SURFACE TENSION IN THE DILUTE ISING MODEL. THE WULFF CONSTRUCTION.

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ABSTRACT. We study the surface tension and the phenomenon of phase coexistence for the Ising model on $\mathbb{Z}^d$ ($d \geqslant 2$) with ferromagnetic but random couplings. We prove the convergence in probability (with respect to random couplings) of surface tension and analyze its large deviations : upper deviations occur at volume order while lower deviations occur at surface order. We study the asymptotics of surface tension at low temperatures and relate the quenched value τ^q of surface tension to maximal flows (first passage times if $d = 2$). For a broad class of distributions of the couplings we show that the inequality $\tau^a \leqslant \tau^q$ – where τ^a is the surface tension under the averaged Gibbs measure – is strict at low temperatures. We also describe the phenomenon of phase coexistence in the dilute Ising model and discuss some of the consequences of the media randomness. All of our results hold as well for the dilute Potts and random cluster models.

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A considerable amount of work permitted to understand on a rigorous basis the phenomenon of phase coexistence in models of statistical mechanics like the Ising model. Phase coexistence in the Ising model was first described in the pioneer work [20], in the two dimensional case and at low temperatures. The construction was then simplified [37] and extended up to the critical temperature [28, 29, 30], still in the two dimensional case. The generalization to higher dimensions was achieved later thanks to the L^1-approach [5, 9, 11]. The interested reader will find pedagogical presentations of the problem and the methods in the course [10] and the review [6].

The present work is concerned with the phenomenon of phase coexistence for the dilute model with random (ferromagnetic) couplings. The random couplings model either *rare defects* in the media, either *intrinsic randomness*. As an example, quenched alloys made of magnetic materials have intrinsic randomness since the strength of the interaction between two spins depends on the nature of the two corresponding atoms.

In order to describe rigorously the phenomenon of phase coexistence in presence of phase coexistence, we followed the same plan as in the above mentioned works. In a first step we established a *coarse graining* for the model [44]. In a second step – the present one – we study *surface tension*. The combination of these tools allow us describe the phenomenon of phase coexistence in the presence of random media.

Before we turn to the presentation of the model and of our results, we would like to stress two consequences of the media randomness on the phenomenon of phase coexistence: first, it is the case that the shape of crystals are smoother than in presence of uniform couplings. Second, we give an insight to the expected *localization* phenomenon of the crystal which is determined by the realization of the media under averaged Gibbs measure.

The organization of the paper is as follows. In Section 1 below we introduce the model and give a complete summary of our results on surface tension, its low temperature asymptotics (maximal flows) and phase coexistence. Proofs and intermediate results are given in the three corresponding Sections 2, 3 and 4.

1. The model and our main results

1.1. **The dilute Ising model.** The canonical vectors of $\mathbb{R}^d$ are denoted $(\boldsymbol{e}_i)_{i=1\dots d}$ and for any $x = \sum_{i=1}^n x_i \boldsymbol{e}_i = (x_1, \dots, x_d) \in \mathbb{R}^d$ we consider the following norms on $\mathbb{R}^d$:

$$\|x\|_1 = \sum_{i=1}^d |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^d x_i^2\right)^{1/2} \quad \text{and} \quad \|x\|_\infty = \max_{i=1}^d |x_i|. \tag{1.1}$$

Given $x, y \in \mathbb{Z}^d$ we say that x, y are nearest neighbors (which we denote $x \sim y$) if they are at Euclidean distance 1, i.e. if $\|x-y\|_2 = 1$. To any domain $\Lambda \subset \mathbb{Z}^d$ we associate the edge sets

$$E(\Lambda) = \left\{\{x,y\} : x, y \in \Lambda \text{ and } x \sim y\right\} \tag{1.2}$$

$$\text{and} \quad E^w(\Lambda) = \left\{\{x,y\} : x \in \Lambda, y \in \mathbb{Z}^d \text{ and } x \sim y\right\}. \tag{1.3}$$

We consider in this paper the dilute Ising model on $\mathbb{Z}^d$ for $d \geqslant 2$. It is defined in two steps : first, the couplings between adjacent spins are represented by a random sequence $J = (J_e)_{e \in E(\mathbb{Z}^d)}$ of law $\mathbb{P}$, such that the $(J_e)_{e \in E(\mathbb{Z}^d)}$ are independent, identically distributed in $[0,1]$ under $\mathbb{P}$. Then, given $\Lambda \subset \mathbb{Z}^d$ a finite domain and a spin configuration $\sigma \in \Sigma_\Lambda^+$, where

$$\Sigma_\Lambda^+ = \left\{\sigma : \mathbb{Z}^d \to \{\pm 1\} : \sigma_z = 1, \forall z \notin \Lambda\right\},$$

we let

$$H_\Lambda^{J,+}(\sigma) = -\sum_{e=\{x,y\}\in E^w(\Lambda)} J_e \sigma_x \sigma_y \tag{1.4}$$

the Hamiltonian with plus boundary condition on Λ. The dilute Ising model on Λ with plus boundary condition, given a realization J of the couplings, is the probability measure $\mu_\Lambda^{J,+}$ on Σ_Λ^+ that satisfies

$$\mu_\Lambda^{J,+}(\{\sigma\}) = \frac{1}{Z_{\Lambda,\beta}^{J,+}} \exp\left(-\frac{\beta}{2} H_\Lambda^{J,+}(\sigma)\right), \quad \forall \sigma \in \Sigma_\Lambda^+ \tag{1.5}$$

where $\beta \geqslant 0$ is the inverse temperature and $Z_{\Lambda,\beta}^{J,+}$ is the partition function

$$Z_{\Lambda,\beta}^{J,+} = \sum_{\sigma\in\Sigma_\Lambda^+} \exp\left(-\frac{\beta}{2} H_\Lambda^{J,+}(\sigma)\right). \tag{1.6}$$

Consider

$$m_\beta = \lim_{N\to\infty} \mathbb{E}\mu_{\hat\Lambda_N,\beta}^{J,+}(\sigma_0) \tag{1.7}$$

the magnetization in the thermodynamic limit, where $\hat\Lambda_N$ is the symmetric box $\hat\Lambda_N = \{-N,\dots,N\}^d$ and $\mathbb{E}$ the expectation associated with $\mathbb{P}$. When $m_\beta > 0$ the boundary condition has an influence on the spins at an arbitrary distance. In the region $m_\beta > 0$ we say that the Ising model has two phases because the structure of the spins under $\mathbb{E}\mu_\Lambda^{J,+}$ (the plus phase) is not the same as the structure of the spins under $\mathbb{E}\mu_\Lambda^{J,-}$ (the minus phase), where $\mu_\Lambda^{J,-}$ corresponds to the minus boundary condition.

It is shown in [1] that the dilute Ising model undergoes a phase transition at low temperature when the random interactions percolate. In our settings, this means that the critical inverse temperature

$$\beta_c = \inf\{\beta \geqslant 0 : m_\beta > 0\}, \tag{1.8}$$

which is never smaller than β_c^{pure} – the critical inverse temperature for the pure Ising model ($J \equiv 1$) – is finite if and only if $\mathbb{P}(J_e > 0) > p_c(d)$ where $p_c(d)$ is the threshold for bond percolation on $\mathbb{Z}^d$.

The aim of the paper is to understand the mechanism of phase coexistence in the dilute Ising model, hence we will consider in the following a distribution $\mathbb{P}$ of the couplings such that $\mathbb{P}(J_e > 0) > p_c(d)$ and an inverse temperature $\beta > \beta_c$. However, some of our results hold on a possibly stronger assumption $\beta > \hat\beta_c \geqslant \beta_c$ where $\hat\beta_c$ is the critical inverse temperature for slab percolation – see (1.14) below – as this assumption allows us to use the renormalization framework of [44].

1.2. The Fortuin-Kasteleyn representation.

The study of surface tension for the dilute Ising model will be led under the random-cluster model that corresponds to the measure $\mu_{\Lambda,\beta}^{J,+}$. We call

$$\Omega = \left\{\omega : E(\mathbb{Z}^d) \to \{0,1\}\right\}$$

the set of cluster configurations on $E(\mathbb{Z}^d)$, and for any $\omega \in \Omega$ and $E \subset E(\mathbb{Z}^d)$ we call $\omega_{|E}$ the restriction of ω to E, defined by

$$(\omega_{|E})_e = \begin{cases} \omega_e & \text{if } e \in E \\ 0 & \text{else.} \end{cases}$$

The set of cluster configurations on E is $\Omega_E = \{\omega_{|E}, \omega \in \Omega\}$. Given a parameter $q \geqslant 1$ and an inverse temperature $\beta \geqslant 0$, a realization of the random couplings $J : E(\mathbb{Z}^d) \to [0,1]$, a finite edge set $E \subset E(\mathbb{Z}^d)$ and a boundary condition $\pi \in \Omega_{E^c}$ we consider the random cluster model $\Phi_{E,\beta}^{J,\pi,q}$ on Ω_E defined by

$$\Phi_{E,\beta}^{J,\pi,q}(\{\omega\}) = \frac{1}{Z_{E,\beta}^{J,\pi,q}} \prod_{e\in E} p_e^{\omega_e}(1-p_e)^{1-\omega_e} \times q^{C_E^\pi(\omega)}, \quad \forall \omega \in \Omega_E \tag{1.9}$$

where $p_e = 1 - \exp(-\beta J_e)$, $C_E^\pi(\omega)$ is the number of clusters of the set of vertices in $\mathbb{Z}^d$ attained by E under the wiring $\omega \vee \pi$ such that $(\omega \vee \pi)_e = \max(\omega_e, \pi_e)$, and $Z_{E,\beta}^{J,\pi,q}$ is the renormalization constant making $\Phi_{E,\beta}^{J,\pi,q}$ a probability measure.

For convenience we use the same notation for the probability measure $\Phi_{E,\beta}^{J,\pi,q}$ and for its expectation. Most often we will take either $\pi = f$, where f is the free boundary condition : $f_e = 0, \forall e \in E^c$, or $\pi = w$ where w is the wired boundary condition : $w_e = 1, \forall e \in E^c$. When the parameters q and β are clear from the context we omit them. Given $\mathcal{R}$ a compact subset of $\mathbb{R}^d$ (usually a rectangular parallelepiped) we denote by $\Phi_{\mathcal{R}}^{J,\pi}$ the measure $\Phi_{E(\dot{\mathcal{R}}\cap\mathbb{Z}^d)}^{J,\pi}$ on the cluster configurations on $E(\dot{\mathcal{R}} \cap \mathbb{Z}^d)$, where $\dot{\mathcal{R}}$ stands for the interior of $\mathcal{R}$. In particular, for any $g, h : \Omega \to \mathbb{R}$ the quantities $\Phi_{\mathcal{R}_1}^{J,\pi}(g)$ and $\Phi_{\mathcal{R}_2}^{J,\pi}(h)$ are independent under $\mathbb{P}$.

The connection between the dilute Ising model $\mu_{\Lambda,\beta}^{J,+}$ and the random-cluster model was made explicit in [22]. Consider the joint probability measure

$$\Psi_{\Lambda,\beta}^{J,+}(\{(\sigma,\omega)\}) = \frac{\mathbf{1}_{\{\sigma \prec \omega\}}}{\tilde{Z}_{\Lambda,\beta}^{J,+}} \prod_{e \in E^w(\Lambda)} (p_e)^{\omega_e} (1-p_e)^{1-\omega_e}, \quad \forall (\sigma,\omega) \in \Sigma_\Lambda^+ \times \Omega_{E(\Lambda)}$$

where $p_e = 1 - \exp(-\beta J_e)$, $\sigma \prec \omega$ is the event that σ and ω are compatible, namely that $\omega_e = 1 \Rightarrow \sigma_x = \sigma_y, \forall e = \{x,y\} \in E^w(\Lambda)$, and $\tilde{Z}_{\Lambda,\beta}^{J,+}$ is the corresponding normalizing factor. Then,

i. The marginal of $\Psi_{\Lambda,\beta}^{J,+}$ on the variable σ is the Ising model $\mu_\Lambda^{J,+}$,
ii. Its marginal on the variable ω is the random-cluster model $\Phi_{E(\Lambda),\beta}^{J,w,2}$ with wired boundary condition w and parameter $q = 2$.
iii. Conditionally on ω, the spin σ of each connected component of Λ for ω (now *cluster*) is constant, and equal to $+1$ if the cluster is connected to Λ^c. The spin of all clusters not touching Λ^c are independent and equal to $+1$ with a probability $1/2$.
iv. Conditionally on σ, the edges are open (i.e. $\omega_e = 1$ for $e = \{x,y\}$) independently, with respective probabilities $p_e \delta_{\sigma_x,\sigma_y}$.

According to point *ii* and *iii* we can study surface tension for the Ising model under the Fortuin-Kasteleyn representation. This representation allows to study at the same time the surface tension and the phenomenon of phase coexistence for the dilute Ising model ($q = 2$), but also for dilute percolation ($q = 1$) and for the dilute Potts model ($q \in \{3, 4, \dots\}$).

An important benefit of the representation is that it makes possible the use of the comparison inequalities for the random cluster model. We say that a function $f : \Omega_E \to \mathbb{R}^+$ is increasing if, for all $\omega, \omega' \in \Omega_E$ one has $\omega \leqslant_\Omega \omega' \Rightarrow f(\omega) \leqslant f(\omega')$ where $\leqslant_\Omega$ stands for the product order on Ω_E. It was shown in [2] that:

i. For any $h : \Omega_E \to \mathbb{R}^+$ increasing, $\Phi_{E,\beta}^{J,\pi,q}(h)$ is a non-decreasing function of J, β and π.
ii. (FKG inequality) For any $g, h : \Omega_E \to \mathbb{R}^+$ increasing,

$$\Phi_{E,\beta}^{J,\pi,q}(gh) \geqslant \Phi_{E,\beta}^{J,\pi,q}(g)\Phi_{E,\beta}^{J,\pi,q}(h). \tag{1.10}$$

iii. (DLR Equation) For any $E' \subsetneq E$ and $\omega' \in \Omega_{E'}$,

$$\Phi_{E,\beta}^{J,\pi,q}\left(.|\omega_{|E'} = \omega'\right) = \Phi_{E\setminus E',\beta}^{J,\pi\vee\omega',q}. \tag{1.11}$$

Finally, let us recall the assumption of slab percolation, that is the basis for a renormalization framework in the dilute Ising model [44]. When $d \geqslant 3$, we say that slab percolation occurs under $\mathbb{E}\Phi_\beta^{J,f,q}$ if, for large enough H,

$$\inf_{L \in \mathbb{N}^\star} \inf_{x,y \in S_{L,H}} \mathbb{E}\Phi_{S_{L,H}}^{J,f}\left(x \overset{\omega}{\leftrightarrow} y\right) > 0 \tag{1.12}$$

where $S_{L,H}$ is the slab $S_{L,H} = \{1,\dots,L\}^{d-1} \times \{1,\dots H\}$. When $d = 2$, we say that slab percolation occurs when there exists $\kappa : \mathbb{N}^\star \mapsto \mathbb{N}^\star$ with $\lim_{N\to\infty} \kappa(N)/N = 0$ such that

$$\lim_{N\to\infty} \mathbb{E}\Phi_{S_{N,\kappa(N)}}^{J,f}(\text{there is an horizontal crossing for } \omega) > 0. \tag{1.13}$$

The critical inverse temperature for slab percolation is

$$\hat{\beta}_c = \inf \left\{ \beta \geqslant 0 : \text{slab percolation occurs under } \mathbb{E}\Phi_\beta^{J,f,q} \right\}, \tag{1.14}$$

it satisfies $\hat{\beta}_c \geqslant \beta_c$ where β_c is the critical inverse temperature for phase transition in the dilute Ising (resp. Potts) model. We believe that $\hat{\beta}_c$ and β_c do coincide. Upper bounds on $\hat{\beta}_c$ are derived in [44] from the argument of [1]. The technical assumption $\beta > \hat{\beta}_c$ allows us to use a coarse graining, which is a fundamental tool at the moment of defining the local phase of the dilute Ising model (see Theorem 5.7 in [44], or Section 4 below).

1.3. **Surface tension.** One of the main issue we address in this paper is the behavior of surface tension and the influence of the random couplings. We consider the surface tension in large rectangular parallelepiped oriented along some direction $\boldsymbol{n} \in S^{d-1}$, where S^{d-1} is the set of unit vectors of $\mathbb{R}^d$. The other axes of the parallelepiped are represented by $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$, where

$$\mathbb{S}_{\boldsymbol{n}} = \left\{ \sum_{k=1}^{d-1} [\pm 1/2] \boldsymbol{u}_k ; (\boldsymbol{u}_1, \dots, \boldsymbol{u}_{k-1}, \boldsymbol{n}) \text{ is an orthonormal basis of } \mathbb{R}^d \right\}$$

is the set of $d-1$ dimensional hypercubes of side-length 1, centered at 0, orthogonal to $\boldsymbol{n} \in S^{d-1}$. Finally, we call $x \in \mathbb{R}^d$ the center of the rectangular parallelepiped and L, H its side-lengths, and denote finally by

$$\mathcal{R}_{x,L,H}(\mathcal{S}, \boldsymbol{n}) = x + L\mathcal{S} + [-H, H]\boldsymbol{n} \tag{1.15}$$

the rectangular parallelepiped centered at x, with basis $x + L\mathcal{S}$ and extension $2H$ in the direction $\boldsymbol{n}$ (see Figure 1). The discrete version of $\mathcal{R}$ is $\hat{\mathcal{R}} = \dot{\mathcal{R}} \cap \mathbb{Z}^d$ and the inner discrete boundary of $\mathcal{R}$ is

$$\partial \hat{\mathcal{R}} = \left\{ y \in \hat{\mathcal{R}} : \exists z \in \mathbb{Z}^d \setminus \hat{\mathcal{R}}, z \sim y \right\}.$$

For any $\mathcal{R}$ as in (1.15) we decompose $\partial \hat{\mathcal{R}}$ into its *upper* and *lower* parts $\partial^+ \hat{\mathcal{R}} = \{ y \in \partial \hat{\mathcal{R}} : (y - x) \cdot \boldsymbol{n} \geqslant 0 \}$ and $\partial^- \hat{\mathcal{R}} = \{ y \in \partial \hat{\mathcal{R}} : (y - x) \cdot \boldsymbol{n} < 0 \}$.

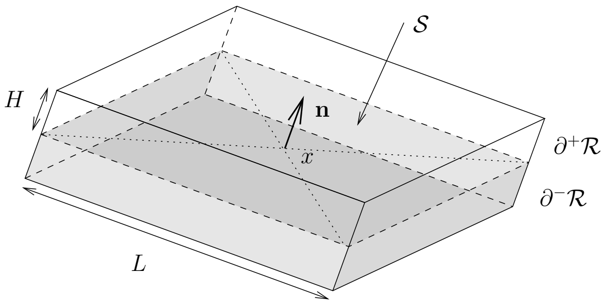

FIGURE 1. The rectangular parallelepiped $\mathcal{R}_{x,L,H}(\mathcal{S}, \boldsymbol{n})$.

In the context of statistical physics, the surface tension is the excess free energy per surface unit due to the presence of an interface. The surface tension in $\mathcal{R}$ thus quantifies the probability of observing the plus phase in the upper part of $\mathcal{R}$ and the minus phase in the opposite part under the measure $\mu^J_{\mathcal{R}}$ with free boundary condition. It is more convenient to formulate the definition under the random cluster model, where we translate the event of *phase coexistence* into an event of *disconnection*.

Definition 1.1. Let $\mathcal{R}$ be a rectangular parallelepiped as in (1.15). The event of disconnection between the upper and lower parts of $\partial \hat{\mathcal{R}}$ is

$$\mathcal{D}_{\mathcal{R}} = \left\{ \omega \in \Omega : \partial^+ \hat{\mathcal{R}} \overset{\omega}{\nleftrightarrow} \partial^- \hat{\mathcal{R}} \right\} \tag{1.16}$$

and the surface tension in $\mathcal{R}$ is

$$\tau^J_{\mathcal{R}} = -\frac{1}{L^{d-1}} \log \Phi^{J,w}_{\mathcal{R}} (\mathcal{D}_{\mathcal{R}}) . \tag{1.17}$$

We denote by $J^{\min}$ and $J^{\max}$ the lowest and largest values of the couplings according to the support of $\mathbb{P}$, that is to say :

$$\begin{array}{rcl} J^{\min} & = & \inf\{\lambda \geqslant 0 : \mathbb{P}(J_e < \lambda) > 0\} \\ \text{and} \quad J^{\max} & = & \sup\{\lambda \geqslant 0 : \mathbb{P}(J_e > \lambda) > 0\}. \end{array}$$

We also denote by $\tau^{\min}_{\mathcal{R}}$ (resp. $\tau^{\max}_{\mathcal{R}}$) the value of the surface tension in $\mathcal{R}$ corresponding to the constant couplings $J \equiv J^{\min}$ (resp. $J \equiv J^{\max}$). We have:

Proposition 1.2. *Let $\mathcal{R}$ be a rectangular parallelepiped as in (1.15), with $L, H \geqslant 2\sqrt{d}$. The surface tension $\tau^J_{\mathcal{R}}$ is a non-decreasing function of J and β. It is a non-increasing function of H. With probability one under $\mathbb{P}$,*

$$0 \leqslant \tau^{\min}_{\mathcal{R}}(\boldsymbol{n}) \leqslant \tau^J_{\mathcal{R}}(\boldsymbol{n}) \leqslant \tau^{\max}_{\mathcal{R}}(\boldsymbol{n}) \leqslant c_d \beta J^{\max}$$

where $c_d < \infty$ depends on d only.

As in the uniform case [36], surface tension is sub-additive (see Theorem 2.1), and this implies convergence in probability of $\tau^J_{\mathcal{R}}(\boldsymbol{n})$.

Theorem 1.3. *There exists $\tau^q_\beta(\boldsymbol{n}) \geqslant 0$, the* quenched *surface tension, such that, for all $\beta \geqslant 0$ and $\boldsymbol{n} \in S^{d-1}$,*

$$\lim_{N \to \infty} \tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})} = \tau^q_\beta(\boldsymbol{n}) \quad \text{in } \mathbb{P}\text{-probability}$$

whatever is $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$ and $\delta > 0$.

Similarly, the surface tension for the constant couplings $J^{\min}$ and $J^{\max}$ also converge and we denote by $\tau^{\min}(\boldsymbol{n})$ and $\tau^{\max}(\boldsymbol{n})$ their respective limits.

The sub-additivity is of much help for controlling the order of deviations from the quenched value of surface tension $\tau^q_\beta(\boldsymbol{n})$. Upper large deviations happen at a volume order hence they have no influence on the phenomena we study here:

Theorem 1.4. *For any $\varepsilon > 0$ and $\delta > 0$,*

$$\limsup_N \frac{1}{N^d} \log \mathbb{P}\left(\tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})} \geqslant \tau^q(\boldsymbol{n}) + \varepsilon \right) < 0.$$

We will be more concerned with lower large deviations. These are possible when $\tau^{\min}(\boldsymbol{n}) < \tau^q(\boldsymbol{n})$. The inequality is known to be strict only in two specific cases: when $J^{\min} = 0$ and $\beta > \hat{\beta}_c$, it is the case that $\tau^{\min}(\boldsymbol{n}) < \tau^q(\boldsymbol{n})$ because $\tau^{\min}(\boldsymbol{n}) = 0$, while the coarse graining [44] implies:

Proposition 1.5. *Assume $\beta > \hat{\beta}_c$. For any $\boldsymbol{n} \in S^{d-1}$, $\tau^q(\boldsymbol{n}) > 0$.*

When $\mathbb{P}(J_e > J^{\min}) > p_c(d)$ and β is large enough, the strict inequality $\tau^{\min}(\boldsymbol{n}) < \tau^q(\boldsymbol{n})$ also holds, cf. Corollary 1.14 below.

When lower large deviations occur, they have at most surface order. It is another consequence of sub-additivity that:

Theorem 1.6. *For every $\boldsymbol{n} \in S^{d-1}$ and $\beta \geqslant 0$, $\tau > \tau^{\min}(\boldsymbol{n})$, the limit*

$$I_{\boldsymbol{n}}(\tau) = \lim_N -\frac{1}{N^{d-1}} \log \mathbb{P}\left(\tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})} \leqslant \tau \right) \tag{1.18}$$

exists in $[0, +\infty)$ and does not depend on $\delta > 0$, nor on $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$. $I_{\boldsymbol{n}}$ is continuous, convex non-increasing, and $I_{\boldsymbol{n}}(\tau) = 0$ for $\tau \geqslant \tau^q(\boldsymbol{n})$.

