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Groups of finite Morley rank with solvable local subgroups

Adrien Deloro*and Eric Jaligot†

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Abstract

We lay down the foundations of the theory of groups of finite Morley rank in which local subgroups are solvable and we proceed to the local analysis of these groups. We prove a main Uniqueness Theorem, analogous to the Bender method in finite group theory, and derive its corollaries. We also consider homogeneous cases and study torsion.

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1 Introduction

In the Classification of the Finite Simple Groups [GLS94], the study of minimal simple groups has been a fundamental minimal case for the whole process. The local analysis of these finite simple groups, in which each proper subgroup is solvable, has been delineated by J. Thompson, originally for the Odd Order Theorem [FT63, BG94]. This work has later been used to get a classification of minimal simple groups in presence of elements of order 2, and this classification has then been slightly generalized to the case of finite “locally solvable” groups, that is finite groups in which each normalizer of a nontrivial solvable subgroup is also solvable. The simplicity assumption was replaced by a mere nonsolvability assumption. This full classification, with only very few extra groups in addition to the minimal simple ones, has been published in the series of papers [Tho68, Tho70, Tho71, Tho73].

The present paper is the first of a series containing the same transfer of arguments from the minimal simple case to the locally solvable case in the context of groups of finite Morley rank. Indeed, a large body of work has been accomplished in the last years about *minimal connected simple* groups of finite Morley rank, that is connected simple groups of finite Morley rank in which every proper definable connected subgroup is solvable, and we propose here to transfer this work to the more general class of *locally solvable* groups of finite Morley rank, that is groups of finite Morley rank in which $N(A)$ is solvable for each nontrivial definable abelian subgroup A .

As we prefer most of the time with groups of finite Morley rank to work in the connected category, we will actually weaken this definition of local solvability in the following three possible ways, by assuming solvability of the *connected components* only of normalizers of nontrivial definable abelian groups A , in which case we will use the terminology *locally[◦]*, and/or by considering nontrivial definable *connected* abelian subgroups A only, in which case we will use the terminology *solvable[◦]*. In particular, we will most of the time work with the weakest definition of *local[◦] solvability[◦]*, i.e. assuming only that $N^{\circ}(A)$ is solvable for each nontrivial definable connected abelian subgroup A of the ambient group. In *local[◦] solvability* we consider all nontrivial definable abelian (not necessarily connected) subgroups A .

The only known infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields, and a long-standing conjecture postulates that there are no other such groups. Local solvability is a “smallness” condition and as in Thompson’s final classification, the simplicity assumption is replaced here generally by a mere *nonsolvability* assumption. In particular, the only known nonsolvable connected locally[◦] solvable groups of finite Morley rank are of the form PSL_2 over some algebraically closed field K , and of the form SL_2 in the slightly more general locally[◦] solvable[◦] case. For example, if we consider

in $\mathrm{SL}_3(K)$ the definable connected abelian subgroup

$$A = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} : t \in K^\times \right\},$$

then $N^\circ(A)$ is a central product $A \cdot E$ where E is a definable connected subgroup isomorphic to $\mathrm{SL}_2(K)$, so that $N^\circ(A)$ is not solvable. More precisely, for connected locally[◦] solvable and locally[◦] solvable[◦] groups of finite Morley rank there are in the classical algebraic case no other groups than PSL_2 and SL_2 , and in particular no groups of Lie rank 2 and more.

All the classes of locally solvable groups of finite Morley rank defined here contain of course all solvable groups of finite Morley rank, groups of the form PSL_2 or SL_2 , but also many hypothetic configurations of semisimple so-called bad groups of finite Morley rank which appear as potential counterexamples to the main conjecture on simple groups.

Hence, all the results of the present papers will lead to a kind of trichotomy (in a very large sense) for locally[◦] solvable[◦] groups as follows.

- Solvable groups.
- PSL_2 or SL_2 .
- Semisimple bad groups.

In particular, the present work encapsulates the existing theory of solvable groups of finite Morley rank on the one hand, and of minimal connected simple groups on the other.

In this first paper we are going to recast all the theory of solvable and minimal connected simple groups in this general context. In our second paper [DJ07] we are going to concentrate on the case of groups with involutions. Contrarily to the finite case, we cannot jump directly as in [Tho68] in the finite case to the case of groups with involutions, as no analog of the Feit-Thompson theorem is available in the context of groups of finite Morley rank. This is mostly due to the possible existence of bad groups, and we refer to [Jal01a] for the connection between the two problems. Our results towards algebraicity will only be partial, even in presence of involutions, but with a very severe limitation of nonalgebraic configurations. We refer to the introduction of [DJ07] for a more precise description of the case with involutions.

The present paper contains a collection of results concerning the local analysis of locally[◦] solvable[◦] and locally[◦] solvable groups of finite Morley rank which will be fully exploited in [DJ07]. That's why it also contains no theorem easily stated in the present introduction. The whole theory is naturally recasted in terms of generous Carter subgroups with the appeal of [Jal06].

We will not consider the locally solvable/solvable[◦] cases, which boil down rather to finite group theory and hence to Thompson's classification [Tho68]. We will however insist on the differences between local[◦] solvability, which in general offers no new substantial phenomena compared to the minimal connected simple

case, and the weaker local^o solvability^o, where new phenomena can occur. This is at least explained by the alternative SL_2 to PSL_2 .

Terminology. A word should be said about the terminology adopted, as it might be confusing with the more classical notion of local solvability. In general group theory this refers usually to groups in which all finitely generated subgroups are solvable. In finite group theory, a subgroup is called *local* if it is the normalizer of a nontrivial p -subgroup for some prime p . This terminology goes back to Alperin. In [Tho68] a group in which each local subgroup is solvable was called an *N-group*, and Thompson’s classification was stated for nonsolvable *N-groups*. We borrow the term “local” to speak of subgroups normalizing subgroups similar to p -groups, and hence we hope that “locally solvable” is clear enough in this context. We also note that a group of finite Morley rank in which every finitely generated subgroup is solvable — the usual group theoretic notion — must be solvable, and hence is locally solvable in our sense.

Historical remarks. A few historical remarks are necessary. Solvable groups of finite Morley rank have been highly investigated, notably by Nesin and Frécon. As mentioned already, this theory becomes incorporated to the present one.

With the ongoing work on simple groups of finite Morley rank with involutions, it became clear as corollaries of [Jal99] and [Jal01b] that there was no “small” simple groups of finite Morley rank of mixed type, and that the only specimen in even type was $\mathrm{PSL}_2(K)$, with K an algebraically closed field of characteristic 2.

Then it was time to start the study of “small” simple groups of odd type, even though there was almost nothing to start with. The foundations, notably the notion of minimal connected simple group, were laid down in the preprint [Jal00] which remained unpublished. It contained the first recognition of PSL_2 in characteristic different from 2 in this context, though under strong assumptions at that time. It also contained the embryo of local analysis of minimal connected simple groups of finite Morley rank. The original lemma, which turned out later to be an analog of the Bender method in finite group theory, was there given in any characteristic. It has unfortunately been disseminated between different characteristics later, and we will give here global forms and the general Uniqueness Theorem in Section 4.1.

Because of the absence of a unipotence theory in characteristic zero at that time, and in order to reduce the size of an overambitious project to manageable size, the second author adopted the so-called “tameness” assumption for the recognition of PSL_2 with the weakest expectable hypothesis in this context. The nonalgebraic configurations were also studied in this tame context, and the full analysis algebraic/nonalgebraic appeared in [CJ04].

In the meantime Cherlin suggested to develop a robust unipotence theory in characteristic 0 for attacking certain problems concerning large groups of odd type without the tameness assumption. This became the main tool in Burdges’ thesis [Bur04a] and this application corresponds to [Bur04b]. This new abstract unipotence theory then allowed one to develop the local analysis of minimal connected simple groups where the above mentioned uniqueness theorem fails

[Bur07]. It was also Cherlin’s idea to use this in presence of involutions to study other nonalgebraic configurations without tameness [BCJ07, Case II].

With a nice unipotence theory then available in any characteristic, the recognition of PSL_2 started again in the context of minimal connected simple groups of odd type without tameness, in the thesis of the first author [Del07b]. The recognition of PSL_2 has then been obtained as in the tame case under the weakest expectable assumptions and appeared in [Del07a]. Using this new experience for the algebraic case, the nonalgebraic configurations were studied in [Del08], reaching essentially all the conclusions of [CJ04] in the general case. The paper [DJ07] will at the same time improve and linearize the sequence of arguments represented by [BCJ07, Del07a, Del08], and also greatly simplify those in [Del08].

