

## Error bounds on the non-normal approximation of Hermite power variations of fractional Brownian motion

Jean-Christophe Breton<sup>1</sup> and Ivan Nourdin<sup>2</sup>  
*Université de La Rochelle and Université Paris VI*

**Abstract:** Let  $q \geq 2$  be a positive integer,  $B$  be a fractional Brownian motion with Hurst index  $H \in (0, 1)$ ,  $Z$  be an Hermite random variable of index  $q$ , and  $H_q$  denote the Hermite polynomial having degree  $q$ . For any  $n \geq 1$ , set  $V_n = \sum_{k=0}^{n-1} H_q(B_{k+1} - B_k)$ . The aim of the current paper is to derive, in the case when the Hurst index verifies  $H > 1 - 1/(2q)$ , an upper bound for the total variation distance between the laws  $\mathcal{L}(Z_n)$  and  $\mathcal{L}(Z)$ , where  $Z_n$  stands for the correct renormalization of  $V_n$  which converges in distribution towards  $Z$ . Our results should be compared with those obtained recently by Nourdin and Peccati (2007) in the case when  $H < 1 - 1/(2q)$ , corresponding to the situation where one has normal approximation.

*Key words:* Total variation distance; Non-central limit theorem; Fractional Brownian motion; Hermite power variation; Multiple stochastic integrals; Hermite random variable.

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### 1 Introduction

Let  $q \geq 2$  be a positive integer and  $B$  be a fractional Brownian motion (fBm) with Hurst index  $H \in (0, 1)$ . The asymptotic behavior of the  $q$ -Hermite power variations of  $B$  with respect to  $\mathbb{N}$ , defined as

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1} - B_k), \quad n \geq 1, \quad (1.1)$$

has recently received a lot of attention, see e.g. [9], [10] and references therein. Here,  $H_q$  stands for the Hermite polynomial with degree  $q$ , given by  $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} (e^{-x^2/2})$ . We have  $H_2(x) = x^2 - 1$ ,  $H_3(x) = x^3 - 3x$ , and so on. The analysis of the asymptotic behavior of (1.1) is motivated, for instance, by the traditional applications of quadratic variations to parameter estimation problems (see e.g. [1, 4, 8, 13] and references therein).

In the particular case of the standard Brownian motion (that is when  $H = \frac{1}{2}$ ), the asymptotic behavior of (1.1) can be immediately deduced from the classical central limit theorem. When  $H \neq \frac{1}{2}$ , the increments of  $B$  are not independent anymore and the asymptotic behavior of (1.1) is consequently more difficult to reach. However, thanks to the seminal works of Breuer and Major [3], Dobrushin and Major [6], Giraitis and Surgailis [7] and Taqqu [12], it is well-known that we have, as  $n \rightarrow \infty$ :

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<sup>1</sup>Université de La Rochelle, Laboratoire Mathématiques, Image et Applications, Avenue Michel Crépeau, 17042 La Rochelle Cedex, France, [jean-christophe.breton@univ-lr.fr](mailto:jean-christophe.breton@univ-lr.fr)

<sup>2</sup>Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie (Paris VI), Boîte courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France, [ivan.nourdin@upmc.fr](mailto:ivan.nourdin@upmc.fr)

1. If  $0 < H < 1 - 1/(2q)$  then

$$Z_n := \frac{V_n}{\sigma_{q,H} \sqrt{n}} \xrightarrow{\text{Law}} \mathcal{N}(0, 1). \quad (1.2)$$

2. If  $H = 1 - 1/(2q)$  then

$$Z_n := \frac{V_n}{\sigma_{q,H} \sqrt{n \log n}} \xrightarrow{\text{Law}} \mathcal{N}(0, 1). \quad (1.3)$$

3. If  $H > 1 - 1/(2q)$  then

$$Z_n := \frac{V_n}{n^{1-q(1-H)}} \xrightarrow{\text{Law}} Z \sim \text{“Hermite random variable”}. \quad (1.4)$$

Here,  $\sigma_{q,H} > 0$  denotes an (explicit) constant depending only on  $q$  and  $H$ . Moreover, the Hermite random variable  $Z$  appearing in (1.4) is defined as the value at time 1 of the Hermite process, i.e.

$$Z = I_q^W(L_1), \quad (1.5)$$

where  $I_q^W$  denotes the  $q$ -multiple stochastic integral with respect to a Wiener process  $W$ , while  $L_1$  is the symmetric kernel defined as

$$L_1(y_1, \dots, y_q) = \frac{1}{q!} \mathbf{1}_{[0,1]^q}(y_1, \dots, y_q) \int_{y_1 \vee \dots \vee y_q}^1 \partial_1 K_H(u, y_1) \dots \partial_1 K_H(u, y_q) du,$$

with  $K_H$  the square integrable kernel given by (2.2). We refer to [10] for a complete discussion of this subject.

