RANKIN-COHEN BRACKETS ON QUASIMODULAR FORMS

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Abstract. We give the algebra of quasimodular forms a collection of Rankin-Cohen operators. These operators extend those defined by Cohen on modular forms and, as for modular forms, the first of them provides a Lie structure on quasimodular forms. They also satisfy a "Leibniz rule" for the usual derivation. Rankin-Cohen operators are useful for proving arithmetical identities. In particular, we explain why Chazy equation has the exact shape it has.

Introduction

The purpose of this paper is to present a generalisation for quasimodular forms of the Rankin-Cohen brackets for modular forms: for each $n \geq 0$, k, ℓ, s, t positive integers, we define bilinear differential operators $[\,,\,]_n$ sending $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t}$. We have denoted $\widetilde{M}_k^{\leq s}$ the vector space of quasimodular forms of weight k and depth less or equal than s on $\mathrm{SL}(2,\mathbb{Z})$ (see section 1.1 for the definitions).

We give a quite precise description of the image of this bilinear form in terms of modular and parabolic forms. This allows us to obtain efficiently classical differential equations and arithmetical identities.

Then we prove that the Rankin-Cohen brackets satisfy the "Leibniz rule" for the normalized usual derivation ($\mathrm{D} := \frac{\mathrm{d}}{2\pi\mathrm{i}\mathrm{d}z}$): $\mathrm{D}[f,g]_n = [\mathrm{D}\, f, g]_n + [f, \mathrm{D}\, g]_n$.

The first section is a presentation of the definitions and classical results concerning quasimodular forms and Rankin-Cohen brackets on modular forms.

In the second section, we prove the following theorem.

Theorem 1. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0, \dots, \lfloor k/2 \rfloor\}$, $t \in \{0, \dots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{\geq 0}$. Define*

$$\Phi_{n;k,s;\ell,t}(f,g) := \sum_{r=0}^{n} (-1)^r \binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r} \mathrm{D}^r f \, \mathrm{D}^{n-r} g.$$

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Then
$$\Phi_{n;k,s;\ell,t}\left(\widetilde{M}_k^{\leq s},\widetilde{M}_\ell^{\leq t}\right)\subset\widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

In some case we get a more precise description in terms of the spaces of modular forms M_k and the spaces of parabolic forms S_k.

Proposition 2. *Under the hypothesis of theorem 1, if $n>0$ then*
$$\Phi_{n;k,s;\ell,t}\left(\widetilde{M}_k^{\leq s},\widetilde{M}_\ell^{\leq t}\right)\in S_{k+\ell+2n}\oplus\bigoplus_{j=1}^{s+t}\mathrm{D}^j M_{k+\ell+2n-2j}.$$
If moreover $n>s+t$, then
$$\Phi_{n;k,s;\ell,t}\left(\widetilde{M}_k^{\leq s},\widetilde{M}_\ell^{\leq t}\right)\in S_{k+\ell+2n}\oplus\bigoplus_{j=1}^{s+t-1}\mathrm{D}^j M_{k+\ell+2n-2j}\oplus\mathrm{D}^{s+t} S_{k+\ell+2n-2s-2t}.$$
The same conclusion holds if $n=s+t$ and f or g vanishes at infinity.

Remark 1. This notion is consistent with the one for modular forms, the standard Rankin-Cohen bracket of $f\in M_k$ and $g\in M_\ell$ is $\Phi_{n;k,0;\ell,0}(f,g)$ (see paragraph 1.2).

Remark 2. For $n\geq 0$, a bilinear differential operator Ψ sending $\widetilde{M}_k^{\leq s}\times\widetilde{M}_\ell^{\leq t}$ to $\bigcup_v\widetilde{M}_{k+\ell+2n}^{\leq v}$ is necessarily (for weight compatibility reasons) a linear combination of $(f,g)\mapsto\mathrm{D}^r f\,\mathrm{D}^{n-r} f$, $r\in\{0,\dots,n\}$. Such a differential operator sends in principle $\widetilde{M}_k^{\leq s}\times\widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t+n}$ (see lemma 7). So the operator Φ introduced before has the advantage of reducing the depth of the quasimodular form obtained, and it was not obvious that such an operator was existing.

Remark 3. Theorem 1 is valid for quasimodular forms on any subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$.

In the third section, we show that the behaviour of this operator under derivation is natural.

Theorem 3. *Under the hypothesis of theorem 1, for all $f\in\widetilde{M}_k^{\leq s}$ and $g\in\widetilde{M}_\ell^{\leq t}$,*
$$\mathrm{D}\,\Phi_{n;k,s;\ell,t}(f,g)=\Phi_{n;k,s;\ell+2,t+1}(f,\mathrm{D}\,g)+\Phi_{n;k+2,s+1;\ell,t}(\mathrm{D}\,f,g).$$

Remark 4. For f of weight k and *exact depth* s and g of weight ℓ and *exact depth* t, we write $[f,g]_n$ instead of $\Phi_{n;k,s;\ell,t}(f,g)$. Recall (see proposition 6) that if h has weight $w>0$ and depth d then $\mathrm{D}\,h$ has weight $w+2$ and depth $d+1$. The following theorem may then be rewritten as
$$\mathrm{D}[f,g]_n=[\mathrm{D}\,f,g]_n+[f,\mathrm{D}\,g]_n.$$

For modular forms, Zagier, Cohen and Manin showed [CMZ97] that the sum of Rankin-Cohen brackets defines an associative product on the algebra $M=\prod_{k\geq 0}M_k$. In a recent paper, Bieliavski, Tang and Yao [BTY07] showed that this sum is isomorphic to the standard Moyal product. Do the Rankin-Cohen brackets for quasimodular forms introduced here have such a geometric interpretation ?