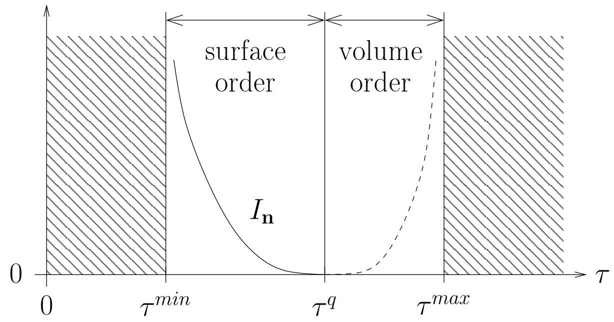

Figure 2. Large deviations of surface tension.

For convenience, we extend the definition of $I_{\boldsymbol{n}}$ letting

$$I_{\boldsymbol{n}}(\tau) = \begin{cases} +\infty & \text{for } \tau < \tau^{\min}(\boldsymbol{n}) \\ \lim_{\varepsilon\to 0^+} I_{\boldsymbol{n}}(\tau+\varepsilon) & \text{at } \tau = \tau^{\min}(\boldsymbol{n}). \end{cases} \tag{1.19}$$

In order to show that lower deviations are exactly of surface order, we need to prove that $I_{\boldsymbol{n}}$ is positive on the left of τ^q. We developed an argument based on measure concentration coupled with a control of the length of the interface. In the case of the Ising model at low temperatures we could establish a first control:

Theorem 1.7. *Assume $q = 2$ and $J^{\min} > 0$. Then, for β large enough there exists $c > 0$ such that, for all $r > 0$,*

$$I_{\boldsymbol{n}}(\tau^q(\boldsymbol{n}) - r) \geqslant cr^2. \tag{1.20}$$

In the general case a careful adaptation of the method yields:

Theorem 1.8. *For every $\boldsymbol{n} \in S^{d-1}$, for Lebesgue-almost all $\beta \geqslant 0$,*

$$\limsup_{r\to 0^+} \frac{I_{\beta,\boldsymbol{n}}(\tau^q_\beta(\boldsymbol{n}) - r)}{r^2} > 0. \tag{1.21}$$

These quadratic lower bounds on the rate function generalize common controls for directed polymers models [16, 8], which were introduced in order to represent interfaces in the two-dimensional Ising model with random couplings at low temperatures [27, 16]. Its is probable however that the quadratic order in the former Theorems is not optimal in two dimensions, as the comparison with directed polymers suggests that

$$I_{\boldsymbol{n}}(\tau^q(\boldsymbol{n}) - r) \underset{r\to 0^+}{\sim} cr^{3/2} \quad \text{and} \quad \xi = \frac{2}{3}. \tag{1.22}$$

This scaling has been established rigorously for the zero-temperature limit of directed polymers: *last passage percolation*, for a geometric distribution of the passage times, see Theorem 1.1 and (2.23) in [31].

Some of our results on phase coexistence and on the dynamics of the dilute Ising model [45] require that lower large deviations are actually of surface order. An easy but important consequence of Theorem 1.8 is:

Corollary 1.9. *The lower large deviations are of surface order when $\beta \mapsto \tau^q_\beta(\boldsymbol{n})$ is left continuous. Hence the set*

$$\mathcal{N}_I = \Big\{\beta \geqslant 0 : \exists \boldsymbol{n} \in S^{d-1} \text{ and } r > 0 \text{ such that } I_{\beta,\boldsymbol{n}}(\tau^q_\beta(\boldsymbol{n}) - r) = 0\Big\} \tag{1.23}$$

is at most countable.

We end the presentation on surface tension with the definition of the surface tension under the averaged Gibbs measure. It is the Fenchel-Legendre transform of $I_{\boldsymbol{n}}$,

$$\tau^\lambda(\boldsymbol{n}) = \inf_{\tau\in\mathbb{R}} \{\lambda\tau + I_{\boldsymbol{n}}(\tau)\} \tag{1.24}$$

which coincides with the surface tension under an average of Gibbs measures:

$$\tau^\lambda(\boldsymbol{n}) = \lim_{N\to\infty} -\frac{1}{N^{d-1}} \log \mathbb{E}\left(\left[\Phi^{J,w}_{\mathcal{R}^N}(\mathcal{D}_{\mathcal{R}^N}) \right]^\lambda \right) \qquad \lambda > 0, \boldsymbol{n} \in S^{d-1} \tag{1.25}$$

where $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$, as shown in Proposition 2.2. The particular case $\lambda = 1$ corresponds to the usual notion of surface tension under the averaged measure (or *annealed* surface tension) and we denote $\tau^a(\boldsymbol{n}) = \tau^{\lambda=1}(\boldsymbol{n})$.

The asymptotics of τ^λ/λ as $\lambda \to 0$ or $+\infty$ are given in Proposition 2.3. An important question about the surface tension under the averaged Gibbs measure is whether the random media is able to turn Jensen's inequality

$$\tau^\lambda(\boldsymbol{n}) \leqslant \lambda\tau^q(\boldsymbol{n}) \tag{1.26}$$

into a strict inequality. A partial answer to this question is given in the next Section. Let us explicit the connection between the strict inequality $\tau^\lambda(\boldsymbol{n}) \leqslant \lambda\tau^q(\boldsymbol{n})$ and the asymptotics of $I_{\boldsymbol{n}}$ on the left of $\tau^q(\boldsymbol{n})$: the opposite of the slope of $I_{\boldsymbol{n}}$ on the left of $\tau^q(\boldsymbol{n})$ is exactly

$$\alpha_{\boldsymbol{n}} = \sup\left\{ \lambda > 0 : \tau^\lambda(\boldsymbol{n}) = \lambda\tau^q(\boldsymbol{n}) \right\}, \tag{1.27}$$

with the convention that $\sup\emptyset = 0$.

Finally, as in the non-random case, the homogeneous extension of each of these notions of surface tension τ^q, τ^λ, $\tau^{\min}$ and $\tau^{\max}$ are convex and continuous (Proposition 2.4).

1.4. **Low temperature asymptotics.** The low temperature asymptotics of surface tension permit to give a more precise insight into the properties of surface tension in random media. First we need to introduce the concept of *maximal flow* through the capacities J, where $J = (J_e)_{e\in E(\mathbb{Z}^d)}$ is the family of random couplings introduced in the former section. Here we give only a brief overview of maximal flows. The reader is invited to consult [33, 32] for a pedagogical introduction. Recent results on maximal flows, including large deviations, can be found in [42, 39, 47].

We will use an analogy for describing maximal flows. Imagine a liquid which has to cross a lattice made of tubes with limited capacity. Then, the maximal flow, in a given direction, is the quantity of liquid that can flow through the lattice, per unit of surface.

Given a rectangular parallelepiped $\mathcal{R}$ as in (1.15) and $I \subset E(\hat{\mathcal{R}})$, we consider the event that ω is closed on I:

$$\mathcal{Z}_I = \{\omega_e = 0, \forall e \in I\}.$$

We say that I is an *interface* for $\mathcal{R}$ if $\mathcal{Z}_I \subset \mathcal{D}_{\mathcal{R}}$ and $\forall e \in I, \mathcal{Z}_{I\setminus\{e\}} \not\subset \mathcal{D}_{\mathcal{R}}$. In other words, I is an interface for $\mathcal{R}$ if the disconnection on I is enough for disconnecting $\partial^+\hat{\mathcal{R}}$ from $\partial^-\hat{\mathcal{R}}$ and if there is no superfluous edge in I. This notion of interface corresponds to the geometrical notion of interface if, to the edges of I we associate their dual $d-1$ dimensional facets.

According to the max-flow min-cut Theorem [7], the maximum flow from $\partial^-\hat{\mathcal{R}}$ to $\partial^+\hat{\mathcal{R}}$ by the edges of capacities J_e is also the flow through the interface of minimal capacity. We use this characterization for our definition. Given a rectangular parallelepiped $\mathcal{R} = \mathcal{R}_{x,L,H}(\mathcal{S}, \boldsymbol{n})$ as in (1.15) we call $\mathcal{I}(\mathcal{R})$ the set of interfaces for $\mathcal{R}$ and define the maximal flow in $\mathcal{R}$, for a realization J of the media, as

$$\mu^J_{\mathcal{R}} = \frac{1}{L^{d-1}} \inf_{I\in\mathcal{I}(\mathcal{R})} \sum_{e\in I} J_e. \tag{1.28}$$

This quantity has the same properties as surface tension since it is as well sub-additive. In particular, the maximal flow in $\mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$ converges in $\mathbb{P}$-probability, upper deviations occur at volume order and lower deviations occur at surface order [41, 15]. We will make use of the following results:

Theorem 1.10. *There exists $\mu(\boldsymbol{n}) \in [0,+\infty)$, the maximal flow for the distribution $\mathbb{P}$ in the direction $\boldsymbol{n} \in S^{d-1}$, such that, for any $\delta > 0$ and $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$,*

$$\mu^J_{\mathcal{R}_{0,N,\delta N}(\boldsymbol{n},\mathcal{S})} \underset{N\to\infty}{\longrightarrow} \mu(\boldsymbol{n}) \quad \text{in } \mathbb{P}\text{-probability.}$$

It is positive if and only if $\mathbb{P}(J_e > 0) > p_c(d)$. Furthermore,

$$J^{\min}\|\boldsymbol{n}\|_1 \leqslant \mu(\boldsymbol{n}) \tag{1.29}$$

and the inequality is strict when $\mathbb{P}(J_e > J^{\min}) > p_c(d)$.

The convergence of the maximal flow is a consequence of the sub-additivity. It is shown in [46] that the maximal flow is 0 when $\mathbb{P}(J_e > 0) \leqslant p_c(d)$, while its positivity was established in [13], under the conjecture that the critical threshold for percolation and slab percolation do coincide, proved later on in [26]. The inequality (1.29) is easily obtained from the remark that minimal interfaces have cardinal of order $N^{d-1}\|\boldsymbol{n}\|_1$. When $\mathbb{P}(J_e > J^{\min}) > p_c(d)$, for small $\varepsilon > 0$ there is a percolating net of edges with values $J_e \geqslant J^{\min} + \varepsilon$ [26, 38], which is responsible for the strict inequality. See also Proposition 4.1 in [39].

It turns out that the maximal flow determines the asymptotics of the quenched value of surface tension at low temperatures. Precisely, we show that:

Proposition 1.11. *Let $\mathbb{P}$ be a product measure on $[0,1]^d$ such that $\mathbb{P}(J_e > 0) = 1$. Then, uniformly over $\boldsymbol{n} \in S^{d-1}$,*

$$\lim_{\beta\to\infty} \frac{\tau^q_\beta(\boldsymbol{n})}{\beta} = \mu(\boldsymbol{n}). \tag{1.30}$$

Clearly, (1.30) also holds in the case $\mathbb{P}(J_e > 0) \leqslant p_c(d)$ since $\tau^q_\beta(\boldsymbol{n}) \leqslant \beta\mu(\boldsymbol{n}) = 0$ (cf. Lemma 3.1 and Theorem 1.10). When $\mathbb{P}(J_e > 0) > p_c(d)$ a renormalization argument allows us to prove that:

Proposition 1.12. *Assume that $\mathbb{P}(J_e > 0) > p_c(d)$. Then,*

$$\liminf_{\beta\to+\infty} \frac{\tau^q_\beta(\boldsymbol{n})}{\beta} > 0, \tag{1.31}$$

uniformly over $\boldsymbol{n} \in S^{d-1}$.

On the other hand, the surface tension under the averaged Gibbs measure is asymptotically determined by $J^{\min}$:

Proposition 1.13. *For all product measure $\mathbb{P}$ on $[0,1]^d$ and all $\lambda > 0$, uniformly over $\boldsymbol{n} \in S^{d-1}$,*

$$\lim_{\beta\to+\infty} \frac{\tau^\lambda_\beta(\boldsymbol{n})}{\beta} = \lambda J^{\min}\|\boldsymbol{n}\|_1. \tag{1.32}$$

In the case $J^{\min} = 0$ and $\mathbb{P}(J_e > 0) = 1$, an equivalent to $\tau^\lambda_\beta(\boldsymbol{n})$ is given in Proposition 3.2.

These asymptotics have consequences on the shape of the crystals under both the quenched and the averaged Gibbs measure (see Proposition 3.3 below). They also immediately imply that the inequality $\tau^\lambda_\beta(\boldsymbol{n}) \leqslant \lambda\tau^q_\beta(\boldsymbol{n})$ is strict at low temperatures in a number of cases:

Corollary 1.14. *Assume that $\mathbb{P}(J_e > J^{\min}) > p_c(d)$. Then, for any $\lambda > 0$ there is $\beta^\lambda_c < \infty$ such that*

$$\tau^\lambda_\beta(\boldsymbol{n}) < \lambda\tau^q_\beta(\boldsymbol{n}), \quad \forall \boldsymbol{n} \in S^{d-1}, \forall\beta > \beta^\lambda_c. \tag{1.33}$$

In particular if $J^{\min} = 0$ and if there is still a phase transition (i.e. $\mathbb{P}(J_e > 0) > p_c(d)$), then the inequality is always strict at low temperatures.

One consequence of (1.33) is the strict inequality $\tau^{\min}_\beta(\boldsymbol{n}) < \tau^q_\beta(\boldsymbol{n})$ under the same assumptions than in the Corollary, as $\lambda\tau^{\min}_\beta(\boldsymbol{n}) \leqslant \tau^\lambda_\beta(\boldsymbol{n})$ (Proposition 2.3).

Let us conclude on a comparison with the directed polymer model in $1+1$ dimensions: for the latter model, it was proved recently [17] that the Lyapunov exponent is positive at all $\beta \geqslant 0$, which corresponds in our settings to the strict inequality $\tau^a_\beta(\boldsymbol{n}) = \tau^{\lambda=1}_\beta < \tau^q_\beta(\boldsymbol{n})$.

1.5. **Phase coexistence.** We describe finally the phenomenon of phase coexistence in the dilute Ising model. Phase coexistence occurs when both the plus and the minus phase are present at the same time and occupy (distinct) regions of the domain. This phenomenon does not occur naturally in the Ising model. One way of obtaining phase coexistence is by conditioning the measure μ_Λ^J on the event that the overall magnetization

$$m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x \tag{1.34}$$

is smaller than $m < m_\beta$. Under this conditional measure, we will show that the two phases do coexist and that the minus phases occupies a fraction of the volume v such that $(1-2v)m_\beta = m$. Furthermore, the shape U of the region containing the minus phase is deterministic : if τ is the surface tension of the model, the observed shape minimizes the *surface energy*

$$\mathcal{F}(U) = \int_{\partial U}\tau(\boldsymbol{n})ds$$

under the volume constraint $\mathrm{Vol}(U)\geqslant v$, and this implies that U is a translated of $v^{1/d}\mathcal{W}$ where $\mathcal{W}$ is the renormalized Wulff crystal associated to τ:

$$\mathcal{W} = \lambda\left\{x\in\mathbb{R}^d : x\cdot\boldsymbol{n}\leqslant\tau(\boldsymbol{n}), \forall\boldsymbol{n}\in S^{d-1}\right\} \tag{1.35}$$

where $\lambda>0$ is chosen such that $\mathrm{Vol}(\mathcal{W})=1$.

Before we state our results, let us recall that $\mathcal{N}_I$ stands for the at-most-countable set of β at which lower large deviations for surface tension are possible at less than surface order (Corollary 1.9) and $\hat\beta_c$ is the slab percolation threshold (1.14). Another important notation is

$$\mathcal{N} = \left\{\beta\geqslant 0 : \lim_{N\to\infty}\mathbb{E}\Phi^{J,f}_{\hat\Lambda_N} \neq \lim_{N\to\infty}\mathbb{E}\Phi^{J,w}_{\hat\Lambda_N}\right\} \tag{1.36}$$

the set of β such that infinite volume averaged FK measures are not unique. $\mathcal{N}$ is at most countable (Theorem 2.3 in [44]).

We denote by $\mathcal{W}^q$ (resp. $\mathcal{W}^\lambda$) the Wulff crystal associated with the surface tension τ^q (resp. τ^λ) as in (1.35), and $v=\alpha^d$ the fraction of the volume occupied by the minus phase. The Wulff crystal $\alpha\mathcal{W}$ of volume v fits into the unit box $[0,1]^d$ only if $\alpha\,\mathrm{diam}_\infty(\mathcal{W})\leqslant 1$, where

$$\mathrm{diam}_\infty(A) = \sup_{x,y\in A}\|x-y\|_\infty, \qquad A\subset\mathbb{R}^d.$$

Our first theorem concerns the cost of the lower large deviations for the magnetization. In the sequel, $\Lambda_N = \{1,\ldots,N\}^d$.

Theorem 1.15. *Assume $\beta>\hat\beta_c$ with $\beta\notin\mathcal{N}$. Then, for all $0\leqslant\alpha<1/\mathrm{diam}_\infty(\mathcal{W}^q)$,*

$$\frac{1}{N^{d-1}}\log\mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta}\leqslant 1-2\alpha^d\right)\underset{N\to\infty}{\longrightarrow} -\mathcal{F}^q(\alpha\mathcal{W}^q) \quad \textit{in } \mathbb{P}\textit{-probability.} \tag{1.37}$$

Then we describe the geometry of the two phases. We consider a mesoscopic scale $K\in\mathbb{N}^\star$ and define the magnetization profile $\mathcal{M}_K$ as

$$\begin{array}{cccc}\mathcal{M}_K: & [0,1]^d & \longrightarrow & [-1,1] \\ & x & \longmapsto & \frac{1}{K^d}\sum_{z\in\Lambda_N\cap\Delta_{i(x)}}\sigma_z\end{array} \tag{1.38}$$

where

$$i(x) = \left(\left[\frac{Nx_1}{K}\right],\ldots,\left[\frac{Nx_d}{K}\right]\right) \quad\text{and}\quad \Delta_i = Ki+\{1,\ldots,K\}^d. \tag{1.39}$$

Hence, unless x is too close to the border of $[0,1]^d$, $\mathcal{M}_K(x)$ is the magnetization in a block of side-length K that contains Nx. Theorem 5.7 in [44] provides a strong stochastic control on $\mathcal{M}_K$ when $\beta>\hat\beta_c$. In particular, when K is large enough, at every x the probability that $\mathcal{M}_K(x)$ is close to either m_β or $-m_\beta$ is close to one under the averaged measure $\mathbb{E}\mu^{J,+}_{\Lambda_N}$. Hence $\mathcal{M}_K/m_\beta$

describes the geometry of the phases in the Ising model: it is close to one on the plus phase region, close to minus one on the minus phase region.

We need a few more notations. To $U \subset \mathbb{R}^d$ Borel measurable, we associate the profile

$$\chi_U : x \in \mathbb{R}^d \mapsto \begin{cases} 1 & \text{if } x \notin U \\ -1 & \text{else} \end{cases} \tag{1.40}$$

and denote by $\|.\|_{L^1}$ the norm of the L^1-space $L^1\left([0,1]^d;\mathbb{R}\right)$. We also consider the set of vectors z such that the translate $z+U$ fits into $[0,1]^d$:

$$\mathcal{T}(U) = \left\{ z \in \mathbb{R}^d : z + U \subset [0,1]^d \right\}. \tag{1.41}$$

Our second theorem describes the geometrical structure of the two phases when they coexist:

Theorem 1.16. *Assume that $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$. For all $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^q)$ and $\varepsilon > 0$, for any K large enough one has*

$$\lim_{N\to\infty} \mu^{J,+}_{\Lambda_N}\left(\inf_{z\in\mathcal{T}(\alpha\mathcal{W}^q)} \left\| \frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^q} \right\|_{L^1} \leqslant \varepsilon \,\middle|\, \frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d \right) = 1 \tag{1.42}$$

in $\mathbb{P}$-probability ($\mathbb{P}$-a.s. when $\beta \notin \mathcal{N}_I$).

Note that, although we state our theorems for the Ising model, they could easily be adapted to the Potts model with random interactions, or to random-cluster models, as the two fundamental tools for the study of phase coexistence, the coarse graining [44] and the study of surface tension, were developed in the more general setting of the random-cluster model ($q \geqslant 1$) with random couplings.

The fact that we consider K large but finite is a slight improvement with respect to former works. In general, one can take any $K = K_N$ such that $1 \ll K_N \ll N$ because on the one hand, $\mathcal{M}_{K_N}$ is close to the local mean of $\mathcal{M}_K$ as $K_N \gg 1$, and this local mean is close to $\chi_{z+\alpha\mathcal{W}}$ because the K_N-blocks intersecting $N\partial(z+\alpha\mathcal{W})$ contribute to a negligible volume as $K_N \ll N$.

Let us conclude this paragraph on a first consequence of the presence of random couplings : the limit shape of the droplet at low temperatures is smoother. In the case of the pure Ising model, the Wulff crystal converges to the unit hypercube $[\pm 1/2]^d$ as the temperature goes to zero. Here, $\mathcal{W}^q$ converges to the Wulff crystal associated with the maximal flow μ when $\mathbb{P}(J_e > 0) = 1$ (Proposition 3.3). Little is known on the crystal $\mathcal{W}^\mu$ associated to the maximal flow μ. Yet, as discussed in Section 3.4, an argument by Durrett and Liggett [21] shows that $\mathcal{W}^\mu$ is not a square when, for instance, $d = 2$,

$$\mathbb{P}(J_e = 1/2) = p \quad \text{and} \quad \mathbb{P}(J_e = 1) = 1 - p$$

with $\vec{p_c} < p < 1$, where $\vec{p_c}$ is the critical threshold for oriented bond percolation.

1.6. **Phase coexistence under averaged Gibbs measures.** Now we consider the issue of phase coexistence under averaged Gibbs measures, that is, when phase coexistence is imposed on both the spin configuration and the random couplings. Before we go further, we would like to remark that averaged Gibbs measures do not have the physical meaning of the quenched measure: in quenched ferromagnets, the disorder is frozen and thus cannot be influenced by the spin configuration itself. However, the analysis presented here gives an insight on the phenomenon of *localization* which can occur in models with media randomness.

First we remark that the cost for phase coexistence is here determined by the surface tension $\tau^\lambda(\boldsymbol{n})$:

Theorem 1.17. *For all $\lambda > 0$ and $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^\lambda)$,*

$$\frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d \right) \right)^\lambda \right] \underset{N\to\infty}{\longrightarrow} -\mathcal{F}^\lambda(\alpha\mathcal{W}^\lambda). \tag{1.43}$$

The inequality $\tau^\lambda < \lambda\tau^q$ at low temperatures (Corollary 1.14) implies that

$$\mathcal{F}^\lambda(\mathcal{W}^\lambda) \leqslant \mathcal{F}^\lambda(\mathcal{W}^q) < \lambda\mathcal{F}^q(\mathcal{W}^q),$$

in other words the cost for phase coexistence is *strictly smaller* under the averaged Gibbs measure than under the quenched Gibbs measure. One can go further and analyze the cost for reducing the cost for phase coexistence under averaged Gibbs measures: the functional

$$\mathcal{J}(f) = \sup_{\lambda>0}\{\mathcal{F}^\lambda(\mathcal{W}^\lambda) - \lambda f\} \in [0,\infty], \qquad f \in \mathbb{R}$$

is the rate function for lower deviations of the cost for phase coexistence. If $\mathcal{W}^{\min}$ and $\mathcal{F}^{\min}$ stand respectively for the Wulff crystal and the surface energy associated to $\tau^{\min}$, then $\mathcal{J}$ is infinite on the left of $\mathcal{F}^{\min}(\mathcal{W}^{\min})$, finite on the right of $\mathcal{F}^{\min}(\mathcal{W}^{\min})$ and zero on the right of $\mathcal{F}^q(\mathcal{W}^q)$, and:

Corollary 1.18. *For any $f \neq \mathcal{F}^{\min}(\mathcal{W}^{\min})$ and $\alpha \geqslant 0$ small enough,*

$$\lim_N \frac{1}{N^{d-1}} \log \mathbb{P}\left(\frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d\right) \geqslant -\alpha^{d-1} f\right) = -\alpha^{d-1}\mathcal{J}(f).$$

Upper deviations for the cost of phase coexistence, on the other hand, happen at volume order (cf. the proof of Proposition 4.14).

