

# ON THE BASE LOCUS OF THE LINEAR SYSTEM OF GENERALIZED THETA FUNCTIONS

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ABSTRACT. Let  $\mathcal{M}_r$  denote the moduli space of semi-stable rank- $r$  vector bundles with trivial determinant over a smooth projective curve  $C$  of genus  $g$ . In this paper we study the base locus  $\mathcal{B}_r \subset \mathcal{M}_r$  of the linear system of the determinant line bundle  $\mathcal{L}$  over  $\mathcal{M}_r$ , i.e., the set of semi-stable rank- $r$  vector bundles without theta divisor. We construct base points in  $\mathcal{B}_{g+2}$  over any curve  $C$ , and base points in  $\mathcal{B}_4$  over any hyperelliptic curve.

## 1. INTRODUCTION

Let  $C$  be a complex smooth projective curve of genus  $g$  and let  $\mathcal{M}_r$  denote the coarse moduli space parametrizing semi-stable rank- $r$  vector bundles with trivial determinant over the curve  $C$ . Let  $\mathcal{L}$  be the determinant line bundle over the moduli space  $\mathcal{M}_r$  and let  $\Theta \subset \text{Pic}^{g-1}(C)$  be the Riemann theta divisor in the degree  $g-1$  component of the Picard variety of  $C$ . By [BNR] there is a canonical isomorphism  $|\mathcal{L}|^* \xrightarrow{\sim} |r\Theta|$ , under which the natural rational map  $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$  is identified with the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|, \quad E \mapsto \theta(E) \subset \text{Pic}^{g-1}(C).$$

The underlying set of  $\theta(E)$  consists of line bundles  $L \in \text{Pic}^{g-1}(C)$  with  $h^0(C, E \otimes L) > 0$ . For a general semi-stable vector bundle  $E$ ,  $\theta(E)$  is a divisor. If  $\theta(E) = \text{Pic}^{g-1}(C)$ , we say that  $E$  has no theta divisor. We note that the indeterminacy locus of the theta map  $\theta$ , i.e., the set of bundles  $E$  without theta divisor, coincides with the base locus  $\mathcal{B}_r \subset \mathcal{M}_r$  of the linear system  $|\mathcal{L}|$ .

Over the past years many authors [A], [B2], [He], [Hi], [P], [R], [S] have studied the base locus  $\mathcal{B}_r$  of  $|\mathcal{L}|$  and their analogues for the powers  $|\mathcal{L}^k|$ . For a recent survey of this subject we refer to [B1].

It is natural to introduce for a curve  $C$  the integer  $r(C)$  defined as the minimal rank for which there exists a semi-stable rank- $r(C)$  vector bundle with trivial determinant over  $C$  without theta divisor (see also [B1] section 6). It is known [R] that  $r(C) \geq 3$  for any curve  $C$  and that  $r(C) \geq 4$  for a generic curve  $C$ . Our main result shows the existence of vector bundles of low ranks without theta divisor.

**Theorem 1.1.** *We assume that  $g \geq 2$ . Then we have the following bounds.*

- (1)  $r(C) \leq g + 2$ .
- (2)  $r(C) \leq 4$ , if  $C$  is hyperelliptic.

The first part of the theorem improves the upper bound  $r(C) \leq \frac{(g+1)(g+2)}{2}$  given in [A]. The statements of the theorem are equivalent to the existence of a semi-stable rank- $(g+2)$  (resp.

rank-4) vector bundle without theta divisor — see section 2.1 (resp. 2.2). The construction of these vector bundles uses ingredients which are already implicit in [Hi].

Theorem 1.1 seems to hint towards a dependence of the integer  $r(C)$  on the curve  $C$ .

*Notations:* If  $E$  is a vector bundle over  $C$ , we will write  $H^i(E)$  for  $H^i(C, E)$  and  $h^i(E)$  for  $\dim H^i(C, E)$ . We denote the slope of  $E$  by  $\mu(E) := \frac{\deg E}{\operatorname{rk} E}$ , the canonical bundle over  $C$  by  $K$  and the degree  $d$  component of the Picard variety of  $C$  by  $\operatorname{Pic}^d(C)$ .

## 2. PROOF OF THEOREM 1.1

**2.1. Semi-stable rank- $(g+2)$  vector bundles without theta divisor.** We consider a line bundle  $L \in \operatorname{Pic}^{2g+1}(C)$ . Then  $L$  is globally generated,  $h^0(L) = g+2$  and the evaluation bundle  $E_L$ , which is defined by the exact sequence

$$(1) \quad 0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0,$$

is stable (see e.g. [Bu]), with  $\deg E_L = 2g+1$ ,  $\operatorname{rk} E_L = g+1$  and  $\mu(E_L) = 2 - \frac{1}{g+1}$ .