The existence of Rankin-Cohen brackets (thanks to proposition 2) provides a new tool to obtain arithmetical identities. For example, we recover the Ramanujan differential equations, Chazy differential equation (and explain why such a differential equation has to exist), van der Pol equality and Niebur equality. As usual, define for $h \geq 2$ the Eisenstein series:

$$E_h(z) := 1 - \frac{2h}{B_h} \sum_{n=1}^{+\infty} \sigma_h(n) \exp(2\pi \mathrm{i} n z) \tag{1}$$

where B_h is the Bernoulli number and

$$\sigma_h(n) := \sum_{d|n} d^h.$$

One of the three Ramanujan equations is

$$\mathrm{D}\, E_2 = -\frac{1}{12}(E_4 - E_2^2).$$

It is a direct consequence of

$$[E_2, \Delta]_1 = \Delta E_4$$

where Δ is the unique primitive form of weight 12 on $\mathrm{SL}(2, \mathbb{Z})$. If we write $\tau(n)$ for the nth coefficient of Δ, Niebur [Nie75] equality is

$$\tau(n) = n^4 \sigma_1(n) - 24 \sum_{a=1}^{n-1} (35a^4 - 52a^3 n + 18a^2 n^2) \sigma_1(a) \sigma_1(n-a)$$

and it follows from

$$[E_2, E_2]_4 = -48\Delta.$$

Van der Pol [vdP51] equality is

$$\tau(n) = n^2 \sigma_3(n) + 60 \sum_{a=1}^{n-1} a(9a - 5n) \sigma_3(a) \sigma_3(n-a).$$

It follows from

$$[E_4, \mathrm{D}\, E_4]_1 = 960\Delta.$$

Many examples of the two previous type are given in [RS07]. Finally, a quite astonishing equality is Chazy differential equation. Its usual form is

$$\mathrm{D}^3 E_2 = E_2 \mathrm{D}^2 E_2 - \frac{3}{2}(\mathrm{D}\, E_2)^2$$

and it follows from

$$[[K, \Delta]_1, \Delta]_1 = 24\Delta K^2 \tag{2}$$

where $K = [E_2, \Delta]_1$. The most outer bracket is on modular forms since it may be shown that $[K, \Delta]_1$ has depth 0. That such a differential equation has to exist is a consequence of the following proposition that we prove using Rankin-Cohen brackets.

Proposition 4. *Let $n \geq 0$ and $r \in \{0, \dots, n\}$. Then*

$$\mathrm{D}^r E_2 \mathrm{D}^{n-r} E_2 \in \bigoplus_{\substack{j=0 \\ j \equiv n \pmod 2}}^{n-4} \mathrm{D}^j S_{2n+4-2j} \oplus \mathbb{C}\mathrm{D}^n E_4 \oplus \mathbb{C}\mathrm{D}^{n+1} E_2.$$

In particular, $[E_2, E_2]_0 \in \mathbb{C}E_4 + \mathbb{C}\,\mathrm{D}\,E_2$, $[E_2, E_2]_2 \in \mathbb{C}\mathrm{D}^2\,E_4$, $[E_2, E_2]_4 \in \mathbb{C}\Delta$ *and*

$$[E_2, E_2]_{2n} \in S_{4(n+1)} \oplus \mathrm{D}^2\,S_{4n} \quad \textit{if } n \geq 3.$$

Indeed for $n = 2$, this proposition implies that both quasimodular forms $E_2\,\mathrm{D}^2\,E_2$ and $(\mathrm{D}\,E_2)^2$ are in $\mathbb{C}\mathrm{D}^2\,E_4 \oplus \mathbb{C}\mathrm{D}^3\,E_2$. Hence $\mathrm{Vect}(E_2\,\mathrm{D}^2\,E_2, (\mathrm{D}\,E_2)^2) = \mathrm{Vect}(\mathrm{D}^2\,E_4, \mathrm{D}^3\,E_2)$ and $\mathrm{D}^3\,E_2$ is a linear combination of $E_2\,\mathrm{D}^2\,E_2$ and $(\mathrm{D}\,E_2)^2$: this is the shape of Chazy equation.

1. Overview

1.1. **Quasimodular forms.** In this section, we introduce usual definitions and notations and recall some useful properties of quasimodular forms. For a more detailed introduction, see [MR05, §17].

We introduce the following notations: let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathcal{H}$, we define

$$\mathrm{X}(\gamma, z) := \frac{c}{cz + d}$$

and

$$\mathrm{X}(\gamma)\colon z \mapsto \mathrm{X}(\gamma, z).$$

As usual, the complex upper half-plane is denoted by $\mathcal{H}$. For $k \geq 0$, $f\colon \mathcal{H} \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ the function $(f \underset{k}{|} \gamma)$ is defined by $(f \underset{k}{|} \gamma)(z) = (cz + d)^{-k} f(\gamma z)$.

Definition 5. *Let $k \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$. An holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ is a quasimodular form of weight k, depth s (over $\mathrm{SL}(2,\mathbb{Z})$) if there exist holomorphic functions $Q_0(f)$, $Q_1(f)$, …, $Q_s(f)$ on $\mathcal{H}$ such that*

$$(f \underset{k}{|} \gamma) = \sum_{i=0}^{s} Q_i(f)\,\mathrm{X}(\gamma)^i \tag{3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and such that $Q_s(f)$ is not identically vanishing and f has no negative terms in its Fourier expansion. By convention, the 0 function is a quasimodular form of depth $-\infty$ and any weight.

Remark 5. Taking $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in (3) implies that f is periodic of period 1 hence has a Fourier expansion. The definition requires this Fourier expansion to be of the shape

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n) e^{2\pi i n z}.$$

The set of quasimodular forms of weight k and depth s is denoted by $\widetilde{M}_k^s$. It is often more convenient to use the $\mathbb{C}$-vectorial space of quasimodular forms of weight k and depth less or equal than s, which is denoted by $\widetilde{M}_k^{\leq s}$. It can be shown that there are no quasimodular forms (except 0) of negative weight or of depth $s > k/2$ [MR05, lemme 120]. Hence we extend our notation by defining $M_k^{\leq s} = \{0\}$ if $k < 0$ and $M_k^{\leq s} = M_k^{\leq k/2}$ if $s > k/2$.

Remark 6. With this definition, the space M_k of modular forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$ is exactly the space $\widetilde{M}_k^{\leq 0}$.

Remark 7. A basic example of quasimodular form which is not a modular form is E_2 defined in (1). It satisfies for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ the transformation property

$$(E_2 \underset{2}{|} \gamma) = E_2 + \frac{6}{\pi \mathrm{i}} \mathrm{X}(\gamma),$$

proving that $E_2 \in \widetilde{M}_2^1$ (see e.g., [MR05, lemme 19]).

