$ is countable if and only if $\boldsymbol{c} > R^n(\sigma(\boldsymbol{t'}))$ and $\dim_H F(\boldsymbol{c}, q) = 0$ if and only if $\boldsymbol{c} \geq R^n(\sigma(\boldsymbol{t'}))$.*
- *If q is odd let $\boldsymbol{v}$ be Thue's sequences from (15.1). Then $F(\boldsymbol{c}, q)$ is countable if and only if $\boldsymbol{c} > R^n(\boldsymbol{v}')$ and $\dim_H F(\boldsymbol{c}, q) = 0$ if and only if $\boldsymbol{c} \geq R^n(\boldsymbol{v}')$.*

Proof: For the case with q even, the result follows from Allouche and Cosnard's Theorem 14.4, Moreira's Theorem 14.5 and Theorem 16.5. The case with q odd is obtained from Corollary 15.10 and Theorem 16.5. □

Theorem 16.10 *If q is even and $i \in \{0, 1\}$, or q is odd and $i = 0$ then for a finite $e_{q,i}$-minimal sequence $\boldsymbol{c}$ the system $\sigma : F(\boldsymbol{c}, q) \to F(\boldsymbol{c}, q)$ is topologically mixing.*

Proof: Let $U = [\boldsymbol{u}] \cap F(\boldsymbol{c}, q)$ and $V = [\boldsymbol{v}] \cap F(\boldsymbol{c}, q)$ and assume they are both non-empty. By Lemma 12.17 there is a k such that $[\boldsymbol{u}[1, k]\boldsymbol{c}] \cap U$ is non-empty and by Lemma 12.18 there exists a finite sequence $\boldsymbol{w}$ such that $\boldsymbol{u}[1, k]\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^\infty \in U$, where $\boldsymbol{z} = c_1(q - c_1 - 1)$.

For the case when q is even, let $\boldsymbol{a} = c_1$ if $v_1 = q - c_1 - 1$ otherwise let $\boldsymbol{a}$ be void. Then there exists an N_1 such that

$$\big[\boldsymbol{u}[1, k]\,\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^{n_1}\boldsymbol{a}\,\boldsymbol{v}\big] \cap U \neq \emptyset \qquad (16.1)$$

for $n_1 > N_1$. As $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequence there exists N_2 and N_3 such that

$$\big[\boldsymbol{u}[1, k]\,\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^{n_2}c_1\,\boldsymbol{z}^{n_3}\boldsymbol{a}\,\boldsymbol{v}\big] \cap U \neq \emptyset \qquad (16.2)$$

for $n_2 > N_2$ and $n_3 > N_3$. Combining (16.1) and (16.2) gives $\sigma^n(U) \cap V \neq \emptyset$ for all n larger than some N_4.

Now assume that q is odd. Since $\boldsymbol{c}$ is an $e_{q,0}$-sequence we have that $\boldsymbol{c} < ([\frac{q}{2}]-1)[\frac{q}{2}]^k = e([\frac{q}{2}],q)[1,k+1]$ for some $k > K$. But then also $\boldsymbol{c}' > ([\frac{q}{2}]+1)[\frac{q}{2}]^k$. Hence there exists an N_5 and an N_6 such that

$$\left[\boldsymbol{u}[1,k]\,\boldsymbol{c}\,\boldsymbol{w}\,\boldsymbol{z}^{n_5}[\tfrac{q}{2}]^{n_6}\,\boldsymbol{v}\right] \cap U \neq \emptyset$$

for $n_5 > N_5$ and $n_6 > N_6$. This gives $\sigma^n(U) \cap V \neq \emptyset$ for all n larger than some N_7. □

Corollary 16.11 *If q is even and $i \in \{0,1\}$, or q is odd and $i = 0$ then for a finite $e_{q,i}$-minimal sequence $\boldsymbol{c}$ the transition matrix $A_{\boldsymbol{c}}$ corresponding to $F(\boldsymbol{c},q)$ is primitive.*

The mixing result in Theorem 16.10 can not be extend to be valid for any $e_{q,i}$-minimal sequence other than those stated in the Theorem 16.10. We give an example to illustrate this.

Example 16.12 By Example 12.19 and Example 16.6 it follows that we can not have topologically mixing in the case for finite $e_{q,i}$-minimal sequence, for any $q \geq 2$ and any $i \geq 2$.