The shape of crystals under averaged Gibbs measures is as well determined by the surface tension $\tau^\lambda(\boldsymbol{n})$:

Theorem 1.19. *For any $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^{\lambda=1})$ and $\varepsilon > 0$, for any K large enough one has*

$$\lim_{N\to\infty} \left(\mathbb{E}\mu^{J,+}_{\Lambda_N}\right)\left(\inf_{z\in\mathcal{T}(\alpha\mathcal{W}^{\lambda=1})} \left\| \frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^{\lambda=1}} \right\|_{L^1} \leqslant \varepsilon \,\middle|\, \frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d\right) = 1. \tag{1.44}$$

This result extends in fact to all $\lambda > 0$, at the price however of heavier notations, because $\mathbb{E}[(\mu^{J,+}_{\Lambda_N}(.))^\lambda]$ is not a measure when $\lambda \neq 1$:

Theorem 1.20. *For any $\lambda > 0$, any $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^\lambda)$ and $\varepsilon > 0$, for any K large enough one has*

$$\lim_{N\to\infty} \frac{\mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\inf_{z\in\mathcal{T}(\alpha\mathcal{W}^\lambda)} \left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1} \leqslant \varepsilon \text{ and } \frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d\right)\right)^\lambda\right]}{\mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d\right)\right)^\lambda\right]} = 1.$$

We conclude the summary of our results with a description of a phenomenon of *localization*. First, let us characterize the typical value of the surface tension $\tau^J_{\mathcal{R}}$ under the averaged measure, conditioned to phase coexistence. For any $\beta \geqslant 0, \lambda > 0$ and $\boldsymbol{n} \in S^{d-1}$, this value stands between

$$\hat{\tau}^{\lambda,-}(\boldsymbol{n}) = \inf\left\{\tau \geqslant 0 : I_{\boldsymbol{n}}(\tau) + \lambda\tau = \tau^\lambda(\boldsymbol{n})\right\} \tag{1.45}$$

$$\text{and } \hat{\tau}^{\lambda,+}(\boldsymbol{n}) = \sup\left\{\tau \geqslant 0 : I_{\boldsymbol{n}}(\tau) + \lambda\tau = \tau^\lambda(\boldsymbol{n})\right\}. \tag{1.46}$$

The equality $\hat{\tau}^{\lambda,-}(\boldsymbol{n}) = \hat{\tau}^{\lambda,+}(\boldsymbol{n})$ holds whenever there is at most one τ at which the slope of $I_{\boldsymbol{n}}$ equals λ, that is, for all but at most countably many values of $\lambda > 0$. Note also that the strict inequality $\tau^\lambda(\boldsymbol{n}) < \lambda\tau^q(\boldsymbol{n})$ implies $\hat{\tau}^{\lambda,+}(\boldsymbol{n}) < \tau^q(\boldsymbol{n})$. We also consider a similar quantity for the quenched value of surface tension:

$$\tilde{\tau}^q(\boldsymbol{n}) = \inf\{\tau : I_{\boldsymbol{n}}(\tau) = 0\} \tag{1.47}$$

which coincides with $\tau^q(\boldsymbol{n})$ for all $\boldsymbol{n} \in S^{d-1}$, for all but at most countably many $\beta \geqslant 0$, see Corollary 1.9.

Our last Theorem describes the value of the surface tension τ^J conditionally on the position of the crystal: we prove that the typical value of surface tension is τ^q outside the boundary of the crystal, while on the boundary it is reduced to $\hat{\tau}^\lambda$. When $\tau^\lambda(\boldsymbol{n}) < \lambda\tau^q(\boldsymbol{n})$ for some $\boldsymbol{n} \in S^{d-1}$, the location of the Wulff crystal under averaged Gibbs measures is thus *determined* by the realization of the media: the boundary of the crystal coincides with the place where surface tension is reduced.

Theorem 1.21. *Let $\lambda > 0$, $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^\lambda)$ and $z \in \mathcal{T}(\alpha\mathcal{W}^\lambda)$. Consider $h, \delta, \gamma > 0$ and a parallelepiped rectangle $\mathcal{R} = \mathcal{R}_{x,h,\delta h}(\boldsymbol{n}, \mathcal{S}) \subset (0,1)^d$ as in (1.15). Call $\mathcal{R}^N = N\mathcal{R} + z_N(\mathcal{R})$ where $z_N(\mathcal{R}) \in (-1/2, 1/2]^d$ is chosen such that the center of $\mathcal{R}^N$ belongs to $\mathbb{Z}^d$. For $\varepsilon > 0$ small enough and K large enough,*

$$\lim_{N\to\infty} \frac{\mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\tau^J_{\mathcal{R}^N} \in \mathcal{A} \text{ and } \left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1} \leqslant \varepsilon\right)\right)^\lambda\right]}{\mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1} \leqslant \varepsilon\right)\right)^\lambda\right]} = 1$$

when

i. $\mathcal{R} \cap z + \alpha\partial\mathcal{W}^\lambda = \emptyset$ *and* $\mathcal{A} = [\tilde{\tau}^q(\boldsymbol{n}) - \gamma, \tau^q(\boldsymbol{n}) + \gamma]$

ii. or $x \in z + \alpha\partial\mathcal{W}^\lambda$, $\boldsymbol{n}$ *is the outer local normal to* $z + \alpha\mathcal{W}^\lambda$ *at* x, h *is small enough and* $\mathcal{A} = \left[\hat{\tau}^{\lambda,-}(\boldsymbol{n}) - \gamma, \hat{\tau}^{\lambda,+}(\boldsymbol{n}) + \gamma\right]$.

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2. Surface tension

As announced in the former Section, surface tension is a fundamental tool for understanding the mechanism of phase coexistence. It quantifies the free energy per surface unit of an interface separating the plus and minus phases in the dilute Ising model. In this Section, we prove the convergence of surface tension in dilute models and study its large deviations.

2.1. **Sub-additivity and convergence.** In many aspects the surface tension for the dilute Ising model is similar to the one of the Ising model with deterministic couplings. It has the crucial property of being *sub-additive*, as in the uniform case [36]: this is shown in Theorem 2.1 below. We present here the proof of Proposition 1.2, Theorem 2.1 and finally Theorem 1.3. We also explain why surface tension is positive under the assumption that $\beta > \hat{\beta}_c$ (Proposition 1.5).

Proof (Proposition 1.2). The surface tension $\tau^J_{\mathcal{R}}$ is a non-decreasing function of J and β because $\mathcal{D}_{\mathcal{R}}$ is a decreasing event while the measure $\Phi^J_{\mathcal{R}}$ stochastically increases with $p = 1 - \exp(-\beta J_e)$.

Now we consider $H' \geqslant H$ and call $\mathcal{R} = \mathcal{R}_{x,L,H}(\mathcal{S}, \boldsymbol{n})$ and $\mathcal{R}' = \mathcal{R}_{x,L,H'}(\mathcal{S}, \boldsymbol{n})$. In view of the DLR equation and of the monotonicity of $\Phi^{J,\pi}_{\mathcal{R}}$ along π, the measure $\Phi^{J,w}_{\mathcal{R}'}$ restricted to $E(\hat{\mathcal{R}})$ is stochastically smaller than $\Phi^{J,w}_{\mathcal{R}}$. On the other hand, it is clear that $\mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}'}$, and because $\mathcal{D}_{\mathcal{R}}$ is a decreasing event we conclude that

$$\Phi^{J,w}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) \leqslant \Phi^{J,w}_{\mathcal{R}'}(\mathcal{D}_{\mathcal{R}}) \leqslant \Phi^{J,w}_{\mathcal{R}'}(\mathcal{D}_{\mathcal{R}'}),$$

which shows that $\tau^J_{\mathcal{R}}$ is a non-increasing function of H.

It is clear from the definition that $\tau^{\min}_{\mathcal{R}} \geqslant 0$. The inequality $\tau^{\min}_{\mathcal{R}} \leqslant \tau^J_{\mathcal{R}} \leqslant \tau^{\max}_{\mathcal{R}}$ is a consequence of the monotony in J. We conclude with the upper bound on $\tau^{\max}_{\mathcal{R}}$. Because of the monotony in H we can take $H = 2\sqrt{d}$ (which ensures that disconnection is still possible). We have: $\tau^{\max}_{\mathcal{R}} \leqslant \tau^{\max}_{\mathcal{R}'}$ where $\mathcal{R}' = \mathcal{R}_{x,L,2\sqrt{d}}(\mathcal{S}, \boldsymbol{n})$. It is enough to close all the edges of $\widehat{\mathcal{R}'}$ to realize the disconnection in $\widehat{\mathcal{R}'}$. The DLR equation, combined with the monotonicity of $\Phi^{J^{\max},\pi}_{\{e\}}$ along the boundary condition π yields:

$$\tau^{\max}_{\mathcal{R}'} \leqslant -\frac{1}{L^{d-1}} \log \prod_{e \in E(\widehat{\mathcal{R}'})} \Phi^{J^{\max},w}_{\{e\}}(\{\omega_e = 0\}) = \beta J^{\max} \frac{|E(\widehat{\mathcal{R}'})|}{L^{d-1}}.$$

Finally, $|E(\widehat{\mathcal{R}'})|$ is not larger than $2d$ times the cardinal of $\widehat{\mathcal{R}'}$, which is itself not larger than the volume of $V = \bigcup_{x\in\widehat{\mathcal{R}'}} \left(x+[0,1]^d\right) \subset \mathcal{R}_{0,L+2\sqrt{d},3\sqrt{d}}(\mathcal{S},\boldsymbol{n})$. Consequently,

$$\tau^{J^{\max}}_{\mathcal{R}'} \leqslant \beta J^{\max} \times 2d \times \frac{\left(L+2\sqrt{d}\right)^{d-1} \times 6\sqrt{d}}{L^{d-1}} \leqslant \beta J^{\max} \times 6 \times 2^d d^{3/2}.$$

□

Now we address the issue of sub-additivity. It is a fundamental tool not only for proving the convergence of surface tension, but also for establishing the large deviations principles in the next Section.

Theorem 2.1. *Consider $\boldsymbol{n} \in S^{d-1}$, $\mathcal{S},\mathcal{S}' \subset \mathbb{S}_{\boldsymbol{n}}$ and $H,l \geqslant 2\sqrt{d}$, $L \geqslant 4\sqrt{d}l$. Let $\mathcal{R} = \mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})$. There is a collection $(\mathcal{R}_i)_{i\in\mathcal{C}}$ of rectangular parallelepipeds $\mathcal{R}_i = \mathcal{R}_{z_i,l,H}(\mathcal{S}',\boldsymbol{n})$ that are disjoint subsets of $\mathcal{R}$, centered at $z_i \in \mathbb{Z}^d$, with*

$$1 - c_d\left(\frac{l}{L}+\frac{1}{l}\right) \leqslant \left(\frac{l}{L}\right)^{d-1} |\mathcal{C}| \leqslant 1 \tag{2.1}$$

such that, for any $J : E(\hat{\mathcal{R}}) \to [0,1]$:

$$\tau^J_{\mathcal{R}} \leqslant \frac{1}{|\mathcal{C}|}\sum_{i\in\mathcal{C}} \tau^J_{\mathcal{R}_i} + \beta c_d\left(\frac{l}{L}+\frac{1}{l}\right) \tag{2.2}$$

where $c_d < \infty$ is a constant that depends on d only.

Let us make a few comments on this Theorem. First, a key feature of the sub-additivity as formulated in Theorem 2.1 is the *independence* of the $\tau^J_{\mathcal{R}_i}$ under $\mathbb{P}$ since the $\mathcal{R}_i$ are disjoint. Note that as well, the $\tau^J_{\mathcal{R}_i}$ have the same law as the $\mathcal{R}_i$ are all centered at lattice points. Three error terms appear in Theorem 2.1. Their origins are as follows (see also Figure 3):

i. the term $\beta c_d/l$ stands for the cost of disconnection in the middle section of $\mathcal{R}$ between adjacent $\mathcal{R}_i$,
ii. the term $\beta c_d l/L$ represents the cost of disconnection in the area not covered by the $\mathcal{R}_i$
iii. and the increase of H by $\sqrt{d}/2$ for $\mathcal{R}$ with respect to the $\mathcal{R}_i$ is a consequence of the requirement that the $\mathcal{R}_i$ be all centered at lattice points.

The last error term could be avoided for *rational* directions $\boldsymbol{n} \in S^{d-1}$, yet (as the two others) it will soon disappear when we take the limit $H \to \infty$.

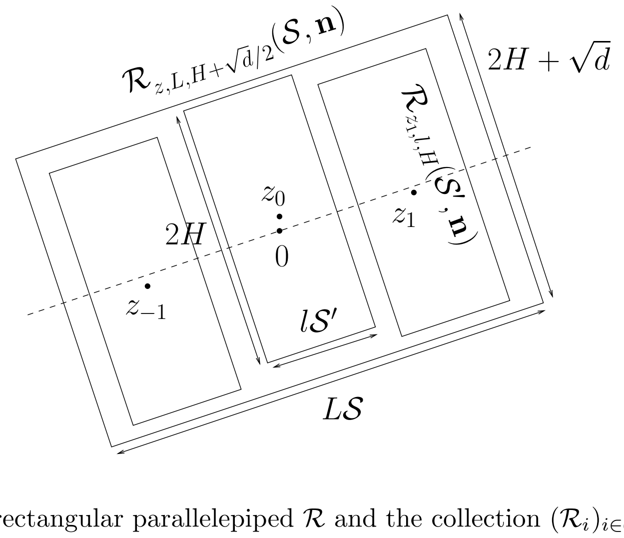

FIGURE 3. The rectangular parallelepiped $\mathcal{R}$ and the collection $(\mathcal{R}_i)_{i\in\mathcal{C}}$ in Theorem 2.1.

The reader will notice that the use of the FK representation permits to give a relatively short proof of Theorem 2.1.

Proof (Theorem 2.1). We begin with the definition of z_i and $\mathcal{C}$. We call $(\boldsymbol{e}'_k)_{k=1\dots d-1}$ the edges of $\mathcal{S}'$ and $\boldsymbol{e}'_d = \boldsymbol{n}$, so that $(\boldsymbol{e}'_k)_{k=1\dots d}$ is an orthonormal basis of $\mathbb{R}^d$. For all $i = (i_k)_{k=1\dots d-1} \in \mathbb{Z}^{d-1}$ we define z_i as the unique point of $\mathbb{Z}^d$ such that

$$\left(l+\sqrt{d}\right)\sum_{k=1}^{d-1} i_k \boldsymbol{e}'_k \in z_i + \left[-\frac{1}{2},\frac{1}{2}\right)^d$$

and call

$$\mathcal{C} = \left\{i \in \mathbb{Z}^{d-1} : \mathcal{R}_i \subset \mathcal{R}\right\}$$

letting

$$\mathcal{R} = \mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n}) \quad \text{and} \quad \mathcal{R}_i = \mathcal{R}_{z_i,l,H}(\mathcal{S}',\boldsymbol{n}).$$

We proceed with the proof of (2.1) first. We call $\mathcal{H}_{\boldsymbol{n}}$ the hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{n}$ that contains 0 and remark that the orthogonal projections of $z_i + l\mathcal{S}'$ (for all $i \in \mathcal{C}$) on $\mathcal{H}_{\boldsymbol{n}}$ are disjoint and all included in $L\mathcal{S}$. Hence their total surface $|\mathcal{C}|l^{d-1}$ does not exceed the surface of $L\mathcal{S}$, namely L^{d-1}, and the upper bound in (2.1) follows. Reusing the previous notations we call

$$z'_i = \left(l+\sqrt{d}\right)\sum_{k=1}^{d-1} i_k \boldsymbol{e}'_k, \quad \forall i \in \mathbb{Z}^{d-1}$$

so that $z'_i \in \mathcal{H}_{\boldsymbol{n}}$. We consider then

$$\mathcal{C}' = \left\{i \in \mathbb{Z}^{d-1} : z'_i + \left(l+\sqrt{d}\right)\mathcal{S}' \subset L\mathcal{S}\right\}.$$

In view of the inequality $d(z_i, z'_i) \leqslant \sqrt{d}/2$ it follows that $z_i + l\mathcal{S}' \subset \mathcal{R}$, for all $i \in \mathcal{C}'$, hence $\mathcal{C}' \subset \mathcal{C}$. On the other hand, for any $i \in \mathbb{Z}^{d-1}$ such that $z'_i + (l+\sqrt{d})\mathcal{S}' \cap (L - 2\sqrt{d}(l+\sqrt{d}))\mathcal{S} \neq \emptyset$ we have $i \in \mathcal{C}'$, hence

$$\left(\frac{L-2\sqrt{d}(l+\sqrt{d})}{l+\sqrt{d}}\right)^{d-1} \leqslant |\mathcal{C}'| \leqslant |\mathcal{C}|$$

and

$$\begin{aligned}
\left(\frac{l}{L}\right)^{d-1}|\mathcal{C}| &\geqslant \left(\frac{l}{l+\sqrt{d}} - 2\sqrt{d}\frac{l}{L}\right)^{d-1} \\
&\geqslant \left(1 - \frac{\sqrt{d}}{l} - 2\sqrt{d}\frac{l}{L}\right)^{d-1} \\
&\geqslant 1 - (d-1)\left(\frac{\sqrt{d}}{l} + 2\sqrt{d}\frac{l}{L}\right)
\end{aligned}$$

which yields the lower bound for (2.1). We pass now to the proof of (2.2) and call

$$\mathcal{E} = \left\{e \in E(\hat{\mathcal{R}}) \setminus \bigcup_{i\in\mathcal{C}} E(\hat{\mathcal{R}}_i) : d(e,\mathcal{H}_{\boldsymbol{n}}) \leqslant \frac{\sqrt{d}}{2}\right\}$$

where $d(e,\mathcal{H}_{\boldsymbol{n}})$ stands for the shortest distance between one extremity of e and $\mathcal{H}_{\boldsymbol{n}}$. The inclusion

$$\left(\bigcap_{i\in\mathcal{C}} \mathcal{D}_{\mathcal{R}_i}\right) \bigcap \{\omega_e = 0, \forall e \in \mathcal{E}\} \subset \mathcal{D}_{\mathcal{R}}$$

holds: consider ω that belongs to the left-hand side and let c an ω-open path issued from $\partial^+\hat{\mathcal{R}}$. Every times c enters some $\hat{\mathcal{R}}_i$ by the upper boundary $\partial^+\hat{\mathcal{R}}$, it also exits by the same upper boundary since $\omega \in \mathcal{D}_{\hat{\mathcal{R}}_i}$. As c cannot use the edges of $\mathcal{E}$ it is not able to cross the middle hyperplane $\mathcal{H}_{\boldsymbol{n}}$ elsewhere than in the $\hat{\mathcal{R}}_i$, and in particular it cannot reach $\partial^-\hat{\mathcal{R}}$. Since the $\mathcal{D}_{\hat{\mathcal{R}}_i}$ as well as the

$\{\omega_e = 0\}$ are decreasing events, the DLR equations and the monotonicity along the boundary condition for Φ^J imply that

$$\begin{aligned} \Phi^{J,w}_{\hat{\mathcal{R}}}(\mathcal{D}_{\mathcal{R}}) &\geqslant \prod_{i\in\mathcal{C}} \Phi^{J,w}_{\mathcal{R}_i}(\mathcal{D}_{\mathcal{R}_i}) \times \prod_{e\in\mathcal{E}} \Phi^{J,w}_{\{e\}}(\{\omega_e = 0\}) \\ &\geqslant \prod_{i\in\mathcal{C}} \Phi^{J,w}_{\mathcal{R}_i}(\mathcal{D}_{\mathcal{R}_i}) \times \exp\left(-\beta|\mathcal{E}|\right) \end{aligned} \tag{2.3}$$

as $\Phi^{J,w}_{\{e\}}(\{\omega_e = 0\} = 1 - p_e = \exp\left(-\beta J_e\right) \geqslant \exp(-\beta)$. We proceed then with an estimate over the cardinality of $\mathcal{E}$: we call $F = \left\{x \in \mathbb{Z}^d : \exists y, \{x, y\} \in \mathcal{E}\right\}$ the set of extremities of some $e \in \mathcal{E}$ and remark that $|\mathcal{E}| \leqslant d \operatorname{Vol}(V)$ where $V = \bigcup_{x\in F} x + [0,1]^d$. We have

$$V \subset \mathcal{R}_{0,L+2\sqrt{d},3\sqrt{d}/2}(\mathcal{S}, \boldsymbol{n}) \quad \text{while} \quad V \cap \mathcal{R}_{z_i,l-2\sqrt{d},\infty}(\mathcal{S}', \boldsymbol{n}) = \emptyset, \ \ \forall i \in \mathcal{C},$$

hence

$$|\mathcal{E}| \leqslant d \times \frac{3\sqrt{d}}{2} \times \left(\left(L + 2\sqrt{d}\right)^{d-1} - |\mathcal{C}| \left(l - 2\sqrt{d}\right)^{d-1} \right) \leqslant c_d L^{d-1} \left(\frac{l}{L} + \frac{1}{l} \right)$$

in view of the lower bound in (2.1). Taking logarithms in (2.3) and dividing by $-L^{d-1}$ we obtain the inequality

$$\tau^J_{\mathcal{R}} \leqslant \left(\frac{l}{L}\right)^{d-1} \sum_{i\in\mathcal{C}} \tau^J_{\mathcal{R}_i} + c_d\beta \left(\frac{l}{L} + \frac{1}{l} \right)$$

and (2.2) follows from the upper bound in (2.1).

We conclude with a word on the structure of the sequence $(\tau^J_{\mathcal{R}_i})_{i\in\mathcal{C}}$. The $\mathcal{R}_i$ are disjoint by construction, hence so are the edge sets $E(\hat{\mathcal{R}}_i)$, hence the $\tau^J_{\mathcal{R}_i}$ are independent. They are identically distributed as the $\mathcal{R}_i$ are all centered at lattice points, $\mathbb{P}$ being translation invariant as a product measure. □

Now we establish the convergence for surface tension and prove Theorem 1.3. The proof of this Theorem is based on the sub-additivity of surface tension. We do not apply directly Kingman's sub-additive Theorem [34] as we want to show that τ^q does not depend on $\mathcal{S}$, nor on δ.

Proof Taking the expectation $\mathbb{E}$ in the sub-additivity inequality (2.2) we get

$$\mathbb{E}\tau^J_{\mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})} \leqslant \mathbb{E}\tau^J_{\mathcal{R}_{0,l,H}(\mathcal{S}',\boldsymbol{n})} + \beta c_d \left(\frac{l}{L} + \frac{1}{l} \right).$$

Applying $\limsup_{L\to\infty}$, then $\liminf_{l\to\infty}$ and taking the decreasing limit in H we obtain

$$\lim_{H\to\infty} \limsup_{L\to\infty} \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \leqslant \lim_{H\to\infty} \liminf_{L\to\infty} \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S}',\boldsymbol{n})}$$

which proves that

$$\tau^q(\boldsymbol{n}) = \lim_{H\to\infty} \liminf_{L\to\infty} \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} = \lim_{H\to\infty} \limsup_{L\to\infty} \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \tag{2.4}$$

exists and does not depend on $\mathcal{S} \in \mathbb{S}_n$.

We prove now the convergence $\tau^J_{\mathcal{R}^N} \to \tau^q(\boldsymbol{n})$ in $\mathbb{P}$-probability, where $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$. The sub-additivity (2.2) yields: for any $\delta > 0$ and N large enough,

$$\tau^J_{\mathcal{R}^N} \leqslant \tau^J_{\mathcal{R}_{0,N,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})} \leqslant \frac{1}{|\mathcal{C}|} \sum_{i\in\mathcal{C}} \tau^J_{\mathcal{R}_{z_i,L,H}} + \beta c_d \left(\frac{L}{N} + \frac{1}{L} \right)$$

Taking $\limsup_{N\to\infty}$ and applying the strong law of large numbers give:

$$\limsup_{N\to\infty} \tau^J_{\mathcal{R}^N} \leqslant \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} + \frac{\beta c_d}{L} \qquad \mathbb{P}\text{-a.s.}$$

and after $\liminf_{L\to\infty}$ and $\lim_{H\to\infty}$ we see that, for all $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$ and $\delta > 0$,

$$\limsup_{N\to\infty} \tau^J_{\mathcal{R}^N} \leqslant \tau^q(\boldsymbol{n}) \qquad \mathbb{P}\text{-a.s.} \tag{2.5}$$

On the other hand, the sub-additivity (2.2) is also responsible for the convergence of $\mathbb{E}\tau^J_{\mathcal{R}^N}$: remark that

$$\mathbb{E}\tau^J_{\mathcal{R}_{0,L,\delta N+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})} \leqslant \mathbb{E}\tau^J_{\mathcal{R}^N} + \beta c_d \left(\frac{N}{L}+\frac{1}{N}\right),$$

hence $\limsup_{L\to\infty}$ followed by $\liminf_{N\to\infty}$ give:

$$\tau^q(\boldsymbol{n}) \leqslant \liminf_{N\to\infty} \mathbb{E}\tau^J_{\mathcal{R}^N}. \tag{2.6}$$

Together with (2.5) and (2.6), the boundedness of $\tau^J_{\mathcal{R}^N}$ ensures the convergence in probability. □

Let us sketch now a proof of Proposition 1.5, namely that the quenched surface tension $\tau^q(\boldsymbol{n})$ is *positive* for any $\beta > \hat{\beta}_c$: thanks to the renormalization argument of [44], one can compare the surface tension $\tau^a = \tau^{\lambda=1}$ under the averaged Gibbs measure to the surface tension of high density site percolation, which is positive. The claim follows as $\tau^q \geqslant \tau^a$ by Jensen's inequality.