The space $\widetilde{M}_* = \bigcup_{k,s} \widetilde{M}_k^{\leq s}$ is equipped with a natural filtered-graded algebra structure (the graduation according to the weight, the filtration according to the depth). The canonical multiplication $(f,g) \longmapsto fg$ defines a morphism $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t} \longrightarrow \widetilde{M}_{k+\ell}^{\leq s+t}$.

If $f \in \widetilde{M}_k^{\leq s}$, the sequence $(Q_i(f))_{i\in\mathbb{Z}}$ is defined by the quasimodularity condition (3), if $i \in \{0,\ldots,s\}$, and $Q_i(f) = 0$ for $i \notin \{0,\ldots,s\}$. One can show that $Q_0(f) = f$ and $Q_i(f) \in \widetilde{M}_{k-2i}^{\leq s-i}$ [MR05, Lemme 119].

Quasimodular forms are the natural extension of modular forms into a stable by derivation space, because of the following proposition.

Proposition 6. *If $k > 0$, the normalized derivation* $\mathrm{D} := \dfrac{\mathrm{d}}{2\pi\mathrm{i}\mathrm{d}z}$ *maps* $\widetilde{M}_k^s$ *to* $\widetilde{M}_{k+2}^{s+1}$.

For $r \in \mathbb{Z}_{\geq 0}$, write $f^{(r)} := \mathrm{D}^r(f)$ and $f' = f^{(1)}$. The following lemma connects the transformation equation of f and $f^{(r)}$.

Lemma 7. *Let $f \in \widetilde{M}_k^{\leq s}$. Then,*

(4)
$$(\mathrm{D}^r f \underset{k+2r}{|} \gamma) = \sum_{i=0}^{s+r} \left[\sum_{j=0}^{r} \frac{1}{(2\pi\mathrm{i})^j} j! \binom{r}{j} \binom{k+r-i+j-1}{j} \mathrm{D}^{r-j} Q_{i-j}(f) \right] \mathrm{X}(\gamma)^i$$

for all $r \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \Gamma$.

Proof. The result is obtained inductively on r: it is obvious for $r = 0$, and for the induction suppose that for $r \geq 0$, formula (4) holds. Let $g = f^{(r)}$. For $i \in \mathbb{Z}$ we have

$$(5)\quad Q_i(g) = \sum_{j=0}^{r} \frac{1}{(2\pi\mathrm{i})^j} j! \binom{r}{j} \binom{k+r-i+j-1}{j} Q_{i-j}(f)^{(r-j)} \in \widetilde{M}_{k+2r-2i}^{\leq s+r-i}.$$

Then using proposition 6 (which implies that $f^{(r+1)} \in \widetilde{M}_{k+2r+2}^{\leq r+s+1}$) and lemma 118 of [MR05] we find

$$(f^{(r+1)} \underset{k+2r+2}{|} \gamma) = \sum_{i=0}^{s+r+1} \left(Q_i(g)' + \frac{k+2r-i+1}{2\pi\mathrm{i}} Q_{i-1}(g) \right) \mathrm{X}(\gamma)^i.$$

From (5) we compute

$$Q_i(g)' + \frac{k+2r-i+1}{2\pi\mathrm{i}}Q_{i-1}(g) =$$
$$Q_i(f)^{(r+1)} + \frac{k+2r-i+1}{(2\pi\mathrm{i})^{r+1}}r!\binom{k+2r-i}{r}Q_{i-r-1}(f) + \sum_{j=1}^{r}\frac{1}{(2\pi\mathrm{i})^j}Q_{i-j}(f)^{(r+1-j)}$$
$$\times\left(\frac{r!}{(r-j)!}\binom{k+r-i+j-1}{j} + \frac{(k+2r-i+1)r!}{(r+1-j)!}\binom{k+r-i+j-1}{j-1}\right).$$

Formula (4) for $r+1$ instead of r follows by expanding the binomial coefficients. □

Finally, we shall need the following structure result. For completness, we provide a short proof that should convince that the theory requires E_2.

Proposition 8. *Quasimodular forms can be expressed as linear combinations of derivatives of modular forms and E_2 :*

$$\widetilde{M}_k^{\leq k/2} = \bigoplus_{i=0}^{k/2-1} \mathrm{D}^i\, M_{k-2i} \oplus \mathbb{C}\mathrm{D}^{k/2-1}\, E_2.$$

Proof. We proceed by descent on the depth. If f has weight k and depth s, we would like to have a modular form g such that $f - \mathrm{D}^s\, g$ has depth strictly less than s. For any $g \in M_{k-2s}$, multiple use of differentiation theorem [MR05, Lemme 118] lead to

$$Q_s(D^s g) = \left(\frac{1}{2\pi\mathrm{i}}\right)^s s!\binom{k-s-1}{s}g. \tag{6}$$

If $\binom{k-s-1}{s} \neq 0$, which happens if $s < k/2$, we can choose

$$g = (2\pi\mathrm{i})^s\frac{(k-2s-1)!}{(k-s-1)!}Q_s(f) \in M_{k-2s}.$$

For $s = k/2$, we use

$$Q_{k/2}(D^{k/2-1}E_2) = \left(\frac{1}{2\pi\mathrm{i}}\right)^{k/2-1}\left(\frac{k}{2}-1\right)!\frac{6}{\pi\mathrm{i}}$$

and choose

$$\alpha = \frac{\pi\mathrm{i}}{6}\cdot\frac{(2\pi\mathrm{i})^{k/2-1}}{\left(\frac{k}{2}-1\right)!}Q_{k/2}(f) \in M_0 = \mathbb{C}$$

to obtain

$$f - \alpha\,\mathrm{D}^{k/2}\, E_2 \in \widetilde{M}_k^{\leq k/2-1}.$$

□

1.2. **Usual Rankin-Cohen brackets for modular forms.** The Rankin-Cohen brackets have been introduced by Cohen after a work of Rankin. These are bilinear differential operators, whose main property is to preserve modular forms. More precisely, let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We write $M_k(\Gamma)$ for the space of modular forms of weight k over Γ. For each

$n \geq 0$, $(f,g) \in M_k(\Gamma) \times M_\ell(\Gamma)$, define the n-Rankin-Cohen bracket of f and g by

$$[f,g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} \mathrm{D}^r f \, \mathrm{D}^{n-r} g. \tag{7}$$

Then $[f,g]_n \in M_{k+\ell+2n}(\Gamma)$. Moreover, if Φ is a bilinear differential operator sending $M_k(\Gamma) \times M_\ell(\Gamma)$ to $M_{k+\ell+2n}(\Gamma)$ for all $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ a finite index subgroup, then (up to constant) $\Phi(f,g) = [f,g]_n$. For an overview of Rankin-Cohen brackets including a proof of these results[1], see for instance [Zag94], [Zag92] or [MR05].