The final generalization from minimal connected simple groups to locally solvable groups has been suggested by Borovik by analogy with finite group theory.

Organization of the paper. Section 2 will contain background material, with notably an emphasis on the abstract unipotence theory in groups of finite Morley rank in Section 2.1 in continuation of [Bur06] and [FJ08]. We shall formalize the notion of *soapy* subgroups, the finest approximation of unipotent subgroups where all the finest computations will be done in [DJ07].

Section 3 will lay down the foundations concerning locally solvable groups of finite Morley rank. In Sections 3.3 and 3.4 we will focus on the new phenomena which can occur in the locally[◦] solvable[◦] case in comparison to the locally[◦] solvable one.

Section 4 will concern the local analysis of locally solvable groups of finite Morley rank, with in Section 4.1 the main Uniqueness Theorem (usually called “Jaligot’s Lemma”) corresponding to the Bender method in finite group theory. The analysis of a maximal pair of Borel subgroups from [Bur07], a parallel technic, will follow in Section 4.3. We will also derive consequences of the Uniqueness Theorem on generosity as in [CJ04].

Section 5 eventually concludes with several particular aspects concerning homogeneous cases as well as torsion.

Notations and background. For the basic background on groups of finite Morley rank we generally refer to [BN94]. The more recent [ABC08] is also a very complete source. We will try to refer as much as possible to these when needed, but we assume the reader familiar with certain background facts such as Zilber’s generation lemma and its corollaries [Zil77] [BN94, §5.4], notably the definability of subgroups generated by definable connected subgroups and corollaries on commutator subgroups.

Fact 1.1 [BN94, Corollary 5.29] *Let G be a group of finite Morley rank, H a definable connected subgroup, and X an arbitrary subset of G . Then $[H, X]$ is a definable connected subgroup of G .*

We will also assume the reader familiar with the descending chain condition on definable subgroups, the existence of connected components, the unique-

ness of generic types in connected groups [Che79], and its immediate corollary concerning actions on finite sets.

Fact 1.2 *A connected group acting definably on a finite set fixes it pointwise.*

If X is a subset, or a single element, of a group of finite Morley rank, we denote by $H(X)$ the *definable hull* of X , that is the smallest definable subgroup containing X . In the literature it is the notation “ $d(\cdot)$ ” which is commonly used, but we prefer to keep the latter for certain integer-valued unipotence “ d ” egrees, and instead use “ $H(\cdot)$ ” for “ H ”ulls which are definable subgroups.

If x and y are elements of a group, we write x^y for $y^{-1}xy$, and if X and Y are two subsets we denote by X^Y the set of elements x^y . (This notation might be floppy, as we may for example use x^G for the conjugacy class of x in G .) We denote by $N(X)$ the set of elements g such that $X^g = X$ (with an index if one wants to specify in which particular subset elements g are taken).

2 Background

2.1 Unipotence theory

For the following abstract unipotence theory in groups of finite Morley rank [Bur04a, Bur04b, Bur06], we follow essentially the general exposition of [FJ08]. We denote by \mathcal{P} the set of all prime numbers.

A *decent torus* is a divisible abelian group of finite Morley rank which coincides with the definable hull of its (divisible abelian) torsion subgroup. The latter is known to be in the finite Morley rank context a direct product, with p varying in \mathcal{P} , of finite products of the Prüfer p -group \mathbb{Z}_{p^∞} [BP90], and by divisibility decent tori are connected.

If p is a prime, a *p -unipotent* group of finite Morley rank is a definable connected nilpotent p -group of bounded exponent.

A *unipotence parameter* is a pair

$$\tilde{p} = (\text{characteristic } p, \text{unipotence degree } r) \in (\{\infty\} \cup \mathcal{P}) \times (\mathbb{N} \cup \{\infty\})$$

satisfying $p < \infty$ if and only if $r = \infty$. A group of finite Morley rank is a *\tilde{p} -group* if it is nilpotent and of the following form, depending on the value of \tilde{p} .

- if $\tilde{p} = (\infty, 0)$, a decent torus.
- if $\tilde{p} = (\infty, r)$, with $0 < r < \infty$, a group generated by its definable indecomposable subgroups A such that $A/\Phi(A)$ is torsion-free and of rank r . Here a group of finite Morley rank is *indecomposable* if it is abelian and not the sum of two proper definable subgroups. An indecomposable group A must be connected [Bur06, Lemma 1.2], and $\Phi(A)$ denotes its maximal proper definable connected subgroup.
- if $\tilde{p} = (p, \infty)$, with p prime, a p -unipotent subgroup.

We note that nilpotence of \tilde{p} -groups is imposed by definition, and that these groups are in any case generated by definable connected subgroups, and hence always connected by Zilber's generation lemma [BN94, Corollary 5.28]. A *Sylow* \tilde{p} -subgroup of a group of finite Morley rank is a maximal definable (connected) \tilde{p} -subgroup.

The term "characteristic" for p in a unipotence parameter (p, r) clearly refers to the characteristic of the ground field for p -unipotent groups in algebraic groups when p is finite. When p is infinite and $0 < r < \infty$, it refers to nontrivial torsion-free groups, which are potentially additive groups of fields of characteristic 0. When p is infinite and $r = 0$, i.e. for decent tori, it conveys no special meaning. The term "unipotence degree" (one can also speak of "weight") is explained in Fact 2.12 below by the constraints on actions of such groups on others.

A group of finite Morley rank is (p, r) -homogeneous if every definable connected nilpotent subgroup is a (p, r) -group. We say that it is *homogeneous* if it is (p, r) -homogeneous for some unipotence parameter (p, r) . Following [Che05], a divisible abelian $(\infty, 0)$ -homogeneous group of finite Morley rank is usually called a *good torus*.

Fact 2.1 [FJ08, Lemma 2.17] *Depending on the value of \tilde{p} , the \tilde{p} -homogeneity of a \tilde{p} -group is equivalent to the following:*

- (1) *if $\tilde{p} = (\infty, 0)$, to being a good torus.*
- (2) *if $\tilde{p} = (\infty, r)$, with $0 < r < \infty$, to having only \tilde{p} -subgroups as definable connected abelian subgroups.*
- (3) *if $\tilde{p} = (p, \infty)$, with p prime, then a \tilde{p} -group is always \tilde{p} -homogeneous.*

Fact 2.2 [FJ08, Theorem 2.18] *Let G be a connected group of finite Morley rank acting definably on a \tilde{p} -group H . Then $[G, H]$ is a definable \tilde{p} -homogeneous subgroup of H .*

Proof. The main point is when the unipotence degree r of H satisfies $0 < r < \infty$ and is proved in [Fré06a, Theorem 4.11]. When the unipotence degree of H is infinite, this is just Fact 2.1 (3). Decent tori are centralized by any connected group acting on them as an easy consequence of Fact 1.2 called *rigidity* of decent tori (see Fact 2.12 (1) below). Hence $[G, H]$ is trivial when $r = 0$. \square

Corollary 2.3 *Let G be any \tilde{p} -group. Then G^n and $G^{(n)}$ are definable homogeneous \tilde{p} -subgroups for any $n \geq 1$.*

If G is a group of finite Morley rank and $\tilde{\pi}$ is a set of unipotence parameters, we define

$$U_{\tilde{\pi}}(G) = \langle \Sigma \mid \tilde{p} \in \tilde{\pi} \text{ and } \Sigma \text{ is a definable } \tilde{p}\text{-subgroup of } G \rangle.$$

The latter subgroup is always definable and connected by Zilber's generation lemma. When $\tilde{\pi}$ is empty it is trivial and when $\tilde{\pi}$ consists of a single unipotence parameter \tilde{p} we simply write $U_{\tilde{p}}(G)$. If $\tilde{p} = (p, \infty)$ for some prime p , we also write $U_p(G)$ for $U_{\tilde{p}}(G)$. A $U_{\tilde{\pi}}$ -group is a group G such that $U_{\tilde{\pi}}(G) = G$.

Fact 2.4 [FJ08, Lemma 2.13] *Let $f : G \rightarrow H$ be a definable homomorphism between two groups of finite Morley rank. Then*

- (1) (*Push-forward*) $f(U_{\tilde{\pi}}(G)) \leq U_{\tilde{\pi}}(H)$ is a $U_{\tilde{\pi}}$ -group.
- (2) (*Pull-back*) *Assume all unipotence degrees involved in $\tilde{\pi}$ are finite, or that G is solvable. If $U_{\tilde{\pi}}(H) \leq f(G)$, then $f(U_{\tilde{\pi}}(G)) = U_{\tilde{\pi}}(H)$.*

In particular, an extension of a solvable $U_{\tilde{\pi}}$ -group by a solvable $U_{\tilde{\pi}}$ -group is a $U_{\tilde{\pi}}$ -group.