2.2. **Upper large deviations.** Due to the presence of the random couplings, surface tension can *fluctuate* around its typical value. The sub-additivity permits to study the order of the cost of large deviations. First, we examine upper deviations and prove Theorem 1.4. The proof is based on the following argument: we split $\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ into cN rectangular parallelepipeds $\mathcal{R}_i$ with finite height H. In order to increase $\tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}$ one has to increase surface tension in each $\mathcal{R}_i$, but the cost of increasing one $\tau^J_{\mathcal{R}_i}$ is already of surface order by sub-additivity.

Proof (Theorem 1.4). As a first step towards the proof we estimate the cost for upper deviations of surface tension in a rectangular parallelepiped of fixed height, using the sub-additivity of τ^J. From the definition of $\tau^q(\boldsymbol{n})$ at (2.4) it follows that for any H large enough,

$$\limsup_L \mathbb{E}\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \leqslant \tau^q(\boldsymbol{n}) + \frac{\varepsilon}{6}.$$

Given such an H we fix l large enough such that $\mathbb{E}\tau^J_{\mathcal{R}_{0,l,H}(\mathcal{S},\boldsymbol{n})} \leqslant \tau^q(\boldsymbol{n}) + \varepsilon/3$ and $c_d\beta/l \leqslant \varepsilon/4$, where c_d refers to the constant in the sub-additivity equation. With the notations of Theorem 2.1 we have:

$$\tau^J_{\mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})} \leqslant \frac{1}{|\mathcal{C}|}\sum_{i\in\mathcal{C}} \tau^J_{\mathcal{R}_{z_i,l,H}(\mathcal{S},\boldsymbol{n})} + \frac{\varepsilon}{4} + \beta c_d \frac{l}{L} \tag{2.7}$$

and the $\tau^J_{\mathcal{R}_{z_i,l,H}(\mathcal{S},\boldsymbol{n})}$ are i.i.d. variables of mean not larger than $\tau^q(\boldsymbol{n}) + \varepsilon/3$. Hence, Cramér's Theorem tells that

$$\mathbb{P}\left(\frac{1}{|\mathcal{C}|}\sum_{i\in\mathcal{C}} \tau^J_{\mathcal{R}_{z_i,l,H}(\mathcal{S},\boldsymbol{n})} \geqslant \tau^q(\boldsymbol{n}) + \frac{\varepsilon}{2}\right) \leqslant \exp(-c|\mathcal{C}|)$$

for some $c > 0$. Reporting in (2.7) proves that for any $\varepsilon > 0$, for any H large enough:

$$\limsup_{L\to\infty} \frac{1}{L^{d-1}} \log \mathbb{P}\left(\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \geqslant \tau^q(\boldsymbol{n}) + \varepsilon\right) < 0 \tag{2.8}$$

– that is, the cost for increasing $\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}$ is of surface order. We fix such an H and decompose now the rectangular parallelepiped $\mathcal{R} = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ in the direction $\boldsymbol{n}$. Precisely, we let

$$\tilde{x}_i = 2\left(H + \frac{\sqrt{d}}{2}\right) i\boldsymbol{n}, \quad \forall i \in \mathbb{Z} \quad \text{and} \quad \tilde{\mathcal{R}}_i = \mathcal{R}_{\tilde{x}_i,N,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n}).$$

We call $\mathcal{G}$ the set of $i \in \mathbb{Z}$ such that $\tilde{\mathcal{R}}_i \subset \mathcal{R}$ and consider, for all $i \in \mathcal{G}$, x_i the point of $\mathbb{Z}^d$ such that $\tilde{x}_i \in x_i + [-1/2, 1/2)^d$ and let

$$\mathcal{R}_i = \mathcal{R}_{x_i,N-\sqrt{d},H}(\mathcal{S},\boldsymbol{n}).$$

The rectangular parallelepipeds $\mathcal{R}_i$ are disjoint subsets of $\mathcal{R} = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$, all centered at lattice points. Furthermore, if we call $\mathcal{E}_{\mathrm{lat}}$ the set of edges in $E(\hat{\mathcal{R}})$ with one extremity at distance at most $\sqrt{d}$ from the lateral boundary of $\mathcal{R}$, we have:

$$\omega \in \bigcup_{i\in\mathcal{G}} \mathcal{D}_{\mathcal{R}_i} \text{ and } \omega_e = 0, \forall e \in \mathcal{E}_{\mathrm{lat}} \quad \Rightarrow \quad \omega \in \mathcal{D}_{\mathcal{R}}.$$

Hence the DLR equation yields:

$$\begin{aligned}\Phi^{J,w}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) &\geqslant \max_{i\in\mathcal{G}} \Phi^{J,w}_{\mathcal{R}}(\omega_e = 0, \forall e \in \mathcal{E}_{\mathrm{lat}} \text{ and } \omega \in \mathcal{D}_{\mathcal{R}_i}) \\ &\geqslant e^{-\beta|\mathcal{E}_{\mathrm{lat}}|} \times \max_{i\in\mathcal{G}} \Phi^{J,w}_{\mathcal{R}_i}(\omega \in \mathcal{D}_{\mathcal{R}_i}).\end{aligned}$$

As $|\mathcal{E}_{\mathrm{lat}}| \leqslant c_d\delta N^{d-1}$ we conclude finally to the inequality

$$\tau^J_{\mathcal{R}} \leqslant c_d\delta\beta + \min_{i\in\mathcal{G}} \tau^J_{\mathcal{R}_i}. \tag{2.9}$$

Inequality (2.9) states that in order to increase significantly $\tau^J_{\mathcal{R}}$, one must increase each $\tau^J_{\mathcal{R}_i}$. Yet, the cost for increasing one of the $\tau^J_{\mathcal{R}_i}$ is of surface order (2.8), and the $\tau^J_{\mathcal{R}_i}$ are independent variables. Hence for any $\delta > 0$ such that $c_d\delta\beta < \varepsilon$,

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}\left(\tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})} \geqslant \tau^q(\boldsymbol{n}) + 2\varepsilon\right) < 0.$$

As $\tau^J_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}$ decreases with δ, the claim follows for arbitrary $\delta > 0$. □

2.3. **Lower large deviations.** Contrary to upper deviations, lower large deviations occur at surface order. Here we consider the rate function $I_{\boldsymbol{n}}$ for lower large deviations. The fact that deviations occur at the same order as the disconnecting event defining surface tension is responsible for the distinct behavior of surface tension under quenched and averaged measures. Explicit bounds on the rate function $I_{\boldsymbol{n}}$ will be derived in Sections 2.5 and 2.6.

Proof We begin with the definition of the rate function $I_{\mathcal{R}}$ in a rectangular parallelepiped $\mathcal{R} = \mathcal{R}_{0,L,H}(\mathcal{S}, \boldsymbol{n})$ as the surface cost for reducing $\tau^J_{\mathcal{R}}$ to τ:

$$I_{\mathcal{R}}(\tau) = -\frac{1}{L^{d-1}} \log \mathbb{P}\left(\tau^J_{\mathcal{R}} \leqslant \tau\right).$$

According to Proposition 1.2, $I_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}(\tau)$ is a non-increasing function of τ and H. Hence the limit

$$I_{(\mathcal{S},\boldsymbol{n})}(\tau) = \lim_{\varepsilon\to 0^+} \inf_H \limsup_L I_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}(\tau+\varepsilon) \in [0,\infty] \tag{2.10}$$

exists – we introduce the parameter $\varepsilon > 0$ in order to compensate for the error terms in (2.2). It is clearly a non-increasing function of τ. We prove now that it is also convex in τ and that it does not depend on $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$: let $\mathcal{S}' \in \mathbb{S}_n$, $\varepsilon > 0$ and $\alpha \in [0,1]$. Using the notations $\mathcal{R} = \mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S}, \boldsymbol{n})$, $\mathcal{R}_i = \mathcal{R}_{z_i,l,H}(\mathcal{S}', \boldsymbol{n})$ and $\mathcal{C}$ of the sub-additivity Theorem (Theorem 2.1), we have

$$\tau^J_{\mathcal{R}} \leqslant \frac{|\mathcal{C}^1|}{|\mathcal{C}|}\tau^1 + \frac{|\mathcal{C}^2|}{|\mathcal{C}|}\tau^2 + \varepsilon + \beta c_d\left(\frac{l}{L} + \frac{1}{l}\right)$$

if $\mathcal{C}^1 \sqcup \mathcal{C}^2$ is a partition of $\mathcal{C}$ such that

$$\tau^J_{\mathcal{R}_i} \leqslant \begin{cases} \tau^1 + \varepsilon & \text{if } i \in \mathcal{C}^1 \\ \tau^2 + \varepsilon & \text{if } i \in \mathcal{C}^2. \end{cases} \tag{2.11}$$

The probability for realizing condition (2.11) equals

$$\exp\left(-|\mathcal{C}^1| l^{d-1} I_{\mathcal{R}_{0,l,H}(\mathcal{S}',\boldsymbol{n})}\left(\tau^1+\varepsilon\right) - |\mathcal{C}^2| l^{d-1} I_{\mathcal{R}_{0,l,H}(\mathcal{S}',\boldsymbol{n})}\left(\tau^2+\varepsilon\right)\right)$$

and letting $|\mathcal{C}^1|/|\mathcal{C}| \to \alpha$ and $L \to \infty$ we see that

$$\begin{aligned}&\limsup_L I_{\mathcal{R}_{0,L,H+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})}\left(\alpha\tau^1 + (1-\alpha)\tau^2 + 2\varepsilon + \beta c_d/l\right) \leqslant \\ &\alpha I_{\mathcal{R}_{0,l,H}(\mathcal{S}',\boldsymbol{n})}\left(\tau^1+\varepsilon\right) + (1-\alpha) I_{\mathcal{R}_{0,l,H}(\mathcal{S}',\boldsymbol{n})}\left(\tau^2+\varepsilon\right).\end{aligned} \tag{2.12}$$

Taking the superior limit in l, then the limit in H, then $\varepsilon \to 0^+$ we obtain

$$I_{(\mathcal{S},\boldsymbol{n})}\left(\alpha\tau^1 + (1-\alpha)\tau^2\right) \leqslant \alpha I_{(\mathcal{S}',\boldsymbol{n})}\left(\tau^1\right) + (1-\alpha) I_{(\mathcal{S}',\boldsymbol{n})}\left(\tau^2\right)$$

which proves both the independence of $I_{(\mathcal{S},\boldsymbol{n})}$ with respect to $\mathcal{S}$ (take $\alpha = 1$) and the convexity along τ. We let now $I_{\boldsymbol{n}} = I_{(\mathcal{S},\boldsymbol{n})}$ and postpone the proof of (1.18) for a while. The continuity of $I_{\boldsymbol{n}}$ on the interior of the domain of finiteness of $I_{\boldsymbol{n}}$ is a consequence of its convexity. Hence we examine the domain of finiteness of $I_{\boldsymbol{n}}$. Let first $\tau < \tau^{\min}(\boldsymbol{n})$. If $\varepsilon > 0$ is small enough, the event $\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \leqslant \tau + \varepsilon < \tau^{\min}(\boldsymbol{n})$ has a probability zero and consequently, $I_{\boldsymbol{n}}(\tau) = +\infty$. The second easy regime is $\tau \geqslant \tau^q(\boldsymbol{n})$: from Proposition 1.4 we infer that $\lim_{L\to\infty} \mathbb{P}(\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \leqslant \tau + \varepsilon) = 1$ provided that H is large enough and this implies $I_{\boldsymbol{n}}(\tau) = 0$. If at last $\tau > \tau^{\min}(\boldsymbol{n})$, there is H such that

$$\limsup_L \tau^{J^{\min}}_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} < \tau.$$

We will prove that, for $\delta > 0$ small enough we still have:

$$\limsup_L \tau^{J^{\min}+\delta}_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} < \tau. \tag{2.13}$$

If we let $\mathcal{R} = \mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})$ and differentiate along δ, we obtain

$$\frac{\partial \tau^{J^{\min}+\delta}_{\mathcal{R}}}{\partial\delta} = \sum_{e\in E(\hat{\mathcal{R}})} \left.\frac{\partial\tau^J_{\mathcal{R}}}{\partial J_e}\right|_{J=J^{\min}+\delta}$$

yet, (2.22) and Proposition 2.6 indicate that for any $J \in \mathcal{J}$,

$$\frac{L^{d-1}}{\beta}\frac{\partial\tau^J_{\mathcal{R}}}{\partial J_e} \leqslant 1.$$

As a consequence, $\tau^{J^{\min}+\delta}_{\mathcal{R}}$ is a $c_d\beta H$-Lipschitz function of δ. The same is true for $\limsup_L \tau^{J^{\min}+\delta}_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}$, thus (2.13) holds true for $\delta > 0$ small enough. Now we write, for any L large enough:

$$\begin{aligned} I_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}(\tau) &= -\frac{1}{L^{d-1}}\log\mathbb{P}\left(\tau^J_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})} \leqslant \tau\right) \\ &\leqslant -\frac{1}{L^{d-1}}\log\mathbb{P}\left(J_e \leqslant J^{\min}+\delta, \quad \forall e \in E\left(\hat{\mathcal{R}}_{0,L,H}(\mathcal{S},\boldsymbol{n})\right)\right) \\ &\leqslant c_d H \times (-\log\mathbb{P}(J_e \in [J^{\min}, J^{\min}+\delta])) \end{aligned}$$

which is finite thanks to the definition of $J^{\min}$. This ends the proof that $I_{\boldsymbol{n}}(\tau) < \infty$, for any $\tau > \tau^{\min}(\boldsymbol{n})$.

We address at last the convergence (1.18). The inequality $I_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}(\tau) \leqslant I_{\mathcal{R}_{0,N,H}(\mathcal{S},\boldsymbol{n})}(\tau)$ when $N\delta \geqslant H$ yields an upper bound on the superior limit:

$$\limsup_N I_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}(\tau) \leqslant \inf_H \limsup_L I_{\mathcal{R}_{0,L,H}(\mathcal{S},\boldsymbol{n})}(\tau) \leqslant I_{\boldsymbol{n}}(\tau^-) = I_{\boldsymbol{n}}(\tau)$$

for all $\tau > \tau^{\min}(\boldsymbol{n})$, thanks to the continuity of $I_{\boldsymbol{n}}$. For the lower bound we use the sub-additivity of surface tension. Applying (2.12) with $\alpha = 1$, $l = N$, $H = \delta N$ yields: for any $\varepsilon > 0$ and N large enough,

$$\limsup_L I_{\mathcal{R}_{0,L,\delta N+\sqrt{d}/2}(\mathcal{S},\boldsymbol{n})}(\tau + 3\varepsilon) \leqslant I_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}(\tau+\varepsilon)$$

and replacing $\tau + \varepsilon$ with τ, we obtain after the limits $N \to \infty$ and $\varepsilon \to 0^+$ the lower bound

$$I_{\boldsymbol{n}}(\tau) \leqslant \liminf_N I_{\mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})}(\tau), \quad \forall\tau\in\mathbb{R}.$$

□

2.4. **Surface tension under averaged Gibbs measures.** The rate function $I_{\boldsymbol{n}}$ can be analyzed through a dual quantity: the surface tension under the averaged Gibbs measure defined at (1.24). The duality of Fenchel-Legendre transforms for convex functions (Lemma 4.5.8 in [18]) implies that $\lambda \mapsto \tau^\lambda(\boldsymbol{n})$ is concave and that

$$I_{\boldsymbol{n}}(\tau) = \sup_{\lambda>0}\{\tau^\lambda(\boldsymbol{n}) - \lambda\tau\}. \tag{2.14}$$

As we said at (1.25), $\tau^\lambda(\boldsymbol{n})$ can be interpreted as the surface tension under an average of $\Phi^{J,w}_{\mathcal{R}}$. Indeed, if we let

$$\tau^\lambda_{\mathcal{R}} = -\frac{1}{L^{d-1}}\log\mathbb{E}\left(\left[\Phi^{J,w}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}})\right]^\lambda\right) = -\frac{1}{L^{d-1}}\log\mathbb{E}\left(\exp\left(-\lambda L^{d-1}\tau^J_{\mathcal{R}}\right)\right), \tag{2.15}$$

for any rectangular parallelepiped $\mathcal{R}$ of side-length L as in (1.15), then Varadhan's Lemma yields:

Proposition 2.2. *For any $\lambda > 0$ and $\boldsymbol{n} \in S^{d-1}$, for any sequence of rectangular parallelepipeds $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ with $\delta > 0$ and $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$, the quantity $\tau^\lambda_{\mathcal{R}^N}$ converges to $\tau^\lambda(\boldsymbol{n})$:*

$$\lim_N \tau^\lambda_{\mathcal{R}^N} = \tau^\lambda(\boldsymbol{n}). \tag{2.16}$$

Thus, the limit does not depend on $\delta > 0$ nor on $\mathcal{S} \in \mathbb{S}_n$.

We defined at (1.47) the value $\tilde\tau^q(\boldsymbol{n})$ of the surface tension at which $I_{\boldsymbol{n}}(\tau)$ becomes zero. Below are some immediate consequences of the definition of $\tau^\lambda(\boldsymbol{n})$ at (1.24) together with (2.16), which allow to sketch the graph of $\lambda \mapsto \tau^\lambda(\boldsymbol{n})$ on Figure 4:

Proposition 2.3. *The following inequalities hold:*

$$\lambda\tau^{\min}(\boldsymbol{n}) \leqslant \tau^\lambda(\boldsymbol{n}) \leqslant \lambda\tilde\tau^q(\boldsymbol{n}), \quad \forall \boldsymbol{n} \in S^{d-1}, \lambda > 0 \tag{2.17}$$

while:

$$\frac{\tau^\lambda(\boldsymbol{n})}{\lambda} \underset{\lambda\to 0^+}{\longrightarrow} \tilde\tau^q(\boldsymbol{n}) \quad \textit{and} \quad \frac{\tau^\lambda(\boldsymbol{n})}{\lambda} \underset{\lambda\to+\infty}{\longrightarrow} \tau^{\min}(\boldsymbol{n}), \qquad \forall \boldsymbol{n} \in S^{d-1}. \tag{2.18}$$

Hence, $\tau^\lambda(\boldsymbol{n})$ is positive if and only if $\tilde\tau^q(\boldsymbol{n}) > 0$. Furthermore:

$$\tau^\lambda(\boldsymbol{n}) \underset{\lambda\to+\infty}{\longrightarrow} \lim_{\tau\to 0^+} I_{\boldsymbol{n}}(\tau) \in [0,\infty]. \tag{2.19}$$

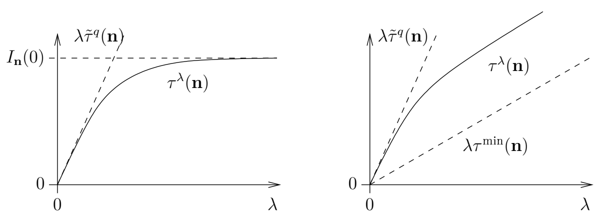

FIGURE 4. The graph of $\lambda \mapsto \tau^\lambda(\boldsymbol{n})$ in the case of dilution ($\tau^{\min} = 0$ and $I_{\boldsymbol{n}}(0) < \infty$, left) and distributions with $\tau^{\min} > 0$ (right).

Another important yet classical fact is the *convexity* of surface tension [36]. The proposition below is a consequence of the weak triangle inequality for $\tau^J_{\mathcal{R}}$ (see [36] or [10] for the uniform case, or Appendix 2.5.2 in [45]).

Proposition 2.4. *Let f^q be the homogeneous extension of τ^q to $\mathbb{R}^d$, namely:*

$$f^q(x) = \begin{cases} \|x\|\tau^q(x/\|x\|) & \textit{if } x \in \mathbb{R}^d \setminus \{0\} \\ 0 & \textit{if } x = 0, \end{cases}$$

and let f^λ (resp. $\tilde f^q$) be the homogeneous extension of τ^λ (resp. $\tilde\tau^q$) to $\mathbb{R}^d$. Then, f^q, f^λ and $\tilde f^q$ are convex and τ^q, τ^λ and $\tilde\tau^q$ are continuous on S^{d-1}.

2.5. **Concentration at low temperatures.** In this Section and the next one we establish respectively Theorems 1.7 and 1.8. In both cases we use concentration of measure theory, which is a very efficient tool for analyzing the fluctuations of product measures. In the case of polymers or even spin glasses it yields relevant bounds on the probabilities of deviations, see [35] for a review. Concerning the Ising (or random-cluster) model with random couplings, its application to the deviations of surface tension requires a control over the surface of the interface, and this is the point where the proofs of Theorems 1.7 and 1.8 differ: at low temperatures one can control rather easily the length of the interface, while under the only assumptions of Theorem 1.8 the same control is not immediate.

The surface tension $\tau^\lambda(\boldsymbol{n})$ under averaged Gibbs measure plays an important role here, as well as the modified measure $\mathbb{E}_\lambda$ defined at (2.25) below. We will obtain lower bounds on $\tau^\lambda(\boldsymbol{n})$, which correspond to lower bounds on $I_{\boldsymbol{n}}(\tau)$ by (2.14).

Rather than making the assumption that the product measure $\mathbb{P}$ satisfies a logarithmic Sobolev inequality as in [45][1], we use general bounds on product measure (Corollary 5.8 in [35]). The author thanks Raphaël Rossignol for pointing out this improvement. The proof of Theorem 1.7 is made of four steps, the first three being common with the proof of Theorem 1.8.

The first step consists in relating the derivative of the surface tension $\tau^\lambda_{\mathcal{R}}(\boldsymbol{n})$ in a rectangular parallelepiped $\mathcal{R}$ as in (1.15), with a basis of side-length L, to the entropy of the positive function $\exp(f_\lambda)$ where

$$f_\lambda = -\lambda L^{d-1}\tau^J_{\mathcal{R}}.$$

We recall that the *entropy* of a positive measurable function f with $\mathbb{E}(f\log(1+f))<\infty$ is

$$\mathrm{Ent}_{\mathbb{P}}(f) = \mathbb{E}(f\log f) - \mathbb{E}(f)\log\mathbb{E}(f). \tag{2.20}$$

With these notations, it is immediate that:

Lemma 2.5. *For any $\lambda>0$,*

$$-\frac{\partial}{\partial\lambda}\left(\frac{\tau^\lambda_{\mathcal{R}}}{\lambda}\right) = \frac{1}{\lambda^2L^{d-1}}\frac{\mathrm{Ent}_{\mathbb{P}}(\exp(f_\lambda))}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)} \tag{2.21}$$

As a second step we study the quantity

$$a^J_e = \frac{L^{d-1}}{\beta}\frac{\partial\tau^J_{\mathcal{R}}}{\partial J_e}. \tag{2.22}$$

The proposition below provides an interpretation of a^J_e as the probability that the disconnecting interface due to the event $\mathcal{D}_{\mathcal{R}}$ passes through the edge e. We prove also, and this is crucial for our construction, that the actual value of J_e does not influence too much that of a^J_e:

Proposition 2.6. *For any e, a^J_e is a $\mathcal{C}^\infty$ function of the $J_{e'}$. For any $J\in[0,1]^{E(\mathbb{Z}^d)}$, one has*

$$a^J_e = \frac{1}{p_e}\left(\Phi^{J,w}_{\mathcal{R}}(\omega_e) - \Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{D}_{\mathcal{R}})\right) \quad \textit{if} \;\; J_e>0 \tag{2.23}$$

together with the following inequalities:

$$0\leqslant a^J_e\leqslant 1 \quad \textit{and} \quad \sup_{J_e} a^J_e \leqslant e^\beta \inf_{J_e} a^J_e. \tag{2.24}$$

The controls (2.24), together with Corollary 5.8 in [35], permit to establish the third step. Given a rectangular parallelepiped $\mathcal{R}$ as in (1.15) and $\lambda\geqslant 0$, we introduce the probability measure $\mathbb{P}_\lambda$ that to any bounded measurable $h: J\mapsto h(J)\in\mathbb{R}$ gives expectation

$$\mathbb{E}_\lambda(h(J)) = \mathbb{E}\left(h(J)\frac{\exp\left(-\lambda L^{d-1}\tau^J_{\mathcal{R}}\right)}{\mathbb{E}\exp\left(-\lambda L^{d-1}\tau^J_{\mathcal{R}}\right)}\right). \tag{2.25}$$

[1] Usual measures such as dilution $\mathbb{P}(J_e\in\{0,1\})=1$, or J_e with positive density on $[0,1]$ do satisfy a logarithmic Sobolev inequality, cf. [35] or Theorems 4.2, 6.6 and Section 6.3 in [12].