Rankin-Cohen brackets appear to be useful in various mathematical domains as for instance invariant theory ([UU96] and [CMS01]) or non-commutative geometry [Yao07].

2. Rankin-Cohen brackets

We prove our main result (theorem 1). For $n \geq 0$ and any sequence $\mathbf{a} = (a_r)_{0 \leq r \leq n}$, the bilinear forms we study take the form

$$\Phi_{\mathbf{a}}(f,g) = \sum_{r=0}^{n} a_r \, \mathrm{D}^r f \, \mathrm{D}^{n-r} g.$$

We first establish a sufficient condition on $\mathbf{a}$ (lemma 9). For s, t and n nonnegative integers, we introduce the set

$$\mathcal{E}(s,t,n) = \left\{ (u,v,\alpha,\beta) \in \mathbb{Z}^4_{\geq 0} \colon u \leq s,\ v \leq t,\ \alpha+\beta \leq u+v+n-s-t-1 \right\}.$$

Lemma 9. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0, \ldots, \lfloor k/2 \rfloor\}$, $t \in \{0, \ldots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{>0}$. For $\mathbf{a} = (a_r)_{0 \leq r \leq n}$ satisfying*

$$\sum_{r=0}^{n} a_r \binom{r}{\alpha} \binom{n-r}{\beta} (k+r-u-1)!(\ell+n-r-v-1)! = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$, one has

$$\Phi_{\mathbf{a}}\left(\widetilde{M}_k^{\leq s}, \widetilde{M}_\ell^{\leq t} \right) \subset \widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. Let $f \in \widetilde{M}_k^{\leq s}$ and $g \in \widetilde{M}_\ell^{\leq t}$. From lemma 7 we deduce

$$\begin{aligned} (\Phi_{\mathbf{a}}(f,g) \underset{k+\ell+2n}{|} \gamma) &= \sum_{r=0}^{n} a_r (f^{(r)} \underset{k+2r}{|} \gamma)(g^{(n-r)} \underset{\ell+2(n-r)}{|} \gamma) \\ &= \sum_{i=0}^{s+t+n} C(\mathbf{a}; i)(f,g)\, \mathrm{X}(\gamma)^i \end{aligned}$$

[1]The uniqueness result needs explanations: it is proved by using only algebraic arguments, the demonstration does not depend on the group Γ or on growth conditions. Of course, it is possible that for some fixed group Γ the uniqueness result does not hold (for instance if $M_k(\Gamma) = \{0\}$!).

with

(8)
$$C(\mathbf{a};i)(f,g)=\sum_{\substack{(i_1,i_2)\in\mathbb{Z}^2_{\geq 0}\\ i_1+i_2=i}}\sum_{r=0}^{n}a_r\sum_{j_1=0}^{r}\left(\frac{1}{2\pi\mathrm{i}}\right)^{j_1}j_1!\binom{r}{j_1}\binom{k+r-i_1+j_1-1}{j_1}$$
$$\times\sum_{j_2=0}^{n-r}\left(\frac{1}{2\pi\mathrm{i}}\right)^{j_2}j_2!\binom{n-r}{j_2}\binom{\ell+n-r-i_2+j_2-1}{j_2}Q_{i_1-j_1}(f)^{(r-j_1)}Q_{i_2-j_2}(g)^{(n-r-j_2)}.$$

It follows that $\Phi_{\mathbf{a}}(f,g)\in\widetilde{M}^{\leq s+t}_{k+\ell+2n}$ if and only if $C(\mathbf{a};s+t+i)=0$ for all $i\in\{1,\dots,n\}$. This is easily seen to be equivalent to

$$\sum_u\sum_v\sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2_{\geq 0}\\ \alpha+\beta=n+u+v-s-t-i}}\left(\frac{1}{2\pi\mathrm{i}}\right)^{n-\alpha-\beta}\sum_r a_r(r-\alpha)!(n-r-\beta)!$$
$$\times\binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{r-\alpha}\binom{\ell+n-r-v-1}{n-r-\beta}Q_u(f)^{(\alpha)}Q_v(g)^{(\beta)}=0$$

for all $i\in\{1,\dots,n\}$, the sets of summation being determined by the binomial coefficients. Hence, $\Phi_{\mathbf{a}}\left(\widetilde{M}^{\leq s}_k,\widetilde{M}^{\leq t}_\ell\right)\subset\widetilde{M}^{\leq s+t}_{k+\ell+2n}$ is implied by

$$\sum_r a_r\binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)!=0 \tag{9}$$

for all $(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)$. □

Remark 8. The statement of the previous lemma is in fact an equivalence, if we ask $\Phi_{\mathbf{a}}$ to satisfy $\Phi_{\mathbf{a}}\left(\widetilde{M}^{\leq s}_k(\Gamma),\widetilde{M}^{\leq t}_\ell(\Gamma)\right)\subset\widetilde{M}^{\leq s+t}_{k+\ell+2n}(\Gamma)$ for each finite index subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$: indeed for $\{a(u,v,\alpha,\beta)\}$ a non identically zero family of complex numbers, if $\Psi\colon (f,g)\mapsto\sum_{(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)}a(u,v,\alpha,\beta)Q_u(f)^{(\alpha)}Q_v(g)^{(\beta)}$ satisfy $\Psi(\widetilde{M}^{\leq s}_k(\Gamma),\widetilde{M}^{\leq t}_\ell(\Gamma))=0$, then exists $M>0$ such that the minimum of $\dim(\widetilde{M}^{\leq s}_k(\Gamma))$ and $\dim(\widetilde{M}^{\leq t}_\ell(\Gamma))$ is strictly smaller than M. However, as for modular forms, for each $A>0$ exists Γ a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ such that $\dim\widetilde{M}^{\leq s}_k(\Gamma)>A$ and $\dim\widetilde{M}^{\leq t}_\ell(\Gamma)>A$ (recall that $k,\ell\in\mathbb{Z}_{>0}$).