Fact 2.5 [Bur06, §3] *Let G be a nilpotent group of finite Morley rank.*

- (1) *G is the central product of its Sylow p -subgroups and its Sylow (∞, r) -subgroups.*
- (2) *If G is connected, then G is the central product of its Sylow \tilde{p} -subgroups.*

Proof. The connected case corresponds to [FJ08, Theorem 2.7]. Without connectedness we refer to the decomposition of G as the central product of a definable divisible (connected) subgroup D and a definable subgroup B of bounded exponent of [Nes91] [BN94, Theorem 6.8], and to the decomposition of a nilpotent group of bounded exponent as the central product of its (definable) Sylow p -subgroups. \square

Fact 2.6 *A \tilde{p} -group of finite Morley rank cannot be a \tilde{q} -group when $\tilde{q} \neq \tilde{p}$.*

Proof. It suffices to use the commutation provided by Fact 2.5 (2) to reduce the problem to abelian groups. Then it follows easily from the definitions. \square

The following fact is a variation on the usual *normalizer condition* in finite nilpotent groups.

Fact 2.7 ([Bur06, Lemma 2.4], [FJ08, Proposition 2.8]) *Let G be a \tilde{p} -group and $H < G$ a proper definable subgroup. If S_1 and S_2 denote the Sylow \tilde{p} -subgroups of H and of $N_G(H)$ respectively, then $S_1 < S_2$.*

Fact 2.8 [FJ08, Lemma 2.9] *Let G be a group of finite Morley rank, S a subset of G , and H a definable \tilde{p} -subgroup of G normalized by S . Then $[H, S]$ is a \tilde{p} -subgroup of H .*

Fact 2.9 *Let \tilde{p} be a unipotence parameter and q a prime number. Let H be a \tilde{p} -group of finite Morley rank without elements of order q , and assume K is a definable solvable q -group of automorphisms of H of bounded exponent. Then $C_H(K)$ is a definable \tilde{p} -subgroup of H .*

Proof. By descending chain condition on centralizers, $C_H(K)$ is the centralizer of a finitely generated subgroup of K , and by local finiteness of the latter we may assume K finite. In particular $C_H(K)$ is connected by [Bur04b, Fact 3.4].

When $\tilde{p} = (\infty, 0)$, H is a good torus, and in particular $(\infty, 0)$ -homogeneous, and the connected subgroup $C_H(K)$ is also a good torus. Otherwise, $C_H(K)$ is also a \tilde{p} -group, by [Bur04a, Lemma 3.18] [Bur04b, Lemma 3.6] when the unipotence parameter is finite or Fact 2.1 (3) when the characteristic is finite. \square

Definition 2.10 *Let G be a group of finite Morley rank.*

- (1) *We say that G admits the unipotence parameter \tilde{p} if $U_{\tilde{p}}(G) \neq 1$.*
- (2) *We denote by $d_{\infty}(G)$ the maximal unipotence degree in characteristic ∞ , i.e. the maximal integer $r \in \mathbb{N}$ such that G admits the unipotence parameter (∞, r) , and -1 if G admits none such.*
- (3) *If p is a prime, we denote by $d_p(G)$ the maximal unipotence degree in characteristic p , i.e. the ∞ symbol if G admits the unipotence parameter (p, ∞) , and -1 otherwise.*
- (4) *A unipotence parameter $\tilde{p} = (p, r)$ is maximal in its characteristic for G if $d_p(G) = r$ (notice here that the characteristic p can be ∞ or prime). This is equivalent to saying that r is the maximal unipotence parameter in characteristic p .*
- (5) *Finally, we define the absolute unipotence degree $d(G)$ of G as the maximum of $d_{\infty}(G)$ and $\max_{p \in \mathcal{P}} \{d_p(G)\}$.*

We say that a unipotence parameter (p, r) is *absolutely maximal* for G if $d(G) = d_p(G) = r$, i.e. if G contains nontrivial p -unipotent subgroups if $p < \infty$ and otherwise admits (∞, r) and contains no nontrivial definable connected nilpotent subgroup of bounded exponent and no nontrivial definable (∞, r') -subgroup with $r' > r$.

We say that a unipotence parameter (p, r) is *maximal* for G if $d(G) = 0$ whenever $r = 0$, or $d_p(G) = r$ otherwise. This has essentially the effect of not considering good tori of PSL_2 over a pure field of positive characteristic as having maximal unipotence degree. We will often mention this special example separately.

The following lemma makes known facts more transparent in our notation.

Lemma 2.11 *Let G be a group of finite Morley rank.*

- (1) G is finite if and only if $d(G) = -1$.
- (2) G is a good torus if and only if G is connected solvable and $d(G) \leq 0$.

Proof. If $d(G) \geq 0$, then G has a nontrivial definable connected nilpotent subgroup, and hence it cannot be finite. Conversely, if G is infinite, then its minimal infinite definable subgroups are abelian by Reineke's Theorem [BN94, Theorem 6.4]. As such subgroups are also connected, they contain a nontrivial Sylow \tilde{p} -subgroup for some unipotence parameter $\tilde{p} = (p, r)$ by Fact 2.5 (2), and hence $d(G) \geq r \geq 0 > -1$.

If G is a good torus, then it is abelian and connected, and any definable connected subgroup is a good torus, in particular a decent torus, and by Fact 2.6 $d(G) \leq 0$. Conversely, if G is a connected solvable group which admits no unipotence parameter $\tilde{p} = (p, r)$ with $r \geq 1$, then G is a good torus by [Bur04b, Theorem 2.15]. \square

For any group G of finite Morley rank we define, similarly to $U_p(G)$, the *unipotent radical in characteristic ∞* as

$$U_\infty(G) = U_{(\infty, d_\infty(G))}(G).$$

One can also define the *absolute unipotent radical* $U(G)$ as

$$U(G) = \langle U_p(G) \mid p \text{ prime} \rangle \text{ if it is nontrivial and } U_\infty(G) \text{ otherwise.}$$

Finally, a unipotent radical $U_{(p,r)}(G)$ is *maximal* for G if (p, r) is maximal for G .

2.2 Carter and soapy subgroups

The preceding abstract unipotence theory in groups of finite Morley rank gives important approximations of semisimple and unipotent subgroups of algebraic groups. On the one hand it gives a good approximation of maximal tori in any group of finite Morley rank via the notion of Carter subgroup. On the other hand it detects, and it is a more difficult task, approximations of unipotent subgroups in locally solvable groups via the notion of soapy subgroups.

All this is due to a good understanding of possible actions of \tilde{p} -subgroups onto each other in groups of finite Morley rank. These constraints can be summarized as follows. The first item is often called *rigidity* of decent tori.

Fact 2.12 *Let G be a group of finite Morley rank, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ two sets of unipotence parameters, and $r \in \mathbb{N} \cup \{\infty\}$.*

- (1) *Assume $G = TH$ where T is a definable decent torus of G and H is a definable connected subgroup normalizing T . Then $T \leq Z(G)$. In particular, if T is a definable decent torus in a group of finite Morley rank, then $C^\circ(T) = N^\circ(T)$.*

- (2) Assume $G = U_1U_2$ where each $U_i = U_{\tilde{\pi}_i}(U_i)$ is a definable nilpotent subgroup and U_1 is normal. Assume that all unipotence degrees involved in $\tilde{\pi}_1$ are $\leq r$ and that all unipotence degrees involved in $\tilde{\pi}_2$ are $\geq r$. Then U_1U_2 is nilpotent.
- (3) Assume $G = H_1H_2$ where each $H_i = U_{\tilde{\pi}_i}(H_i)$ is definable and H_1 is normal and nilpotent. Assume that all unipotence degrees involved in $\tilde{\pi}_1$ are $\leq r$ and that all unipotence degrees involved in $\tilde{\pi}_2$ are $> r$. Then $G = H_1C^\circ(H_1)$.
- (4) Assume $G = U_1U_2$ where U_1 is a normal nilpotent subgroup such that $U_1 = U_{\tilde{\pi}_1}(U_1)$, will all unipotence degrees involved in $\tilde{\pi}_1$ infinite, and $U_2 = U_{\tilde{\pi}_1}(U_2)$, where all unipotence degrees r involved in $\tilde{\pi}_2$ satisfy $0 < r < \infty$. Then $U_2 \leq C(U_1)$.