In the case $q = 3$ and $i \geq 1$ we have that any finite $e_{3,1}$-minimal sequence $\boldsymbol{c}$ must have a prefix $\boldsymbol{p}$ of the form 01^k2 for some $k > 0$. If letting $\boldsymbol{u} = 1 \in S^*(3)$ we see that $\boldsymbol{p} = \tilde{\boldsymbol{u}}(\boldsymbol{u}^*)^k\boldsymbol{u}'$. Hence Lemma 10.2 gives that $\sigma^n([01^k2]) \cap [1^{k+1}] = \emptyset$ for all n and therefore is $\sigma : F(\boldsymbol{c},3) \to F(\boldsymbol{c},3)$ not topologically mixing. Theorem 16.5 generalises this property for any finite $e_{3,1}$-minimal sequence to any $e_{q,i}$-minimal sequence with odd $q \geq 3$ and $i \geq 1$. □

There are $\boldsymbol{s}$ which are not $e_{q,i}$-minimal sequences but the system $\sigma : F(\boldsymbol{s},q) \to F(\boldsymbol{s},q)$ is topologically mixing. To see this, assume that $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequences such that we have that $\sigma : F(\boldsymbol{c},q) \to F(\boldsymbol{c},q)$ is topologically mixing. Then Lemma 10.1 gives that for any $\boldsymbol{s} \in [\boldsymbol{c},\boldsymbol{c}^\infty]$ we have $F(\boldsymbol{c},q) = F(\boldsymbol{s},q)$. Hence we can find a sequence which is not $e_{q,i}$-minimal but still gives the mixing property.

Definition 16.13 *For an $e_{q,i}$-sequence $\boldsymbol{c}$ we define the interval $I(\boldsymbol{c},q)$ to be the set*

$$I(\boldsymbol{c},q) = \{\boldsymbol{x} \in S^\infty(q) : e(g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c}),q) \leq \boldsymbol{x} \leq (g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c}))^\infty\}\,.$$

The next theorem says that the definition of $I(\boldsymbol{c}, q)$ is independent of the choice of the representative $\boldsymbol{c}$.

Theorem 16.14 *For any $\boldsymbol{a} \in I(\boldsymbol{c}, q)$ we have $I(\boldsymbol{a}, q) = I(\boldsymbol{c}, q)$.*

Proof: We may assume that $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequence. First consider the case $\boldsymbol{c} < \boldsymbol{a} < \boldsymbol{c}^\infty$. If $m_{\boldsymbol{c}} > m_{\boldsymbol{a}}$ then the $e_{q,i}$-minimality of $\boldsymbol{c}$ and Lemma 4.6 gives

$$\boldsymbol{c} < \boldsymbol{a} \le g_{m_{\boldsymbol{a}},k_{\boldsymbol{a}}}(\boldsymbol{a})^\infty = g_{m_{\boldsymbol{a}},k_{\boldsymbol{a}}}(\boldsymbol{c})^\infty < \boldsymbol{c},$$

a contradiction. If $m_{\boldsymbol{c}} < m_{\boldsymbol{a}}$ then similarly,

$$\boldsymbol{c}^\infty = g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})^\infty = g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{a})^\infty < g_{m_{\boldsymbol{a}},k_{\boldsymbol{a}}}(\boldsymbol{a}) \le \boldsymbol{a},$$

a contradiction to $\boldsymbol{a} \in I(\boldsymbol{c}, q)$. Let us turn to the case $e(\boldsymbol{c}) < \boldsymbol{a} < \boldsymbol{c}$. If $m_{\boldsymbol{c}} > m_{\boldsymbol{a}}$ then

$$\boldsymbol{a} < e(g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{a})) = e(g_{m_{\boldsymbol{c}},k_{\boldsymbol{c}}}(\boldsymbol{c})),$$

which contradicts $\boldsymbol{a} \in I(\boldsymbol{c}, q)$. Finally, if $m_{\boldsymbol{c}} < m_{\boldsymbol{a}}$ then we would have

$$\boldsymbol{c} < e(g_{m_{\boldsymbol{a}},k_{\boldsymbol{a}}}(\boldsymbol{c})) = e(g_{m_{\boldsymbol{a}},k_{\boldsymbol{a}}}(\boldsymbol{a})) \le \boldsymbol{a} < \boldsymbol{c},$$

again a contradiction. □

Theorem 16.15 *The interval $I(\boldsymbol{c}, q)$ is the largest interval I on which we have $\dim_H F(\boldsymbol{c}, q) = \dim_H F(\boldsymbol{a}, q)$ for $\boldsymbol{a} \in I$.*

Proof: We may assume that $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequence. By Lemma 10.1 we have that $F(\boldsymbol{c}, q) = F(\boldsymbol{c}^\infty, q)$ and therefore

$$\dim_H F(\boldsymbol{c}, q) = \dim_H F(\boldsymbol{c}^\infty, q).$$

Lemma 11.16 gives that $\dim_H F(\boldsymbol{c}, q) = \dim_H F(f(\boldsymbol{c}), q)$. Let $\boldsymbol{x}$ be an element in $F(e(\boldsymbol{c}, q), q) \setminus F(f(\boldsymbol{c}, q), q)$. As $f(\boldsymbol{c}, q) = \tilde{\boldsymbol{c}}\,\boldsymbol{c}'$ we have from Lemma 10.2 that $\boldsymbol{x}$ must end with a sequence in $\mathcal{A}(\boldsymbol{c})$. Moreover, since $\boldsymbol{c}$ is a finite $e_{q,i}$-sequence we have

$$\begin{aligned}
\dim_H \mathcal{A}(\boldsymbol{c}) &= \frac{1}{|\boldsymbol{c}|}\frac{\log 2}{\log q} \\
&\le \frac{1}{2^i}\frac{\log 2}{\log q} \\
&= \dim_H F(\boldsymbol{e}_{q,i}, q) \\
&\le \dim_H F(\boldsymbol{c}, q).
\end{aligned}$$

Combining the above chain of inequalities with Lemma 10.2 shows that $\dim_H F(\boldsymbol{c}, q) = \dim_H F(e(\boldsymbol{c}, q), q)$.