A cohomology class  $e \in \operatorname{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1})$  determines a rank- $(g+2)$  vector bundle  $\mathcal{E}_e$  given as an extension

$$(2) \quad 0 \longrightarrow E_L \longrightarrow \mathcal{E}_e \longrightarrow LK^{-1} \longrightarrow 0.$$

**Proposition 2.1.** *For any non-zero class  $e$ , the rank- $(g+2)$  vector bundle  $\mathcal{E}_e$  is semi-stable.*

*Proof.* Consider a proper subbundle  $A \subset \mathcal{E}_e$ . If  $A \subset E_L$ , then  $\mu(A) \leq \mu(E_L) = 2 - \frac{1}{g+1}$  by stability of  $E_L$ , so the subbundles of  $E_L$  cannot destabilize  $\mathcal{E}_e$ . If  $A \not\subset E_L$ , we introduce  $S = A \cap E_L \subset E_L$  and consider the exact sequence

$$0 \longrightarrow S \longrightarrow A \longrightarrow LK^{-1}(-D) \longrightarrow 0,$$

where  $D$  is an effective divisor. If  $\operatorname{rk} S = g+1$  or  $S = 0$ , we easily conclude that  $\mu(A) < \mu(\mathcal{E}_e) = 2$ . If  $\operatorname{rk} S < g+1$  and  $S \neq 0$ , then stability of  $E_L$  gives the inequality  $\mu(S) < \mu(E_L) = 2 - \frac{1}{g+1}$ . We introduce the integer  $\delta = 2\operatorname{rk} S - \deg S$ . Then the previous inequality is equivalent to  $\delta \geq 1$ . Now we compute

$$\mu(A) = \frac{\deg S + \deg LK^{-1}(-D)}{\operatorname{rk} S + 1} \leq \frac{2\operatorname{rk} S - \delta + 3}{\operatorname{rk} S + 1} = 2 + \frac{1 - \delta}{\operatorname{rk} S + 1} \leq 2 = \mu(\mathcal{E}_e),$$

which shows the semi-stability of  $\mathcal{E}_e$ . □

We tensorize the exact sequence (1) with  $L$  and take the cohomology

$$(3) \quad 0 \longrightarrow H^0(E_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0.$$

Note that  $h^1(E_L^* \otimes L) = h^0(E_L \otimes KL^{-1}) = 0$  by stability of  $E_L$ . The second map  $\mu$  is the multiplication map and factorizes through  $\operatorname{Sym}^2 H^0(L)$ , i.e.,

$$\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L) = \ker \mu.$$

By Serre duality a cohomology class  $e \in \operatorname{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1}) = H^0(E_L^* \otimes L)^*$  can be viewed as a hyperplane  $H_e \subset H^0(E_L^* \otimes L)$ . Then we have the following

**Proposition 2.2.** *If  $\Lambda^2 H^0(L) \subset H_e$ , then the vector bundle  $\mathcal{E}_e$  satisfies*

$$h^0(\mathcal{E}_e \otimes \lambda) > 0, \quad \forall \lambda \in \text{Pic}^{g-3}(C).$$

*Proof.* We tensorize the exact sequence (2) with  $\lambda \in \text{Pic}^{g-3}(C)$  and take the cohomology

$$0 \longrightarrow H^0(E_L \otimes \lambda) \longrightarrow H^0(\mathcal{E}_e \otimes \lambda) \longrightarrow H^0(LK^{-1}\lambda) \xrightarrow{\cup_e} H^1(E_L \otimes \lambda) \longrightarrow \dots$$

Since  $\deg LK^{-1}\lambda = g$ , we can write  $LK^{-1}\lambda = \mathcal{O}_C(D)$  for some effective divisor  $D$ . It is enough to show that  $h^0(\mathcal{E}_e \otimes \lambda) > 0$  holds for  $\lambda$  general. Hence we can assume that  $h^0(LK^{-1}\lambda) = h^0(\mathcal{O}_C(D)) = 1$ .

If  $h^0(E_L \otimes \lambda) > 0$ , we are done. So we assume  $h^0(E_L \otimes \lambda) = 0$ , which implies  $h^1(E_L \otimes \lambda) = 1$  by Riemann-Roch. Hence we obtain that  $h^0(\mathcal{E}_e \otimes \lambda) > 0$  if and only if the cup product map

$$\cup_e : H^0(\mathcal{O}_X(D)) \longrightarrow H^1(E_L \otimes \lambda) = H^0(E_L^* \otimes L(-D))^*$$

is zero. Furthermore  $\cup_e$  is zero if and only if  $H^0(E_L^* \otimes L(-D)) \subset H_e$ . Now we will show the inclusion

$$(4) \quad H^0(E_L^* \otimes L(-D)) \subset \Lambda^2 H^0(L).$$

We tensorize the exact sequence (1) with  $L(-D)$  and take cohomology

$$0 \longrightarrow H^0(E_L^* \otimes L(-D)) \longrightarrow H^0(L) \otimes H^0(L(-D)) \xrightarrow{\mu} H^0(L^2(-D)) \longrightarrow \dots$$

Since  $h^0(E_L^* \otimes L(-D)) = 1$ , we conclude that  $h^0(L(-D)) = 2$  and  $H^0(E_L^* \otimes L(-D)) = \Lambda^2 H^0(L(-D)) \subset \Lambda^2 H^0(L)$ .