The exact expression of the distribution function of  $Z_n$  is very complicated when  $H \neq \frac{1}{2}$ . That is why, when  $n$  is assumed to be large, it is common to use (1.2)–(1.4) as a justification to replace, in any computation involving it, the distribution function of  $Z_n$  by that of the corresponding limit. Of course, if one applies this strategy without any care (in particular, if one has no idea, even imprecise, of an error bounds in terms of  $n$ ), then it is easy to imagine that the obtained result could be very far from the reality (as a “concrete” example, see (1.6) below, which represents the worst case we will obtain here). To the best of our knowledge, in all the works using (1.2)–(1.4) with statistical applications in mind (for instance [1, 4, 8, 13]), never their author(s) considered this problem. The current paper, together with [11], seem to be the first attempt in such direction.

Recall that the *total variation distance* between the laws of two real-valued random variables  $Y$  and  $X$  is defined as

$$d_{TV}(\mathcal{L}(Y), \mathcal{L}(X)) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P(Y \in A) - P(X \in A)|$$

where  $\mathcal{B}(\mathbb{R})$  denotes the set of Borelian of  $\mathbb{R}$ . In [11], by combining Stein’s method with Malliavin calculus (see also Theorem 1.3 below), the following result is shown:

**Theorem 1.1.** *If  $H < 1 - 1/(2q)$  then, for some constant  $c_{q,H} > 0$  depending uniquely on  $q$  and  $H$ , we have:*

$$d_{TV}(\mathcal{L}(Z_n), \mathcal{N}(0, 1)) \leq c_{q,H} \begin{cases} n^{-1/2} & \text{if } H \in (0, \frac{1}{2}] \\ n^{H-1} & \text{if } H \in [\frac{1}{2}, \frac{2q-3}{2q-2}] \\ n^{qH-q+\frac{1}{2}} & \text{if } H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}) \end{cases}$$

for  $Z_n$  defined by (1.2).

Here, we deal with the remaining cases, that is when  $H \in [1 - \frac{1}{2q}, 1)$ . Our main result is as follows:

**Theorem 1.2.** *1. If  $H = 1 - 1/(2q)$  then, for some constant  $c_{q,H} > 0$  depending uniquely on  $q$  and  $H$ , we have*

$$d_{TV}(\mathcal{L}(Z_n), \mathcal{N}(0, 1)) \leq \frac{c_{q,H}}{\sqrt{\log n}} \quad (1.6)$$

for  $Z_n$  defined by (1.3).

*2. If  $H \in (1 - 1/(2q), 1)$  then, for some constant  $c_{q,H} > 0$  depending uniquely on  $q$  and  $H$ , we have*

$$d_{TV}(\mathcal{L}(Z_n), \mathcal{L}(Z)) \leq c_{q,H} n^{1-\frac{1}{2q}-H} \quad (1.7)$$

for  $Z_n$  and  $Z$  defined by (1.4).

Actually, the case when  $H = 1 - 1/(2q)$  can be tackled by mimicking the proof of Theorem 1.1. Only minor changes are required: we will also conclude thanks to the following general result by Nourdin and Peccati.

**Theorem 1.3.** *(cf. [11]) Fix an integer  $q \geq 2$  and let  $\{f_n\}_{n \geq 1}$  be a sequence of  $\mathfrak{H}^{\odot q}$ . Then we have*

$$d_{TV}(\mathcal{L}(I_q(f_n)), \mathcal{N}(0, 1)) \leq 2 \sqrt{E \left( 1 - \frac{1}{q} \|DI_q(f_n)\|_{\mathfrak{H}}^2 \right)^2},$$

where  $D$  stands for the Malliavin derivative with respect to  $X$ .

Here, and for the rest of the paper,  $X$  denotes a centered Gaussian isonormal process on a real separable Hilbert space  $\mathfrak{H}$  and, as usual,  $\mathfrak{H}^{\odot q}$  (resp.  $I_q$ ) stands for the  $q$ th symmetric tensor product of  $\mathfrak{H}$  (resp. the multiple Wiener-Itô integral of order  $q$  with respect to  $X$ ). See Section 2 for more precise definitions and properties.

When  $H \in (1 - 1/(2q), 1)$ , Theorem 1.3 can not be used (the limit in (1.4) being not Gaussian), and another argument is required. Our new idea is as follows. First, using the scaling property (2.1) of fBm, we construct, for every fixed  $n$ , a copy  $S_n$  of  $Z_n$  that converges in  $L^2$ . Then, we use the following result by Davydov and Martynova.

**Theorem 1.4.** *(cf. [5]; see also [2]) Fix an integer  $q \geq 2$  and let  $f \in \mathfrak{H}^{\odot q} \setminus \{0\}$ . Then, for any sequence  $\{f_n\}_{n \geq 1} \subset \mathfrak{H}^{\odot q}$  converging to  $f$ , there exists a constant  $c_{q,f}$ , depending only on  $q$  and  $f$ , such that:*

$$d_{TV}(\mathcal{L}(I_q(f_n)), \mathcal{L}(I_q(f))) \leq c_{q,f} \|f_n - f\|_{\mathfrak{H}^{\odot q}}^{1/q}.$$

The rest of the paper is organized as follows. In Section 2, some preliminary results on fractional Brownian motion and Malliavin calculus are presented. Section 3 deals with the case  $H \in (1 - 1/(2q), 1)$ , while the critical case  $H = 1 - 1/(2q)$  is considered in Section 4.

