Prolate Spheroidal Wave Functions In q-Fourier Analysis

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Abstract

In this paper we introduce a new version of the Prolate spheroidal wave function using standard methods of q-calculus and we formulate some of its properties. As application we give a q-sampling theorem which extrapolates functions defined on q^n and $0 < q < 1$.

Keywords : q-Prolate spheroidal wave function, q-sampling,

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1 Introduction

The prolate spheroidal wave functions, which are a special case of the spheroidal wave functions, possess a very surprising and unique property [7]. They are an orthogonal basis of both $L^2(-1,1)$ and the Paley-Wiener space of bandlimited functions. They also satisfy a discrete orthogonality relation. No other system of classical orthogonal functions is known to possess this strange property. We prove that there are new systems possessing this property in q-Fourier analysis. In the following we discuss some properties of the q-Prolate spheroidal wave function using news developments and technics in q-Fourier analysis. In particular we prove that these functions forms an orthogonal basis of the q-Paley-Wiener space $PW^v_{q,a}$. Finally and as application we give a constructive q-sampling formula having as sampling points q^n where $n \in \mathbb{Z}$. In the end, we cit the reference [1], where the reproducing kernel for the q-Paley-Wiener space was already discussed, and the explicit formula for the kernel was given, similar to the formula in Remark 3. However, the paper [1] proceeds with a q-sampling theorem which extrapolates functions defined on the zeros of the q-Bessel function. These zeros are given in the following form

$$\left\{q^{-n+\epsilon_n}\right\}_{n\in\mathbb{N}},$$

where $0 < \epsilon_n < 1$, but it is not explicitly evaluated.

2 Preliminary

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [3]. We put

$$\mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

and if $a = q^n, \quad n \in \mathbb{Z}$ put

$$[0, a]_q = \{q^s, \quad s \in \mathbb{Z}, \quad s \geq n\}.$$

For complex z, let

$$(z; q)_0 = 1, \quad (z; q)_n = \prod_{i=0}^{n-1}(1 - zq^i), \quad n = 1...\infty.$$

Jackson's q-integral in the interval $[0, a]$ and in the interval $[0, \infty[$ are defined, respectively, by(see [4])

$$\int_0^a f(x)d_qx = (1 - q)a\sum_{n=0}^{\infty} q^n f(aq^n),$$

$$\int_0^{\infty} f(x)d_qx = (1 - q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

For $v > -1$, let $\mathcal{L}_{q,p,v}$ be the space of even functions f defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p,v} = \left[\int_0^{\infty} |f(x)|^p x^{2v+1} d_qx\right]^{1/p} < \infty.$$

The set $\mathcal{L}_{q,2,v}$ is an Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^{\infty} f(t)g(t)t^{2v+1}d_qt.$$

We consider $\mathcal{L}_{q,v,a}$ the space of function defined on $[0,a]_q$ which satisfies

$$\int_0^a |f(x)|^2 x^{2v+1} d_q x < \infty,$$

and $\mathcal{L}_{q,a}^v$ the subspace of $\mathcal{L}_{q,2,v}$ given by the natural embedding of $\mathcal{L}_{q,v,a}$ in $\mathcal{L}_{q,2,v}$.

The normalized Hahn-Exton q-Bessel function of order $v > -1$ (see [6]) is defined by

$$j_v(z,q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q,q)_n (q^{v+1},q)_n} z^{2n}.$$

It is an entire analytic function in z.

Proposition 1 *For $\Re(v) > -1, a > 0$ and $y, z \in \mathbb{C}\backslash\{0\}$ we have*

$$\int_0^a j_v(yt,q^2) j_v(zt,q^2) t^{2v+1} d_q t$$
$$= \frac{1-q}{1-q^{2v+2}} a^{2v+2} \quad \frac{y^2 j_{v+1}(ay,q^2) j_v(aq^{-1}z,q^2) - z^2 j_{v+1}(az,q^2) j_v(aq^{-1}y,q^2)}{y^2 - z^2}.$$

Proof. See [5] (Proposition 1.3) ■

The following results in this section were proved in [2].