For the maximallity, assume first that q is even and $i \in \{0, 1\}$, or q is odd and $i = 0$. Let $A_{\boldsymbol{c}}$ be a transition matrix corresponding to $F(\boldsymbol{c}, 2)$. Lemma 12.15 gives that there is a sequence $\{\boldsymbol{a}_k\}$ of finite $e_{q,i}$-minimal sequences growing to $e(\boldsymbol{c}, q)$. Let A_k be the transition matrix corresponding to $F(\boldsymbol{a}_k, q)$. From Corollary 16.11 it follows that $A_{\boldsymbol{c}}$ and A_k are primitive matrices. As $(\boldsymbol{a}_k)^\infty \in F(\boldsymbol{a}_k, q) \setminus F(\boldsymbol{c}, q)$ we have that $A_k \geq A_{\boldsymbol{c}}$, entry by entry, (we may rescale the matrices to have the same size), and where the inequality is strict for at least one pair of indices r, s. As A_k is primitive it follows from the Perron-Frobenius Theorem 2.2 and Theorem 2.8 that

$$\dim_H F(\boldsymbol{a}_k, q) > \dim_H F(\boldsymbol{c}, q),$$

and therefore the interval $I(\boldsymbol{c}, q)$ cannot be extended leftward. Similarly, we use the sequence $\{\boldsymbol{b}_k\}$ from Lemma 12.16 to show that $\boldsymbol{c}^\infty$ is the right endpoint of the interval I.

By Lemma 14.8 and Theorem 14.9 we may now extend the result to be valid in any interval $[\boldsymbol{e}_{2,i-1}, \boldsymbol{e}_{q,i})$ for $i \geq 2$ and Lemma 15.8 settles the case of $q = 3$ and $i \geq 1$. Finally Theorem 16.5 implies the result for any $q \geq 2$ and $i \geq 0$. □

Example 16.16 For the $e_{2,1}$-minimal sequence $\boldsymbol{c} = 001$ we have the interval $I(\boldsymbol{c}, 2) = [000(110)^\infty, (001)^\infty]$, which corresponds to the real interval $I(\frac{1}{8}, 2) = [\frac{3}{28}, \frac{1}{7}]$. Note that set $F(\boldsymbol{c}, 2)$ is the set of infinite binary sequence $\boldsymbol{s}$ which contains at most 2 consecutive zeros or ones. The corresponding transition matrix $A_{\boldsymbol{c}}$ to $F(\boldsymbol{c}, 2)$ is

$$A_{\boldsymbol{c}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad (16.3)$$

The spectral radius of the transition matrix $A_{\boldsymbol{c}}$ gives

$$\dim_H F(\boldsymbol{c}, q) = \dim_H F(001, 2) = \frac{\log \frac{1+\sqrt{5}}{2}}{\log 2} \approx 0.69424.$$

□

Example 16.17 For the $e_{7,2}$-minimal sequence $\boldsymbol{c} = 234243$ it would be hard to directly calculate the spectral radius of the transition matrix $A_{\boldsymbol{c}}$ associated to $F(\boldsymbol{c}, 7)$, as $A_{\boldsymbol{c}}$ would be of size $7^5 \times 7^5$. The re-alphabetisation R function gives the $e_{3,2}$-minimal sequence $R^{-2}(\boldsymbol{c}) = 012021$. Then the map T gives $T(R^{-2}(\boldsymbol{c})) = 0010\,1100\,1101$, and if we let $\boldsymbol{u} = 0011$, the function $\mu_{\boldsymbol{u}}$ simplifies the problem substantially. We get $\mu_{\boldsymbol{u}}(T(R^{-2}(\boldsymbol{c}))) = 001$, and hence we have the same transition matrix as (16.3). This gives

$$\begin{aligned}
\dim_H F(\boldsymbol{c}, 7) &= \frac{\log 3}{\log 7} \dim_H F(R^{-2}(\boldsymbol{c}), 3) \\
&= \frac{2 \log 2}{\log 7} \dim_H F(T(R^{-2}(\boldsymbol{c})), 2) \\
&= \frac{1}{2} \frac{\log 2}{\log 7} \dim_H F(\mu_{\boldsymbol{u}}(T(R^{-2}(\boldsymbol{c}))), 2) \\
&= \frac{1}{2} \frac{\log \frac{1+\sqrt{5}}{2}}{\log 7} \approx 0.12365.
\end{aligned}$$