Proposition 2.7. *Denote* $m_{\mathbb{P}} = \mathbb{E}(J_e)$. *For any* $\lambda \geqslant 0$, *we have both*

$$\frac{\mathrm{Ent}_{\mathbb{P}}(\exp(f_\lambda))}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)} \leqslant \lambda^2 \frac{\beta^2 e^{\beta(1+\lambda)}}{4} \begin{cases} \mathbb{E}_\lambda\left(\sum_{e\in E(\hat{\mathcal{R}})} a_e^J\right) \\ \frac{1}{m_\beta} L^{d-1} \frac{1}{\lambda} \frac{\partial \tau^\lambda_{\mathcal{R}}}{\partial \beta}. \end{cases} \tag{2.26}$$

The second majoration leads to Theorem 1.8, while the first one yields Theorem 1.7 after a last step: using Peierls' argument we show that, in the Ising model ($q = 2$) with couplings $J_e \geqslant \varepsilon > 0$, the length of the interface is of order N^{d-1}.

Proposition 2.8. *Let* $q = 2$ *and* $\varepsilon > 0$. *There exists* $c_d < \infty$ *such that, for* β *large enough, for* $\mathcal{R} = \mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$ *with* $\delta \in (0, 1)$, *for any realization* J *of the random couplings such that* $J_e \geqslant \varepsilon$ *and* N *large enough,*

$$\sum_{e\in E(\widehat{\mathcal{R}^N})} a_e^J \leqslant \frac{c_d}{\varepsilon} N^{d-1}. \tag{2.27}$$

We give now the proofs of all the propositions, followed by that of Theorem 1.7.

Proof (Proposition 2.6). The fact that a_e^J is a $\mathcal{C}^\infty$ function of $J_{e'}$ is a consequence of the same property for $\tau^J_{\mathcal{R}}$, the quantity $\Phi^{J,w}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}})$ being always positive. We introduce next a few notations: we let

$$w^J_{\mathcal{R}}(\omega) = \prod_{e\in E(\hat{\mathcal{R}})} \left(\frac{p_e}{1-p_e}\right)^{\omega_e} q^{C^w_{E(\hat{\mathcal{R}})}(\omega)} \quad \text{and} \quad Z^J_{\mathcal{R}}(\mathcal{A}) = \sum_{\omega\in\mathcal{A}} w^J_{\mathcal{R}}(\omega) \tag{2.28}$$

for any $\omega \in \Omega_{E(\hat{\mathcal{R}})}$ and $\mathcal{A} \subset \Omega_{E(\hat{\mathcal{R}})}$, see (1.9) for the definition of $C^w_{E(\hat{\mathcal{R}})}(\omega)$. For all J with $J_e > 0$, we have

$$\frac{\partial \log w^J_{\mathcal{R}}(\omega)}{\partial J_e} = \beta \frac{\omega_e}{p_e}$$

and as a consequence, for all J with $J_e > 0$,

$$\begin{aligned} a_e^J &= -\frac{1}{\beta}\frac{\partial}{\partial J_e} \log \frac{Z^J_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}})}{Z^J_{\mathcal{R}}(\Omega_{E(\hat{\mathcal{R}})})} \\ &= \frac{1}{p_e}\left(\Phi^{J,w}_{\mathcal{R}}(\omega_e) - \Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{D}_{\mathcal{R}})\right). \end{aligned}$$

Under this formulation, the FKG inequality and the bound $\Phi^{J,w}_{\mathcal{R}}(\omega_e) \leqslant p_e$ imply that $0 \leqslant a_e^J \leqslant 1$ for any $J \in \mathcal{J}$ with $J_e > 0$, and the inequality extends by continuity to the whole of $\mathcal{J}$. We now calculate the derivative of a_e^J along J_e for $J_e > 0$ and obtain, as

$$\frac{\partial}{\partial J_e}\left[\frac{\Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{A})}{p_e}\right] = \beta\left[\frac{\Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{A})}{p_e} - \frac{\Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{A})^2}{p_e^2}\right],$$

that, for any $J \in \mathcal{J}$ with $J_e > 0$,

$$\frac{\partial a_e^J}{\partial J_e} = \beta a_e^J \left(1 - \frac{\Phi^{J,w}_{\mathcal{R}}(\omega_e)}{p_e} - \frac{\Phi^{J,w}_{\mathcal{R}}(\omega_e|\mathcal{D}_{\mathcal{R}})}{p_e}\right).$$

This implies in particular that

$$\left|\frac{\partial a_e^J}{\partial J_e}\right| \leqslant \beta a_e^J$$

and the comparison $\sup_{J_e\in[0,1]} a_e^J \leqslant e^\beta \inf_{J_e\in[0,1]} a_e^J$ follows. □

Proof (Proposition 2.7). According to Corollary 5.8 in [35] and to the Mean Value Theorem, we have

$$\frac{\mathrm{Ent}_{\mathbb{P}}(\exp(f_\lambda))}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)} \leqslant \frac{1}{4}\sum_{e\in E(\hat{\mathcal{R}})} \frac{\mathbb{E}\left(\left(\sup_{J_e\in[0,1]} \frac{\partial f_\lambda}{\partial J_e}\right)^2 \exp(\sup_{J_e\in[0,1]} f_\lambda)\right)}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)}.$$

It is clear that

$$\frac{\partial f_\lambda}{\partial J_e} = -\lambda\beta a_e^J.$$

On the other hand, Proposition 2.6 yields

$$\sup_{J_e\in[0,1]} (a_e^J)^2 \leqslant \sup_{J_e\in[0,1]} a_e^J \leqslant e^\beta \inf_{J_e\in[0,1]} a_e^J$$

and

$$\sup_{J_e\in[0,1]} \exp(f_\lambda) \leqslant e^{\beta\lambda} \inf_{J_e\in[0,1]} \exp(f_\lambda),$$

hence

$$\frac{\mathrm{Ent}_{\mathbb{P}}(\exp(f_\lambda))}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)} \leqslant \frac{\lambda^2\beta^2 e^{\beta(1+\lambda)}}{4} \sum_{e\in E(\hat{\mathcal{R}})} \mathbb{E}\left(\inf_{J_e\in[0,1]} a_e^J \times \frac{\inf_{J_e\in[0,1]}\exp(f_\lambda)}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)}\right)$$

and the first bound follows. For the second one, remark that as we take infimums over J_e we in fact obtain a quantity that is *independent* of J_e. Thus

$$\begin{aligned}\frac{\mathrm{Ent}_{\mathbb{P}}(\exp(f_\lambda))}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)} &\leqslant \frac{\lambda^2\beta^2 e^{\beta(1+\lambda)}}{4} \sum_{e\in E(\hat{\mathcal{R}})} \mathbb{E}\left(\frac{J_e}{m_{\mathbb{P}}}\inf_{J_e\in[0,1]} a_e^J \times \frac{\inf_{J_e\in[0,1]}\exp(f_\lambda)}{\mathbb{E}\left(\exp\left(f_\lambda\right)\right)}\right)\\ &\leqslant \frac{\lambda^2\beta^2 e^{\beta(1+\lambda)}}{4m_{\mathbb{P}}}\mathbb{E}_\lambda\left(\sum_{e\in E(\hat{\mathcal{R}})} J_e a_e^J\right)\end{aligned}$$

which ends the proof as

$$\frac{1}{\lambda}\frac{\partial \tau_{\mathcal{R}}^\lambda}{\partial\beta} = \frac{1}{L^{d-1}}\mathbb{E}_\lambda\left(\sum_{e\in E(\hat{\mathcal{R}})} J_e a_e^J\right).$$

□

Proof (Proposition 2.8). As $\mathcal{R} = \mathcal{R}_N = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ is centered at the origin, we consider

$$\begin{aligned}\Sigma_{\mathcal{R}}^{+} &= \left\{\sigma : \mathbb{Z}^d \to \{\pm1\} : \sigma_x = 1, \forall x \notin \hat{\mathcal{R}}\setminus\partial\hat{\mathcal{R}}\right\}\\ \Sigma_{\mathcal{R}}^{\pm} &= \left\{\sigma : \mathbb{Z}^d \to \{\pm1\} : \sigma_x = \begin{cases} 1 & \text{if } x\cdot\boldsymbol{n}\geqslant 0\\ -1 & \text{else}\end{cases}, \forall x \notin \hat{\mathcal{R}}\setminus\partial\hat{\mathcal{R}}\right\}\end{aligned}$$

the set of spin configurations on $\hat{\mathcal{R}}$ with plus or mixed boundary conditions. The correspondence between the random-cluster representation (with $q = 2$) and Ising model gives

$$\tau_{\mathcal{R}}^J = \frac{1}{N^{d-1}}\log\frac{Z_{\mathcal{R}}^{J,+}}{Z_{\mathcal{R}}^{J,\pm}}$$

where $Z_{\mathcal{R}}^{J,+}$ and $Z_{\mathcal{R}}^{J,\pm}$ are the partition functions

$$\begin{aligned}Z_{\mathcal{R}}^{J,+} &= \sum_{\sigma\in\Sigma_{\mathcal{R}}^+}\exp\left(\frac{\beta}{2}\sum_{e=\{x,y\}\in E(\hat{\mathcal{R}})} J_e\sigma_x\sigma_y\right)\\ \text{and}\quad Z_{\mathcal{R}}^{J,\pm} &= \sum_{\sigma\in\Sigma_{\mathcal{R}}^\pm}\exp\left(\frac{\beta}{2}\sum_{e=\{x,y\}\in E(\hat{\mathcal{R}})} J_e\sigma_x\sigma_y\right),\end{aligned}$$

leading thus to

$$a_e^J = \mu_{\mathcal{R}}^{J,+}(\sigma_x\sigma_y) - \mu_{\mathcal{R}}^{J,\pm}(\sigma_x\sigma_y),\ \ \forall e=\{x,y\}\in E(\hat{\mathcal{R}}) \tag{2.29}$$

where $\mu_{\mathcal{R}}^{J,\pm}$ is the Ising model on $\hat{\mathcal{R}}$ with mixed boundary condition (plus on $\partial^+\hat{\mathcal{R}}$, minus on $\partial^-\hat{\mathcal{R}}$). We consider now an interface I for $\mathcal{R}$ as in Section 1.4. We recall that it is a minimal set of edges such that connections from $\partial^+\hat{\mathcal{R}}$ to $\partial^-\hat{\mathcal{R}}$ through $E(\hat{\mathcal{R}})\setminus I$ are impossible. We consider I^+ the upper part of the interface I:

$$I^+ = \{x : \exists y \in \mathbb{Z}^d : \{x,y\} \in I \text{ and } x \nleftrightarrow \partial^-\hat{\mathcal{R}} \text{ in } E(\hat{\mathcal{R}})\setminus I\}$$

and define symmetrically the set I^-. We call then $\mathcal{S}_I$ the event that I is the spin interface between $\partial^+\hat{\mathcal{R}}$ and $\partial^-\hat{\mathcal{R}}$ under the measure $\mu^{J,\pm}_{\mathcal{R}}$:

$$\mathcal{S}_I = \left\{ \sigma \in \Sigma^{\pm}_{\mathcal{R}} : \begin{array}{l} \sigma(x) = +1, \forall x \in I^+ \\ \sigma(x) = -1, \forall x \in I^- \end{array} \right\}.$$

Conditionally on $\mathcal{S}_I$, the restriction of $\mu^{J,\pm}_{\mathcal{R}}$ to the upper (resp. lower) parts of $\hat{\mathcal{R}}$ equals the Ising measure with uniform plus (resp. minus) boundary condition. Hence, for any $\{x,y\} \notin I$ we have

$$\mu^{J,\pm}_{\mathcal{R}}(\sigma_x\sigma_y|\mathcal{S}_I) \geqslant \mu^{J,+}_{\mathcal{R}}(\sigma_x\sigma_y), \tag{2.30}$$

and consequently

$$\sum_{e\in E(\hat{\mathcal{R}})} a^J_e \leqslant \sum_{I \text{ interface}} \mu^{J,\pm}_{\mathcal{R}}(\mathcal{S}_I) \times 2|I|. \tag{2.31}$$

Thus it remains only to bound the average interface length, in the Ising sense, under $\mu^{J,\pm}_{\mathcal{R}}$. We remark that $\mu^{J,\pm}_{\mathcal{R}}(\mathcal{S}_I)$ can also be written as

$$\mu^{J,\pm}_{\mathcal{R}}(\mathcal{S}_I) = \frac{Z^{J,+}_{\mathcal{R}\setminus I} \exp\left(-\beta\sum_{e\in\Gamma} J_e\right)}{Z^{J,\pm}_{\mathcal{R}}}$$

where $Z^{J,+}_{\mathcal{R}\setminus I}$ stands for the partition function associated to the set of configurations with plus boundary condition on I^+, I^- and on $\partial\hat{\mathcal{R}}$. Thanks to the assumption $J_e \geqslant \varepsilon$ and to the remarks that

$$\begin{aligned} Z^{J,+}_{\mathcal{R}\setminus I} &\leqslant Z^{J,+}_{\mathcal{R}} \\ \text{and } Z^{J,\pm}_{\mathcal{R}} &\geqslant Z^{J,+}_{\mathcal{R}} \exp\left(-\beta|\partial^-\hat{\mathcal{R}}|\right), \end{aligned}$$

we have

$$\mu^{J,\pm}_{\mathcal{R}}(\mathcal{S}_I) \leqslant \exp\left(-\beta\varepsilon|I| + \beta c_d N^{d-1}\right)$$

as $\delta < 1$. We conclude with a Peierls estimate and bound the number of interfaces of cardinal $n \geqslant 2c_d N^{d-1}/\varepsilon$ by $(c_d)^n$:

$$\sum_{e\in E(\hat{\mathcal{R}})} a^J_e \leqslant \frac{2c_d N^{d-1}}{\varepsilon} + \sum_{n\geqslant 2c_d N^{d-1}/\varepsilon} n(c_d)^n e^{-\beta\varepsilon n + \beta c_d N^{d-1}}$$

The second term goes to 0 with $N \to \infty$ for β large enough. □

Proof (Theorem 1.7). The combination of Lemma 2.5, Propositions 2.7 and 2.8 implies that in the setting of Theorem 1.7,

$$-\frac{\partial}{\partial\lambda}\left(\frac{\tau^\lambda_{\mathcal{R}}}{\lambda}\right) \leqslant \frac{c_d\beta^2 e^{\beta(1+\lambda)}}{4J^{\min}}$$

for β large enough, $\mathcal{R} = \mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$, $\delta \in (0,1)$ and N large enough. Integrating over λ we obtain, as $\lim_{\lambda\to 0^+} \tau^\lambda_{\mathcal{R}}/\lambda = \mathbb{E}\tau^J_{\mathcal{R}}$, the inequality

$$\tau^\lambda_{\mathcal{R}} \geqslant \lambda\mathbb{E}\tau^J_{\mathcal{R}} - \lambda^2 \frac{c_d\beta^2 e^{\beta(1+\lambda)}}{4J^{\min}}.$$

Letting $N \to \infty$ gives

$$\tau^\lambda(\boldsymbol{n}) \geqslant \lambda\tau^q(\boldsymbol{n}) - \lambda^2 \frac{c_d\beta^2 e^{\beta(1+\lambda)}}{4J^{\min}}$$

and the duality formula (2.14) yields the claim with $c = c_d J^{\min}/(\beta^2 \exp(2\beta))$, for large enough β. □

2.6. **Concentration in a general setting.** We give now the proof of Theorem 1.8, which is based on Herbst's argument, together with the controls of Lemma 2.5 and Proposition 2.7. We will then give the proof of Corollary 1.9.

First we give an immediate consequence of the duality formula (2.14):

Lemma 2.9. *Assume that*

$$\limsup_{\lambda\to 0^+} \frac{\tau^\lambda(\boldsymbol{n}) - \lambda\tau^q(\boldsymbol{n})}{\lambda^2} \geqslant -c \quad \textit{for some } c \in [0,\infty]. \tag{2.32}$$

Then,

$$\limsup_{r\to 0^+} \frac{I_{\boldsymbol{n}}(\tau^q(\boldsymbol{n}) - r)}{r^2} \geqslant \frac{1}{4c} \in [0,\infty]. \tag{2.33}$$

Proof (Theorem 1.8). Given $\delta > 0$ and $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$, we denote $\mathcal{R}^N$ the rectangular parallelepiped $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\boldsymbol{n}, \mathcal{S})$ and introduce

$$K_{\boldsymbol{n}}^{\mathbb{P},\beta} = \liminf_{\lambda\to 0^+} \liminf_{N\to\infty} \frac{1}{\lambda} \int_0^\lambda \frac{\partial \tau^{\lambda'}_{\mathcal{R}^N}}{\partial\beta} \frac{d\lambda'}{\lambda'} \in [0,\infty]$$

In view of Theorem 1.3 and Proposition 2.2 we have

$$\begin{aligned}
\tau^\lambda(\boldsymbol{n}) - \lambda\tau^q(\boldsymbol{n}) &= \lim_{N\to\infty} \tau^\lambda_{\mathcal{R}^N} - \lambda \mathbb{E}\tau^J_{\mathcal{R}^N} \\
&= \lim_{N\to\infty} \lambda \int_0^\lambda \frac{\partial}{\partial\lambda'} \left(\frac{\tau^{\lambda'}_{\mathcal{R}^N}}{\lambda'} \right) d\lambda'
\end{aligned}$$

as $\mathbb{E}\tau^J_{\mathcal{R}^N} = \lim_{\lambda\to 0^+} \tau^\lambda_{\mathcal{R}^N}/\lambda$ for any N finite. Lemma 2.5 and Proposition 2.7 yield, for any $\varepsilon > 0$:

$$\begin{aligned}
\limsup_{\lambda\to 0^+} \frac{\tau^\lambda(\boldsymbol{n}) - \lambda\tau^q(\boldsymbol{n})}{\lambda^2} &\geqslant -\frac{\beta^2 e^{\beta(1+\varepsilon)}}{4m_{\mathbb{P}}} \liminf_{\lambda\to 0^+} \liminf_{N\to\infty} \frac{1}{\lambda} \int_0^\lambda \frac{\partial \tau^{\lambda'}_{\mathcal{R}^N}}{\partial\beta} \frac{d\lambda'}{\lambda'} \\
&= -\frac{\beta^2 e^{\beta(1+\varepsilon)}}{4m_{\mathbb{P}}} K_{\boldsymbol{n}}^{\mathbb{P},\beta}
\end{aligned}$$

and an immediate application of Lemma 2.9 gives, after the limit $\varepsilon \to 0$, the lower bound:

$$\limsup_{r\to 0^+} \frac{I_{\beta,\boldsymbol{n}}(\tau^q_\beta(\boldsymbol{n}) - r)}{r^2} \geqslant \frac{m_{\mathbb{P}}}{\beta^2 e^\beta K_{\boldsymbol{n}}^{\mathbb{P},\beta}}. \tag{2.34}$$

The lower bound is positive when $K_{\boldsymbol{n}}^{\mathbb{P},\beta} < \infty$. In order to show that this is the case for Lebesgue almost all β, we evaluate the integral of $K_{\boldsymbol{n}}^{\mathbb{P},\beta}$ on some interval $[\beta_1, \beta_2]$. For any $\delta > 0$ and $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$, Fatou's Lemma and Fubini Theorem imply that

$$\begin{aligned}
\int_{\beta_1}^{\beta_2} K_{\boldsymbol{n}}^{\mathbb{P},\beta} d\beta &\leqslant \liminf_{\lambda\to 0^+} \liminf_{N\to\infty} \frac{1}{\lambda} \int_0^\lambda \int_{\beta_1}^{\beta_2} \frac{\partial \tau^{\lambda'}_{\mathcal{R}_N}}{\partial\beta} \frac{d\lambda'}{\lambda'} \\
&= \liminf_{\lambda\to 0^+} \liminf_{N\to\infty} \frac{1}{\lambda} \int_0^\lambda \frac{\tau^{\lambda'}_{\beta_2,\mathcal{R}_N} - \tau^{\lambda'}_{\beta_1,\mathcal{R}_N}}{\lambda'} d\lambda'.
\end{aligned}$$

The convergence as $N \to \infty$ is uniformly dominated (by Jensen's inequality and Proposition 1.2, $0 \leqslant \tau^\lambda_{\mathcal{R}_N} \leqslant \lambda c_d \beta$) hence we finally obtain

$$\begin{aligned}
\int_{\beta_1}^{\beta_2} K_{\boldsymbol{n}}^{\mathbb{P},\beta} d\beta &\leqslant \liminf_{\lambda\to 0^+} \frac{1}{\lambda} \int_0^\lambda \frac{\tau^{\lambda'}_{\beta_2}(\boldsymbol{n}) - \tau^{\lambda'}_{\beta_1}(\boldsymbol{n})}{\lambda'} d\lambda' \\
&= \tilde{\tau}^q_{\beta_2}(\boldsymbol{n}) - \tilde{\tau}^q_{\beta_1}(\boldsymbol{n}).
\end{aligned} \tag{2.35}$$

in view of (2.18). In particular, $K_{\boldsymbol{n}}^{\mathbb{P},\beta}$ is *finite* for Lebesgue almost all $\beta \geqslant 0$. □

We would like to make a remark on $K_{\boldsymbol{n}}^{\mathbb{P},\beta}$. In view of Corollary 1.9, for Lebesgue almost every β_1, β_2 with $\beta_1 \leqslant \beta_2$ one can replace $\tilde{\tau}^q_{\beta_2}(\boldsymbol{n}) - \tilde{\tau}^q_{\beta_1}(\boldsymbol{n})$ in (2.35) with $\tau^q_{\beta_2}(\boldsymbol{n}) - \tau^q_{\beta_1}(\boldsymbol{n})$. As a consequence, whenever $\tau^q_\beta(\boldsymbol{n})$ is derivable on some interval, $K_{\boldsymbol{n}}^{\mathbb{P},\beta} \leqslant \partial\tau^q_\beta(\boldsymbol{n})/\partial\beta$ for Lebesgue almost every β in that interval.

Proof (Corollary 1.9). We denote by

$$\tau^q_{\beta^-}(\boldsymbol{n}) = \lim_{\varepsilon\to 0^+} \tau^q_{\beta-\varepsilon}(\boldsymbol{n})$$

the left limit of $\tau^q_\beta(\boldsymbol{n})$. For any $\tau \in \mathbb{R}$, $\beta \mapsto I_{\boldsymbol{n}}(\tau)$ is non-decreasing hence $\tilde{\tau}^q_\beta(\boldsymbol{n})$ (defined at (1.47)) does not decrease with β. According to Theorem 1.8, $\tilde{\tau}^q_\beta(\boldsymbol{n})$ coincides with $\tau^q_\beta(\boldsymbol{n})$ for almost all β, hence

$$\tau^q_{\beta^-}(\boldsymbol{n}) \leqslant \tilde{\tau}^q_\beta(\boldsymbol{n}) \leqslant \tau^q_\beta(\boldsymbol{n}), \forall \beta \geqslant 0$$

hence the left continuity of $\tau^q_\beta(\boldsymbol{n})$ at a particular β implies that $\tilde{\tau}^q_\beta(\boldsymbol{n}) = \tau^q_\beta(\boldsymbol{n})$, in other words that lower deviations are (at least) of surface order. This is the first part of the claim. Now we consider

$$\mathcal{D} = \left\{ \beta \in \mathbb{R}^+, \exists \boldsymbol{n} \in S^{d-1} : \tau^q_{\beta^-}(\boldsymbol{n}) \neq \tau^q_\beta(\boldsymbol{n}) \right\}$$

and prove that $\mathcal{D}$ is at most countable. The homogeneous extension of $\tau^q_{\beta^-}(\boldsymbol{n})$ to $\mathbb{R}^d$ is convex as the pointwise limit of the $f^q_{\beta-\varepsilon}$, hence $\tau^q_{\beta^-}(\boldsymbol{n})$ is a continuous function of $\boldsymbol{n} \in S^{d-1}$. Consequently, for any dense sequence $(\boldsymbol{n}_n)_{n\in\mathbb{N}}$ in S^{d-1}, we have

$$\mathcal{D} \quad \subset \quad \bigcup_{n\in\mathbb{N}} \left\{ \beta \in \mathbb{R}^+ : \tau^q_\beta(\boldsymbol{n}_n) \neq \tau^q_{\beta^-}(\boldsymbol{n}_n) \right\}$$

which is at most countable. □

3. Low temperatures asymptotics

Here we study the low temperature asymptotics of surface tension and prove the results presented in Section 1.4. We begin with upper bounds on surface tension which hold in all generality, and then establish lower bounds with the help of Peierls arguments.