Proposition 2

$$|j_v(q^n,q^2)| \leq \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \begin{cases} 1 & if \quad n \geq 0 \\ q^{n^2+(2v+1)n} & if \quad n < 0 \end{cases}.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ introduced in [2],[4] as follow

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt,q^2) t^{2v+1} d_q t,$$

where

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

The $q-$Bessel translation operator is defined as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt,q^2) j_v(yt,q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q, \forall f \in \mathcal{L}_{q,1,v},$$

Recall that $T^v_{q,x}$ is said positive if $T^v_{q,x}f \geq 0$ for $f \geq 0$. In the following we tack $q \in Q_v$ where

$$Q_v = \{q \in]0,1[, \quad T^v_{q,x} \quad \text{is positive for all} \quad x \in \mathbb{R}_q\}.$$

The $q-$convolution product of both functions $f, g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y.$$

Theorem 1 *The operator $\mathcal{F}_{q,v}$ satisfying*

1. *For all functions $f \in \mathcal{L}_{q,2,v}$, $\quad \mathcal{F}^2_{q,v} f(x) = f(x), \quad \forall x \in \mathbb{R}_q$.*
2. *For all functions $f, g \in \mathcal{L}_{q,2,v}$, $\quad \langle \mathcal{F}_{q,v} f, g \rangle = \langle f, \mathcal{F}_{q,v} g \rangle$.*
3. *For all functions $f \in \mathcal{L}_{q,2,v}$, $\quad \|\mathcal{F}_{q,v} f\|_{q,v,2} = \|f\|_{q,v,2}$.*
4. *For all functions $f, g \in \mathcal{L}_{q,1,v}$,*

$$\mathcal{F}_{q,v}(f *_q g)(x) = \mathcal{F}_{q,v} f(x) \times \mathcal{F}_{q,v} g(x), \quad \forall x \in \mathbb{R}_q.$$

In the end we consider $PW^v_{q,a}$ the q-Paley Wiener space

$$PW^v_{q,a} = \left\{ f(x) = \int_0^a u(t) j_v(xt, q^2) t^{2v+1} d_q t, \quad u \in \mathcal{L}^v_{q,a} \right\},$$

the set of q-bandlimited signal.

3 Main Results

We introduce the q-analogue of the Prolate Spheroidal Wave Functions ψ_i as the eigenfunction of the integral operator T^v_a acting on the Hilbert space $\mathcal{L}_{q,v,a}$ as follows

$$T^v_a u(x) = c_{q,v} \int_0^a u(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

then we have

$$T^v_a \psi_i = \lambda_i \psi_i.$$

It's easy to see that the operator T^v_a is symmetric and compact

$$\int_0^a T^v_a u(t) w(t) t^{2v+1} d_q t = \int_0^a u(t) T^v_a w(t) t^{2v+1} d_q t,$$

then the sequence $\{\psi_i\}_{i \in \mathbb{N}}$ forme an orthogonal basis of the Hilbert space $\mathcal{L}_{q,v,a}$ and any eigenvalue λ_i is real.

Proposition 3 *The sequence of eigenvalue* $\{\lambda_i\}_{i\in\mathbb{N}}$ *satisfying*

$$\lambda_0^2 \geq \lambda_1^2 \geq \ldots > 0.$$

Proof. The operator T_a^v is compact, then the spectrum is a countably infinite subset of $\mathbb{R}$ (T_a^v is symmetric) which has 0 as its only limit point. If we denote by

$$\Lambda = \{\lambda_0, \lambda_1, \ldots\},$$

the spectrum of T_a^v then we can write

$$|\lambda_0| \geq |\lambda_1| \geq \ldots \geq 0.$$

To finish the proof, if suffice to prove that $0 \notin \Lambda$. In fact if $T_a^v \psi = 0$ then $\mathcal{F}_{q,v}\psi$ is an entire function which vanishes on $[0, a]_a$. By the identity theorem for analytic functions, $\mathcal{F}_{q,v}\psi = 0$ everywhere and thus $\psi = 0$. ■

Remark 1 *Consider the operator*

$$k_a^v = T_a^v \circ T_a^v,$$

then K_a^v *is an integral operator acting on the Hilbert space* $\mathcal{L}_{q,v,a}$ *as follows*

$$k_a^v u(x) = \int_0^a u(y) k(x, y) y^{2v+1} d_q y,$$

where

$$k(x, y) = c_{q,v}^2 \int_0^a j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} t^{2v+1} d_q t.$$

The function ψ_i *is an eigenfunction of* k_a^v

$$k_a^v \psi_i = \lambda_i^2 \psi_i.$$

Lemma 1 *The function* ψ_i *initially defined on* $\mathbb{R}_q$ *can be extended as an analytic function on* $\mathbb{C}$.