We shall now give a necessary condition for $\mathbf{a}$ satisfying the condition of lemma 9.

Lemma 10. *Let k,ℓ in $\mathbb{Z}_{>0}$, $s\in\{0,\dots,\lfloor k/2\rfloor\}$, $t\in\{0,\dots,\lfloor \ell/2\rfloor\}$ and $n\in\mathbb{Z}_{>0}$. If $\mathbf{a}=(a_r)_{0\leq r\leq n}$ satisfies*

$$\sum_{r=0}^{n}a_r\binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)!=0$$

for all $(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)$, then there exists $\lambda\in\mathbb{C}$ such that

$$a_r=\lambda(-1)^r\binom{k+n-s-1}{n-r}\binom{\ell+n-t-1}{r}$$

for all $r\in\{0,\dots,n\}$.

Proof. Define $\mathbf{b}=(b_r)_{0\leq r\leq n}$ by

$$b_r=a_r(k+r-s-1)!(\ell+n-r-t-1)!$$

for all r. Then

$$\sum_{r=0}^{n} b_r\binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{s-u}\binom{\ell+n-r-v-1}{t-v}=0$$

for all $(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)$. Choosing $u=s$, $t=v$ and $\beta=0$ leads to $F^{(\alpha)}(1)=0$ for all $\alpha\in\{0,\dots,n-1\}$ where F is the generating (polynomial) function of $\mathbf{b}$ defined by

$$F(x)=\sum_{r=0}^{n} b_r x^r.$$

This implies the existence of $\mu\in\mathbb{C}$ such that $F(x)=\mu(x-1)^n$ and thus $b_r=\mu(-1)^r\binom{n}{r}$. The result follows by defining

$$\lambda=\mu\frac{n!}{(k-s+n-1)!(\ell-t+n-1)!}.$$

□

We obtain the existence of the Rankin-Cohen operator for quasimodular forms in showing that the vector $\mathbf{a}$ we found in lemma 10 is admissible.

Lemma 11. *Let k,ℓ in $\mathbb{Z}_{>0}$, $s\in\{0,\dots,\lfloor k/2\rfloor\}$, $t\in\{0,\dots,\lfloor \ell/2\rfloor\}$ and $n\in\mathbb{Z}_{>0}$. Let $\mathbf{a}=(a_r)_{1\leq r\leq n}$ be defined by*

$$a_r=(-1)^r\binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r}.$$

Then

$$\Phi_{\mathbf{a}}\left(\widetilde{M}_k^{\leq s},\widetilde{M}_\ell^{\leq t}\right)\subset\widetilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. By lemma 9 it suffices to check that

(10)
$$\sum_{\substack{(r_1,r_2)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\\ r_1+r_2=n}}\frac{(-1)^{r_1}}{r_1!r_2!}\binom{r_1}{\alpha}\binom{r_2}{\beta}\binom{k-u-1+r_1}{s-u}\binom{\ell-v-1+r_2}{t-v}=0$$

for all $(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)$. Fix $(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)$, then (10) is the coefficient of order n in the product $P_1(X)P_2(X)$ where

$$P_1(X)=\sum_{r_1=0}^{+\infty}\frac{(-1)^{r_1}}{r_1!}\binom{r_1}{\alpha}\binom{k-u-1+r_1}{s-u}X^{r_1}$$

$$P_2(X)=\sum_{r_2=0}^{+\infty}\frac{1}{r_2!}\binom{r_2}{\beta}\binom{\ell-v-1+r_2}{t-v}X^{r_2}.$$

We have

$$P_1(X)=\frac{X^\alpha}{\alpha!}Q_1^{(\alpha)}(X)$$

with

$$Q_1(X)=\sum_{r_1=0}^{+\infty}\frac{(-1)^{r_1}}{r_1!}\binom{k-u-1+r_1}{s-u}X^{r_1}$$

and

$$Q_1(X) = \frac{X^{-k+s+1}}{(s-u)!} R_1^{(s-u)}(X)$$

with

$$\begin{aligned} R_1(X) &= \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!} X^{r_1+k-u-1} \\ &= X^{k-u-1} e^{-X}. \end{aligned}$$

We therefore may write $P_1(X) = \Pi_1(X)e^{-X}$ where Π_1 is a polynomial of degree $\alpha + s - u$. Similary, $P_2(X) = \Pi_2(X)e^X$ where Π_2 is a polynomial of degree $\beta + t - v$. It follows that P_1P_2 is a polynomial of degree $\alpha + \beta + s + t - u - v$. Finally, since, by definition, $\alpha + \beta - u - v < n - s - t$ we get (10). □

Remark 9. With the help of the hypergeometric methods [PWZ96, Chapter 3], we obtain that

$$\Pi_1(X) = (-1)^\alpha \sum_{r=\alpha}^{s-u+\alpha} \binom{k+\alpha-u-1}{k+r-s-1} \binom{r}{\alpha} \frac{X^r}{r!}$$

and

$$\Pi_2(X) = (-1)^\beta \sum_{r=\beta}^{t-v+\beta} (-1)^r \binom{\ell+\beta-v-1}{\ell+r-t-1} \binom{r}{\beta} \frac{X^r}{r!}$$

Previous lemmas prove theorem 1.