Proof. The first item, which was the main key tool in [Che05], is a mere application of Fact 1.2 together with the fact that Prüfer p -ranks of decent tori are finite for any prime p [BP90].

The second item is [FJ08, Proposition 2.10]. See also [FJ05, §3] and [Bur06, §4] for earlier versions of the same fact.

For the third item, we notice that if $\tilde{p} \in \tilde{\pi}_2$ and Σ is any definable connected \tilde{p} -subgroup of H_2 , then $H_1 \cdot \Sigma$ is nilpotent by the second point, and both factors commute by our assumption on the unipotence degrees involved and Fact 2.5 (2). In particular $U_{\tilde{p}}(H_2) \leq C^\circ(H_1)$ and as $H_2 = \langle U_{\tilde{p}}(H_2) \mid \tilde{p} \in \tilde{\pi}_2 \rangle$, our claim follows.

For the last item we refer to [Bur06, Lemma 4.3] for the fact that an (∞, r) -group, with $0 < r < \infty$, which normalizes a p -unipotent group must centralize it. This is essentially a corollary of [Wag01, Corollary 8]. Then one can argue as in the third point. \square

Fact 2.12 has as a general consequence the existence of a very good approximation of semisimple subgroups of algebraic groups in the context of groups of finite Morley rank. If $\tilde{\pi}$ is a set of unipotence parameters, a *Carter $\tilde{\pi}$ -subgroup* of a group of finite Morley rank is a definable connected nilpotent subgroup $Q_{\tilde{\pi}}$ such that $U_{\tilde{\pi}}(N(Q)) = Q$. A *Carter subgroup* of a group of finite Morley rank is a definable connected nilpotent subgroup Q such that $N^\circ(Q) = Q$. By Fact 2.4 this corresponds to a Carter $\tilde{\pi}$ -subgroup for the set $\tilde{\pi}$ of *all* unipotence parameters, or merely the set of unipotence parameters admitted by the ambient group.

The existence of Carter subgroups in arbitrary groups of finite Morley rank, which appeared in [FJ05], has been looked for by the second author originally in the context of minimal connected simple groups in order to generalize [CJ04]. It follows essentially from Fact 2.12, by considering \tilde{p} -subgroups from the least to the most unipotent.

Fact 2.13 [FJ08, Theorem 3.3] *Let G be a group of finite Morley rank and $\tilde{\pi}$ a set of unipotence parameters. Let r be the smallest unipotence degree involved in $\tilde{\pi}$. Then any Sylow (p, r) -subgroup of G is contained in a Carter $\tilde{\pi}$ -subgroup of G .*

A definable subset X of a group G of finite Morley rank is *generous* in G if the union X^G of its G -conjugates is generic in G . In simple algebraic groups maximal tori are generous. In groups of finite Morley rank we only have equivalent conditions to this property.

Fact 2.14 [Jal06, Corollary 3.8] *Let G be a group of finite Morley rank and Q a Carter subgroup of G . Then the following are equivalent.*

- (1) Q is generous in G .
- (2) There exists a definable generic subset Y of Q such that, for each $y \in Y$, Q is the unique maximal definable connected nilpotent subgroup containing y .
- (3) Q is generically disjoint from its conjugates.
- (4) There exists a definable generic subset of Q all of whose elements are contained in only finitely many conjugates of Q .

At the opposite of semisimple groups, we pass now to the approximations of unipotent subgroups. We denote by $F(G)$ the *Fitting* subgroup of any group G , i.e. the subgroup generated by all normal nilpotent subgroups. It is always definable and nilpotent in the finite Morley rank case [BN94, Theorem 7.3]. A consequence of Fact 2.12 dual to Fact 2.13 is the following.

Fact 2.15 *Let H be a connected solvable group of finite Morley rank and $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$. Assume $d_p(H) \leq r$. Then $U_{\tilde{p}}(H) \leq F^\circ(H)$.*

Proof. See [FJ08, Lemma 2.11], and [Bur04b, Theorem 2.16] for the original version. It suffices to use Fact 2.12 (2) and (4) to conclude that $F^\circ(H) \cdot U_{\tilde{p}}(H)$ is nilpotent, and then to use the fact that $H/F^\circ(H)$ is abelian (Fact 2.22 below). \square

We note that the assumption $r > 0$ is necessary in Fact 2.15. In the standard Borel subgroup B of PSL_2 in positive characteristic, $d_\infty(B) = 0$, but maximal tori of B are not in the unipotent radical of B .

Unipotent subgroups are usually not generous in linear algebraic groups, and thus in general more difficult to detect. Every *nontrivial* subgroup $U_{\tilde{p}}(H)$ as in Fact 2.15 is generally a good approximation of unipotent radical, at least much finer than the Fitting subgroup. We will need even finer approximations when considering locally solvable groups of finite Morley rank, notably the property

of being homogeneous and central in the Fitting subgroup. This issues from the minimal subgroups used originally in [Jal00], after the considerable reworking in [Del07a, Del08].

Recall that, for every connected solvable group H of finite Morley rank, a unipotence parameter $\tilde{q} = (q, d)$ is *maximal* for H if $d(H) = 0$ whenever $d = 0$, or $d_q(H) = d$ otherwise. By Lemma 2.11, a nontrivial connected solvable group H is a good torus if and only if its unique maximal unipotence parameter is $(\infty, 0)$. Otherwise, maximal unipotence parameters are all the (p, ∞) such that $U_p(H) \neq 1$ and the (∞, d) with $d \geq 1$ and $d_\infty(H) = d$ if it exists.

Definition 2.16 *Let H be a connected solvable group of finite Morley rank. A subgroup U of H is soapy (resp. characteristically soapy) in H if the two following conditions hold.*

- (1) U is a nontrivial definable connected subgroup of $Z(F^\circ(H))$, \tilde{q} -homogeneous for some unipotence parameter \tilde{q} maximal for H .
- (2) U is normal (resp. definably characteristic) in H .

We haven't found a better name for these subgroups. We will see in Section 4.1.4 that in locally[◦] solvable[◦] groups these subgroups have a strong tendency to escape from intersections of distincts Borel subgroups, like unipotent subgroups in PSL_2 and like a soap between to hands. Another not less serious reason for this name is that these groups were born near Marseilles, which is famous for its soap.

We could also specify a set of maximal unipotence parameters for H , and define these interesting subgroups as products of the present ones. In practice only one unipotence parameter will suffice for us.

The next lemma says that the existence of soapy subgroups is not essentially weaker than that of characteristically soapy subgroups.

Lemma 2.17 *Let H be a connected solvable group of finite Morley rank and \tilde{q} a unipotence parameter maximal for H . If H contains a \tilde{q} -homogeneous soapy subgroup, then it contains a \tilde{q} -homogeneous characteristically soapy subgroup as well.*

Proof. If $\tilde{q} = (\infty, 0)$ then H is a good torus, and H itself is the desired group.

In general one can proceed as follows. Let U be a \tilde{q} -homogeneous soapy subgroup of H . Let \tilde{U} be the subgroup of $Z(F^\circ(H))$ generated by all \tilde{q} -homogeneous soapy subgroups of H . It is nontrivial, definable and connected as the product of finitely many soapy subgroups by Zilber's generation lemma, and one sees easily that it is \tilde{q} -homogeneous with Fact 2.4 (see also [Fré06a, Corollary 3.5]). It is clearly definably characteristic in H . Hence \tilde{U} is characteristically soapy in H . \square

We finish this section with a general criterion for building characteristically soapy subgroups.

Lemma 2.18 *Let H be a connected solvable group of finite Morley rank and \tilde{q} a unipotence parameter maximal for H . If $U_{\tilde{q}}(Z(F^\circ(H)))$ is not central in H , then H contains a \tilde{q} -homogeneous characteristically soapy subgroup.*

Proof. Set $U = [U_{\tilde{q}}(Z(F^\circ(H))), H]$. By assumption U is nontrivial. It is a definable connected homogeneous \tilde{q} -subgroup by Fact 2.2, contained in $Z(F^\circ(H))$ as the latter is normal in H , and obviously definably characteristic in H . \square

2.3 Conjugacy theorems

As far as unipotence theory is concerned, there are two general conjugacy theorems in groups of finite Morley rank. The first one has a nontrivial content only in presence of divisible torsion.