Proof. The result follows from the relation

$$\psi_i(z) = \frac{1}{\lambda_i} c_{q,v} \int_0^a \psi_i(t) j_v(zt, q^2) t^{2v+1} d_q t,$$

and the fact that $j_v(., q^2)$ is an entire function. ■

Proposition 4 *The function* ψ_i *belonging to the Paley-Wiener space* $PW_{q,a}^v$

Proof. Let

$$\phi_i(x) = \frac{1}{\lambda_i}\psi_i(x)\chi_{[0,a]}(x),$$

then

$$\begin{aligned}\mathcal{F}_{q,v}\phi_i(x) &= c_{q,v}\int_0^\infty \phi_i(t)j_v(xt,q^2)t^{2v+1}d_qt \\ &= \frac{c_{q,v}}{\lambda_i}\int_0^a \psi_i(t)j_v(xt,q^2)t^{2v+1}d_qt = \psi_i(x),\end{aligned}$$

which implies that $\psi_i \in PW^v_{q,a}$. ■

In the following we assume that

$$\|\psi_i\|^2_{q,2,v} = \langle\psi_i,\psi_i\rangle = 1.$$

Proposition 5 *The sequence* $\{\psi_i\}_{i\in\mathbb{N}}$ *forme an orthonormal basis of* $PW^v_{q,a}$.

Proof. The q-Bessel Fourier transform

$$\mathcal{F}_{q,v} : \mathcal{L}^v_{q,a} \to PW^v_{q,a},$$

define an isomorphism, and the sequence $\{\phi_i\}_{i\in\mathbb{N}}$ form an orthogonal basis of the Hilbert space $\mathcal{L}^v_{q,a}$, which lead to the result. ■

Proposition 6 *Let*

$$k_x : y \mapsto k(x,y),$$

then

$$f \in PW^v_{q,a} \Leftrightarrow f(x) = \langle f, k_x\rangle, \quad \forall x \in \mathbb{R}_q.$$

Proof. Let

$$\sigma_a(y) = \mathcal{F}_{q,v}\left(\chi_{[0,a]}\right)(x) = c_{q,v}\int_0^a j_v(ty,q^2)t^{2v+1}d_qt,$$

therefore

$$T^v_{q,x}\sigma_a(y) = c_{q,v}\int_0^a j_v(tx,q^2)j_v(ty,q^2)t^{2v+1}d_qt = \frac{1}{c_{q,v}}k(x,y),$$

and then

$$\begin{aligned}f \in PW^v_{q,a} &\Leftrightarrow \mathcal{F}_{q,v}f(x) = \mathcal{F}_{q,v}f(x)\chi_{[0,a]}(x) = \mathcal{F}_{q,v}f(x)\mathcal{F}_{q,v}\sigma_a(x) \\ &\Leftrightarrow f(x) = f *_q \sigma_a(x) = c_{q,v}\langle f, T^v_{q,x}\sigma_a\rangle = \langle f, k_x\rangle.\end{aligned}$$

This finish the proof ■

Corollary 1 *We have*

$$k(x,y) = \sum_{i=0}^{\infty} \psi_i(x)\psi_i(y), \quad \forall x, y \in \mathbb{R}_q.$$

Proof. In fact $k_x \in PW_{q,a}^v$. Then

$$k_x(y) = \sum_{i=0}^{\infty} \langle k_x, \psi_i \rangle \psi_i(y).$$

On the other hand

$$\psi_i \in PW_{q,a}^v \Leftrightarrow \langle \psi_i, k_x \rangle = \psi_i(x),$$

which prove the result. ∎

Lemma 2 *For* $i, j \in \mathbb{N}$

$$\int_0^a \psi_i(x)\psi_j(x)x^{2v+1}d_qx = \lambda_i\lambda_j\delta_{ij}.$$

Proof. In fact

$$\langle \phi_i, \phi_j \rangle = \langle \mathcal{F}_{q,v}\phi_i, \mathcal{F}_{q,v}\phi_j \rangle = \langle \psi_i, \psi_j \rangle,$$

and

$$\langle \phi_i, \phi_j \rangle = \frac{1}{\lambda_i\lambda_j} \int_0^a \psi_i(x)\psi_j(x)x^{2v+1}d_qx.$$