3. Rankin-Cohen brackets and derivation

In this section, we prove theorem 3. First, we remark that

$$\begin{aligned} \Phi_{n;k,s;\ell,t}(f,g)' = \sum_{r=0}^{n-1} (-1)^r \Bigg[&\binom{k-s+n-1}{n-r} \binom{\ell-t+n-1}{r} \\ &- \binom{k-s+n-1}{n-r-1} \binom{\ell-t+n-1}{r+1} \Bigg] f^{(r+1)} g^{(n-r)} \\ &+ \binom{k-s+n-1}{n} f g^{(n+1)} + (-1)^n \binom{\ell-t+n-1}{n} f^{(n+1)} g. \end{aligned} \tag{11}$$

Next,

$$\begin{aligned} \Phi_{n;k,s;\ell+2,t+1}(f,g') = &\binom{k-s+n-1}{n} f g^{(n+1)} \\ &- \sum_{r=0}^{n-1} (-1)^r \binom{k-s+n-1}{n-r-1} \binom{\ell-t+n}{r+1} f^{(r+1)} g^{(n-r)} \end{aligned}$$

so that

$$
\begin{aligned}
(12)\quad & \Phi_{n;k+2,s+1;\ell,t}(f',g)+\Phi_{n;k,s;\ell+2,t+1}(f,g') = \\
& \binom{k-s+n-1}{n} fg^{(n+1)} + (-1)^n \binom{\ell-t+n-1}{n} f^{(n+1)}g \\
& \quad + \sum_{r=0}^{n-1}(-1)^r \left[\binom{k-s+n}{n-r}\binom{\ell-t+n-1}{r} \right. \\
& \qquad \left. - \binom{k-s+n-1}{n-r-1}\binom{\ell-t+n}{r+1} \right] f^{(r+1)}g^{(n-r)}
\end{aligned}
$$

and equality from (11) and (12) follows by expanding the binomial coefficients.

4. A more precise structure result

In this section, we prove proposition 2. Let $n>0$. If $f\in \widetilde{M}_k^s$ and $g\in\widetilde{M}_\ell^t$ then $\Phi_{n;k,s;\ell,t}(f,g)$ has weight $k+\ell+2n$ and depth less than $s+t$. Since $n>0$ this depth is not maximal since

$$s+t\leq \frac{k}{2}+\frac{\ell}{2}<\frac{k+\ell+2n}{2}.$$

Then it follows from proposition 8 that

$$\Phi_{n;k,s;\ell,t}(f,g)\in M_{k+\ell+2n}\oplus\bigoplus_{j=1}^{s+t}\mathrm{D}^j\, M_{k+\ell+2n-2j}.$$

However, the definition of $\Phi_{n;k,s;\ell,t}(f,g)$ implies that its Fourier coefficient at 0 is 0 and since this is also true for derivatives of modular forms we get

$$\Phi_{n;k,s;\ell,t}(f,g)\in S_{k+\ell+2n}\oplus\bigoplus_{j=1}^{s+t}\mathrm{D}^j\, M_{k+\ell+2n-2j}.$$

The contribution to $\Phi_{n;k,s;\ell,t}(f,g)$ coming from

$$S_{k+\ell+2n}\oplus\bigoplus_{j=1}^{s+t-1}\mathrm{D}^j\, M_{k+\ell+2n-2j}$$

has depth strictly less than $s+t$. Hence

$$Q_{s+t}\left(\Phi_{n;k,s;\ell,t}(f,g)\right)=Q_{s+t}(\mathrm{D}^{s+t}\,g)$$

where $g\in M_{k+\ell+2n-2s-2t}$. Since

$$Q_{s+t}(\mathrm{D}^{s+t}\,g)=(2\pi\mathrm{i})^{-s-t}\frac{(k+\ell+2n-s-t-1)!}{(k+\ell+2n-2s-2t-1)!}g$$

(see (6)), to prove that g is parabolic we shall prove that the Fourier coefficient at 0 of $Q_{s+t}\left(\Phi_{n;k,s;\ell,t}(f,g)\right)$ is 0. From (8) we get

(13)
$$Q_{s+t}\left(\Phi_{n;k,s;\ell,t}(f,g)\right)=\sum_u\sum_v\sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2_{\geq 0}\\ \alpha+\beta=n+u+v-s-t}}\left(\frac{1}{2\pi\mathrm{i}}\right)^{n-\alpha-\beta}\sum_r a_r(r-\alpha)!(n-r-\beta)!$$
$$\times\binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{r-\alpha}\binom{\ell+n-r-v-1}{n-r-\beta}Q_u(f)^{(\alpha)}Q_v(g)^{(\beta)}.$$

Since derivatives of quasimodular forms have Fourier coefficients vanishing at 0, the only contribution to the Fourier coefficient of $Q_{s+t}\left(\Phi_{n;k,s;\ell,t}(f,g)\right)$ at 0 is given by $(\alpha,\beta)=(0,0)$ in (13). However, the summation is on (α,β) such that $\alpha+\beta=n+u+v-s-t$ and we have $n+u+v-s-t>0$ if $n>s+t$. Thanks to (13) we also see that if $f\in\widetilde{M}_k^{\leq s}$ and $g\in\widetilde{M}_\ell^{\leq t}$ satisfies $s+t>0$ and $\widehat{g}(0)=0$ then

$$\Phi_{s+t;k,s;\ell,t}(f,g)\in S_{k+\ell+2s+2t}\oplus\bigoplus_{j=1}^{s+t-1}\mathrm{D}^j M_{k+\ell+2s+2t-2j}\oplus\mathrm{D}^{s+t}S_{k+\ell}.$$

5. Applications

An easy but useful consequence of the fact that $\mathrm{D}\,\Delta=\Delta E_2$ is the following lemma.

Lemma 12. *Let $n\geq 0$. Let $f\in\widetilde{M}_k^{\leq s}$ and $g\in\widetilde{M}_\ell^{\leq t}$. There exists $h\in\widetilde{M}_{k+\ell+2n}^{\leq s+t}$ such that*
$$\Phi_{n;k,s;\ell,t}(f,\Delta g)=\Delta h.$$

For example, we have
$$\Phi_{1;k+12,s;12,0}(\Delta f,\Delta)=\Delta\Phi_{1;k,s;12,0}(f,\Delta).$$

5.1. **Homogeneous products of derivatives of E_2.** In this section we prove proposition 4 by recursion on n. For $n=0$ we have $E_2^2=E_4+12\,\mathrm{D}\,E_2\in\mathbb{C}E_4\oplus\mathbb{C}\,\mathrm{D}\,E_2$. Assume that:
$$\mathrm{D}^r E_2\,\mathrm{D}^{n-r}E_2\in\bigoplus_{\substack{j=0\\ j\equiv n\pmod 2}}^{n-4}\mathrm{D}^j S_{2n+4-2j}\oplus\mathbb{C}\,\mathrm{D}^n E_4\oplus\mathbb{C}\,\mathrm{D}^{n+1}E_2\quad(0\leq r\leq n).$$