□

Example 16.18 The $e_{q,0}$-minimal sequences of length 1 are the sequences $\boldsymbol{c}_r = r$ with $0 \le r < \frac{q}{2}$. A transition matrix $A_{\boldsymbol{c}_r}$ of size $q \times q$ associated to $F(\boldsymbol{c}_r, q)$ is

$$A_{\boldsymbol{c}_r} = \begin{pmatrix} \overbrace{0 \quad \dots \quad 0}^{r} & \overbrace{1 \quad \dots \quad 1}^{q-2r} & \overbrace{0 \quad \dots \quad 0}^{r} \\ \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \\ 0 \quad \dots \quad 0 & 1 \quad \dots \quad 1 & 0 \quad \dots \quad 0 \end{pmatrix}$$

It is not hard to see that the spectral radius of $A_{\boldsymbol{c}_r}$ is $q - 2r$. Hence we have

$$\dim_H F(\boldsymbol{e}^r_{q,0}, q) = \dim_H F(e(r, q), q) = \frac{\log(q - 2r)}{\log q} \tag{16.4}$$

for $0 \le r < \frac{q}{2}$. □

Denote by $IEM(q,i)$ the set of infinite $e_{q,i}$-minimal sequences and let us define the set of all $e_{q,i}$-minimal sequences for all $i \geq 0$ by

$$IEM(q) = \bigcup_{i=0}^{\infty} IEM(q,i). \tag{16.5}$$

From Lemma 11.17 we have the following corollary

Corollary 16.19 *The set $IEM(q)$ has Lebesgue measure* 0.

Proof: From Lemma 12.13 we have that $IEM(q)$ is contained in the set of infinite shift-bounded sequences $ISB(q)$, from (11.3). Hence the result follows from Lemma 11.17. □

Theorem 16.20 *The derivative of ϕ_q is zero Lebesque a.e.*

Proof: The sequences which give rise to one-point intervals $I(\boldsymbol{c},q) = \{\boldsymbol{c}\}$ are precisely the sequences $\boldsymbol{c} \in IEM(q)$. As $IEM(q)$ has Lebesgue measure 0 we must have that the complementary set, the set formed by the intervals, has full Lebesgue measure. □

Let us turn to the question of continuity of the function $\phi_q : \boldsymbol{c} \mapsto \dim_H F(\boldsymbol{c},q)$. In [14], Labarca and Moreira show that the Hausdorff dimension of the set

$$\{\boldsymbol{x} \in S^{\infty}(2) : \boldsymbol{a} \geq \sigma^n(\boldsymbol{x}) \geq \boldsymbol{b} \text{ for all } n \geq 0\}$$

depends continuously on the parameters $\boldsymbol{a}$ and $\boldsymbol{b}$. This clearly implies the continuity of ϕ_2. We will extend this result and show that the continuity of ϕ_q for any integer $q \geq 2$. For the next counting lemma recall that by $|\cdot|$ we mean the cardinality of a set.

Lemma 16.21 *Let $\boldsymbol{c}$ be a finite $e_{q,i}$-minimal sequence and let $\boldsymbol{a}_k = \boldsymbol{a}_k(\boldsymbol{c},q)$ and $\boldsymbol{b}_k = \boldsymbol{b}_k(\boldsymbol{c},q)$.*

1. *There is a constant C such that $|F(\boldsymbol{a}_k,q)[1,k]| \leq C|F(\boldsymbol{c},q)[1,k]|$ for all $k \geq 1$.*

2. *Given k large enough, there is a constant C such that for all $n \geq 1$ we have*

$$\left|F(\boldsymbol{b}_k, q)[1,n]\right| \geq C\left|F(\boldsymbol{c}, q)[1,n]\right| \left(1 - \frac{1}{k}\right)^n.$$

Proof:(1). Let $\boldsymbol{x} \in F(\boldsymbol{a}_k, q)[1,k] \setminus F(\boldsymbol{c}, q)[1,k]$. Then $\boldsymbol{x}$ may start with a prefix of a sequence in $F(\boldsymbol{c}, q)$ and must end with a prefix of $\boldsymbol{a}_k$ Hence we have

$$\begin{aligned} \left|F(\boldsymbol{a}_k, q)[1,k]\right| &\leq \sum_{i=1}^{k} \left|F(\boldsymbol{c}, q)[1,i]\right| \\ &\leq k_1 \sum_{i=1}^{k} \lambda_{\boldsymbol{c}}^i \\ &\leq k_2 \lambda_{\boldsymbol{c}}^k \\ &\leq C\left|F(\boldsymbol{c}, q)[1,k]\right| \end{aligned}$$

for some constants k_1 and k_2 and where $\log \lambda_{\boldsymbol{c}}$ is the topological entropy of $F(\boldsymbol{c}, q)$.