3.1. Upper bounds on surface tension.

Relevant upper bounds on surface tension are easily established:

Lemma 3.1. *Let $\mathbb{P}$ be a product measure $\mathbb{P}$ on $[0,1]^d$, $\boldsymbol{n} \in S^{d-1}$ and $\lambda > 0$. Then,*

$$\tau^q_\beta(\boldsymbol{n}) \quad \leqslant \quad \beta\mu(\boldsymbol{n}) \tag{3.1}$$

$$\text{and} \quad \tau^\lambda_\beta(\boldsymbol{n}) \quad \leqslant \quad \|\boldsymbol{n}\|_1 \times \log \frac{1}{\mathbb{E}\exp(-\lambda\beta J_e)} \tag{3.2}$$

Proof We begin with the proof of (3.1) and consider a rectangular parallelepiped $\mathcal{R}$. With the notations of Section 1.4, for all interface $I \in \mathcal{I}(\mathcal{R})$, the DLR equation yields

$$\Phi^{J,w}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) \geqslant \Phi^{J,w}_{\mathcal{R}}(\mathcal{Z}_I) \geqslant \prod_{e\in I} \Phi^{J,w}_{\{e\}}(\omega_e = 0) = \exp\left(-\beta \sum_{e\in I} J_e \right)$$

and consequently $\tau^J_{\beta,\mathcal{R}} \leqslant \beta\mu^J_{\mathcal{R}}$, which implies (3.1) taking $\mathcal{R} = \mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$ and $N \to \infty$. Similarly, in view of the definition (2.15) we have

$$\begin{aligned} \tau^\lambda_{\beta,\mathcal{R}} \quad &\leqslant \quad -\frac{1}{L^{d-1}} \log \mathbb{E}\left(\prod_{e\in I} \Phi^{J,w}_{\{e\}}(\omega_e = 0)^\lambda \right) \\ &\leqslant \quad \frac{|I|}{L^{d-1}} \log \frac{1}{\mathbb{E}\exp(-\lambda\beta J_e)} \end{aligned}$$

which yields (3.2) if we choose for I the interface of smallest cardinal in $\mathcal{I}(\mathcal{R})$, which has a cardinal approximately $\|\boldsymbol{n}\|_1 L^{d-1}$. □

3.2. **Quenched surface tension and maximal flows.** We present here the proof of Proposition 1.11, which is based on a control of the length of the interface, using a Peierls argument, and then the proof of Proposition 1.12 which uses a renormalization argument.

Proof (Proposition 1.11). Given a rectangular parallelepiped $\mathcal{R}$, we have

$$\Phi^J_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) \leqslant \sum_{I\in\mathcal{I}(\mathcal{R})} \Phi^J_{\mathcal{R}}(\mathcal{Z}_I) \leqslant \sum_{I\in\mathcal{I}(\mathcal{R})} \prod_{e\in I} q e^{-\beta J_e}.$$

We decompose the sum according to the length of the interface: for any $c > \|\boldsymbol{n}\|_1$,

$$\Phi^J_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) \leqslant \sum_{I\in\mathcal{I}(\mathcal{R}):|I|<cL^{d-1}} q^{|I|} e^{-\beta L^{d-1}\mu^J_{\mathcal{R}}} + \sum_{I\in\mathcal{I}(\mathcal{R}):|I|\geqslant cL^{d-1}} q^{|I|} e^{-\beta\sum_{e\in I} J_e} \tag{3.3}$$

The first term is not larger than

$$(c_d q)^{cL^{d-1}} \exp\left(-\beta L^{d-1}\mu^J_{\mathcal{R}}\right)$$

and the expectation of the second one is

$$\mathbb{E}\left(\sum_{I\in\mathcal{I}(\mathcal{R}):|I|\geqslant cL^{d-1}} q^{|I|} e^{-\beta\sum_{e\in I} J_e}\right) \leqslant \frac{1}{1-c_d q\mathbb{E}(e^{-\beta J_e})} \times \left[c_d q \mathbb{E}(e^{-\beta J_e})\right]^{cL^{d-1}}$$

if $\rho_\beta = c_d q\mathbb{E}(e^{-\beta J_e}) < 1$, which is the case for β large as $\mathbb{P}(J_e = 0) = 0 < (c_d q)^{-1}$. For any such β, applying Markov's inequality we obtain, for any $\varepsilon > 0$:

$$\mathbb{P}\left(\sum_{I\in\mathcal{I}(\mathcal{R}):|I|\geqslant cL^{d-1}} q^{|I|} e^{-\beta\sum_{e\in I} J_e} \geqslant (\rho_\beta)^{(1-\varepsilon)cL^{d-1}}\right) \leqslant \frac{1}{1-\rho_\beta} \times (\rho_\beta)^{\varepsilon cL^{d-1}}.$$

Hence (3.3) shows that, for J typical under $\mathbb{P}$ – up to large deviations of surface order –

$$\Phi^J_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) \leqslant (c_d q)^{cL^{d-1}} \exp\left(-\beta L^{d-1}\mu^J_{\mathcal{R}}\right) + (\rho_\beta)^{(1-\varepsilon)cL^{d-1}}$$

which proves that

$$\tau^q_\beta(\boldsymbol{n}) \geqslant \min\left(\beta\mu(\boldsymbol{n}) - c\log(c_d q), c\log\frac{1}{\rho_\beta}\right)$$

for any $\beta \geqslant 0$ such that $\rho_\beta < 1$. The lower bound is optimal for

$$c = \frac{\beta\mu(\boldsymbol{n})}{\log(c_d q) + \log\frac{1}{\rho_\beta}}$$

which is negligible with respect to β in the limit $\beta \to +\infty$, as $\log(1/\rho_\beta) \to +\infty$. The limit (1.30) follows – the uniformity over $\boldsymbol{n} \in S^{d-1}$ is a consequence of the fact that μ is bounded. If $J_{\min} > 0$, then we even have, for some $C < \infty$, that for β large enough (independent of $\boldsymbol{n} \in S^{d-1}$),

$$\tau^q_\beta(\boldsymbol{n}) \geqslant \beta\mu(\boldsymbol{n}) - C.$$

□

Proof (Proposition 1.12). The proof for (1.31) exploits a renormalization argument similar to the one used in [14]. As $\mathbb{P}(J_e > 0) > p_c(d)$, for small enough $\varepsilon > 0$ it is still the case that $\mathbb{P}(J_e \geqslant \varepsilon) > p_c(d)$. We say that $e \in \mathbb{E}(\mathbb{Z}^d)$ is open for J if $J_e \geqslant \varepsilon$, and consider the connected components for these definition of open edges. A block $B^K_i = Ki + \{1, \ldots, 2K\}^d$ ($i \in \mathbb{Z}^d$) of side-length $2K$, is said *good* when, in B^K_i,

i. there is a unique connected component of diameter larger or equal to K
ii. and this connected component touches all the faces of the block.

The work of Pisztora [38] together with the knowledge that, in the case of independent bond percolation, the slab percolation threshold coincides with the threshold for percolation [26] imply that the sequence of random variables

$$\eta_i = \mathbf{1}_{\{\text{The block } B_i^K \text{ is good}\}}$$

stochastically dominate a site percolation process of parameter ρ close to one, provided that K is large enough. Provided that ρ (hence K) is large enough, a Peierls argument shows that there is a probability $1-\exp(cN^{d-1})$ (for some $c>0$, for N large enough) that no K-block interface in $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ contains less than half of good blocks.

Given a suitable K we now establish a lower bound on the quenched surface tension. Consider a realization J of the couplings with the property that no K-block interface in $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S},\boldsymbol{n})$ contains less than half of good blocks. For any such J, the event of disconnection requires the choice of a block surface of cardinality at least $(N/K)^{d-1}$, and that, in each good block, at least one edge with $J_e \geqslant \varepsilon$ be closed. Hence: for N large and J typical up to surface order large deviations,

$$\Phi^{J,w}_{\mathcal{R}^N}(\mathcal{D}_{\mathcal{R}^N}) \quad\leqslant \sum_{n\geqslant (N/K)^{d-1}} (c_d)^n \left[c_d K^d q e^{-\beta\varepsilon}\right]^{(1-\varepsilon)n}$$

leading to $\tau^q_\beta \geqslant \left[(1-\varepsilon)\beta\varepsilon - \log\left(c_d^2 K^d q\right)\right]/K^{d-1}$ for large enough β. The claim follows. □

3.3. **Surface tension under the averaged Gibbs measure.** Under the assumption $\mathbb{P}(J_e > 0) = 1$, we establish a lower bound on $\tau^\lambda_\beta(\boldsymbol{n})$ which is equivalent to the upper bound:

Proposition 3.2. *Assume that* $\mathbb{P}(J_e>0)=1$. *Then, uniformly over* $\boldsymbol{n}\in S^{d-1}$,

$$\tau^\lambda_\beta(\boldsymbol{n}) \quad\geqslant\quad (1-o_{\beta\to\infty}(1)) \times \|\boldsymbol{n}\|_1 \times \log\frac{1}{\mathbb{E}\exp\left(-\lambda\beta J_e\right)}. \tag{3.4}$$

Before we address the proof of Proposition 3.2, let us remark that Proposition 1.13 is a clear consequence of Lemma 3.1 and Proposition 3.2.

Proof (Proposition 3.2). Remark that for any $I\subset\mathcal{I}(\mathcal{R})$,

$$\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{Z}_I\right) \leqslant \prod_{e\in I}\Phi^{J,f}_{\mathcal{R}}\left(\omega_e=0\right) \leqslant \prod_{e\in I} q e^{-\beta J_e}.$$

If $\lambda\leqslant 1$, the inequality $(\sum_{i=1}^n x_i)^\lambda \leqslant \sum_{i=1}^n x_i^\lambda$ for non-negative x_i yields

$$\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{D}_{\mathcal{R}}\right)^\lambda \leqslant \left(\sum_{I\in\mathcal{I}(\mathcal{R})}\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{Z}_I\right)\right)^\lambda \leqslant \sum_{I\in\mathcal{I}(\mathcal{R})}\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{Z}_I\right)^\lambda$$

hence

$$\begin{aligned}\mathbb{E}\left[\left(\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{D}_{\mathcal{R}}\right)\right)^\lambda\right] &\leqslant \sum_{I\in\mathcal{I}(\mathcal{R})}\prod_{e\in I} q^\lambda \mathbb{E}e^{-\lambda\beta J_e}\\ &= \sum_{I\in\mathcal{I}(\mathcal{R})}\left(q^\lambda\mathbb{E}e^{-\lambda\beta J_e}\right)^{|I|}\end{aligned}$$

As $\mathbb{E}e^{-\lambda\beta J_e}\to 0$ as $\beta\to\infty$ under the assumption $\mathbb{P}(J_e>0)=1$, the Peierls argument gives a relevant lower bound : there is c_d depending only on the dimension d, such that the number of interfaces of cardinal n in $\mathcal{I}(\mathcal{R})$ is not larger than $(c_d)^n$. Hence,

$$\begin{aligned}\mathbb{E}\left[\left(\Phi^{J,w}_{\mathcal{R}}\left(\mathcal{D}_{\mathcal{R}}\right)\right)^\lambda\right] &\leqslant \sum_{n\geqslant \min_{I\in\mathcal{I}(\mathcal{R})}|I|} \left(c_d q^\lambda \mathbb{E}e^{-\lambda\beta J_e}\right)^n\\ &\leqslant \frac{1}{1-c_d q^\lambda\mathbb{E}e^{-\lambda\beta J_e}} \times \left[c_d q^\lambda \mathbb{E}e^{-\lambda\beta J_e}\right]^{\min_{I\in\mathcal{I}(\mathcal{R})}|I|}\end{aligned}$$

for β large enough, thus

$$\tau_\beta^\lambda \geqslant \|\boldsymbol{n}\|_1 \times \left(\log \frac{1}{\mathbb{E}e^{-\lambda\beta J_e}} - (1+\lambda)\log c_d - \lambda \log q \right) \tag{3.5}$$

for all $\lambda \leqslant 1$ and β large enough. If $\lambda \geqslant 1$, Minkowski's inequality yields:

$$\begin{aligned} \left[\mathbb{E}\left[\left(\Phi_{\mathcal{R}}^{J,w}(\mathcal{D}_{\mathcal{R}}) \right)^\lambda \right] \right]^{1/\lambda} &\leqslant \sum_{I\in\mathcal{I}(\mathcal{R})} \left[\mathbb{E}\left[\left(\Phi_{\mathcal{R}}^{J,w}(\mathcal{Z}_I) \right)^\lambda \right] \right]^{1/\lambda} \\ &\leqslant \sum_{I\in\mathcal{I}(\mathcal{R})} \prod_{e\in I} q \left[\mathbb{E}e^{-\lambda\beta J_e} \right]^{1/\lambda} \end{aligned}$$

and we conclude similarly that (3.5) holds again for all $\lambda \geqslant 1$ and β large enough. The claim (3.4) follows from the divergence $\lim_{\beta\to+\infty} \log(1/\mathbb{E}e^{-\lambda\beta J_e}) = +\infty$ under the assumption $\mathbb{P}(J_e > 0) = 1$, and the convergence is uniform in $\boldsymbol{n} \in S^{d-1}$ as (3.5) holds for any β large enough independent of $\boldsymbol{n}$. □

3.4. **Limit shapes at low temperatures.** The limit shape of Wulff crystals (1.35) are immediately inferred from the uniform limits for surface tension. The Proposition below is a consequence of Proposition 1.11 for the first point, Lemma 3.1 and Proposition 3.2 for the second one:

Proposition 3.3. *Let $\mathbb{P}$ be a product measure on $[0,1]^d$ with $\mathbb{P}(J_e > 0) = 1$.*

i. The Wulff crystal $\mathcal{W}^q$ converges to the Wulff crystal $\mathcal{W}^\mu$ associated with the maximal flow μ.
ii. For any $\lambda > 0$, the Wulff crystal $\mathcal{W}^\lambda$ converges to the hypercube $\mathcal{W}^{\|\cdot\|_1} = [\pm 1/2]^d$.

We remarked above that the maximal flow determines the limit shape of the crystal in the (quenched) dilute Ising model as the temperature goes to zero, while the crystals under the averaged Gibbs measure converges to the unit hypercube.

Let us explain how the result of Durrett and Liggett [21] for site first passage percolation can be used to show that $\mathcal{W}^\mu$ is not in general an hypercube. We consider $d = 2$ and $\mathbb{P}$ such that

$$\mathbb{P}\left(J_e = \frac{1}{2} \right) = p \quad \text{and} \quad \mathbb{P}(J_e = 1) = 1 - p \tag{3.6}$$

with $\overrightarrow{p_c} < p < 1$, where $\overrightarrow{p_c}$ is the critical threshold for oriented bond percolation. In the two dimensional case, one can also interpret $\mu(\boldsymbol{n})$ as the limit ratio over N of the time needed for reaching the position $N\boldsymbol{n}^\perp$, if to every edge e we associate a passage time J_e. If $\boldsymbol{n}$ belongs to the cone of oriented percolation (modulo the symmetries of the lattice $\mathbb{Z}^2$) one can reach the position $N\boldsymbol{n}^\perp$ following a directed path with all edges (except finitely many at the origin) satisfying $J_e = 1/2$. Hence, in those directions,

$$\mu(\boldsymbol{n}) = \frac{1}{2}\|\boldsymbol{n}\|_1.$$

However, the argument of [21] (applied to bond in place of site first passage percolation) shows that when $\boldsymbol{n}$ is close enough to the axis, one has to use edges $J_e = 1$ with a positive frequency, thus $\mu(\boldsymbol{n}) > \|\boldsymbol{n}\|_1/2$ for those directions. The reciprocity formula

$$\tau(\boldsymbol{n}) = \sup_{x\in\mathcal{W}^\tau} x \cdot \boldsymbol{n}$$

for Wulff crystals, where $\mathcal{W}^\tau = \{x : x\cdot\boldsymbol{n} \leqslant \tau(\boldsymbol{n}), \forall \boldsymbol{n}\}$ is the un-normalized Wulff crystal for τ, shows that $\mathcal{W}^\mu$ is not a square as $\mu(\boldsymbol{n})$ is not proportional to $\|\boldsymbol{n}\|_1$.

4. Phase coexistence

4.1. **Profiles of bounded variation and surface energy.** The coarse graining for the dilute Ising model (Theorem 5.10 in [44]) implies that at every position, the local magnetization $\mathcal{M}_K$ is close to $\pm m_\beta$ with large probability. In order to describe the geometrical structure of the phases, we estimate the probability that $\mathcal{M}_K/m_\beta$ be close, in L^1-distance, to a given Borel measurable function $u : [0,1]^d \to \{\pm 1\}$. As a first step towards the description of phase coexistence, we define here the set of profiles we consider, define surface energy and the associated isoperimetric problem.

In the following, $\mathcal{L}^d$ stands for the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^{d-1}$ for the $d-1$ dimensional Hausdorff measure, which gives to any Borel set $X \subset \mathbb{R}^d$ the weight

$$\mathcal{H}^{d-1}(X) = \lim_{\delta\to 0^+} \frac{\alpha_{d-1}}{2^{d-1}} \inf\left\{ \sum_{i\in I} [\mathrm{diam}(E_i)]^{d-1} : \sup_{i\in I} \mathrm{diam}(E_i) < \delta, X \subset \bigcup_{i\in I} E_i \right\}$$

where the infimum takes into account finite or countable coverings $(E_i)_{i\in I}$, and α_{d-1} is the volume of the unit ball of $\mathbb{R}^{d-1}$. The L^1-distance between two Borel measurable functions $u, v : [0,1]^d \to \mathbb{R}$ is

$$\|u - v\|_{L^1} = \int_{[0,1]^d} |u - v| d\mathcal{L}^d,$$

and the set L^1 is

$$\left\{ u : [0,1]^d \to \mathbb{R} \text{ Borel measurable}, \|u\|_{L^1} < \infty \right\}.$$

In order that L^1 be a Banach space for the L^1-norm, we identify $u : [0,1]^d \to \mathbb{R}$ with the class of functions $v : \|u-v\|_{L^1} = 0$ that coincide with u on a set of full measure. We also denote by $\mathcal{V}(u,\delta)$ the neighborhood of radius $\delta > 0$ in L^1 around $u \in L^1$.

For the study of phase coexistence, we have to consider virtually any $u \in L^1$ taking values in $\{\pm 1\}$. Before we can define the surface energy for such profiles, a description of the boundary of these profiles is necessary. It is done conveniently in the framework of bounded variation profiles (Chapter 3 in [4]). Given a Borel subset $U \subset \mathbb{R}^d$, the variation (or perimeter) of U is

$$\mathcal{P}(U) = \sup\left\{ \int_U \mathrm{div}\, f d\mathcal{L}^d, f \in \mathcal{C}_c^\infty(\mathbb{R}^d, [-1,1]) \right\} \in [0,\infty]$$

where $\mathcal{C}_c^\infty(\mathbb{R}^d, [-1,1])$ is the set of $\mathcal{C}^\infty$ functions from $\mathbb{R}^d$ to $[-1,1]$ with compact support, and div the divergence operator:

$$\mathrm{div}\, f = \frac{\partial f}{\partial x_1} + \ldots + \frac{\partial f}{\partial x_n}.$$

To $U \subset \mathbb{R}^d$ Borel measurable, we associate $u = \chi_U$ as in (1.40) and define the set of bounded variation profiles BV as follows:

$$\mathrm{BV} = \left\{ u = \chi_U : U \subset (0,1)^d \text{ is a Borel set and } \mathcal{P}(U) < \infty \right\}.$$

Bounded variations profiles $u = \chi_U \in \mathrm{BV}$ have a *reduced boundary* $\partial^\star u$ and an outer normal $\boldsymbol{n}^u_\cdot : \partial^\star u \to S^{d-1}$ with, in particular, $\mathcal{H}^{d-1}(\partial^\star u) = \mathcal{P}(U)$.

This allows us to define the *surface energy* of bounded variation profiles. As the outer normal $\boldsymbol{n}^u_\cdot$ defined on $\partial^\star u$ is Borel measurable, we can consider

$$\mathcal{F}^q(u) = \int_{\partial^\star u} \tau^q(\boldsymbol{n}^u_x) d\mathcal{H}^{d-1}(x), \qquad \forall u \in \mathrm{BV} \tag{4.1}$$

and

$$\mathcal{F}^\lambda(u) = \int_{\partial^\star u} \tau^\lambda(\boldsymbol{n}^u_x) d\mathcal{H}^{d-1}(x), \qquad \forall u \in \mathrm{BV}, \forall \lambda > 0. \tag{4.2}$$

where τ^q (resp. τ^λ) stands for the quenched surface tension of the dilute Ising model (resp. surface tension under the averaged Gibbs measure), see Theorem 1.3 and (1.24). Because the homogeneous extension of the surface tensions τ^q and τ^λ are convex (Proposition 2.4), $\mathcal{F}^q$ and $\mathcal{F}^\lambda$ are lower semi-continuous with respect to the L^1-norm. See Chapter 14 in [10] or Theorem 2.1 in [3]. For commodity, when $u = \chi_U \in \mathrm{BV}$ we also denote by $\mathcal{F}^q(U)$ (resp. $\mathcal{F}^\lambda(U)$) the surface energy of u.

When surface tension is positive, the level sets of $\mathcal{F}^q$ and $\mathcal{F}^\lambda$ are compact since, for all $a \geqslant 0$, the set

$$\mathrm{BV}_a = \{u = \chi_U \in \mathrm{BV} : \mathcal{P}(U) \leqslant a\} \tag{4.3}$$

is itself compact for the L^1-norm, cf. Theorem 3.23 in [4]. Consequently, $\mathcal{F}^q$ and $\mathcal{F}^\lambda$ are good rate functions.

Let us conclude with a word on the solutions to the *isoperimetric problem* of finding the $u \in \mathrm{BV}$ such that

$$\int_{[0,1]^d} u\, d\mathcal{L}^d \leqslant \frac{m}{m_\beta} \text{ and } \mathcal{F}^q(u) \text{ is minimal ?} \tag{4.4}$$

The renormalized Wulff crystal $\mathcal{W}^q$ (1.35) is known to be the solution to the same problem *without* the constraint that $U \subset (0,1)^d$. Precisely, the solutions to $U \subset \mathbb{R}^d$ Borel set with

$$\mathcal{L}^d(U) = 1 \text{ and } \mathcal{F}^q(U) \text{ minimal}$$

are the translates of $\mathcal{W}^q$, as the homogeneous extension of τ^q is convex (Proposition 2.4) – see [40], [24] and [25].