Finally the proposition follows: if  $\Lambda^2 H^0(L) \subset H_e$ , then by (4)  $H^0(E_L^* \otimes L(-D)) \subset H_e$  for general  $D$ , or equivalently  $h^0(\mathcal{E}_e \otimes \lambda) > 0$  for general  $\lambda \in \text{Pic}^{g-3}(C)$ .  $\square$

We introduce the linear subspace  $\Gamma \subset \text{Ext}^1(LK^{-1}, E_L)$  defined by

$$\Gamma := \ker \left( \text{Ext}^1(LK^{-1}, E_L) = H^0(E_L^* \otimes L)^* \longrightarrow \Lambda^2 H^0(L)^* \right),$$

which has dimension  $\frac{g(g-1)}{2} > 0$ . Then for any non-zero cohomology class  $e \in \Gamma$  and any  $\gamma \in \text{Pic}^2(C)$  satisfying  $\gamma^{g+2} = L^2 K^{-1} = \det \mathcal{E}_e$ , the rank- $(g+2)$  vector bundle

$$\mathcal{E}_e \otimes \gamma^{-1}$$

has trivial determinant, is semi-stable by Proposition 2.1 and has no theta divisor by Proposition 2.2.

**2.2. Hyperelliptic curves.** In this subsection we assume that  $C$  is hyperelliptic and we denote by  $\sigma$  the hyperelliptic involution. The construction of a semi-stable rank-4 vector bundle without theta divisor has been given in [Hi] section 6 in the case  $g = 2$ , but it can be carried out for any  $g \geq 2$  without major modification. For the convenience of the reader, we recall the construction and refer to [Hi] for the details and the proofs.

Let  $w \in C$  be a Weierstrass point. Any non-trivial extension

$$0 \longrightarrow \mathcal{O}_C(-w) \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is a stable,  $\sigma$ -invariant, rank-2 vector bundle with  $\deg G = -1$ . By [Hi] Theorem 4 a cohomology class  $e \in H^1(\text{Sym}^2 G)$  determines a symplectic rank-4 bundle

$$0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

Moreover it is easily seen that, for any non-zero class  $e$ , the vector bundle  $\mathcal{E}_e$  is semi-stable. By [Hi] Lemma 16 the composite map

$$D_G : \mathbb{P}H^1(\mathrm{Sym}^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_G : H^1(\mathrm{Sym}^2 G) \longrightarrow H^0(\mathrm{Pic}^{g-1}(C), 4\Theta).$$

The involution  $i(L) = KL^{-1}$  on  $\mathrm{Pic}^{g-1}(C)$  induces a linear involution on  $|4\Theta|$  with eigenspaces  $|4\Theta|_{\pm}$ . Note that  $4\Theta \in |4\Theta|_+$ . We now observe that  $\theta(\mathcal{E}) \in |4\Theta|_+$  for any symplectic rank-4 vector bundle  $\mathcal{E}$  — see e.g. [B2]. Moreover we have the equality  $\theta(\sigma^*\mathcal{E}) = i^*\theta(\mathcal{E})$  for any vector bundle  $\mathcal{E}$ . These two observations imply that the linear map  $\widetilde{D}_G$  is equivariant with respect to the induced involutions  $\sigma$  and  $i$ . Since  $\mathrm{im} \widetilde{D}_G \subset H^0(\mathrm{Pic}^{g-1}(C), 4\Theta)_+$ , we obtain that one of the two eigenspaces  $H^1(\mathrm{Sym}^2 G)_{\pm}$  is contained in the kernel  $\ker \widetilde{D}_G$ , hence give base points for the theta map. We now compute as in [Hi] using the Atiyah-Bott-fixed-point formula

$$h^1(\mathrm{Sym}^2 G)_+ = g - 1, \quad h^1(\mathrm{Sym}^2 G)_- = 2g + 1.$$

One can work out that  $H^1(\mathrm{Sym}^2 G)_+ \subset \ker \widetilde{D}_G$ . Hence any  $\mathcal{E}_e$  with non-zero  $e \in H^1(\mathrm{Sym}^2 G)_+$  is a semi-stable rank-4 vector bundle without theta divisor.

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