(2). Similarly, a sequences $\boldsymbol{d}$ in the set $F(\boldsymbol{c}, q)[1,n] \setminus F(\boldsymbol{b}_k, q)[1,n]$ must contain, at least once, the subsequence $\boldsymbol{u} = \boldsymbol{c}^k\, \boldsymbol{c}[1, |p(\boldsymbol{c}, q)|]$, (or the inverse $\boldsymbol{u}^*$). To see this, the sequence $\boldsymbol{b}_k$ is strictly larger than $\boldsymbol{u}$ and hence no sequence in $F(\boldsymbol{b}_k, q)[1,n]$ can contain the subsequence $\boldsymbol{u}$. Conversely, as $\boldsymbol{u}$ is larger than $\left(\boldsymbol{c}[1, n_{\boldsymbol{c}}]^\infty\right)[1, |\boldsymbol{u}|]$ we have that $\boldsymbol{u}$ is an allowed pattern in sequences in $F(\boldsymbol{c}, q)[1,n]$. The same argumentation hold for $\boldsymbol{u}^*$. The number of sequences of length n that contains the patterns $\boldsymbol{u}$ or $\boldsymbol{u}^*$ precisely r times, is bounded by

$$2^r \binom{n - r|\boldsymbol{u}|}{r} \left|F(\boldsymbol{b}_k, q)\left[1, n - r|\boldsymbol{u}|\right]\right|,$$

this, by looking at the number of places the pattern $\boldsymbol{u}$ can be placed in. We have by summing up for a k large enough

$$C\big|F(\boldsymbol{c},q)[1,n]\big| \leq$$

$$\leq \lambda_{\boldsymbol{b}_k}^n + 2\binom{n-|\boldsymbol{u}|}{1}\lambda_{\boldsymbol{b}_k}^{n-|\boldsymbol{u}|} + 2^2\binom{n-2|\boldsymbol{u}|}{2}\lambda_{\boldsymbol{b}_k}^{n-2|\boldsymbol{u}|} + \dots$$

$$\leq \lambda_{\boldsymbol{b}_k}^n \left(1 + \binom{n}{1}\frac{2}{\lambda_{\boldsymbol{b}_k}^{|\boldsymbol{u}|}} + \binom{n}{2}\frac{2^2}{\lambda_{\boldsymbol{b}_k}^{2|\boldsymbol{u}|}} + \dots\right)$$

$$\leq \lambda_{\boldsymbol{b}_k}^n \left(1 + \frac{2}{\lambda_{\boldsymbol{b}_k}^{|\boldsymbol{u}|}}\right)^n$$

$$\leq \lambda_{\boldsymbol{b}_k}^n \left(1 + \frac{1}{k}\right)^n,$$

as by Moreira's Theorem 14.5 we have $\lambda_{\boldsymbol{b}_k} > 1$, which concludes the lemma. □

Theorem 16.22 *The map ϕ_q is continuous.*

Proof: By Theorem 2.8 we just have to show that the entropy of $F(\boldsymbol{c},q)$ depends continuously on $\boldsymbol{c}$. It is clear that for any sequence $\boldsymbol{c}$ the estimate $|F(\boldsymbol{c},q)[1,rn]| \leq |F(\boldsymbol{c},q)[1,n]|^r$ holds. Let $\boldsymbol{a}_k = \boldsymbol{a}_k(\boldsymbol{c},q)$ and $\boldsymbol{b}_k = \boldsymbol{b}_k(\boldsymbol{c},q)$. Hence by Lemma 16.21 it follows that

$$\begin{aligned}
h_{\text{top}}\big(F(\boldsymbol{c},q)\big) &\leq \lim_{k\to\infty} h_{\text{top}}\big(F(\boldsymbol{a}_k,q)\big) \\
&= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\big|F(\boldsymbol{a}_k,q)[1,n]\big| \\
&= \lim_{k\to\infty}\lim_{r\to\infty}\frac{1}{rk}\log\big|F(\boldsymbol{a}_k,q)[1,rk]\big| \\
&\leq \lim_{k\to\infty}\lim_{r\to\infty}\frac{1}{rk}\log\big|F(\boldsymbol{a}_k,q)[1,k]\big|^r \\
&\leq \lim_{k\to\infty}\frac{1}{k}\log C\big|F(\boldsymbol{c},q)[1,k]\big| \\
&= h_{\text{top}}\big(F(\boldsymbol{c},q)\big),
\end{aligned}$$

which shows the left-continuity of the entropy in the left endpoint of the interval $I(\boldsymbol{c}, q)$. The right-continuity follows trivially as the entropy is constant in a neighbourhood to the right of this point. In the same way the left-continuity in the right endpoint of $I(\boldsymbol{c}, q)$ is also clear. Again by Lemma 16.21 we have

$$\begin{aligned} h_{\text{top}}\big(F(\boldsymbol{c},q)\big) &\geq \lim_{k\to\infty} h_{\text{top}}\big(F(\boldsymbol{b}_k,q)\big) \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\big|F(\boldsymbol{b}_k,q)[1,n]\big| \\ &\geq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\left(C\big|F(\boldsymbol{c},q)[1,n]\big|\left(1-\frac{1}{k}\right)^n\right) \\ &= h_{\text{top}}\big(F(\boldsymbol{c},q)\big)+\lim_{k\to\infty}\log\left(1-\frac{1}{k}\right) \\ &= h_{\text{top}}\big(F(\boldsymbol{c},q)\big), \end{aligned}$$

and the right-continuity in the right endpoints follows and concludes the theorem. □

For the rest of this section we turn our interest to the set of all infinite $e_{q,i}$-minimal sequences for all $i \geq 0$, the set $IEM(q)$, defined in (16.5).