Deal first with the case where $n=2m$ is even. By recursion hypothesis, we have
$$\begin{aligned}\mathrm{D}\left(\mathrm{D}^r E_2\,\mathrm{D}^{n-r}E_2\right)&=\mathrm{D}^r E_2\,\mathrm{D}^{n+1-r}E_2+\mathrm{D}^{r+1}E_2\,\mathrm{D}^{n-r}E_2\\&\in\bigoplus_{\substack{j=0\\ j\equiv n\pmod 2}}^{n-4}\mathrm{D}^{j+1}S_{2n+4-2j}\oplus\mathbb{C}\,\mathrm{D}^{n+1}E_4\oplus\mathbb{C}\,\mathrm{D}^{n+2}E_2.\end{aligned}$$

The set $\{\mathrm{D}^r E_2\,\mathrm{D}^{n-r}E_2,\ 0\leq r\leq n\}$ has $m+1$ distinct terms (corresponding to $0\leq r\leq m$). The set $\{\mathrm{D}^r E_2\,\mathrm{D}^{n+1-r}E_2,\ 0\leq r\leq n+1\}$ has also $m+1$

distinct terms (corresponding to $0 \leq r \leq m$). It follows that

$$\left\{\mathrm{D}^r E_2 \,\mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \,\mathrm{D}^{n-r} E_2,\ r \in \{0,\dots,m\}\right\}$$

and

$$\left\{\mathrm{D}^r E_2 \,\mathrm{D}^{n+1-r} E_2,\ r \in \{0,\dots,m\}\right\}$$

are basis of the same space with change of basis matrix given by

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}.$$

It follows that for any $r \in \{0,\dots,m\}$ (hence any $r \in \{0,\dots,n\}$) we have

$$\mathrm{D}^r E_2 \,\mathrm{D}^{n+1-r} E_2 \in \bigoplus_{\substack{j=0 \\ j \equiv n+1 \pmod 2}}^{n-3} \mathrm{D}^j S_{2n+6-2j} \oplus \mathbb{C}\mathrm{D}^{n+1} E_4 \oplus \mathbb{C}\mathrm{D}^{n+2} E_2.$$

We now deal with the case where $n = 2m-1$ is odd. Again, by recursion hypothesis, we have

$$\begin{aligned}\mathrm{D}\left(\mathrm{D}^r E_2 \,\mathrm{D}^{n-r} E_2\right) &= \mathrm{D}^r E_2 \,\mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \,\mathrm{D}^{n-r} E_2 \\ &\in \bigoplus_{\substack{j=0 \\ j \equiv n \pmod 2}}^{n-4} \mathrm{D}^{j+1} S_{2n+4-2j} \oplus \mathbb{C}\mathrm{D}^{n+1} E_4 \oplus \mathbb{C}\mathrm{D}^{n+2} E_2.\end{aligned}$$

The subspace generated by all the quasimodular forms $\mathrm{D}^r E_2 \,\mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \,\mathrm{D}^{n-r} E_2$ when r runs over $\{0,\dots,2m-1\}$ is the hyperplane

$$\left\{\sum_{r=0}^{2m} \alpha_r \,\mathrm{D}^r E_2 \,\mathrm{D}^{2m-r} E_2 \middle| \sum_{r=0}^{2m} (-1)^r \alpha_r = 0\right\}$$

hence it is sufficient for the proof of our recursion step to find a linear combination

$$\sum_{r=0}^{2m} \alpha_r \,\mathrm{D}^r E_2 \,\mathrm{D}^{2m-r} E_2 \in \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{2m-4} \mathrm{D}^j S_{4m+4-2j} \oplus \mathbb{C}\mathrm{D}^{2m} E_4 \oplus \mathbb{C}\mathrm{D}^{2m+1} E_2$$

with

$$\sum_{r=0}^{2m} (-1)^r \alpha_r \neq 0.$$

This is the step where we use Rankin-Cohen brackets. Since $[E_2, E_2]_{2m+2} \in \widetilde{M}_{4m+8}^{\leq 2}$ we have $Q_2\left([E_2, E_2]_{2m+2}\right) \in S_{4m+4}$ (see (13) for the cuspidality).

Equation (8) combined with the fact that $Q_1(E_2)$ is constant implies that

$$(14)\quad Q_2\left([E_2,E_2]_{2m+2}\right) = \frac{24}{(2\pi\mathrm{i})^2}(2m+2)\,\mathrm{D}^{2m+1}E_2 + \frac{4}{(2\pi\mathrm{i})^2}\Bigg[\sum_{r=2}^{2m+2}(-1)^r\binom{2m+2}{r}^2\binom{r}{2}\binom{r+1}{2}\mathrm{D}^{r-2}E_2\,\mathrm{D}^{2m+2-r}E_2 + \sum_{r=1}^{2m+1}(-1)^r\binom{2m+2}{r}^2\binom{r+1}{2}\binom{2m+3-r}{2}\mathrm{D}^{r-1}E_2\,\mathrm{D}^{2m+1-r}E_2\Bigg].$$

Let

$$\alpha_r(N) = 2(-1)^r\binom{r}{2}\binom{N}{r}\binom{N}{r-1}(N+1-2r).$$

Equation (14) gives

$$\sum_{r=2}^{2m+2}\alpha_r(2m+2)\,\mathrm{D}^{r-2}E_2\,\mathrm{D}^{2m+2-r}E_2 = (2\pi\mathrm{i})^2Q_2\left([E_2,E_2]_{2m+2}\right) - 24(2m+2)\,\mathrm{D}^{2m+1}E_2 \in S_{4m+4}\oplus\mathbb{C}\mathrm{D}^{2m+1}E_2.$$

Let $\beta_r(N) = (-1)^r\alpha_r(N)$. We prove that

$$A(N) = \sum_{r=2}^{N}(-1)^r\alpha_r(N) = \sum_{r\in\mathbb{Z}}\beta_r(N)$$

is strictly negative (hence differs from 0). Zeilberger's algorithm (*e.g.*, on the open-source computer algebra system Maxima) [PWZ96, Chapter 6] provides a function $K(N,r)$ such that[2]

$$2(N+1)(2N-1)\beta_r(N) - N(N-1)\beta_r(N+1) = K(N,r+1)\beta_{r+1}(N) - K(N,r)\beta_r(N).$$

More precisely

$$(15)\quad K(N,r) = \frac{(r-2)(r-1)(N+1)[3N^3+8N^2(1-r)+N(4r^2-6r+3)-2r^2+4r-2]}{(N-2r+1)(N-r+1)(N-r+2)(N-1)}.$$

We deduce the recursive formula

$$\frac{A(N+1)}{A(N)} = \frac{2(N+1)(2N-1)}{N(N-1)}$$

which, since $A(2)=4$, implies

$$A(N) = -N(N-1)\binom{2N-2}{N-1} < 0.$$

Finally, we have found a function which belongs to the hyperplane. This completes the proof.