## 2 Preliminaries

Let  $B = \{B_t, t \geq 0\}$  be a fBm with Hurst index  $H \in (0, 1)$ , that is a centered Gaussian process, started from zero and with covariance function  $E(B_s B_t) = R(s, t)$ , where

$$R(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}); \quad s, t \geq 0.$$

In particular, it is immediately shown that  $B$  has stationary increments and is selfsimilar of index  $H$ . Precisely, for any  $h, c > 0$ , we have

$$\{B_{t+h} - B_h, t \geq 0\} \stackrel{\text{Law}}{=} \{B_t, t \geq 0\} \quad \text{and} \quad \{c^{-H} B_{ct}, t \geq 0\} \stackrel{\text{Law}}{=} \{B_t, t \geq 0\}. \quad (2.1)$$

For any choice of the Hurst parameter  $H \in (0, 1)$ , the Gaussian space generated by  $B$  can be identified with an isonormal Gaussian process of the type  $B = \{B(h) : h \in \mathfrak{H}\}$ , where the real and separable Hilbert space  $\mathfrak{H}$  is defined as follows: (i) denote by  $\mathcal{E}$  the set of all  $\mathbb{R}$ -valued step functions on  $[0, \infty)$ , (ii) define  $\mathfrak{H}$  as the Hilbert space obtained by closing  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R(t, s).$$

In particular, with such a notation, one has that  $B_t = B(\mathbf{1}_{[0,t]})$ .

From now, assume on one hand that  $B$  is defined on  $[0, 1]$  and on the other hand that the Hurst index verifies  $H > \frac{1}{2}$ . The covariance kernel  $R$  can be written as

$$R(t, s) = \int_0^{s \wedge t} K_H(t, r) K_H(s, r) dr,$$

where  $K_H$  is the square integrable kernel given by

$$K_H(t, s) = \Gamma\left(H + \frac{1}{2}\right)^{-1} (t - s)^{H - \frac{1}{2}} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right), \quad (2.2)$$

$F(a, b, c, z)$  being the Gauss hypergeometric function. Consider the linear operator  $K_H^*$  from  $\mathcal{E}$  to  $L^2([0, 1])$  defined by

$$(K_H^* \varphi)(s) = K_H(1, s) \varphi(s) + \int_s^1 (\varphi(r) - \varphi(s)) \partial_1 K_H(r, s) dr.$$

For any pair of step functions  $\varphi$  and  $\psi$  in  $\mathcal{E}$ , we have  $\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2} = \langle \varphi, \psi \rangle_{\mathfrak{H}}$ . As a consequence, the operator  $K_H^*$  provides an isometry between the Hilbert spaces  $\mathfrak{H}$  and  $L^2([0, 1])$ . Hence, the process  $W = (W_t)_{t \in [0, 1]}$  defined by

$$W_t = B((K_H^*)^{-1}(\mathbf{1}_{[0,t]})) \quad (2.3)$$

is a Wiener process, and the process  $B$  has an integral representation of the form  $B_t = \int_0^t K_H(t, s) dW_s$ , because  $(K_H^* \mathbf{1}_{[0, t]})(s) = K_H(t, s)$ .

The elements of  $\mathfrak{H}$  may be not functions but distributions. However,  $\mathfrak{H}$  contains the subset  $|\mathfrak{H}|$  of all measurable functions  $f : [0, 1] \rightarrow \mathbb{R}$  such that

$$\int_{[0, 1]^2} |f(u)||f(v)||u - v|^{2H-2} dudv < \infty.$$

Moreover, for  $f, g \in |\mathfrak{H}|$ , we have

$$\langle f, g \rangle_{\mathfrak{H}} = H(2H - 1) \int_{[0, 1]^2} f(u)g(v)|u - v|^{2H-2} dudv.$$

In the sequel, we note  $\mathfrak{H}^{\otimes q}$  and  $\mathfrak{H}^{\odot q}$ , respectively, the tensor space and the symmetric tensor space of order  $q \geq 1$ . Let  $\{e_k : k \geq 1\}$  be a complete orthogonal system in  $\mathfrak{H}$ . Given  $f \in \mathfrak{H}^{\odot p}$  and  $g \in \mathfrak{H}^{\odot q}$ , for every  $r = 0, \dots, p \wedge q$ , the  $r$ th contraction of  $f$  and  $g$  is the element of  $\mathfrak{H}^{\odot(p+q-2r)}$  defined as

$$f \otimes_r g = \sum_{i_1=1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