We define the function $\psi_q : S^\infty(q) \to [0,1]$ by

$$\psi_q(\boldsymbol{c}) = \dim_H\big(IEM(q)\cap[\boldsymbol{c},(q-1)^\infty]\big).$$

Note that we equally could have defined the function ψ as a function on the real interval $[0,1]$. In comparison to ϕ_q the function ψ_q is defined on the parameter-space while ϕ_q is a function on the phase-space.

Theorem 16.23 *For $\boldsymbol{c} \in S^\infty(q)$ we have $\psi_q(\boldsymbol{c}) = \phi_q(\boldsymbol{c})$.*

Proof: Since $IEM(q)\cap[\boldsymbol{c},(q-1)^\infty] \subset F(\boldsymbol{c},q)$ we have $\psi_q(\boldsymbol{c}) \leq \phi_q(\boldsymbol{c})$. Let us turn to the reversed inequality. Assume that $\boldsymbol{c}$ is a finite $e_{q,i}$-minimal sequence. From Lemma 12.16 there is a sequence $\{\boldsymbol{b}_k(\boldsymbol{c},q)\}$ of $e_{q,i}$-minimal sequences tending to $\boldsymbol{c}^\infty$. Put $\boldsymbol{v}_k = \boldsymbol{b}_k(\boldsymbol{b}_k(\boldsymbol{c},q),q)$ and define

$$N_k = \big\{\boldsymbol{b}_k(\boldsymbol{c},q)\,\boldsymbol{u} : \boldsymbol{u} \in [\boldsymbol{v}_k]\cap F(\boldsymbol{v}_k,q)\big\}.$$

We have $IEM(q) \cap [\boldsymbol{c}, (q-1)^\infty] \supset N_k$ and

$$\dim_H N_k = \dim_H F(\boldsymbol{v}_k, q).$$

By choosing k sufficiently large we have that $\dim_H N_k$ is arbitrarily close to $\phi_q(\boldsymbol{c})$. □

From Theorem 16.23 we can deduce that the dimensional structure of Allouche's and Cosnard's set $\Gamma(2)$ is the same as the dimensional structure of $F(\boldsymbol{c}, q)$.

Corollary 16.24 *Let $ISB(q)$ be the set of infinite shift-bounded sequences from a q letter alphabet. Then*

$$\dim_H \left(ISB(q) \cap [\boldsymbol{c}, (q-1)^\infty]\right) = \dim_H \left(\Gamma(q) \cap [0, \boldsymbol{c}']\right) = \phi_q(\boldsymbol{c}),$$

where $\Gamma(q)$ is the set defined in (9.4).

Proof: Let $\Gamma(q)'$ be the set $\{\boldsymbol{x}' : \boldsymbol{x} \in \Gamma(q)\}$. Then we have the chain of inclusions

$$\begin{aligned} IEM(q) \cap [\boldsymbol{c}, (q-1)^\infty] &\subset ISB(q) \cap [\boldsymbol{c}, (q-1)^\infty] \\ &\subset \Gamma(q)' \cap [\boldsymbol{c}, (q-1)^\infty] \\ &\subset F(\boldsymbol{c}, q). \end{aligned}$$

Theorem 16.23 now gives the result. □

17 Numerics

By characterising the dimension of $F(c, q)$ via the spectral radius of a primitive transition matrix the problem of numerically calculate an approximative value of ϕ reduces to calculate the eigenvalues of the transition matrix.

The graph of $c \mapsto \dim_H F(c, 2)$, (see figure 5) was calculate by considering $e_{2,1}$-minimal sequence of length at most 8, which gives transition matrices of size 128×128, and then using Theorem 14.9 to obtain the values of $\dim_H F(c, 2)$ for $e_{2,i}$-minimal sequences with $i > 1$.

The graph of ϕ_3, (see figure 6) was calculate by considering $e_{0,3}$-minimal sequence of length at most 6, which gives transition matrices of

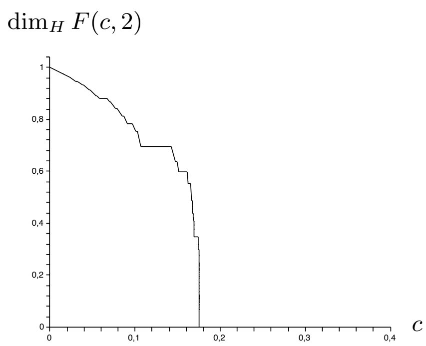


Figure 5: The graph of $c \mapsto \dim_H F(c, 2)$.