[2] Note that no algorithm is needed to check that $K(N,r)$ as defined in (15) works.

5.2. **Niebur formula.** From proposition 4 we obtain

$$\Phi_{4;2,1;2,1}(E_2, E_2) \in S_{12} = \mathbb{C}\Delta.$$

The computation of the first coefficients gives $\Phi_{4;2,1;2,1}(E_2, E_2) = -48\Delta$. This is the differential equation proved by Niebur in [Nie75] :

$$2^3 \cdot 3\Delta = 18(\mathrm{D}^2 E_2)^2 + E_2 \mathrm{D}^4 E_2 - 16 \mathrm{D} E_2 \mathrm{D}^3 E_2$$

and comparing the Fourier expansions gives Niebur formula.

5.3. **van der Pol formula.** From proposition 2 we obtain

$$\Phi_{1;4,0;6,1}(E_4, \mathrm{D} E_4) \in S_{12}.$$

The computation of the first coefficient gives $\Phi_{1;4,0;6,1}(E_4, \mathrm{D} E_4) = 960\Delta$. This is the differential equation proved by van der Pol:

$$4E_4 \mathrm{D}^2 E_4 - 5(\mathrm{D} E_4)^2 = 960\Delta.$$

It leads to

$$\begin{aligned}
\tau(n) &= n^2\sigma_3(n) + 60 \sum_{a+b=n} (4b-5a)b\sigma_3(a)\sigma_3(b) \\
&= n^2\sigma_3(n) + 60 \sum_{a=1}^{n-1} (9a^2 - 13an + 4n^2)\sigma_3(a)\sigma_3(n-a) \\
&= n^2\sigma_3(n) + 60 \sum_{b=1}^{n-1} (9b^2 - 5bn)\sigma_3(a)\sigma_3(n-a)
\end{aligned}$$

and the summation of the two last equalities implies the van der Pol formula in its original form [vdP51, eq. (53)]:

$$\tau(n) = n^2\sigma_3(n) + 60 \sum_{a=1}^{n-1} (2n-3a)(n-3a)\sigma_3(a)\sigma_3(n-a).$$

5.4. **Chazy equation.** Recall that we proved at the end of the introduction that an equation of the shape

$$\alpha E_2 \mathrm{D}^2 E_2 + \beta(\mathrm{D} E_2)^2 = \mathrm{D}^3 E_2$$

has to exist. Coefficients α and β can be computed by identifications of the first Fourier coefficients. Our aim in this section is to give an interpretation of this equation in terms of Rankin-Cohen brackets. We have

$$\Phi_{1;2,1;12,0}(E_2, \Delta) \in \Delta \widetilde{M}_4^{\leq 1} = \mathbb{C}\Delta E_4$$

hence

$$\Phi_{1;2,1;12,0}(E_2, \Delta) = \Delta E_4$$

and

$$\Phi_{1;4,0;12,0}(E_4, \Delta) \in \Delta M_6 = \mathbb{C}\Delta E_6$$

hence

$$\Phi_{1;4,0;12,0}(E_4, \Delta) = 4\Delta E_6$$

so that

$$\Phi_{1;16,0;12,0}\left(\Phi_{1;2,1;12,0}(E_2, \Delta), \Delta\right) = \Delta \Phi_{1;4,0;12,0}(E_4, \Delta) = 4\Delta^2 E_6.$$

Next we compute

$$\Phi_{1;30,0;12,0}(\Delta^2 E_6, \Delta) = \Delta^2 \Phi_{1;6,0;12,0}(E_6, \Delta) \in \Delta^3 M_8 = \mathbb{C}\Delta^3 E_4^2$$

hence

$$\Phi_{1;30,0;12,0}(\Delta^2 E_6, \Delta) = 6\Delta^3 E_4^2 = 6\Delta \Phi_{1;2,1;12,0}(E_2, \Delta)^2$$

and

$$\Phi_{1;30,0;12,0}\left(\Phi_{1;16,0;12,0}\left(\Phi_{1;2,1;12,0}(E_2, \Delta), \Delta\right), \Delta\right) = 24\Delta \Phi_{1;2,1;12,0}(E_2, \Delta)^2.$$

This is (2). We deduce the usual form of the Chazy equation in the following way. From

$$K := \Phi_{1;2,1;12,0}(E_2, \Delta) = E_2 \operatorname{D} \Delta - 12 \operatorname{D} E_2 \Delta = \Delta(E_2^2 - 12 \operatorname{D} E_2)$$

we get

$$L := \Phi_{1;16,0;12,0}(K, \Delta) = 16K\Delta - 12 \operatorname{D} K\Delta = 4\Delta^2 (E_2^3 - 18 E_2 \operatorname{D} E_2 + 36 \operatorname{D}^2 E_2)$$

and since

$$\begin{aligned}\Phi_{1;30,0;12,0}(L, \Delta) &= 30L \operatorname{D} \Delta - 12 \operatorname{D} L\Delta \\ &= 24\Delta^3 \left(E_2^4 - 24E_2^2 \operatorname{D} E_2 + 72 E_2 \operatorname{D}^2 E_2 + 36(\operatorname{D} E_2)^2 - 72 \operatorname{D}^3 E_2\right)\end{aligned}$$

the equality $\Phi_{1;30,0;12,0}(L, \Delta) = 24\Delta K^2$ gives the Chazy equation

$$2 \operatorname{D}^3 E_2 - 2E_2 \operatorname{D}^2 E_2 + 3(\operatorname{D} E_2)^2 = 0.$$

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