In particular, note that  $f \otimes_0 g = f \otimes g$  and, when  $p = q$ , that  $f \otimes_p g = \langle f, g \rangle_{\mathfrak{H}}$ . Since, in general, the contraction  $f \otimes_r g$  is not a symmetric element of  $\mathfrak{H}^{\odot(p+q-2r)}$ , we define  $f \widetilde{\otimes}_r g$  as the canonical symmetrization of  $f \otimes_r g$ . When  $f \in \mathfrak{H}^{\odot q}$ , we write  $I_q(f)$  to indicate its  $q$ th multiple integral with respect to  $B$ . The following formula is useful to compute the product of such integrals: if  $f \in \mathfrak{H}^{\odot p}$  and  $g \in \mathfrak{H}^{\odot q}$ , then

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \quad (2.4)$$

Let  $\mathcal{S}$  be the set of cylindrical functionals  $F$  of the form

$$F = \varphi(B(h_1), \dots, B(h_n)), \quad (2.5)$$

where  $n \geq 1$ ,  $h_i \in \mathfrak{H}$  and the function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$  is such that its partial derivatives have polynomial growth. The Malliavin derivative  $DF$  of a functional  $F$  of the form (2.5) is the square integrable  $\mathfrak{H}$ -valued random variable defined as

$$DF = \sum_{i=1}^n \partial_i \varphi(B(h_1), \dots, B(h_n)) h_i,$$

where  $\partial_i \varphi$  denotes the  $i$ th partial derivative of  $\varphi$ . In particular, one has that  $D_s B_t = \mathbf{1}_{[0, t]}(s)$  for every  $s, t \in [0, 1]$ . As usual,  $\mathbb{D}^{1,2}$  denotes the closure of  $\mathcal{S}$  with respect to the norm  $\|\cdot\|_{1,2}$ , defined by the relation  $\|F\|_{1,2}^2 = E|F|^2 + E\|DF\|_{\mathfrak{H}}^2$ . Note that every multiple integral belongs to  $\mathbb{D}^{1,2}$ . Moreover, we have

$$D_t(I_q(f)) = qI_{q-1}(f(\cdot, t))$$

for any  $f \in \mathfrak{H}^{\odot q}$  and  $t \geq 0$ . The Malliavin derivative  $D$  also satisfies the following chain rule formula: if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable with bounded derivatives and if  $(F_1, \dots, F_n)$  is a random vector such that each component belongs to  $\mathbb{D}^{1,2}$ , then  $\varphi(F_1, \dots, F_n)$  is itself an element of  $\mathbb{D}^{1,2}$ , and moreover

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^n \partial_i \varphi(F_1, \dots, F_n) DF_i.$$

### 3 Case $H \in (1 - 1/(2q), 1)$

In this section, we fix  $q \geq 2$ , we assume that  $H > 1 - \frac{1}{2q}$  and we consider  $Z$  defined by (1.5) for  $W$  the Wiener process defined by (2.3). By the scaling property (2.1) of fBm, remark first that  $Z_n$ , defined by (1.4), has the same law, for any fixed  $n \geq 1$ , as

$$S_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q(n^H(B_{(k+1)/n} - B_{k/n})) = I_q(f_n), \quad (3.1)$$

for  $f_n = n^{q-1} \sum_{k=0}^{n-1} \mathbf{1}_{[k/n, (k+1)/n]}^{\otimes q} \in \mathfrak{H}^{\odot q}$ . In [10], Theorem 1 (point 3), it is shown that the sequence  $\{S_n\}_{n \geq 1}$  converges in  $L^2$  towards  $Z$ , or equivalently that  $\{f_n\}_{n \geq 1}$  is Cauchy in  $\mathfrak{H}^{\odot q}$ . Here, we precise the rate of this convergence:

**Proposition 3.1.** *Let  $f$  denote the limit of the Cauchy sequence  $\{f_n\}_{n \geq 1}$  in  $\mathfrak{H}^{\odot q}$ . We have*

$$E|S_n - Z|^2 = E|I_q(f_n) - I_q(f)|^2 = \|f_n - f\|_{\mathfrak{H}^{\odot q}}^2 = O(n^{2q-1-2qH}), \quad \text{as } n \rightarrow \infty.$$

Proposition 3.1, together with Theorem 1.4 above, immediately entails (1.7) so that the rest of this section is devoted to the proof of the proposition.