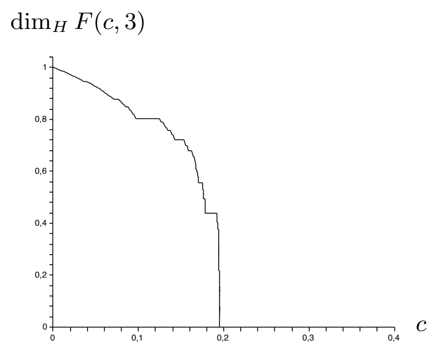


Figure 6: The graph of $c \mapsto \dim_H F(c, 3)$.

size 243×243. Then all $e_{2,2}$-minimal sequence of length 16 was calculated and the graph was obtained via Corollary 15.14.

The graph of ϕ_4, (see figure 7) was calculate by considering $e^1_{4,0}$-minimal sequence of length at most 4, which gives transition matrices of size 256×256. Then all $e_{2,1}$-minimal sequence of length 8 was calculated and the complete graph was obtained via Corollary 16.7 and Theorem 14.9.

A finer subdivision of the interval [0,1], that is, to consider longer minimal sequences, would require harder calculation as the runtime complexity of the computation is exponential in the length of the minimal sequences.

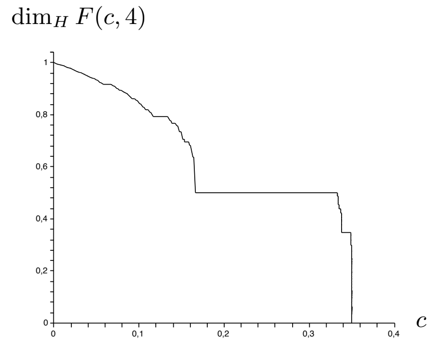


Figure 7: The graph of
$c \mapsto \dim_H F(c,4)$.

18 Conclusion

The work in the thesis has been devoted to the field of diophantine approximation and in particular the study of q-adically badly approximable numbers, BAN. We have considered two different kinds of approximation model, the one-side in Part I and the two-sided in Part II. A main difference between the two models is, in general, the lack of the topologial property of mixing in the two-sided model. The lack of mixing severely complicates the investigation and description of the approximation model. The models have previously been studied in other contexts and different formulations by Urbanski [23], Allouche and Cosnard [1, 2] and Moreira and Labarca [12, 14]. We have used several of their results and in many cases extended and completed them.

To continue the study of diophantine approximation and q-adically badly approximable numbers we think the following questions are of interest,

1. A natural and interesting question is whether the generalised two-sided approximation model

$$F^2(c,\beta) = \left\{ x \in \mathbb{S} : \left\| x - \frac{m}{\beta^n} \right\| < \frac{c}{\beta^n} \text{ finitely often} \right\},$$

   for an arbitrary $\beta > 1$, has the same dimensional properties as $F^2(c,q)$ for an integer.

2. Labarca and Moreira showed in [12] that the map

$$(\boldsymbol{a},\boldsymbol{b}) \mapsto \dim_H\{\boldsymbol{x} \in S^\infty(2) : \boldsymbol{a} \geq \sigma^n(\boldsymbol{x}) \geq \boldsymbol{b} \text{ for all } n \geq 0\}$$

   is continuous in both parameters $\boldsymbol{a}$ and $\boldsymbol{b}$. The question is; can we similarly to $F^2(c,q)$ describe the intervals where the dimension remains unchanged for this map?

3. In Figure 4 the graph of $\beta \mapsto \dim_H F(0.25,\beta)$ is given. What can be said about this map, is it continuous or self-similar?