Fact 2.19 [Che05] *Let G be a group of finite Morley rank. Then $C^\circ(T)$ is generous in G° for every definable decent torus T of G° , and maximal definable decent tori of G° are G° -conjugate.*

The following corollary of Fact 2.19 has been known for a long time in presence of 2-divisible torsion [BN94, Lemma 10.22].

Corollary 2.20 (Control of fusion) *Let G be a group of finite Morley rank, p a prime, and T a p -torus of G . If X and Y are two G -conjugate subsets of G such that $C_T(X)$, $C_T(Y)$, and $C(Y)$ all have the same Prüfer p -ranks, then $Y = X^g$ for some g conjugating $C_T^\circ(X)$ to $C_T^\circ(Y)$. In particular if T is a maximal p -torus of G then any two G -conjugate subsets of $C(T)$ are $N(T)$ -conjugate.*

Proof. First notice that there are always *maximal* p -tori, by finiteness of the Prüfer p -rank [BP90] and compactness.

Assume $Y = X^g$ for some $g \in G$. Then $C_T^\circ(X)^g$ and $C_T^\circ(Y)$ are both contained in the definable subgroup $C^\circ(Y)$. By Fact 2.19 and the assumption, $C_T^\circ(X)^g = C_T^\circ(Y)^\gamma$ for some $\gamma \in C^\circ(Y)$. Then $g\gamma^{-1}$ conjugates $C_T^\circ(X)$ to $C_T^\circ(Y)$ and as $Y^\gamma = Y = X^g$ the element $g\gamma^{-1}$ conjugates X to Y .

When X and Y are two subsets of $C(T)$ and T is maximal we can apply the preceding and the new element g conjugating X to Y will now normalize T . \square

There is no reason why an arbitrary group of finite Morley rank should contain nontrivial torsion as in Fact 2.19. However the next general conjugacy theorem relies on an assumption which is likely to be true in general [Jal06, §4].

Fact 2.21 [Jal06] *Let G be a group of finite Morley rank. Then generous Carter subgroups of G are generous in G° and G° -conjugate.*

In our study of locally solvable groups of finite Morley rank we will of course use much more conjugacy theorems where they are much more abundant, that is in solvable groups.

2.4 Solvable groups

Fact 2.22 [Nes90] *Let H be a connected solvable group of finite Morley rank. Then $H/F^\circ(H)$ is divisible abelian.*

Fact 2.23 ([Fré00, Corollaire 7.15], [BN92]) *Let H be a connected solvable group of finite Morley rank, and π any set of prime numbers. Then Hall π -subgroups of H are connected.*

Fact 2.24 [BP90] *Let p be a prime and S a p -subgroup of a solvable group of finite Morley rank, or more generally a locally finite p -subgroup of any group of finite Morley rank. Then*

- (1) S° is a central product of a p -torus and a p -unipotent subgroup.
- (2) If S is infinite and of bounded exponent, then $Z(S)$ contains infinitely many elements of order p .

Lemma 2.25 *Let H be a connected solvable group of finite Morley rank and p a prime. If $U_p(H) = 1$, then the Sylow p -subgroup of $F(H)$ is central in H .*

Proof. Assume $U_p(H) = 1$, and let S denote the Sylow p -subgroup of $F(H)$. By Fact 2.5, S is the product of a finite p -subgroup and of a p -torus. As each of these two subgroups is normal in H , each is central in H , by Facts 1.2 and 2.12 (1) respectively. \square

The following fact gradually appeared in [Wag94], [Fré00], and [CJ04, 3.5].

Fact 2.26 [FJ08, Theorem 3.11] *Let H be a connected solvable group of finite Morley rank. Then Carter subgroups of H are generous, conjugate and self-normalizing.*

Corollary 2.27 *Let G be a group of finite Morley rank, Q a Carter subgroup and σ an element normalizing Q and not in Q . Then $\sigma \notin C^\circ(X)$ for every $X \subseteq Z(Q)$ such that $C^\circ(X)$ is solvable. In particular such σ and X cannot be in the same definable connected abelian subgroup.*

Proof. Assuming the contrary, then $\sigma \in N_{C^\circ(X)}(Q) = Q$ by the selfnormalization given in Fact 2.26, a contradiction. For the second point we simply notice that otherwise $\sigma \in C^\circ(X)$. \square

Following [FJ08, §4-5] there are nice links between Carter $\tilde{\pi}$ -groups and covering properties in connected solvable groups of finite Morley rank, the so-called connected subformation theory. In particular one knows that the collection \mathcal{N} of connected nilpotent groups of finite Morley rank is a connected subformation. The main link between Carter subgroup theory and subformation theory in connected solvable groups is then a guarantee that Carter subgroups of a connected solvable group G of finite Morley rank are \mathcal{N} -covering subgroups of G , which provides the following important result.

Fact 2.28 [FJ08, Proposition 5.1] *In any connected solvable group of finite Morley rank, Carter subgroups are exactly the \mathcal{N} -covering subgroups and the \mathcal{N} -projectors.*

Here the properties of \mathcal{N} -covering subgroups and \mathcal{N} -projectors which interest us are that these groups cover all nilpotent connected sections containing them.

Fact 2.29 [FJ08, Theorem 5.8] *Let G be a connected solvable group of finite Morley rank and $\tilde{\pi}$ a set of unipotence parameters. Then Carter $\tilde{\pi}$ -subgroups are exactly the $\mathcal{N}_{\tilde{\pi}}$ -projectors and the $\mathcal{N}_{\tilde{\pi}}$ -covering subgroups of G , and are in particular conjugate.*

We note that when $\tilde{\pi}$ is a single unipotence parameter, Carter \tilde{p} -subgroup coincide with Sylow \tilde{p} -subgroups [FJ08, §3.2], so that Sylow \tilde{p} -subgroups are conjugate in connected solvable groups of finite Morley rank. There is also structural information concerning Carter $\tilde{\pi}$ -subgroups of connected solvable groups of finite Morley rank [FJ08, Corollary 5.9], and we will use this only with $\tilde{\pi} = \{\tilde{p}\}$.

Fact 2.30 ([FJ08, Corollary 5.11], [Bur06, Theorem 6.7]) *Let G be a connected solvable group of finite Morley rank. Then the Sylow \tilde{p} -subgroups of G are exactly the subgroups of the form $U_{\tilde{p}}(G')U_{\tilde{p}}(Q)$ for some Carter subgroup Q of G .*

If G is a group of finite Morley rank, we denote by

$$O_{p'}(H)$$

the largest normal definable connected subgroup without p -torsion. It exists by ascending chain condition on definable connected subgroups and elementary properties of lifting of torsion [BN92].

The following facts will be useful when dealing with p -strongly embedded subgroups in Section 5.5 below.

Fact 2.31 (Compare with [CJ04, Lemma 3.2]) *Let H be a connected solvable group of finite Morley rank such that $U_p(H) = 1$. Then $H/O_{p'}(H)$ is divisible abelian.*

Proof. Dividing by $O_{p'}(H)$, we may assume it is trivial and we want to show that H is divisible abelian.

Let $F = F^\circ(H)$. As $O_{p'}(H) = 1$, $O_{p'}(F) = 1$ as well, and $U_q(H) = 1$ for any prime q different from p . By assumption $U_p(H) = 1$ also, and F is divisible by Fact 2.5. As F' is torsion-free, by [BN94, Theorem 2.9] or Fact 2.5 and Corollary 2.3, it must be trivial by assumption. Hence F is divisible abelian.

To conclude it suffices to show that F is central in H , as then H is nilpotent, hence equal to F , and hence divisible abelian, as desired. Let h be any element of H ; we want to show that $[h, F] = 1$. But $[h, F]$ is torsion-free, as the torsion subgroup of F is central in H by Fact 2.12 (1), or using Fact 2.2. Hence $[h, F] \leq O_{p'}(H) = 1$, as desired. \square

Fact 2.32 [Bur04b, Fact 3.7] *Let H be a solvable group of finite Morley rank without elements of order p for some prime p . Let E be a finite elementary abelian p -group acting definably on H . Then*

$$H = \langle C_H(E_0) \mid E_0 \leq E, [E : E_0] = p \rangle.$$

Lemma 2.33 *Let H be a connected solvable group of finite Morley rank such that $U_p(H) = 1$ for some prime p . Assume H contains an elementary abelian p -group E of order p^2 . Then*

$$H = \langle C_H^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

Proof. By assumption and Facts 2.23 and 2.24, Sylow p -subgroups of H are p -tori. Hence E is in a maximal p -torus of H , which is included in a Carter subgroup Q of H by Fact 2.13. By Fact 2.31, $H/O_{p'}(H)$ is abelian. As Carter subgroups cover all abelian quotients in connected solvable groups of finite Morley rank by Fact 2.28, $H = O_{p'}(H) \cdot Q$. As $E \leq Z(Q)$, it suffices to show that

$$O_{p'}(H) = \langle C_{O_{p'}(H)}^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

But the generation by the full centralizers is given by Fact 2.32, and these centralizers are connected by [Bur04b, Fact 3.4]. \square

Corollary 2.34 *Let H be a connected solvable group of finite Morley rank with a toral subgroup E of order p^2 for some prime p . Then*

$$H = \langle C_H^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

Proof. For a connected nilpotent group of finite Morley rank L , we define the “complement” $C_p(L)$ of $U_p(L)$, namely the product of all factors of L as in Fact 2.5 (2), except $U_p(L)$.