*Proof of Proposition 3.1.* We have

$$\begin{aligned} \|f_n\|_{\mathfrak{H}^{\odot q}}^2 &= n^{2q-2} \sum_{k,l=0}^{n-1} \langle \mathbf{1}_{[k/n, (k+1)/n]}, \mathbf{1}_{[l/n, (l+1)/n]} \rangle_{\mathfrak{H}}^q \\ &= H^q (2H-1)^q n^{2q-2} \sum_{k,l=0}^{n-1} \left( \int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv |u-v|^{2qH-2} \right)^q. \end{aligned} \quad (3.2)$$

By letting  $n$  goes to infinity, we obtain

$$\begin{aligned} \|f\|_{\mathfrak{H}^{\odot q}}^2 &= H^q (2H-1)^q \int_{[0,1]^2} |u-v|^{2qH-2q} dudv \\ &= H^q (2H-1)^q \sum_{k,l=0}^{n-1} \int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv |u-v|^{2qH-2q}. \end{aligned} \quad (3.3)$$

Now, let  $\phi \in |\mathfrak{H}|$ . We have

$$\langle f_n, \phi^{\otimes q} \rangle_{\mathfrak{H}^{\odot q}} = n^{q-1} \sum_{l=0}^{n-1} \langle \mathbf{1}_{[l/n, (l+1)/n]}, \phi \rangle_{\mathfrak{H}}^q$$

$$= H^q(2H-1)^q n^{q-1} \sum_{l=0}^{n-1} \left( \int_{l/n}^{(l+1)/n} dv \int_0^1 du \phi(u) |u-v|^{2H-2} \right)^q.$$

By letting  $n$  goes to infinity, we obtain

$$\langle f, \phi^{\otimes q} \rangle_{\mathfrak{H}^{\odot q}} = H^q(2H-1)^q \int_0^1 dv \left( \int_0^1 du \phi(u) |u-v|^{2H-2} \right)^q.$$

Hence, we have

$$\begin{aligned} \langle f, f_n \rangle_{\mathfrak{H}^{\odot q}} &= H^q(2H-1)^q n^{q-1} \sum_{k=0}^{n-1} \int_0^1 dv \left( \int_{k/n}^{(k+1)/n} du |u-v|^{2H-2} \right)^q \\ &= H^q(2H-1)^q n^{q-1} \sum_{k,l=0}^{n-1} \int_{l/n}^{(l+1)/n} dv \left( \int_{k/n}^{(k+1)/n} du |u-v|^{2H-2} \right)^q. \end{aligned} \quad (3.4)$$

Finally, by combining (3.2), (3.3) and (3.4), and by using among others elementary change of variables, we can write:

$$\begin{aligned} \|f_n - f\|_{\mathfrak{H}^{\odot q}}^2 &= H^q(2H-1)^q \sum_{k,l=0}^{n-1} \left\{ n^{2q-2} \left( \int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv |u-v|^{2H-2} \right)^q \right. \\ &\quad - 2n^{q-1} \int_{l/n}^{(l+1)/n} dv \left( \int_{k/n}^{(k+1)/n} du |u-v|^{2H-2} \right)^q \\ &\quad \left. + \int_{k/n}^{(k+1)/n} dw \int_{l/n}^{(l+1)/n} dz |w-z|^{2qH-2q} \right\} \\ &= H^q(2H-1)^q n^{2q-2-2qH} \sum_{k,l=0}^{n-1} \left\{ \left( \int_0^1 du \int_0^1 dv |k-l+u-v|^{2H-2} \right)^q \right. \\ &\quad \left. - 2 \int_0^1 dv \left( \int_0^1 du |k-l+u-v|^{2H-2} \right)^q + \int_0^1 du \int_0^1 dv |k-l+u-v|^{2qH-2q} \right\} \\ &\leq H^q(2H-1)^q n^{2q-1-2qH} \sum_{r \in \mathbb{Z}} \left| \left( \int_0^1 du \int_0^1 dv |r+u-v|^{2H-2} \right)^q \right. \\ &\quad \left. - 2 \int_0^1 dv \left( \int_0^1 du |r+u-v|^{2H-2} \right)^q + \int_0^1 du \int_0^1 dv |r+u-v|^{2qH-2q} \right|. \end{aligned} \quad (3.5)$$

Consequently, to achieve the proof of Theorem 3.1, it remains to ensure that the sum over  $\mathbb{Z}$  in (3.5) is finite. For  $r > 1$ , elementary computations give

$$\begin{aligned} \left( \int_0^1 du \int_0^1 dv |r+u-v|^{2H-2} \right)^q &= (2H(2H-1))^{-q} ((r+1)^{2H} - 2r^{2H} + (r-1)^{2H})^q \\ &= (r^{2H-2} + O(r^{2H-4}))^q \\ &= r^{2qH-2q} + O(r^{2qH-2q-2}) \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \int_0^1 du \int_0^1 dv |r + u - v|^{2qH-2q} &= \frac{(r+1)^{2qH-2q+2} - 2r^{2qH-2q+2} + (r-1)^{2qH-2q+2}}{(2qH-2q+1)(2qH-2q+2)} \\ &= r^{2qH-2q} + O(r^{2qH-2q-2}). \end{aligned} \quad (3.7)$$