19 List of Notation

$\tilde{\boldsymbol{x}}$	decrement of the last symbol, 14
$\hat{\boldsymbol{x}}$	increment of the last symbol, 14
$\boldsymbol{x}^*$	bit-wise inverse, 14
$\boldsymbol{x}'$	real number inverse, 14
$\lvert\boldsymbol{x}\rvert$	length of a sequence, 14
$\lvert E\rvert$	set cardinality, 14
$[x]$	integer part,
$[\boldsymbol{s}]$	cylinder set,
$\boldsymbol{s}[a,b]$	subsequence, 14
$E[a,b]$	set of subsequences, 14
$\Gamma(q)$	Allouche's and Cosnard's unimodal set, 46
$\Gamma_{\boldsymbol{a}}(q)$	Allouche's and Cosnard's generalised set, 46
$\delta_q(\boldsymbol{x},\boldsymbol{y})$	distance, 13
$\mu_{\boldsymbol{u}}$	self-similarity map, 68
ϕ_β	dimension function, 11, 21
ϕ_q	dimension function, 12, 28, 45
$\mathcal{A}$	attractor set, 67
$\boldsymbol{a}_k(\boldsymbol{s},\beta)$	the one-sided increasing accumulation sequence, 26
$\boldsymbol{a}_k(\boldsymbol{s},q)$	the two-sided increasing accumulation sequence, 61
$\boldsymbol{b}_k(\boldsymbol{s},\beta)$	the one-sided decreasing accumulation sequence, 26
$\boldsymbol{b}_k(\boldsymbol{s},q)$	the two-sided decreasing accumulation sequence, 63
$B(c,\beta)$	extremal β-value, 37
BAN	Badly Approximable Numbers, 9
$d(\boldsymbol{s},q)$	the limit under reflection, 51
$d(x,\beta)$	expansion of x in base β, 35
$\dim_H Y$	Hausdorff dimension, 17
$e(\boldsymbol{c},q)$	the extremal map, 54
$\boldsymbol{e}_{q,i}$	subdivision marker of the parameter space, 54
$\boldsymbol{e}^r_{q,0}$	subdivision marker of the parameter space, 54
$f(\boldsymbol{c},q)$	reflection function, 50
$F(c,\beta)$	set of q-adically BAN, 21
$F(c,q)$	set of q-adically BAN, 45
$F^1(c,\beta)$	set of q-adically BAN, 11, 21
$F^2(c,q)$	set of q-adically BAN, 11, 45

References

[1] J. P. Allouche. Théorie des nombres et automates. *Thèse d'État, Université Bordeaux I* (1983).

[2] J. P. Allouche, M. Cosnard. Itérations de foctiones unimodales et suites engendrées par automates. *C. R. Acad. Sci. Paris Sér. I Math.* (3) **296** (1983), 159-162.

[3] J. P. Allouche, M. Cosnard. Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set. *Acta Math. Hungar.* **91** (2001), 325-332.

[4] J. P. Allouche, J. Shallit. The Ubiquitous Prouhet-Thue-Morse Sequence. *In Sequences and Their applications, Proc. SETA'98 (Ed. C. Ding, T. Helleseth, and H. Niederreiter). New York: Springer-Verlag* (1999), 1-16.

[5] J. Berstel. Axel Thue's Papers on Repetition in Words: a Translation *Publications du Laboratoire de Combinatoire et d"Informatique Mathématique, 20, Université du Québec Montréal*, (1995).

[6] M. Dekking. Trancendance du nombre de Thue-Morse. *C. R. Acad. Sci. Paris Sér. I Math.* **285** (1977), 157-160.

[7] A. Dubickas. On the distance from a rational power to the nearest integer. *J. Number Theory* **117** (2006), 222-239.

[8] K. Falconer. Fractal geometry. *Wiley* (1990).

[9] A. Khintchine. Continued Fractions. *P. Noordhoff Ltd* (1963).

[10] B. P. Kitchens. Symbolic dynamics, one-sided, two-sided and countable state Markov shifts. *Springer* (1998).

[11] V. Komornik, P. Loreti. Unique Development in Non-Integer Bases. *Amer. Math. Monthly* **105** (1998), 636-639.

[12] R. Labarca, C. G. Moreira. Essential dynamics for Lorenz maps on the real line and the lexicographical world. *Ann. Inst. H. Poincar Anal. Non Linaire* **23** no. 5, (2006), 683–694.

[13] K. Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. *Math. Ann.* **101** (1929), 342-366, *Corrigendum* **103** (1930), 532.

[14] C. G. Moreira. Maximal invariant sets for restrictions of tent and unimodal maps. *Qual. Theory Dyn. Syst.* **2** no. 2, (2001), 385-398.

[15] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401-416.

[16] O. Perron. Zur Theorie der Matrices *Math. Ann.* **64**, no. 2 (1907), 248-263.

[17] Y. Pesin. Dimension Theory in Dynamical Systems. *The University of Chicago Press* (1997).

[18] T. Persson, J. Schmeling. Dyadic Diophantine Approximation and Katok's Horseshoe Approximation. *Preprints in Mathematical Sciences, Lund University.* no. 3 (2006).

[19] A. Rényi. Representation for real numbers and their ergodic properties. *Acta Math. Acad Sci. Hung.* **8** (1957), 477-493.

[20] J. Schmeling. Symbolic dynamics for β-shifts and self-normal numbers. *Ergod. Th. & Dynam. Sys.* **17** (1997), 675-694.

[21] A. Thue. Über unendliche Zeichenreihern. *Norske vid. Selsk. Skr. Mat. Nat. Kl.*, **7** (1906) 1-22. Reprinted in "Selected mathematical papers of Axel Thue", Nagell T., ed. Universitetsforlaget, Oslo, (1977) 139-158.

[22] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihern. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912) 1-67, Reprinted in "Selected mathematical papers of Axel Thue", Nagell T., ed. Universitetsforlaget, Oslo, (1977) 413-478.

[23] M. Urbanski. On Hausdorff dimension of invariant sets for expanding maps of the circle. *Ergod. Th. & Dynam. Sys.* **6** (1986), 295-309.