On the other hand, if $i \neq j$ then

$$\langle \phi_i, \phi_j \rangle = \int_0^a \phi_i(t)\phi_j(t)t^{2v+1}d_qt = 0.$$

Moreover, $\|\phi_i\|_{q,2,v} = \|\psi_i\|_{q,2,v} = 1$ which prove that $\langle \phi_i, \phi_j \rangle = \delta_{ij}$. This leads to the result. ∎

In order to be more precise about what it means for the energy of a q-bandlimited single $f \in PW_{q,a}^v$ to be mainly concentrated on the interval $[0,a]_q$, we consider the concentration index:

$$\theta_a^v f = \frac{\int_0^a f(x)^2 x^{2v+1} d_qx}{\|f\|_{q,v,2}^2},$$

whose values range from 0 to 1.

Proposition 7 *The maximum value of $\theta_a^v f$ is attained for $f = \psi_0$ and*

$$\theta_a^v f = \frac{\sum_{i=0}^{n} \lambda_i^2 \langle f, \psi_i\rangle^2}{\sum_{i=0}^{n} \langle f, \psi_i\rangle^2} \geq \lambda_n^2, \quad \text{if} \quad f \in span\{\psi_0, \dots, \psi_n\},$$

$$\theta_a^v f = \frac{\sum_{i=n+1}^{\infty} \lambda_i^2 \langle f, \psi_i\rangle^2}{\sum_{i=n+1}^{\infty} \langle f, \psi_i\rangle^2} \leq \lambda_{n+1}^2, \quad \text{if} \quad f \in span\{\psi_0, \dots, \psi_n\}^{\perp}.$$

Proof. With the Parseval equality

$$\int_0^a f(x)^2 x^{2v+1} d_q x = \sum_{i=0}^{\infty} \langle f, \phi_i\rangle^2,$$

and the fact that

$$\begin{aligned}\sum_{i=0}^{\infty} \langle f, \phi_i\rangle^2 &= \sum_{i=0}^{\infty} \langle \mathcal{F}_{q,v} f, \psi_i\rangle^2 \\ &= \sum_{i=0}^{\infty} \lambda_i^2 \langle \mathcal{F}_{q,v} f, \phi_i\rangle^2 = \sum_{i=0}^{\infty} \lambda_i^2 \langle f, \psi_i\rangle^2,\end{aligned}$$

$$\|f\|_{q,v,2}^2 = \sum_{i=0}^{\infty} \langle f, \psi_i\rangle^2,$$

We get

$$\theta_a^v f = \frac{\sum_{i=0}^{\infty} \lambda_i^2 \langle f, \psi_i\rangle^2}{\sum_{i=0}^{\infty} \langle f, \psi_i\rangle^2} \leq \lambda_0^2 = \theta_a^v \psi_0,$$

which leads to the result. ■

Remark 2 *If $b > a$ then*

$$PW_{q,a}^v \subset PW_{q,b}^v,$$

Now let $\{\mu_n\}_{n\in\mathbb{Z}}$ the sequence of eigenvalues of the operator T_b^v then we have

$$\lambda_0^2 = \theta_a^v \psi_0 \leq \theta_b^v \psi_0 \leq \mu_0^2.$$

Proposition 8 *The q-Paley-Wiener space $PW_{q,a}^v$ is a closed subspace of $\mathcal{L}_{q,2,v}$.*

Proof. First we show that $PW_{q,a}^v$ is a subspace of $\mathcal{L}_{q,2,v}$. In fact let

$$f \in PW_{q,a}^v$$

then there exist $u \in \mathcal{L}_{q,a}^v$ such that

$$f(x) = c_{q,v} \int_0^a u(t) j_v(xt, q^2) t^{2v+1} d_q t = \mathcal{F}_{q,v}(u)(x).$$