Now if H is any connected solvable group of finite Morley rank and Q a Carter subgroup of H , then H is the product of the definable connected subgroup $C_p(Q)C_p(F^\circ(H))$ with the normal definable connected subgroup $U_p(H)$, and the first factor has trivial p -unipotent subgroups.

In our particular case, E is by torality contained in a p -torus, and the latter is contained in a Carter subgroup Q of H . By Facts 2.23 and 2.24 E centralizes the normal definable connected subgroup $U_p(H)$, so it suffices to show the generation by centralizers^o in $C_p(Q)C_p(F^\circ(H))$. But this follows from Lemma 2.33. \square

2.5 Genericity

Lemma 2.35 *Let H be a connected solvable group of finite Morley rank generically covered by a uniformly definable family of finite subgroups. Then H is nilpotent and of bounded exponent.*

Proof. We first note that any group generically covered by a uniformly definable family of finite groups is generically of bounded exponent. In fact, by elimination of infinite quantifiers [BC02, Proposition 2.2], there is a uniform bound on the cardinals of the finite groups involved.

Now $H/F^\circ(H)$ is divisible abelian by Fact 2.22. As Prüfer p -ranks are finite for each prime p , there is a finite subgroup of $H/F^\circ(H)$ containing all images modulo $F^\circ(H)$ of the finite groups. This shows by generic covering that $H/F^\circ(H)$ is trivial. Hence H is nilpotent. Now it suffices to use the generic covering again and Fact 2.5 (2) with Fact 2.4 (1). \square

The following lemma has its roots in [Jal00, Lemme 2.13] (see [CJ04, Fact 2.36]).

Lemma 2.36 *Let G be a connected group of finite Morley rank and X a non-empty definable G -invariant subset of G . If M is a definable subgroup of G such that $X \cap M$ is generic in X , then $X \cap M$ contains a definable G -invariant subset generic in X .*

Proof. By assumption X is a union of G -conjugacy classes. By assumption also, $X \cap M$ is nonempty.

Let Y_1 be a definable generic subset of $X \cap M$ consisting of elements of $X \cap M$ whose G -conjugacy classes have traces on $X \cap M$ of constant ranks. Let Y_2 be a definable generic subset of Y_1 consisting of elements of Y_1 whose G -conjugacy classes in G have constant ranks. Both exist as we have, by definability of the rank, finite definable partitions in each case. Now Y_2 is generic in Y_1 which is generic in $X \cap M$, so Y_2 is generic in $X \cap M$ and in X . Replacing X by Y_2^G , one can thus assume that G -conjugacy classes in G of elements of X , as well as their traces on M , are of constant ranks. We also have then that $x^G \cap M$ is nonempty for any x in X .

Now, as X is the union of the G -conjugacy classes of its elements in $X \cap M$ and reduced to the situation where all relevant ranks are constant, the assumption that $X \cap M$ is generic in X implies easily by additivity of the rank that $x^G \cap M$ is generic in x^G for any x in X .

Let $N = \bigcap_{g \in G} M^g$. By descending chain condition on definable subgroups, $N = M^{g_1} \cap \dots \cap M^{g_n}$ for finitely many elements g_1, \dots, g_n of G . As G is connected, x^G , which is in definable bijection with $G/C(x)$, has Morley degree 1 for any x in X . By taking conjugates one also has $x^G \cap M^{g_i}$ generic in x^G for each x in X and each g_i . Hence $x^G \cap N$, which can be written as

$$(x^G \cap M^{g_1}) \cap \dots \cap (x^G \cap M^{g_n}),$$

is also generic in x^G , for any x in X . Now the fact that all ranks involved are constant implies that $X \cap N$ is generic in X as well.

But $X \cap N$ is G -invariant as both sets involved are. Hence $X \cap N$ is the desired definable G -invariant subset of $X \cap M$ generic in X . \square

3 Locally solvable groups

3.1 Foundations

Definition 3.1 We say that a group of finite Morley rank is

- (1) locally solvable if $N(A)$ is solvable for each nontrivial definable abelian subgroup A .
- (2) locally solvable[°] if $N(A)$ is solvable for each nontrivial definable abelian connected subgroup A .
- (3) locally[°] solvable if $N^\circ(A)$ is solvable for each nontrivial definable abelian subgroup A .
- (4) locally[°] solvable[°] if $N^\circ(A)$ is solvable for each nontrivial definable abelian connected subgroup A .

Lemma 3.2 Let G be a group of finite Morley rank.

- (1) If G satisfies one of the Definitions 3.1 (1), (2), (3), or (4), then so does any definable subgroup of G .
- (2) If G is locally solvable, then is it locally solvable[°] and locally[°] solvable, and if G has any of the two latter properties, then it is locally[°] solvable[°].

Proof. Obvious. □

Definition 3.3 Let G be a group of finite Morley rank and H a subgroup of G . We say that a subgroup L of G is

- (1) H -local if $L \leq N(H)$.
- (2) H -local[°] if $L \leq N^\circ(H)$.

Then we say that a subgroup L is *local* if it is H -local for some subgroup H , and *local[°]* if it is H -local[°] for some subgroup H . We can give conditions a priori stronger, but actually equivalent, to Definitions 3.1 (1)–(4) in terms of local subgroups.

Lemma 3.4 Let G be a group of finite Morley rank. Then G is

- (1) locally solvable if and only if X -local subgroups are solvable for every nontrivial solvable subgroup X .
- (2) locally solvable[°] if and only if X -local subgroups are solvable for every infinite solvable subgroup X .
- (3) locally[°] solvable if and only if X -local[°] subgroups are solvable for every nontrivial solvable subgroup X .

- (4) *locally[◦] solvable[◦] if and only if X -local[◦] subgroups are solvable for every infinite solvable subgroup X .*

Proof. Clearly the right conditions are stronger than the left ones.

Assume now a left condition, and suppose X is some nontrivial solvable subgroup of G , and L is an X -local subgroup, i.e. $L \leq N(X)$. Then L normalizes the definable hull $H(X)$ of X , and its connected component $H^\circ(X)$ as well. Now a classical corollary of Zilber's generation lemma on derived subgroups (Fact 1.1) implies that the last nontrivial term of the derived series of $H(X)$, as well as $H^\circ(X)$, is definable. It is abelian by definition, and as it is characteristic in $H(X)$ (resp. $H^\circ(X)$), it is normalized by L . Then one sees in each case which has to be considered that the latter is solvable by the left condition. \square

Nontrivial solvable groups H of finite Morley rank contain certain nontrivial definable characteristic Sylow \tilde{p} -subgroups or Sylow p -subgroups, by Fact 2.5 applied in $F(H)$. Hence H -local subgroups normalize nontrivial \tilde{p} -groups or p -groups, so that our definitions are coherent with the notion due to Alperin of local subgroup in finite group theory [Tho68], as subgroups normalizing nontrivial p -subgroups. Before stating this a little bit more precisely in the locally[◦] solvable[◦] case, we look at quotients.

Lemma 3.5 *Let G be a group of finite Morley rank and N a definable normal solvable subgroup.*

- (1) *If G is locally solvable, then so is G/N .*
- (2) *If G is locally solvable[◦], then so is G/N .*
- (3) *If G is locally[◦] solvable, then so is G/N .*
- (4) *If G is locally[◦] solvable[◦], then so is G/N .*

Proof. We denote by \overline{G} the quotients by N .

(1). Let \overline{A} be a nontrivial definable abelian subgroup of \overline{G} . The preimage of $N_{\overline{G}}(\overline{A})$ normalizes AN , which is solvable and nontrivial, and hence it is solvable by local solvability of G . As N is solvable, $N_{\overline{G}}(\overline{A})$ is also solvable.