Moreover, using the inequality  $|(1+x)^{2H-1} - 1 - (2H-1)x| \leq (2H-1)(H-1)x^2$  for  $x \in [0, 1]$ , we can write

$$\begin{aligned} \int_0^1 dv \left( \int_0^1 du |r + u - v|^{2H-2} \right)^q &= (2H-1)^{-q} \int_0^1 ((r+1-v)^{2H-1} - (r-v)^{2H-1})^q dv \\ &= (2H-1)^{-q} \int_0^1 (r-v)^{2qH-q} \left( \left(1 + \frac{1}{r-v}\right)^{2H-1} - 1 \right)^q dv \\ &= \int_0^1 (r-v)^{2qH-q} \left( \frac{1}{r-v} + R\left(\frac{1}{r-v}\right) \right)^q dv \end{aligned}$$

where the remainder term  $R$  verifies  $|R(u)| \leq (1-H)u^2$ . In particular, for any  $v \in [0, 1]$ , we have

$$(r-v) \left| R\left(\frac{1}{r-v}\right) \right| \leq \frac{1-H}{r-1}.$$

Hence, we deduce:

$$\begin{aligned} \int_0^1 dv \left( \int_0^1 du |r + u - v|^{2H-2} \right)^q &= \int_0^1 (r-v)^{2qH-2q} (1 + O(1/r))^q dv \\ &= r^{2qH-2q+1} \frac{1 - (1-1/r)^{2qH-2q+1}}{2qH-2q+1} (1 + O(1/r)) \\ &= r^{2qH-2q+1} (1/r + O(1/r^2)) (1 + O(1/r)) \\ &= r^{2qH-2q} + O(r^{2qH-2q-1}). \end{aligned} \quad (3.8)$$

By combining (3.6), (3.7) and (3.8), we obtain (since similar arguments also apply for  $r < -1$ ) that

$$\begin{aligned} \left( \int_0^1 du \int_0^1 dv |r + u - v|^{2H-2} \right)^q - 2 \int_0^1 dv \left( \int_0^1 du |r + u - v|^{2H-2} \right)^q \\ + \int_0^1 du \int_0^1 dv |r + u - v|^{2qH-2q} \end{aligned}$$

is  $O(|r|^{2qH-2q-1})$ , so that the sum over  $\mathbb{Z}$  in (3.5) is finite. The proof of Proposition 3.1 is done.  $\square$

#### 4 Case $H = 1 - 1/(2q)$

As we already pointed out in the Introduction, the proof of (1.6) is a slight adaptation of that of Theorem 1.1 (that is Theorem 4.1 in [11]) which was devoted to the case when  $H < 1 - 1/(2q)$ . That is why we will only focus, here, on the differences between the cases

$H < 1 - 1/(2q)$  and  $H = 1 - 1/(2q)$ . In particular, we will freely refer to [11] each time we need an estimate already computed therein.

From now, fix  $H = 1 - 1/(2q)$  and let us evaluate the right-hand side in Theorem 1.3. Once again, instead of  $Z_n$ , we will rather use  $S_n$  defined by

$$S_n = \frac{1}{\sigma_H \sqrt{n \log n}} \sum_{k=0}^{n-1} H_q(n^H(B_{(k+1)/n} - B_{k/n})) = I_q \left( \frac{n^{q-1}}{\sigma_H \sqrt{\log n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k/n, (k+1)/n]}^{\otimes q} \right)$$

in the sequel, in order to facilitate the connection with [11]. First, observe that the covariance function  $\rho_H$  of the Gaussian sequence  $(n^H(B_{(r+1)/n} - B_{r/n}))_{r \geq 0}$ , given by

$$\rho_H(r) = \frac{1}{2} (|r+1|^{2-1/q} - 2|r|^{2-1/q} + |r-1|^{2-1/q}),$$

verifies the following straightforward expansion:

$$\rho_H(r)^q = \left( \left(1 - \frac{1}{2q}\right) \left(1 - \frac{1}{q}\right) \right)^q |r|^{-1} + O(|r|^{-3}), \quad \text{as } |r| \rightarrow \infty. \quad (4.1)$$

Using  $n^{2-1/q} \langle \mathbf{1}_{[k/n, (k+1)/n]}, \mathbf{1}_{[l/n, (l+1)/n]} \rangle_{\mathfrak{H}} = \rho_H(k-l)$ , note that

$$\begin{aligned} \text{Var}(S_n) &= \frac{n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} E [I_q(\mathbf{1}_{[k/n, (k+1)/n]}^{\otimes q}) I_q(\mathbf{1}_{[l/n, (l+1)/n]}^{\otimes q})] \\ &= \frac{q! n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} \langle \mathbf{1}_{[k/n, (k+1)/n]}, \mathbf{1}_{[l/n, (l+1)/n]} \rangle_{\mathfrak{H}}^q = \frac{q!}{\sigma_H^2 n \log n} \sum_{k,l=0}^{n-1} \rho_H(k-l)^q \end{aligned}$$

from which, together with (4.1), we deduce the exact value of  $\sigma_H^2$ :

$$\sigma_H^2 := \lim_{n \rightarrow +\infty} \frac{q!}{n \log n} \sum_{k,l=0}^{n-1} \rho_H(k-l)^q = 2q! \left( \frac{(2q-1)(q-1)}{2q^2} \right)^q. \quad (4.2)$$