As $\mathcal{L}_{q,a}^v \subset \mathcal{L}_{q,2,v}$ and from the Theorem 1 we show that $\mathcal{F}_{q,v}(u) \in \mathcal{L}_{q,2,v}$ which implies

$$PW_{q,a}^v \subset \mathcal{L}_{q,2,v}.$$

Now, given $f \in \mathcal{L}_{q,2,v}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of element of $PW_{q,a}^v$ which converge to f in L^2-norm. For $n \in \mathbb{N}$, there exist $u_n \in \mathcal{L}_{q,a}^v$ such that

$$f_n(x) = c_{q,v} \int_0^a u_n(t) j_v(xt, q^2) t^{2v+1} d_q t.$$

Moreover

$$\lim_{n\to\infty} \|f_n - f\|_{q,2,v} = 0,$$

this give

$$\lim_{n\to\infty} \|\mathcal{F}_{q,v} f_n - \mathcal{F}_{q,v} f\|_{q,2,v} = 0,$$

and then

$$\int_0^a |\mathcal{F}_{q,v} f_n(x) - \mathcal{F}_{q,v} f(x)|^2 x^{2v+1} d_q x + \int_a^\infty |\mathcal{F}_{q,v} f(x)|^2 x^{2v+1} d_q x \to 0,$$

which implies $\mathcal{F}_{q,v} f(x) = 0$ if $x \in \mathbb{R}_q$ and $x > a$ and then $f \in PW_{q,a}^v$. ■

Theorem 2 *For any function $f \in PW_{q,a}^v$ we have*

$$f(z) = (1-q) \sum_{k\in\mathbb{Z}} q^{2k(v+1)} f(q^k) k_z(q^k), \quad \forall z \in \mathbb{C}. \tag{1}$$

Proof. In fact f is an analytic function, and from Proposition 6

$$f(x) = \langle f, k_x \rangle, \quad \forall x \in \mathbb{R}_q.$$

We have

$$\begin{aligned} \langle f, k_x \rangle &= \langle \mathcal{F}_{q,v} f, \mathcal{F}_{q,v} k_x \rangle = c_{q,v} \langle \mathcal{F}_{q,v} f, j_v(x., q^2) \chi_{[0,a]} \rangle \\ &= c_{q,v} \int_0^a \mathcal{F}_{q,v} f(t) j_v(xt, q^2) t^{2v+1} d_q t. \end{aligned}$$

which prove that

$$z \mapsto \langle f, k_z \rangle,$$

is an analytic function. On the other hand

$$\langle f, k_z \rangle = (1-q) \sum_{k \in \mathbb{Z}} q^{2k(v+1)} f(q^k) k_z(q^k),$$

and

$$\langle f, k_{q^k} \rangle = f(q^k), \quad \forall k \in \mathbb{Z}.$$

As $\{0\}$ is an accumulation point of the following set

$$\{q^k, \quad k \in \mathbb{Z}\},$$

we conclude that $\langle f, k_z \rangle = f(z), \quad \forall z \in \mathbb{C}.$ ■

Remark 3 *In many fields , telecommunication in particular, the Whittaker-Shannon-Kotel'nikov sampling theorem plays a central role. It is know that sampling is the process of converting a signal (e.g., a function of continuous time or space) into a numeric sequence (a function of discrete time or space). Namely this theorem say that every function in the cosine Paley-Wiener space:*

$$PW_a^{-\frac{1}{2}} = \left\{ f(x) = \sqrt{\frac{2}{\pi}} \int_0^a u(t) \cos(xt) dt, \quad u \in L^2[0,a] \right\},$$

can be written as

$$f(x) = \sqrt{\frac{2}{\pi}} \sum_{n \in \mathbb{Z}} f\left(\frac{\pi}{a} n\right) \frac{\sin(ax - \pi n)}{ax - \pi n}.$$

Then the above theorem can be viewed as a sampling formula where the sampling points are q^n independent of a. By the use of Proposition 1 we get

$$k_z(q^n) = \frac{(1-q)c_{q,v}^2}{1-q^{2v+2}} a^{2v+2} \times \frac{q^{2n} j_{v+1}(aq^n, q^2) j_v(aq^{-1}z, q^2) - z^2 j_{v+1}(az, q^2) j_v(aq^{-1+n}, q^2)}{q^{2n} - z^2}.$$