(2). One can proceed as in (1), taking A infinite modulo N , and looking at the normalizer of $(AN)^\circ$.

(3). One can proceed as in (1), taking connected components of normalizers throughout.

(4). It suffices to mix the two preceding cases. \square

We continue with trivial remarks. In a group of finite Morley rank we call *Borel* subgroup any maximal definable connected solvable subgroup.

Lemma 3.6 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. Then a subgroup B is a Borel subgroup if and only if B is a maximal X -local[◦] subgroup for some infinite solvable subgroup X . Furthermore X can be chosen to be any, and has to be an infinite normal subgroup of B .*

Proof. Let B be a Borel subgroup of G . Then $B \leq N^\circ(X) \leq N^\circ(H(X))$ for any infinite normal subgroup X of B , and $N^\circ(H(X))$ is solvable by local[◦] solvability[◦] of G . Hence we get equality by maximality of B , and hence B is a connected X -local[◦] subgroup. If B is contained in a Y -local[◦] subgroup L of G for some infinite solvable subgroup Y , then $B \leq N^\circ(H(Y))$ and as $N^\circ(H(Y))$ is solvable by local[◦] solvability[◦] one gets $B = N^\circ(H(Y))$ again by maximality of B . As $B \leq L \leq N^\circ(H(Y))$, $B = L$.

Let now B be a maximal X -local[◦] subgroup of G for some infinite solvable subgroup X . By local[◦] solvability[◦] B is contained in a Borel subgroup B_1 . Now B_1 is Y -local[◦] for some infinite solvable subgroup Y , and the maximality of B implies that $B = B_1$.

Now if a Borel subgroup B normalizes an infinite solvable subgroup X , then $X \cdot B$ is solvable, as well as its definable hull, and by maximality $H^\circ(X) \leq B$, and $H^\circ(X)$ is an infinite normal subgroup of B . \square

Lemma 3.7 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. Then the following are equivalent.*

- (1) $N^\circ(A) < G^\circ$ for each nontrivial definable connected abelian subgroup A .
- (2) G° is not solvable.
- (3) G° has two distinct Borel subgroups.

Proof. If G° is solvable, then $G^\circ \leq N^\circ(A)$ where A is the last nontrivial term of the derived series of G , which is definable and connected by Fact 1.1. Hence the first condition implies the second one.

If G° has two distinct Borel subgroups, then clearly G° cannot be solvable.

Finally, assume $G^\circ = N^\circ(A)$ for some nontrivial definable connected abelian subgroup A . By local[◦] solvability[◦] G° is then solvable, and hence cannot have two distinct Borel subgroups. Hence the last condition implies the first one. \square

Lemma 3.7 can be refined as follows in the locally[◦] solvable case.

Lemma 3.8 *Let G be a locally[◦] solvable group of finite Morley rank. Then the following are equivalent.*

- (1) $N^\circ(A) < G^\circ$ for each nontrivial definable abelian subgroup A of G .
- (2) G° is not solvable.
- (3) G° has two distinct Borel subgroups.

Proof. As in Lemma 3.7. If $N^\circ(A) = G^\circ$ for some nontrivial definable abelian subgroup A of G , then G° is now solvable by local[◦] solvability. \square

In PSL_2 , normalizers[◦] of unipotent subgroups correspond to Borel subgroups. The following is a first approximation of this in locally[◦] solvable[◦] groups.

Lemma 3.9 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. Assume that for q prime or infinite $d_q(G) \geq 1$, and let U be a Sylow $(q, d_q(G))$ -subgroup of G . Then $N^\circ(U)$ is a Borel subgroup of G .*

Proof. By local[◦] solvability[◦] of G , $N^\circ(U) \leq B$ for some Borel subgroup B . Now Fact 2.15 implies $U \leq F^\circ(B)$, and in particular $B \leq N^\circ(U)$ by maximality of U . Hence $N^\circ(U) = B$ is a Borel subgroup of G . \square

Lemma 3.10 *Let G be a locally[◦] solvable[◦] group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and B a Borel subgroup of G such that $d_p(B) = r$. Then $U_{\tilde{p}}(B)$ is a Sylow \tilde{p} -subgroup of G .*

Proof. By Fact 2.15, $U := U_{\tilde{p}}(B)$ is in $F^\circ(B)$, and in particular is a \tilde{p} -group. It is obviously definably characteristic in B . If $U < V$ for some Sylow \tilde{p} -subgroup V of G , then $U < U_{\tilde{p}}(N_V(U))$ by normalizer condition, Fact 2.7. But as $N^\circ(U)$ is solvable by local[◦] solvability[◦] of G , and contains B , it is B by maximality of B . Hence $U < U_{\tilde{p}}(N_V(U)) \leq U_{\tilde{p}}(B) = U$, a contradiction. \square

When $r = 0$ Lemma 3.10 fails. For example, in the standard Borel subgroup B of PSL_2 over a pure algebraically closed field of positive characteristic, $U_{(\infty, 0)}(B) = B$. However the lemma becomes true for $r = 0$ if one assumes that the absolute unipotence degree of B satisfies $d(B) = 0$.

3.2 Semisimple groups

Obviously with locally solvable groups one becomes quickly interested in normal solvable subgroups.

Fact 3.11 [BN94, Theorem 7.3] *Let G be a group of finite Morley rank. Then G has a largest normal solvable subgroup, which is definable. It is denoted by $R(G)$ and called the solvable radical of G .*

Definition 3.12 *Let G be a group of finite Morley rank. We say that*

- (1) G is semisimple if $R(G) = 1$, or equivalently if $N(A) < G$ for each nontrivial abelian subgroup A of G .
- (2) G is semisimple[◦] if $R^\circ(G) = 1$, or equivalently if $N(A) < G$ for each nontrivial connected abelian subgroup A of G .

Of course, if G is any group of finite Morley rank, then $G/R(G)$ is semisimple and $G/R^\circ(G)$ is semisimple^o, as solvable-by-solvable groups are solvable.

Fact 3.13 [BN94, Lemma 6.1] *Let G be a connected group of finite Morley rank with a finite center. Then $G/Z(G)$ has a trivial center.*

The following fact has certainly been implicit in previous arguments, and we just state it precisely.

Fact 3.14 *Let G be a connected group of finite Morley rank with $R(G)$ finite. Then $R(G) = Z(G)$ and $G/R(G)$ is semisimple.*

Proof. The connected group G acts by conjugation on its finite solvable radical $R(G)$, and thus by Fact 1.2 $R(G) \leq Z(G)$. As the center is always contained in the solvable radical one gets $R(G) = Z(G)$. The semisimplicity of $G/R(G)$ is always true. \square

Lemma 3.15 *Let G be a group of finite Morley rank and H a nonsolvable definable connected subgroup of G .*

- (1) *If G is locally^o solvable^o, then H is semisimple^o, $R(H) = Z(H)$ is finite and $H/R(H)$ is semisimple.*
- (2) *If G is locally^o solvable, then H is semisimple.*

Proof. This is obvious by definitions and Fact 3.14. \square

3.3 New configurations

All the work concerning minimal connected simple groups of finite Morley rank generalizes identically to the case of locally^o solvable groups of finite Morley rank. The reason is that in the study of minimal connected simple groups every argument is based on the consideration of normalizers^o of nontrivial subgroups X . If such a subgroup X is finite, then its normalizer^o coincides with its centralizer^o by Fact 1.2.

When dealing with the more general class of locally^o solvable^o groups centralizers^o of elements of finite order might be nonsolvable. In the present papers we try to concentrate exclusively on the more general class of locally^o solvable^o groups, and hence new phenomena can appear. In the present section we try to give an overview of the new pathological configurations which might occur in this context. We see these new configurations as some kind of “speed limits” when generalizing arguments from the minimal connected simple/locally^o solvable case to the more general locally^o solvable^o case.

Recall from Lemma 3.2 that

$$\{\text{locally}^{\circ}\text{ solvable groups}\} \subsetneq \{\text{locally}^{\circ}\text{ solvable}^{\circ}\text{ groups}\},$$

the inclusion being strict. The main (and unique) example in the algebraic category of a connected group which is locally[◦] solvable[◦] but not locally[◦] solvable is $\mathrm{SL}_2(K)$, with K an algebraically closed field of characteristic different from 2: its solvable radical consists of a cyclic group of order 2.

In the context of groups of finite Morley rank there might be other configurations occurring, contradicting even the latter property a priori. In what follows we just make a list of potential pathological configurations of connected locally[◦] solvable[◦] groups of finite Morley rank which are not locally[◦] solvable, and which remain at the end of our classification.