In order to apply Theorem 1.3, we compute the Malliavin derivative of  $S_n$ :

$$DS_n = \frac{qn^{q-1}}{\sigma_H \sqrt{\log n}} \sum_{k=0}^{n-1} I_{q-1}(\mathbf{1}_{[k/n, (k+1)/n]}^{\otimes q-1}) \mathbf{1}_{[k/n, (k+1)/n]}.$$

Hence

$$\|DS_n\|_{\mathfrak{H}}^2 = \frac{q^2 n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} I_{q-1}(\mathbf{1}_{[k/n, (k+1)/n]}^{\otimes q-1}) I_{q-1}(\mathbf{1}_{[l/n, (l+1)/n]}^{\otimes q-1}) \langle \mathbf{1}_{[k/n, (k+1)/n]}, \mathbf{1}_{[l/n, (l+1)/n]} \rangle_{\mathfrak{H}}.$$

The multiplication formula (2.4) yields

$$\|DS_n\|_{\mathfrak{H}}^2 = \frac{q^2 n^{2q-2}}{\sigma_H^2 \log n} \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 \times$$

$$\times \sum_{k,l=0}^{n-1} I_{2q-2-2r} \left( \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r} \tilde{\otimes} \mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r} \right) \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle_{\mathfrak{H}}^{r+1}.$$

We can rewrite

$$1 - \frac{1}{q} \|DS_n\|_{\mathfrak{H}}^2 = 1 - \sum_{r=0}^{q-1} A_r(n)$$

where

$$\begin{aligned} A_r(n) &= \frac{q r! \binom{q-1}{r}^2 n^{2q-2}}{\sigma_H^2 \log n} \times \\ &\times \sum_{k,l=0}^{n-1} I_{2q-2-2r} \left( \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r} \tilde{\otimes} \mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r} \right) \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle_{\mathfrak{H}}^{r+1}. \end{aligned}$$

For the term  $A_{q-1}(n)$ , we have:

$$\begin{aligned} 1 - A_{q-1}(n) &= 1 - \frac{q!}{\sigma_H^2 \log n} n^{2q-2} \sum_{k,l=0}^{n-1} \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle_{\mathfrak{H}}^q \\ &= 1 - \frac{q!}{\sigma_H^2 n \log n} \sum_{k,l=0}^{n-1} \rho_H(k-l)^q = 1 - \frac{q!}{\sigma_H^2 n \log n} \sum_{|r|<n} (n-|r|) \rho_H(r)^q \\ &= 1 - \frac{q!}{\sigma_H^2 \log n} \sum_{|r|<n} \rho_H(r)^q + \frac{q!}{\sigma_H^2 n \log n} \sum_{|r|<n} |r| \rho_H(r)^q = O(1/\log n) \end{aligned}$$

where the last estimate comes from the development (4.1) of  $\rho_H$  and from the exact value (4.2) of  $\sigma_H^2$ .

Next, we show that for any fixed  $r \leq q-2$ , we have  $E|A_r(n)|^2 = O(1/\log n)$ . Indeed:

$$\begin{aligned} E|A_r(n)|^2 &= c(H, r, q) \frac{n^{4q-4}}{\log^2 n} \sum_{i,j,k,l=0}^{n-1} \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle_{\mathfrak{H}}^{r+1} \langle \mathbf{1}_{[i/n,(i+1)/n]}, \mathbf{1}_{[j/n,(j+1)/n]} \rangle_{\mathfrak{H}}^{r+1} \\ &\times \langle \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r} \tilde{\otimes} \mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r}, \mathbf{1}_{[i/n,(i+1)/n]}^{\otimes q-1-r} \otimes \mathbf{1}_{[j/n,(j+1)/n]}^{\otimes q-1-r} \rangle_{\mathfrak{H}^{\otimes 2q-2-2r}} \\ &= \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = q-r-1}} \sum_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta = q-r-1}} c(H, r, q, \alpha, \beta, \gamma, \delta) B_{r,\alpha,\beta,\gamma,\delta}(n) \end{aligned}$$

where  $c(\cdot)$  is a generic constant depending only on its arguments and

$$\begin{aligned} B_{r,\alpha,\beta,\gamma,\delta}(n) &= \frac{n^{4q-4}}{\log n^2} \sum_{i,j,k,l=0}^{n-1} \\ &\langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle_{\mathfrak{H}}^{r+1} \langle \mathbf{1}_{[i/n,(i+1)/n]}, \mathbf{1}_{[j/n,(j+1)/n]} \rangle_{\mathfrak{H}}^{r+1} \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[i/n,(i+1)/n]} \rangle_{\mathfrak{H}}^{\alpha} \end{aligned}$$