For $m < m_\beta$ not too small, $\mathcal{W}^q$ determines as well the optimal profiles in the cube (4.4). Consider $\alpha > 0$ with

$$\alpha^d = \frac{1}{2}\left(1 - \frac{m}{m_\beta}\right). \tag{4.5}$$

The quantity α^d is precisely the least volume of U corresponding to $u = \chi_U \in \mathrm{BV}$ with $\int_{[0,1]^d} ud\mathcal{L}^d \leqslant m/m_\beta$. If some translate of $\alpha\mathcal{W}^q$ fits into the unit cube $[0,1]^d$, that is if $\alpha \operatorname{diam}_\infty(\mathcal{W}^q) \leqslant 1$, then $\mathcal{T}(\alpha\mathcal{W}^q)$ defined at (1.41) is not empty and therefore the infimum of $\mathcal{F}^q(u)$ for $u \in \mathrm{BV}$ with $\int_{[0,1]^d} ud\mathcal{L}^d \leqslant m/m_\beta$ is exactly $\mathcal{F}^q(\alpha\mathcal{W}^q)$. As a consequence, for all α satisfying $\alpha \operatorname{diam}_\infty(\mathcal{W}^q) \leqslant 1$ the optimal phase profiles correspond to the translates of $\alpha\mathcal{W}^q$ that belong to $[0,1]^d$, which are the $z + \alpha\mathcal{W}^q$, for $z \in \mathcal{T}(\alpha\mathcal{W}^q)$. The same remains true if we replace $\mathcal{F}^q$ and $\mathcal{W}^q$ with $\mathcal{F}^\lambda$ and $\mathcal{W}^\lambda$, for any $\lambda > 0$.

4.2. **Covering theorems for BV profiles.** Covering theorems play an essential role in the study of phase coexistence, as they allow to pass from the macroscopic scale (the phase profile u) to the microscopic scale (the dilute Ising model). We give first two definitions:

Definition 4.1. Let $u \in \mathrm{BV}$, $\tau : \mathcal{S}^{d-1} \mapsto [0,\infty]$ continuous, $\delta > 0$ and $\mathcal{R}$ a rectangular parallelepiped as in (1.15), included in $[0,1]^d$. We say that $\mathcal{R}$ is δ-adapted to u and τ at $x \in \partial^\star u$ if the following holds:

i. If $\boldsymbol{n} = \boldsymbol{n}^u_x$ is the outer normal to u at x, there are $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}}$ and $h \in (0,\delta]$ such that, if $\mathcal{R} \subset (0,1)^d$ (we say that $\mathcal{R}$ is interior), then
$$\mathcal{R} = x + h\mathcal{S} + [\pm\delta h]\boldsymbol{n},$$
and if $\mathcal{R} \cap \partial[0,1]^d \neq \emptyset$ (we say that $\mathcal{R}$ is on the border), then $x \in \partial[0,1]^d$, $\boldsymbol{n}$ is also the outer normal to $[0,1]^d$ at x and
$$\mathcal{R} = x + h\mathcal{S} + [-\delta h, 0]\boldsymbol{n}.$$

ii. We have
$$\mathcal{H}^{d-1}(\partial^\star u \cap \partial\mathcal{R}) = 0,$$
$$\left|1 - \frac{1}{h^{d-1}}\mathcal{H}^{d-1}(\partial^\star u \cap \mathcal{R})\right| \leqslant \delta,$$
and
$$\left|\tau(\boldsymbol{n}) - \frac{1}{h^{d-1}}\int_{\partial^\star u \cap \mathcal{R}} \tau(\boldsymbol{n}^u_\cdot)d\mathcal{H}^{d-1}\right| \leqslant \delta.$$

iii. If $\chi : \mathbb{R}^d \to \{\pm 1\}$ is the characteristic function of $\mathcal{R}$ defined by

$$\chi(z) = \begin{cases} +1 & \text{if } (z-x)\cdot \boldsymbol{n} \geqslant 0 \\ -1 & \text{else} \end{cases}, \forall z \in \mathbb{R}^d,$$

then

$$\frac{1}{2\delta h^d}\int_{\mathcal{R}} |\chi - u| d\mathcal{H}^d \leqslant \delta.$$

Definition 4.2. Let $u \in \mathrm{BV}$, $\tau : \mathcal{S}^{d-1} \mapsto [0,\infty]$ continuous and $\delta > 0$. A finite sequence $(\mathcal{R}_i)_{i=1...n}$ of disjoint rectangular parallelepipeds included in $[0,1]^d$ is said to be a δ-covering for $\partial^\star u$ and τ if each $\mathcal{R}_i$ is δ-adapted to u and τ and if

$$\mathcal{H}^{d-1}\left(\partial^\star u \setminus \bigcup_{i=1}^n \mathcal{R}_i\right) \leqslant \delta. \tag{4.6}$$

The Vitali covering theorem (Theorem 13.3 in [10]) is especially well adapted to our purpose. Given a Borel set $E \subset \mathbb{R}^d$, we say that a collection of sets $\mathcal{U}$ is a Vitali class for E if, for each $x \in E$ and $\delta > 0$, there is $U \in \mathcal{U}$ with $0 < \operatorname{diam} U < \delta$ containing x.

Theorem 4.3. [Vitali] *Let $E \subset \mathbb{R}^d$ be $\mathcal{H}^{d-1}$-measurable and consider $\mathcal{U}$ a Vitali class of closed sets for E. Then, there is a countable disjoint sequence $(U_i)_{i\in I}$ in $\mathcal{U}$ such that*

$$\text{either} \quad \sum_{i\in I}(\operatorname{diam} U_i)^{d-1} = \infty \quad \text{or} \quad \mathcal{H}^{d-1}\left(E \setminus \bigcup_{i\in I} U_i\right) = 0.$$

The Vitali Theorem allows us to state a short proof of the following:

Theorem 4.4. *For any $u \in \mathrm{BV}$, $\tau : \mathcal{S}^{d-1} \mapsto [0,\infty]$ continuous and $\delta, h > 0$, there is a δ-covering $(\mathcal{R}_i)_{i=1...n}$ for $\partial^\star u$ and τ.*

Before we give the proof of Theorem 4.4 we recall a property of the reduced boundary (see Theorem 3.59 in [4]):

Lemma 4.5. *Let $u \in \mathrm{BV}$. For all $x \in \partial^\star u$, for all $\delta \in (0,1)$, all $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}_x^u}$ one has*

$$\lim_{h\to 0^+} \frac{1}{h^{d-1}} \mathcal{H}^{d-1}\left(\partial^\star u \cap \dot{\mathcal{R}}_{x,h,\delta h}(\mathcal{S}, \boldsymbol{n}_x^u)\right) = 1.$$

Proof (Theorem 4.4). We design a set E that has zero $\mathcal{H}^{d-1}$-measure and such that the collection of closed rectangular parallelepipeds

$$\mathcal{U}_\delta = \{\mathcal{R}\ \delta\text{-adapted to } u \text{ and } \tau \text{ at } x \in \partial^\star u \setminus E\}$$

is a Vitali class for $\partial^\star u \setminus (E)$. This is enough to prove the claim: thanks to the Vitali covering Theorem, this implies the existence of a countable disjoint sequence $(\mathcal{R}_i)_{i\in I}$ of δ-adapted rectangular parallelepipeds with either

$$\sum_{i\in I}(\operatorname{diam} \mathcal{R}_i)^{d-1} = \infty \quad \text{or} \quad \mathcal{H}^{d-1}\left(\partial^\star u \setminus \bigcup_{i\in I} \mathcal{R}_i\right) = 0.$$

The first case is in contradiction with the inequalities $1/h_i^{d-1}\mathcal{H}^{d-1}(\partial^\star u \cap \mathcal{R}_i) \geqslant 1 - \delta$ and $\mathcal{H}^{d-1}(\partial^\star u) < \infty$, hence the second is realized and the Theorem is proved.

We define the set E by its complement in $\partial^\star u$: $\partial^\star u \setminus E$ is the set of all $x \in \partial^\star u$ such that, for all $\mathcal{S} \in \mathbb{S}_{\boldsymbol{n}_x^u}$, the following holds:

i. If $x \in \partial[0,1]^d$, then $\boldsymbol{n}_x^u$ is the outer normal to $[0,1]^d$ at x.
ii. The set $\left\{h > 0 : \mathcal{H}^{d-1}(\partial^\star u \cap \partial \mathcal{R}_{x,h,\delta h}(\mathcal{S}, \boldsymbol{n}_x^u)) > 0\right\}$ is at most countable.
iii. $\lim_{h\to 0^+} \frac{1}{h^{d-1}} \mathcal{H}^{d-1}\left(\partial^\star u \cap \dot{\mathcal{R}}_{x,h,\delta h}(\mathcal{S}, \boldsymbol{n}_x^u)\right) = 1$.
iv. $\lim_{h\to 0^+} \frac{1}{h^{d-1}} \int_{\partial^\star u \cap \dot{\mathcal{R}}_{x,h,\delta h}(\mathcal{S}, \boldsymbol{n}_x^u)} \tau(\boldsymbol{n}^u_\cdot) d\mathcal{H}^{d-1} = \tau(\boldsymbol{n}_x^u)$.

v. $\lim_{h\to 0^+} \frac{1}{h^d} \int_{\mathcal{R}_{x,h,\delta h}(\mathcal{S},\boldsymbol{n}_x^u)} \left|\chi_{x,\boldsymbol{n}_x^u} - u\right| d\mathcal{L}^d = 0$.

This definition for E implies that $\mathcal{U}_\delta$ is a Vitali class of closed sets for $\partial^\star u \setminus E$. We conclude the proof of Theorem 4.4 showing that E has zero $\mathcal{H}^{d-1}$-measure, and more precisely that each of conditions (i)-(v) is true for (at least) $\mathcal{H}^{d-1}$-almost all $x \in \partial^\star u$:

i. This condition holds for all $x \in \partial^\star u$ because of the inclusion $U \subset (0,1)^d$ if $u = \chi_U$, cf. Theorem 3.59 in [4].
ii. Since the volume of $\partial^\star u$ is zero, (ii) holds for all x.
iii. Condition (iii) holds for all $x \in \partial^\star u$ in view of Lemma 4.5.
iv. It is a consequence of the strong form of the Besicovitch derivation theorem (Theorem 5.52 in [4]) together with Lemma 4.5, that condition (iv) holds for $\mathcal{H}^{d-1}$-almost all $x \in \partial^\star u$.
v. Condition (v) holds for all $x \in \partial^\star u$, cf. Theorem 3.59 in [4].

□

4.3. **Lower bound for phase coexistence.** Here we establish lower bounds for the probability of phase coexistence. In view of the applications, in particular to the control of the dynamics [43] or Chapter 4 in [45], we establish it for a large class of profiles, that include Wulff crystals and shapes with C^1 boundary.

Proposition 4.8 below relates the probability of an event of disconnection along the boundary of a given profile to the surface tension τ^J, for a given realization of the media. In Proposition 4.9 we show that conditionally on this event of disconnection, phase coexistence has large probability. Then we state in Proposition 4.10 a lower bound on the probability of phase coexistence for both quenched and averaged measures.

Given some region $U \subset (0,1)^d$, $N \in \mathbb{N}^\star$ and $\delta > 0$, we consider $\mathcal{E}_U^{N,\delta}$ the set of edges at distance at most $N\sqrt{d}\delta$ from $N\partial U$:

$$\mathcal{E}_U^{N,\delta} = \left\{ e \in E^w(\Lambda_N), d(e, N\partial U) \leqslant N\sqrt{d}\delta \right\}$$

(see Figure 5) and call

$$\mathcal{D}_U^{N,\delta} = \left\{ \omega \in \Omega_{E^w(\Lambda_N)} : x \overset{\omega}{\nleftrightarrow} y, \begin{array}{l} \forall x \in \Lambda_N \setminus NU, y \in \Lambda_N \cap NU \text{ with} \\ d(x/N, \partial U) > \sqrt{d}\delta \text{ and } d(y/N, \partial U) > \sqrt{d}\delta \end{array} \right\}$$

the event that disconnection occurs around ∂U. In order to be able to control the probability of $\mathcal{D}_U^{N,\delta}$, we introduce the following definition:

Definition 4.6. We say that a profile $u = \chi_U$ is regular if

i. U is open and at positive distance from the boundary $\partial[0,1]^d$ of the unit cube,
ii. ∂U is $d-1$ rectifiable and
iii. for small enough $r > 0$, $[0,1]^d \setminus (\partial U + B(0,r))$ has exactly two connected components.

We recall that $E \subset \mathbb{R}^d$ is a $d-1$ rectifiable set if there exists a Lipschitzian function mapping some bounded subset of $\mathbb{R}^{d-1}$ onto E (Definition 3.2.14 in [23]). It is the case in particular of the boundary of non-empty Wulff crystals (Theorem 3.2.35 in [23]) and of bounded polyhedral sets. It follows from Proposition 3.62 in [4] that any $u = \chi_U$ regular belongs to BV and that $\partial U = \partial^\star u$ up to a $\mathcal{H}^{d-1}$-negligible set, so that the covering Theorem applies as well to ∂U. Assumption (ii) in Definition 4.6 has the following consequence:

Lemma 4.7. *Let $u = \chi_U \in$ BV be a regular profile. Then, for any $\delta > 0$, for any δ-covering $(\mathcal{R}_i)_{i=1\dots n}$ of u, one has*

$$\limsup_{r\to 0} \frac{\mathcal{L}^d\left((\partial U \setminus \bigcup_{i=1}^n \mathcal{R}_i) + B(0,r)\right)}{r} \leqslant 2\delta.$$

Proof Clearly, the set

$$E = \partial U \setminus \bigcup_{i=1}^n \dot{\mathcal{R}}_i$$

is a closed, $d-1$ rectifiable set. Thus, the $d-1$ Minkowski content of E equals the $d-1$ dimensional Hausdorff measure of E (Theorem 3.2.39 in [23]). In other words:

$$\lim_{r\to 0} \frac{\mathcal{L}^d\left(E+B\left(0,r\right)\right)}{2r} = \mathcal{H}^{d-1}\left(E\right) \leqslant \delta$$

and the claim follows. □

Before we state Propositions 4.8 and 4.9 we give one more notation. The analysis of surface tension has been done for rectangular parallelepiped centered *at lattice points*. Changing the center of the parallelepipeds does not modify the behavior of surface tension, but this would have led to heavier notations. We prefer to proceed to a small adjustment here: given a macroscopic rectangular parallelepiped $\mathcal{R} \subset (0,1)^d$ and $N \in \mathbb{N}^\star$, we let

$$\mathcal{R}^N = N\mathcal{R} + z_N(\mathcal{R}) \tag{4.7}$$

where $z_N(\mathcal{R}) \in (-1/2, 1/2]^d$ is chosen such that the center of $\mathcal{R}^N$ belongs to $\mathbb{Z}^d$. Still, for any finite collection $(\mathcal{R}_i)_{i=1\ldots n}$ of disjoint rectangular parallelepipeds in $(0,1)^d$ and large enough N, the collection $(\mathcal{R}_i^N)_{i=1\ldots n}$ is disjoint and included in $(0,N)^d$.

Proposition 4.8. *Consider a regular $u = \chi_U$. For any $\delta > 0$ and any δ-covering $(\mathcal{R}_i)_{i=1\ldots n}$ for u, we have*

$$\frac{1}{N^{d-1}} \log \Phi^{J,w}_{\Lambda_N}\left(\mathcal{D}^{N,\delta}_U\right) \geqslant -\sum_{i=1}^n h_i^{d-1} \tau^J_{\mathcal{R}_i^N} - c\beta\delta \tag{4.8}$$

for any N large enough, where $c < \infty$ depends on d and u.

Proposition 4.9. *Assume $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$, and let $u = \chi_U$ regular. For any $\varepsilon > 0$, for small enough $\delta > 0$ there are $K \in \mathbb{N}^\star$ and $c > 0$ such that, for large enough N:*

$$\mathbb{P}\left(\inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,w,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon) \middle| \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \leqslant \frac{1}{2} - e^{-c\sqrt{N}} \right) \leqslant e^{-c\sqrt{N}}. \tag{4.9}$$

Proof (Proposition 4.8). To realize the event of disconnection $\mathcal{D}_U^{N,\delta}$, it is enough to realize all the $\mathcal{D}_{\mathcal{R}_i^N}$ and to close all the edges that are at distance less than $1+\sqrt{d}$ from

$$N\left[\left(\partial U \setminus \bigcup_{i=1}^n \dot{\mathcal{R}}_i\right) \cup \bigcup_{i=1}^n \partial_{\text{lat}} \mathcal{R}_i\right]$$

where $\partial_{\text{lat}}\mathcal{R}$ stands for the lateral boundary of $\mathcal{R}$, that is the faces of $\partial\mathcal{R}$ that are parallel to the orientation $\boldsymbol{n}$ of $\mathcal{R}$. Thanks to Lemma 4.7 and Definition 4.1, there are at most $\delta c_d N^{d-1}\left(1+\mathcal{H}^{d-1}(\partial U)\right)$ such edges for large enough N. An immediate application of the DLR equation yields (4.8). □

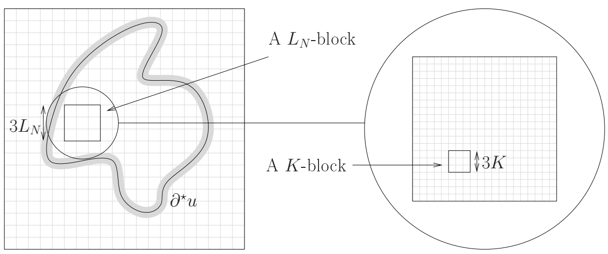

FIGURE 5. The scales K and L_N.

Proof (Proposition 4.9). In order to obtain the claim for a mesoscopic scale K that does not depend on N, we proceed to a coarse grained analysis at two characteristic scales K and $L_N =$

$[\sqrt{N}]$. Given $K \in \mathbb{N}^\star$, we consider $(\Delta_i, \Delta'_i)_{i \in I_{\Lambda_N,K}}$ the (K,K)-covering of Λ_N as in Definition 5.1 in [44] as well as the phase indicator

$$(\phi_i)_{i \in I_{\Lambda_N,K}}$$

given by Theorem 5.10 in [44], for the tolerance δ. We call $F = \{0,1\}^{I_{\Lambda_N,K}}$ the set of site configurations on the index of blocks $I_{\Lambda_N,K}$. In order to apply the stochastic domination Theorem 5.10 (iv) in [44], we will define an increasing function $f : F \to \{0,1\}$ with the appropriate properties. First, we need to describe the L_N-blocks: we call $(\tilde\Delta_j, \tilde\Delta'_j)_{j \in J_{N,K}}$ the (L_N, L_N)-covering for $I_{\Lambda_N,K}$ as in Definition 5.1 in [44]. Then we let

$$J = \left\{ j \in J_{N,K} : \forall i \in \tilde\Delta'_j, E^w\left(\Delta'_i\right) \cap \mathcal{E}_U^{N,\delta} = \emptyset \right\}$$

and

$$I = \bigcup_{j \in J} \tilde\Delta'_j.$$

Given $\rho \in F$ a site configuration on $I_{\Lambda_N,K}$ and $j \in J$, we say that the L_N-block $\tilde\Delta'_j$ is good if there is a crossing cluster of open sites for ρ in $\tilde\Delta'_j$, of density at least $1-\delta$. Then we define $f : F \to \{0,1\}$ letting

$$f(\rho) = \mathbf{1}_{\left\{\text{For all } j \in J,\ \tilde\Delta'_j \text{ is good}\right\}}.$$

Clearly, f is an increasing function. We prove now that its expectation is close to 1 under high-parameter site percolation. Consider $\mathcal{B}_p^I$ the site percolation process on I of density $p \in (0,1)$. According to Theorem 1.1 in [19], for large enough $p < 1$ there is $c > 0$ such that, for large enough N, for all $j \in J$:

$$\mathcal{B}_p^I\left(\left\{ \tilde\Delta'_j \text{ is good} \right\} \right) \geqslant 1 - \exp\left(-2cL_N^{d-1}\right)$$

and consequently (the cardinal of J is bounded by N^d), for $p < 1$ close enough to 1, for large enough N,

$$\mathcal{B}_p^I(f) \geqslant 1 - \exp\left(-c\sqrt{N}\right).$$

Consequently, the stochastic domination for $(|\phi_i|)_{i \in I_{\Lambda_N,K}}$ (see Theorem 5.10 (iv) in [44]) yields the same lower bound on the expectation of $f((|\phi_i|)_{i\in I})$: for large enough K (depending on δ), there is $c > 0$ such that, for any N large enough:

$$\mathbb{E} \inf_\pi \Psi^{J,+}_{\Lambda_N,\beta}\left(f\left((|\phi_i|)_{i\in I}\right) \middle| \omega = \pi \text{ on } E^w(\Lambda_N) \setminus \bigcup_{i\in I} E^w(\Delta'_i) \right) \geqslant 1 - e^{-c\sqrt{N}}. \tag{4.10}$$

The event that $f\left((|\phi_i|)_{i\in I}\right) = 1$ gives a control on the magnetization. For large enough N, the blocks $(\Delta_i)_{i\in I}$ cover a fraction of Λ_N that is close to $1 - \mathcal{L}^d\left(\partial U + B(0, c_d\delta)\right) \underset{\delta\to 0^+}{\longrightarrow} 1$. This and the properties of $(\phi_i)_{i \in I_{\Lambda_N,K}}$ (Theorem 5.10 (i) and (ii) in [44]) imply that, for small enough $\delta > 0$, for large enough N:

$$f\left((|\phi_i|)_{i\in I}\right) = 1 \Rightarrow \frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon) \text{ or } \frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\mathbf{1},\varepsilon).$$

We now consider a boundary condition $\pi \in \mathcal{D}_U^{N,\delta}$. Because of the ω-disconnection, the spin of the clusters touching some $\Delta_i \subset NU$ with $i \in I$ has a symmetric distribution under the conditional measure

$$\Psi^{J,+}_{\Lambda_N,\beta}\left(. \middle| f\left((|\phi_i|)_{i\in I}\right) = 1 \text{ and } \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right).$$

Hence, one has

$$\begin{aligned} &\inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,+}_{\Lambda_N,\beta}\left(\tfrac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon) \middle| \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \\ &\qquad \geqslant \tfrac{1}{2} \inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,+}_{\Lambda_N,\beta}\left(f\left((|\phi_i|)_{i\in I}\right) \middle| \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \end{aligned}$$

The claim follows as (4.10) implies, as $\mathcal{E}_U^{N,\delta} \subset E^w(\Lambda_N) \setminus \bigcup_{i\in I} E^w(\Delta_i')$, that

$$\mathbb{P}\left(\inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,+}_{\Lambda_N,\beta}\left(f\left((|\phi_i|)_{i\in I}\right) \Big| \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \leqslant 1 - e^{-c/2\sqrt{N}} \right) \leqslant e^{-c/2\sqrt{N}}.$$

□

The final formulation of the lower bound for phase coexistence is the following:

Proposition 4.10. *Assume $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$. For any $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^q)$ and $\varepsilon > 0$ there exists $K \in \mathbb{N}^\star$ such that,*

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_{z_0+\alpha\mathcal{W}^q}, \varepsilon) \right) \geqslant -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}) \qquad \mathbb{P}\text{-a.s.} \tag{4.11}$$

where $z_0 = (1/2, \dots 1/2)$. Similarly, for any $\lambda > 0$ and $0 \leqslant \alpha < 1/\operatorname{diam}_\infty(\mathcal{W}^\lambda)$,

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_{z_0+\alpha\mathcal{W}^\lambda}, \varepsilon) \right) \right)^\lambda \right] \geqslant -\mathcal{F}^\lambda(\chi_{\alpha\mathcal{W}^\lambda}). \tag{4.12}$$

Proof Let $U = z_0 + \alpha\mathcal{W}^q$. According to Theorem 3.2.35 in [23], ∂U is rectifiable, hence the profile $u = \chi_U$ is regular. Let $\varepsilon, \delta > 0$. Thanks to Theorem 4.4 there exists a δ-covering $(\mathcal{R}_i)_{i=1}^n$ adapted to the profile χ_U and τ^q. Proposition 4.8 applies and gives, for $\delta > 0$ small enough:

$$\begin{aligned} \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_U, \varepsilon) \right) \quad &\geqslant \quad \inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,w,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_U, \varepsilon) | \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \\ &\quad \times \exp\left(-N^{d-1}\left(\sum_{i=1}^n h_i^{d-1} \tau^J_{\mathcal{R}_i^N} + c\beta\delta \right) \right) \end{aligned} \tag{4.13}$$

where $c < \infty$ depends on d and u. An important remark is that the two factors are *independent* under the product measure $\mathbb{P}$. Proposition 4.9 yields:

$$\mathbb{P}\left(\inf_{\pi \in \mathcal{D}_U^{N,\delta}} \Psi^{J,w,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_U, \varepsilon) | \omega = \pi \text{ on } \mathcal{E}_U^{N,\delta} \right) \leqslant \frac{1}{3} \right) \leqslant e^{-c\sqrt{N}}. \tag{4.14}$$

We prove first (4.11) and consider $\gamma, \xi > 0$. If $\delta > 0$ is small enough, Theorem 1.4 tells that the $\mathbb{P}$-probability that $\tau^J_{\mathcal{R}_i^N} > \tau^q(\boldsymbol{n}_i) + \gamma$ for some $i \in \{1, \dots, n\}$ decays like $\exp(-cN^d)$ where $c > 0$. Hence, with $\mathbb{P}$-probability at least $1 - e^{-c\sqrt{N}/3}$ we have

$$\begin{aligned} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_{z_0+\alpha\mathcal{W}^\tau}, \varepsilon) \right) \quad &\geqslant \quad -\sum_{i=1}^n h_i^{d-1}(\tau^q(\boldsymbol{n}_i) + \gamma) - c\beta\delta \\ &\geqslant \quad -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}) - \xi \end{aligned}$$

for small enough $\delta > 0$ and $\gamma > 0$. Borel-Cantelli Lemma ensures that $\mathbb{P}$-almost surely,

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_{z_0+\alpha\mathcal{W}^\tau}, \varepsilon) \right) \quad \geqslant \quad -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}) - \xi$$

and (4.11) follows letting $\xi \to 0^+$. We conclude with the proof of (4.12), take $\lambda > 0$ and denote here $U = z_0 + \alpha\mathcal{W}^\lambda$. Again, there exists a δ-covering $(\mathcal{R}_i)_{i=1}^n$ adapted to the profile χ_U and τ^λ. For N large enough, the $\mathcal{R}_i^N$ are disjoint and hence the $\tau^J_{\mathcal{R}_i^N}$ are independent under $\mathbb{P}$. Consequently, for N large enough and $\lambda > 0$, (4.13) and (4.14) give

$$\begin{aligned} \mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_U, \varepsilon) \right) \right)^\lambda \right] \quad &\geqslant \quad \frac{1}{2 \times 3^\lambda} \times \prod_{i=1}^l \mathbb{E} \exp\left(-\lambda N^{d-1} h_i^{d-1} \tau^J_{\mathcal{R}_i^N} \right) \\ &\quad \times \exp\left(-\lambda N^{d-1} c\beta\delta \right). \end{aligned}$$

In view of Proposition 2.2, this means

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(\chi_U, \varepsilon)\right)\right)^\lambda\right] \geqslant -\sum_{i=1}^n h_i^{d-1}\tau^\lambda(\boldsymbol{n}_i) - \lambda c\beta\delta$$

and the claim follows as $\delta \to 0$. □

4.4. **Upper bound for phase coexistence.** Here we address the opposite problem of providing an upper bound on the probability of phase coexistence along a given phase profile. Our analysis follows the same line as [5, 6, 11]. The cost of phase coexistence is easily related (Proposition 4.11) to another notion of surface tension (4.15), that uses a L^1-characterization of phase coexistence. Then the L^1-notion of surface tension is related to a percolative definition of surface tension with *free boundary* conditions, with the help of the minimal section argument (Proposition 4.12). As in the uniform setting [11], the surface tension with free boundary condition differs very slightly from the usual notion of surface tension (Proposition 4.13).