Proposition 9 *Given a function $f \in \mathcal{L}_{q,2,v}$ and let*

$$f_a(x) = \langle f, k_x \rangle,$$

then

$$f_a \in PW_{q,a}^v,$$

and for all $\delta > 0$ we have

$$\lim_{a \to \infty} \sup_{x > \delta, x \in \mathbb{R}_q} |f(x) - f_a(x)| = 0.$$

Proof. First

$$|f_a(x)| \leq \|f\|_{q,v,2}\|k_x\|_{q,v,2} < \infty.$$

Now we can write

$$\begin{aligned} f_a(x) = \langle f, k_x\rangle = \langle \mathcal{F}_{q,v}f, \mathcal{F}_{q,v}k_x\rangle &= c_{q,v}\langle \mathcal{F}_{q,v}f, j_v(x., q^2)\chi_{[0,a]}\rangle \\ &= c_{q,v}\int_0^a \mathcal{F}_{q,v}f(t)j_v(xt, q^2)t^{2v+1}d_qt. \end{aligned}$$

which prove that $f_a \in PW^v_{q,a}$. On the other hand

$$f(x) = c_{q,v}\langle \mathcal{F}_{q,v}f, j_v(x., q^2)\rangle,$$

and therefore

$$\begin{aligned} |f(x) - f_a(x)|^2 &= c_{q,v}^2\left|\int_a^\infty \mathcal{F}_{q,v}f(t)j_v(xt, q^2)t^{2v+1}d_qt\right|^2 \\ &\leq c_{q,v}^2\left(\int_a^\infty |\mathcal{F}_{q,v}f(t)||j_v(xt, q^2)|t^{2v+1}d_qt\right)^2 \\ &\leq c_{q,v}^2\int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}d_qt\int_a^\infty |j_v(xt, q^2)|^2t^{2v+1}d_qt \\ &\leq \frac{c_{q,v}^2}{x^{2v+2}}\int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}d_qt\int_{ax}^\infty |j_v(t, q^2)|^2t^{2v+1}d_qt \\ &\leq \frac{c_{q,v}^2\|j_v(., q^2)\|_{q,v,2}^2}{x^{2v+2}}\int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}d_qt. \end{aligned}$$

Using the fact that

$$\int_0^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}d_qt = \|\mathcal{F}_{q,v}f\|_{q,v,2}^2 = \|f\|_{q,v,2}^2 < \infty,$$

we finish the proof. ■

4 Application

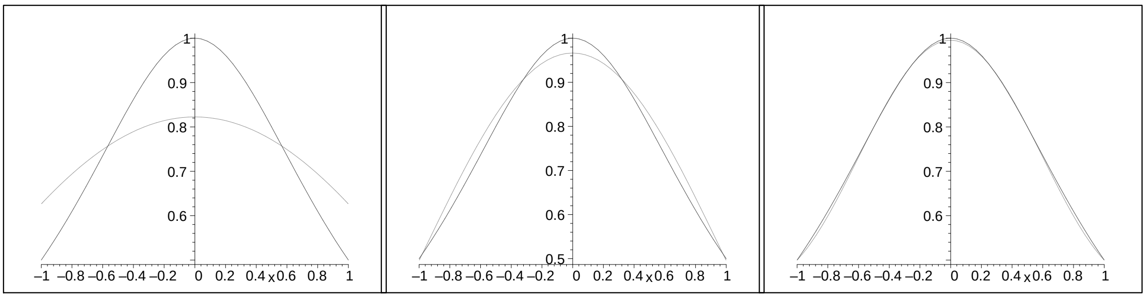
In this section we tack $v = -1/2$ and $q = 0.5$ and we put

$$f(x) = \frac{1}{1+x^2},$$

an even function belong to the space $\mathcal{L}_{q,2,v}$. Using the sampling formula (1) for the function $f_a(x) = \langle f, k_x\rangle$ respectively for $a = 1$, $a = 1/q$ and $a = 1/q^2$ with sampling point

$$q^n, \quad n = -1\ldots 10$$

we obtain



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