$$\begin{aligned}
& \langle \mathbf{1}_{[k/n, (k+1)/n]}, \mathbf{1}_{[j/n, (j+1)/n]} \rangle_{\mathfrak{F}}^\beta \langle \mathbf{1}_{[l/n, (l+1)/n]}, \mathbf{1}_{[i/n, (i+1)/n]} \rangle_{\mathfrak{F}}^\gamma \langle \mathbf{1}_{[l/n, (l+1)/n]}, \mathbf{1}_{[j/n, (j+1)/n]} \rangle_{\mathfrak{F}}^\delta \\
&= \frac{1}{n^2 \log^2 n} \sum_{i,j,k,l=0}^{n-1} \rho_H(k-l)^{r+1} \rho_H(i-j)^{r+1} \rho_H(k-i)^\alpha \rho_H(k-j)^\beta \rho_H(l-i)^\gamma \rho_H(l-j)^\delta.
\end{aligned}$$

As in [11], when  $\alpha, \beta, \gamma, \delta$  are fixed, we can decompose the sum appearing in  $B_{r,\alpha,\beta,\gamma,\delta}(n)$  as follows:

$$\begin{aligned}
& \sum_{i=j=k=l} + \left( \sum_{\substack{i=j=k \\ l \neq i}} + \sum_{\substack{i=j=l \\ k \neq i}} + \sum_{\substack{i=k=l \\ j \neq i}} + \sum_{\substack{j=k=l \\ i \neq j}} \right) + \left( \sum_{\substack{i=j,k=l \\ k \neq i}} + \sum_{\substack{i=k,j=l \\ j \neq i}} + \sum_{\substack{i=l,j=k \\ j \neq i}} \right) \\
& + \left( \sum_{\substack{i=j,k \neq i \\ k \neq l, l \neq i}} + \sum_{\substack{i=k,j \neq i \\ j \neq l, k \neq l}} + \sum_{\substack{i=l,k \neq i \\ k \neq j, j \neq i}} + \sum_{\substack{j=k,k \neq i \\ k \neq l, l \neq i}} + \sum_{\substack{j=l,k \neq i \\ k \neq l, l \neq i}} + \sum_{\substack{k=l,k \neq i \\ k \neq j, j \neq i}} \right) + \sum_{\substack{i,j,k,l \\ \text{are all distinct}}}
\end{aligned}$$

where the indices run over  $\{0, 1, \dots, n-1\}$ . Similar computations as in [11] show that the first, second and third sums are  $O(1/(n \log^2 n))$ ; the fourth and fifth sums are  $O(1/(n^{1/q} \log^2 n))$ ; the sixth, seventh and eighth sums are  $O(1/(n^{2/q} \log^2 n))$ ; the ninth, tenth, eleventh, twelfth, thirteenth and fourteenth sums are also  $O(1/(n^{1/q} \log^2 n))$ . We focus only on the last sum. Actually, it is precisely its contribution which will indicate the true order in (1.6). Once again, we split this sum into 24 sums:

$$\sum_{k>l>i>j} + \sum_{k>l>j>i} + \dots \quad (4.3)$$

We first deal with the first one, for which we have

$$\begin{aligned}
& \frac{1}{(n \log n)^2} \sum_{k>l>i>j} \rho_H(k-l)^{r+1} \rho_H(i-j)^{r+1} \rho_H(k-i)^\alpha \rho_H(k-j)^\beta \rho_H(l-i)^\gamma \rho_H(l-j)^\delta \\
& \leq \frac{1}{(n \log n)^2} \sum_{k>l>i>j} (k-l)^{-1} (i-j)^{-(r+1)/q} (l-i)^{-(q-r-1)/q} \\
& = \frac{1}{(n \log n)^2} \sum_k \sum_{l<k} (k-l)^{-1} \sum_{i<l} (l-i)^{-(q-r-1)/q} \sum_{j<i} (i-j)^{-(r+1)/q} \\
& \leq \frac{1}{n \log^2 n} \sum_{l=1}^{n-1} l^{-1} \sum_{i=1}^{n-1} i^{-(q-r-1)/q} \sum_{j=1}^{n-1} j^{-(r+1)/q} \leq \frac{1}{n \log^2 n} \sum_{l=1}^{n-1} l^{-1} n^{(r+1)/q} n^{1-(r+1)/q} \leq \frac{1}{\log n}
\end{aligned}$$

where the notation  $a_n \leq b_n$  means that  $\sup_{n \geq 1} |a_n|/|b_n| < +\infty$ . Since the other sums in (4.3) are similarly bounded, the fifteenth sum is  $O(1/\log n)$ . Consequently:

$$\sum_{r=0}^{q-2} E[A_r(n)^2] = O(1/\log n).$$

Finally, together with (4.3), we obtain  $E[(1 - \frac{1}{q} \|DS_n\|_{\mathfrak{F}}^2)^2] = O(1/\log n)$  and the proof of (1.6) is achieved thanks to Theorem 1.3.

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