The L^1-definition of surface tension is as follows. Given $\delta > 0$, a rectangular parallelepiped $\mathcal{R} \subset [0,1]^d$ as in Definition 4.1 (i) and $K, N \in \mathbb{N}^\star$ we define

$$\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}} = -\frac{1}{(hN)^{d-1}} \log \sup_{\bar\sigma \in \Sigma^+_{\widehat{N\mathcal{R}}}} \mu^{J,\bar\sigma}_{\widehat{N\mathcal{R}}}\left(\left\|\frac{\mathcal{M}_K}{m_\beta} - \chi\right\|_{L^1(\mathcal{R})} \leqslant 2\delta\mathcal{L}^d(\mathcal{R})\right) \tag{4.15}$$

where χ is the characteristic function of $\mathcal{R}$ as in Definition 4.1 (iii), and $\mu^{J,\bar\sigma}_{\widehat{N\mathcal{R}}}$ the Gibbs measure on $\widehat{N\mathcal{R}}$ with boundary condition $\bar\sigma$. We have:

Proposition 4.11. *Let $u \in \mathrm{BV}$, $\delta > 0$ and assume that $(\mathcal{R}_i)_{i=1\dots n}$ is a δ-covering for u. Then, for any $\varepsilon > 0$ small enough, any $K, N \in \mathbb{N}^\star$ one has:*

$$\frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right) \leqslant -\sum_{i=1}^n h_i^{d-1}\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i}. \tag{4.16}$$

Proof For $\varepsilon > 0$ small enough, the implication

$$\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon) \Rightarrow \left\|\frac{\mathcal{M}_K}{m_\beta} - u\right\|_{L^1(\mathcal{R}_i)} \leqslant \delta\mathcal{L}^d(\mathcal{R}_i), \quad \forall i \in \{1,\dots,n\}$$

holds. Thanks to (iii) in Definition 4.1, for such ε we have

$$\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon) \Rightarrow \left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_i\right\|_{L^1(\mathcal{R}_i)} \leqslant 2\delta\mathcal{L}^d(\mathcal{R}_i), \quad \forall i \in \{1,\dots,n\}.$$

Now, the Gibbs property for $\mu^{J,+}_{\Lambda_N}$ implies that

$$\begin{aligned}
\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right) &\leqslant \mu^{J,+}_{\Lambda_N}\left(\left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_i\right\|_{L^1(\mathcal{R}_i)} \leqslant 2\delta\mathcal{L}^d(\mathcal{R}_i), \forall i \in \{1,\dots,n\}\right) \\
&= \mu^{J,+}_{\Lambda_N}\left(\prod_{i=1}^n \mu^{J,\sigma}_{\widehat{N\mathcal{R}_i}}\left(\left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_i\right\|_{L^1(\mathcal{R}_i)} \leqslant 2\delta\mathcal{L}^d(\mathcal{R}_i)\right)\right) \\
&\leqslant \exp\left(-h_i^{d-1}N^{d-1}\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i}\right)
\end{aligned}$$

thanks to (4.15), and the claim is proved. □

Using the minimal section argument as in [5] one can compare the L^1-surface tension to the surface tension under free boundary condition in $\mathcal{R} = \mathcal{R}_{x,L,H}(\mathcal{S},\boldsymbol{n})$, defined as

$$\tilde{\tau}^J_{\mathcal{R}} = -\frac{1}{L^{d-1}} \log \Phi^{J,f}_{\mathcal{R}}\left(\mathcal{D}_{\tilde{\mathcal{R}}}\right) \tag{4.17}$$

where $\tilde{\mathcal{R}} = \mathcal{R}_{x,L,H/2}(\mathcal{S},\boldsymbol{n})$ is a rectangular parallelepiped twice finer than $\mathcal{R}$.

Proposition 4.12. *Assume $\beta > \hat{\beta}_c$ with $\beta \notin \mathcal{N}$. Then, there exists $c_{d,\delta} \in (0,\infty)$ with $\lim_{\delta\to 0} c_{d,\delta} = 0$ such that, for any $\mathcal{R}$ as in Definition 4.1 (i), for any $\delta > 0$, if K is large enough then:*

$$\limsup_N \frac{1}{N^d} \log \mathbb{P}\left(\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}} \leqslant \tilde{\tau}^J_{\mathcal{R}^N} - c_{d,\delta}\right) < 0. \tag{4.18}$$

We do not detail here the proof of Proposition 4.12 as it is easily adapted from [5]. Then, the argument of [11] let us quantify the influence of the boundary condition on the value of surface tension:

Proposition 4.13. *Assume $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$. Let $\mathcal{R}$ be a rectangular parallelepiped $\mathcal{R}$ as in Definition 4.1 (i), with $\delta \in (0,1)$. Then,*

$$\limsup_N \frac{1}{N^d} \log \mathbb{P}\left(\tilde{\tau}^J_{\mathcal{R}^N} \leqslant \tau^J_{\mathcal{R}^N} - c_d\delta\right) < 0 \tag{4.19}$$

where $c_d < \infty$ depends on d only.

We cannot afford to give here the proof of Proposition 4.13 as the generalization to the random case of the argument of [11] makes it far too long. However, no new ingredient needs to be introduced with respect to the original construction [11], and the interested reader can consult the PhD thesis [45] for a complete development of the proofs of both Propositions 4.12 and 4.13.

The consequence of the three last Propositions, together with Varadhan's Lemma, is a lower bound on the probability of phase coexistence along a given profile under quenched and averaged measures:

Proposition 4.14. *For all $\beta > \hat{\beta}_c$ with $\beta \notin \mathcal{N}$, for every $u \in \mathrm{BV}$ and $\xi, \lambda > 0$, there exists $\varepsilon > 0$ such that, for $K \in \mathbb{N}^\star$ large enough,*

$$\limsup_N \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right) \leqslant -\mathcal{F}^q(u) + \xi \tag{4.20}$$

in $\mathbb{P}$-probability (and $\mathbb{P}$-almost surely if $\beta \notin \mathcal{N}_I$) and

$$\limsup_N \frac{1}{N^{d-1}} \log \mathbb{E}\left[\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right)\right]^\lambda \leqslant -\mathcal{F}^\lambda(u) + \xi. \tag{4.21}$$

Proof We fix $\delta \in (0,1)$ and a δ-covering $(\mathcal{R}_i)_{i=1\dots n}$ for u as in Definition 4.2. We examine first the quenched convergence: according to Propositions 4.12 and 4.13 there is $c > 0$ such that

$$\mathbb{P}\left(\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i} \geqslant \tau^J_{\mathcal{R}^N} - c_{d,\delta} - c_d\delta\right) \geqslant 1 - \exp(-cN^d), \quad \forall i = 1\dots n \tag{4.22}$$

for K and N large enough. On the other hand, for any $\varepsilon > 0$ small enough Propositions 4.11 yields

$$\frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right) \leqslant -\sum_{i=1}^n h_i^{d-1} \tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i}$$

and hence, for K and N large enough,

$$\frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right) \leqslant -\sum_{i=1}^n h_i^{d-1} \left[\tau^J_{\mathcal{R}^N} - c_{d,\delta} - c_d\delta\right]$$

with $\mathbb{P}$-probability greater than $1 - n\exp(-cN^d)$. This implies (4.20) for $\delta > 0$ small enough in view of the convergence $\tau^J_{\mathcal{R}_i^N} \to \tau^q(\boldsymbol{n}_i)$ in $\mathbb{P}$-probability (Theorem 1.3) or of the almost-sure convergence if $\beta \notin \mathcal{N}_I$ (Corollary 1.9). We examine now the averaged convergence: consider $\lambda > 0$ and again, a δ-covering $(\mathcal{R}_i)_{i=1\dots n}$ for u. For K, N large enough and $\varepsilon > 0$ small enough we have

$$\mathbb{E}\left(\left[\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u,\varepsilon)\right)\right]^\lambda\right)$$

$$
\begin{aligned}
&\leqslant \quad \mathbb{E}\exp\left(-\sum_{i=1}^{n}\lambda(h_iN)^{d-1}\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i}\right)\\
&\leqslant \quad n\exp(-cN^d)+\mathbb{E}\exp\left(-\sum_{i=1}^{n}\lambda(h_iN)^{d-1}\tau^J_{\mathcal{R}^N}\right)\\
&\qquad \times\exp\left(\lambda\sum_{i=1}^{n}h_i^{d-1}N^{d-1}\left(c_{d,\delta}+c_d\delta\right)\right)
\end{aligned}
$$

in view of (4.22). Varadhan's Lemma (Proposition 2.2) yields: for any $\varepsilon>0$ small enough, any K large enough,

$$
\begin{aligned}
\limsup_N \frac{1}{N^{d-1}}\log\mathbb{E}\left(\left[\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta}\in\mathcal{V}(u,\varepsilon)\right)\right]^\lambda\right)&\\
\leqslant -\sum_{i=1}^{l}h_i^{d-1}\left[\tau^\lambda(\boldsymbol{n}_i)-c_{d,\delta}-c_d\delta\right]&
\end{aligned}
$$

and the conclusion follows for $\delta>0$ small enough. □

4.5. **Exponential tightness.** The last step towards the proofs of Theorems 1.15 and 1.16 is the exponential tightness property. Note that the compact set BV_a was defined at (4.3).

Proposition 4.15. *For any $\beta>\hat{\beta}_c$ with $\beta\notin\mathcal{N}$, there exists $C>0$ and for every $\delta>0$, for any $K\in\mathbb{N}^\star$ large enough one has*

$$
\limsup_N \frac{1}{N^{d-1}}\log\mathbb{E}\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta}\notin\mathcal{V}(\mathrm{BV}_a,\delta)^c\right)\leqslant -Ca. \tag{4.23}
$$

The proof of Bodineau, Ioffe and Velenik given in [6] applies as well in the present case.

4.6. **Proofs of Theorems 1.15 to 1.20.** Theorems 1.15 and 1.16 are consequences of the large deviations estimates (Propositions 4.10 and 4.14) together with the exponential tightness (Proposition 4.15) in view of the compactness of BV_a. The case of averaged Gibbs measures (Theorems 1.17, 1.19 and 1.20) presents complete similarity with the non-random case and for this reason we focus here only on the quenched case. Furthermore, the proof of Theorem 1.15 is similar to that of Theorem 1.16, which is the reason for which we give the proof of (1.42) only.

Proof (First half of Theorem 1.16). First we establish the lower bound

$$
\liminf_N \frac{1}{N^{d-1}}\log\mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta}\leqslant 1-2\alpha^d\right)\geqslant -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}),\quad \mathbb{P}\text{-almost surely.} \tag{4.24}
$$

The proof goes as follows: for any $\alpha'>\alpha$, for small enough $\varepsilon>0$ one has

$$
\frac{\mathcal{M}_K}{m_\beta}\in\mathcal{V}(\chi_{z_0+\alpha'\mathcal{W}^q},\varepsilon)\Rightarrow\frac{m_{\Lambda_N}}{m_\beta}\leqslant 1-2\alpha^d
$$

hence, Proposition 4.14 gives: for any $\alpha'>\alpha$,

$$
\liminf_N \frac{1}{N^{d-1}}\log\mu^{J,+}_{\Lambda_N}\left(\frac{m_{\Lambda_N}}{m_\beta}\leqslant 1-2\alpha^d\right)\geqslant -\mathcal{F}^q(\chi_{\alpha'\mathcal{W}^q}),\quad \mathbb{P}\text{-almost surely.}
$$

The lower bound (4.24) follows if we let $\alpha'\to\alpha$.

Now we establish the following upper bound: for any $\varepsilon>0$, there is $\delta>0$ such that

$$
\begin{aligned}
\limsup_N \frac{1}{N^{d-1}}\log\mu^{J,+}_{\Lambda_N}\left(\frac{\mathcal{M}_K}{m_\beta}\notin\bigcup_{x\in\mathcal{T}^q_\alpha}\mathcal{V}(\chi_{x+\alpha\mathcal{W}^q},\varepsilon)\text{ and }\frac{m_{\Lambda_N}}{m_\beta}\leqslant 1-2\alpha^d\right)&\\
\leqslant -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q})-\delta&
\end{aligned} \tag{4.25}
$$

in $\mathbb{P}$-probability ($\mathbb{P}$-almost surely if $\beta \notin \mathcal{N}_I$). To begin with, we choose $a > 0$ so large that Ca in Proposition 4.15 is larger than $2\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}) + 2$. Thanks to Markov's inequality, this implies that, for any $\gamma > 0$, for large enough K,

$$\limsup_{N} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N} \left(\frac{\mathcal{M}_K}{m_\beta} \notin \mathcal{V}(\mathrm{BV}_a, \gamma) \right) \leqslant -\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q}) - 1, \tag{4.26}$$

$\mathbb{P}$-almost surely (see (4.3) for the definition of BV_a). Consider $\eta > 0$ and let

$$F = \left\{ u \in \mathrm{BV}_a : \int_{[0,1]^d} u \leqslant 1 - 2\alpha^d + \eta \ \text{ and } \ u \notin \bigcup_{x \in \mathcal{T}^q_\alpha} \mathcal{V}\left(\chi_{x+\alpha\mathcal{W}^q}, \frac{\varepsilon}{2}\right) \right\}.$$

For $\gamma > 0$ small enough, for large enough N the event

$$\frac{\mathcal{M}_K}{m_\beta} \notin \bigcup_{x \in \mathcal{T}^q_\alpha} \mathcal{V}(\chi_{x+\alpha\mathcal{W}^q}, \varepsilon) \text{ and } \frac{m_{\Lambda_N}}{m_\beta} \leqslant 1 - 2\alpha^d$$

implies that

$$\frac{\mathcal{M}_K}{m_\beta} \notin \mathcal{V}(\mathrm{BV}_a, \gamma) \text{ or } \frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(F, \gamma).$$

The probability of the first event is under control (4.26) for any $\gamma > 0$ (and large enough K), hence we focus on the probability of the second one. Given $\xi > 0$, applying Proposition 4.14 we obtain $\varepsilon : u \in \mathrm{BV} \mapsto \varepsilon(u) \in (0, \xi)$ such that, for any $u \in \mathrm{BV}$ and any K large enough:

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N} \left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(u, \varepsilon(u)) \right) \leqslant -\mathcal{F}^q(u) + \xi \tag{4.27}$$

in $\mathbb{P}$-probability ($\mathbb{P}$-almost surely if $\beta \notin \mathcal{N}_I$). The set BV_a is compact for the L^1-norm, thus it can be covered by a finite union $\mathrm{BV}_a \subset \bigcup_{i=1}^n \mathcal{V}(u_i, \varepsilon(u_i))$ with $u_i \in \mathrm{BV}_a$, $i = 1 \dots n$. Since the right-hand side term is open, for $\gamma > 0$ small enough we still have

$$\mathcal{V}(\mathrm{BV}_a, \gamma) \subset \bigcup_{i=1}^n \mathcal{V}(u_i, \varepsilon(u_i)).$$

We consider $(u'_i)_{i=1\dots l}$ the subsequence of the u_i such that $\mathcal{V}(u_i, \varepsilon(u_i))$ intersects $\mathcal{V}(F, \gamma)$. Thanks to the inclusion

$$\mathcal{V}(F, \gamma) \subset \bigcup_{i=1}^l \mathcal{V}(u'_i, \varepsilon(u'_i))$$

and to (4.27), we have: for small enough γ, for large enough K:

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^{J,+}_{\Lambda_N} \left(\frac{\mathcal{M}_K}{m_\beta} \in \mathcal{V}(F, \gamma) \right) \leqslant - \inf_{u \in \mathrm{BV}: u \in \mathcal{V}(F, 2\xi)} \mathcal{F}^q(u) + \xi$$

in $\mathbb{P}$-probability ($\mathbb{P}$-almost surely if $\beta \notin \mathcal{N}_I$). Yet, the limit as $\xi \to 0$ of the right-hand side is bounded from above by $-\inf_{u\in F'} \mathcal{F}^q(u)$ where

$$F' = \left\{ u \in \mathrm{BV}_a : \int_{[0,1]^d} u \leqslant 1 - 2\alpha^d + 2\eta \ \text{ and } \ u \notin \bigcup_{x \in \mathcal{T}^q_\alpha} \mathcal{V}\left(\chi_{x+\alpha\mathcal{W}^q}, \frac{\varepsilon}{4}\right) \right\},$$

for any $\eta > 0$. Yet, $-\inf_{u\in F'} \mathcal{F}^q(u)$ is strictly smaller, in the limit $\eta \to 0$, than $-\mathcal{F}^q(\chi_{\alpha\mathcal{W}^q})$ since the solutions to the isoperimetric problem (4.4) are excluded. Together with (4.26), this implies (4.25) and the conclusion (1.42) follows from (4.24) and (4.25). □

4.7. **Localization of the Wulff crystal under averaged measures.** One consequence of the introduction of the random media is the *localization* of the Wulff crystal if the volume constraint acts on the media as well: the surface tension appears to be reduced on the contour of the crystal. Here we give the proof of Theorem 1.21 after a we state the following immediate consequence of the lower large deviations described in Theorem 1.6:

Lemma 4.16. *Let* $\mathcal{R}^N = \mathcal{R}_{0,N,\delta N}(\mathcal{S}, \boldsymbol{n})$ *and* $\gamma > 0$, $\mathcal{A} = [\hat{\tau}^{\lambda,-}(\boldsymbol{n}) - \gamma, \hat{\tau}^{\lambda,+}(\boldsymbol{n}) + \gamma]$. *Then,*

$$\limsup_N \frac{1}{N^{d-1}} \log \mathbb{E}\left[1_{\left\{\tau^J_{\mathcal{R}^N} \in \mathcal{A}^c\right\}} \times \exp\left(-\lambda N^{d-1} \tau^J_{\mathcal{R}^N}\right)\right] < \tau^\lambda(\boldsymbol{n}).$$

Proof (Theorem 1.21). According to Theorems 1.17 and 1.20, it is enough to prove that

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\mu^{J,+}_{\Lambda_N}\left(\tau^J_{\mathcal{R}^N} \in \mathcal{A}^c \text{ and } \left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1} \leqslant \varepsilon\right)\right)^\lambda\right]$$

$$< -\mathcal{F}^\lambda(\alpha\mathcal{W}^\lambda). \tag{4.28}$$

In the case that the parallelepiped $\mathcal{R}$ does not intersect the crystal $z + \alpha\partial\mathcal{W}^\lambda$, for $\delta > 0$ small enough any δ-covering $(\mathcal{R}_i)_{i=1\dots n}$ for $z+\alpha\mathcal{W}^\lambda$ and τ^q does not intersect $\mathcal{R}$. For $\varepsilon > 0$ small enough and K large enough, Propositions 4.11, 4.12 and 4.13, the definition of the δ-covering and the independence of $\tau^J_{\mathcal{R}}$ from the $\tilde{\tau}^{J,\delta,K}_{N\mathcal{R}_i}$ under the product measure $\mathbb{P}$ imply that the right-hand side of (4.28) is bounded from above by

$$-\mathcal{F}^\lambda(\alpha\mathcal{W}^\lambda) + \underset{\delta\to 0}{o}(1) + \limsup_N \frac{1}{N^{d-1}} \log \mathbb{P}\left(\tau^J_{\mathcal{R}^N} \in \mathcal{A}^c\right)$$

which is strictly smaller than $-\mathcal{F}^\lambda(\alpha\mathcal{W}^\lambda)$ for small enough δ, as the last term is strictly negative.

Now we consider the case when the parallelepiped $\mathcal{R}$ is tangent to the crystal. For $h > 0$ small enough, for $\varepsilon > 0$ small enough, the strict inequality

$$\limsup_N \frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\sup_{\bar\sigma\in\Sigma^+_{\widehat{N\mathcal{R}}}} \mu^{J,\bar\sigma}_{\widehat{N\mathcal{R}}}\left(\tau^J_{\mathcal{R}^N} \in \mathcal{A}^c \text{ and } \left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1(\mathcal{R})} \leqslant \varepsilon\right)\right)^\lambda\right]$$

$$< -\int_{\partial(z+\alpha\mathcal{W}^\lambda)\cap\mathcal{R}} \tau^\lambda(\boldsymbol{n}_\cdot) d\mathcal{H} \tag{4.29}$$

holds according to Propositions 4.12 and 4.13, and Lemma 4.16. Let $(\mathcal{R}_i)_{i=1\dots n}$ be a η-covering for $z + \alpha\mathcal{W}^\lambda$ and τ^λ. Propositions 4.11, 4.12 and 4.13 and the properties of the η-covering imply that for $\varepsilon > 0$ small enough (depending on η) and large enough K, the cost of phase coexistence outside of $\mathcal{R}$ is bounded above by

$$\limsup_N \frac{1}{N^{d-1}} \log \mathbb{E}\left[\left(\prod_{i:\mathcal{R}_i\cap\mathcal{R}=\emptyset} \sup_{\bar\sigma\in\Sigma^+_{\widehat{N\mathcal{R}_i}}} \mu^{J,\bar\sigma}_{\widehat{N\mathcal{R}_i}}\left(\left\|\frac{\mathcal{M}_K}{m_\beta} - \chi_{z+\alpha\mathcal{W}^\lambda}\right\|_{L^1(\mathcal{R})} \leqslant \varepsilon\right)\right)^\lambda\right]$$

$$\leqslant -\int_{\partial(z+\alpha\mathcal{W}^\lambda)\setminus\mathcal{R}} \tau^\lambda(\boldsymbol{n}_\cdot) d\mathcal{H} + \underset{\eta\to 0}{o}(1).$$

Thus, choosing $h > 0$ small enough then $\varepsilon > 0$ small enough and K large enough, the strict inequality holds in (4.28) and the claim follows. □

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