A *full Frobenius* group is a group G with a proper subgroup H such that

$$H \text{ is malnormal in } G \text{ and } G = H^G$$

and we often use the sentence “ $H < G$ is a full Frobenius group” to specify the subgroup H . The existence of such groups of finite Morley rank is the main obstacle to the Algebraicity Conjecture for simple groups of finite Morley rank. We just record a few basic properties of such groups, if they exist.

Fact 3.16 [Jal01a, Propositions 3.3 and 3.4] *Let $H < G$ be a full Frobenius group, with G of finite Morley rank and connected. Then*

- (1) $C(x) \leq H$ and is infinite for each nontrivial element x of H .
- (2) H is definable in the pure group G and connected.
- (3) $HgH \cap Hg^{-1}H = \emptyset$ for any element g in $G \setminus H$.
- (4) $\mathrm{rk}(G) \geq 2\mathrm{rk}(H) + 1$.
- (5) *There exists a nontrivial definable simple subgroup \tilde{G} of G such that $(H \cap \tilde{G}) < \tilde{G}$ is a full Frobenius group.*

We often call a group G as in Fact 3.16 and with H nilpotent a *bad* group. (This notion is floppy.) In any case these groups have no involutions, and hence their torsion can involve only odd primes.

We view the following potential configuration of locally[◦] solvable[◦] group, or any of its natural variations, as a kind of “universal conterexample” to the algebraic case as far as torsion is concerned. Elements belonging to a decent torus are called *toral*.

Configuration 3.17 *G is a connected locally[◦] solvable[◦] group of finite Morley rank with a proper (definable connected) subgroup H such that*

- (1) $H < G$ is a full Frobenius group.
- (2) $R(H) = Z(H)$ is finite and nontrivial, consisting of p -toral elements of H for some prime p .
- (3) $H/Z(H)$ is a full Frobenius group for some proper definable connected solvable subgroup $B/Z(H)$.

- (4) $B/Z(H)$ has nontrivial p -unipotent subgroups, for some prime p dividing $|Z(H)|$, and also nontrivial q -unipotent subgroups for other primes q .

A group G as in Configuration 3.17 would have p -mixed type, i.e. containing both nontrivial p -tori and p -unipotent subgroups, and have nontrivial q -unipotent subgroups for several primes q .

In SL_2 , a generic element belongs to a maximal torus, and in particular to the connected component of its centralizer. Here is another potential new pathological phenomenon with locally[◦] solvable[◦] groups.

Configuration 3.18 G is a connected locally[◦] solvable[◦] group of finite Morley rank with a proper (definable connected) subgroup B such that.

- (1) $B < G$ is a full Frobenius group.
- (2) B is a nilpotent group such that, for x generic in B , $x \notin C^\circ(x)$.

A generic element x of a group G as in Configuration 3.18 would satisfy $x \notin C^\circ(x)$. We note that examples of connected nilpotent groups B of finite Morley rank as in clause (2) of Configuration 3.18 are provided by [BN94, §3.2.3] or the Baudisch 2-nilpotent group [Bau96]. With such subgroups B a group G as in Configuration 3.18 would be locally[◦] solvable, but if G had the prescribed property modulo a nontrivial finite center, then it would not be locally[◦] solvable.

Even with involutions and algebraic subgroups one can imagine the following configuration which seems to remain open at the end of our second paper [DJ07].

Configuration 3.19 G is a connected locally[◦] solvable[◦] group of finite Morley rank with an involution i such that $C(i) < G$ and $C(i) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic different from 2.

In [CJ04] all nonalgebraic configurations are known to have nongenerous Borel subgroups. Even assuming all Borel subgroups generous does not seem to be helpful in [DJ07] toward finding a contradiction in Configuration 3.19. This is a major new phenomenon possibly occurring in the locally[◦] solvable[◦] case as opposed to the minimal connected simple/locally[◦] solvable one.

3.4 Local[◦] solvability/solvability[◦]

In Section 3.3 we saw certain speed limits when considering generalizations to the wider class of locally[◦] solvable[◦] groups, which usually rely on the existence of certain semisimple[◦] but not semisimple groups. We nevertheless intend in this section to start dealing with these aspects in the general class of locally[◦] solvable[◦] groups of finite Morley rank, bearing in mind the speed limits of Section 3.3. For this purpose it is useful to study systematically subgroups of the form $C^\circ(x)$ in locally[◦] solvable[◦] groups. When such a subgroup is not solvable it has a finite solvable radical, which is then the center, and its quotient modulo the center is semisimple. This boils down to the study of semisimple locally[◦] solvable[◦] groups.

We start with some generalities.

Lemma 3.20 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. If H is a nonsolvable definable connected subgroup of G , then $C_G(H)$ is finite.*

Proof. Assume $C_G^\circ(H)$ infinite. Then it contains a nontrivial definable connected solvable subgroup B by Lemma 2.11. We then have $H \leq C^\circ(B) \leq N^\circ(B)$, which must be solvable by local[◦] solvability[◦] of G . \square

In a locally[◦] solvable[◦] group G of finite Morley rank, we call a subset X *exceptional* in G if $C^\circ(X)$ is nonsolvable. Such sets are finite by Lemma 3.20, and as $C(X) = C(\langle X \rangle)$ any such subset X can always be identified with the finite subgroup it generates.

Dually, we call a definable connected subgroup H *exceptional* in G if H is nonsolvable. Then $C(H)$ centralizes the nonsolvable definable connected subgroup H and is an exceptional subset of G .

We denote by \mathcal{E}_f and \mathcal{E}_s the set of finite exceptional subgroups of G and the set of nonsolvable definable connected subgroups respectively (\mathcal{E} stands for “e”xceptional, f for “f”inite, and s for “s”emisimple). Both sets are nonempty if and only if G° is nonsolvable. Of course both sets are naturally ordered by inclusion.

Taking centralizers[◦] $C^\circ(\cdot)$ from \mathcal{E}_f to \mathcal{E}_s and centralizers $C(\cdot)$ from \mathcal{E}_s to \mathcal{E}_f defines a *Galois connection* between \mathcal{E}_f and \mathcal{E}_s (see [Bir67]). That is, and following a similar exposition in [ABC08], they satisfy the following properties.

Lemma 3.21

- (1) *The mappings C° and C are order-reversing.*
- (2) *If $X \in \mathcal{E}_f$ then $X \leq C(C^\circ(X))$ and if $H \in \mathcal{E}_s$ then $H \leq C^\circ(C(H))$.*

As in any Galois connection, this has the following consequence.

Proposition 3.22 *Let $X \in \mathcal{E}_f$ and $H \in \mathcal{E}_s$. Then $C^\circ(X) = C^\circ(C(C^\circ(X)))$ and $C(H) = C(C^\circ(C(H)))$.*

If we denote for X in \mathcal{E}_f and H in \mathcal{E}_s

$$\overline{X} = C(C^\circ(X)) \text{ and } \overline{H} = C^\circ(C(H)),$$

then the two operations $\overline{}$ are *closure operations* on \mathcal{E}_f and \mathcal{E}_s respectively. That is, they satisfy the following.

Corollary 3.23

- (1) *For $X \in \mathcal{E}_f$ and $H \in \mathcal{E}_f$, we have $X \leq \overline{X} = \overline{\overline{X}}$ and $H \leq \overline{H} = \overline{\overline{H}}$.*
- (2) *Monotonicity: For $X_1 \subseteq X_2$ in \mathcal{E}_f and $H_1 \leq H_2$ in \mathcal{E}_s , we have $\overline{X_1} \leq \overline{X_2}$ and $\overline{H_1} \leq \overline{H_2}$.*

The closed elements of \mathcal{E}_f and \mathcal{E}_s are those of the form \overline{X} and \overline{H} respectively.

One can also refine Lemma 3.20 by giving a uniform bound on cardinals of elements of \mathcal{E}_f . We first note the following general fact.

Lemma 3.24 *Let G be a group of finite Morley rank. Then there exists a natural number m such that, for any subset X of G , $|C(X)| \leq m$ or $C(X)$ is infinite.*

Proof. As G is stable, it satisfies the Baldwin-Saxl chain condition [Poi87, §1.3]. This means that there exists a fixed integer k such that, for every subset X of G , $C(X) = C(x_1, \dots, x_k)$ for some elements x_1, \dots, x_k of X , and the family of all subgroups of the form $C(X)$ is uniformly definable (by a formula without parameters).

Now the uniform bound m on the cardinals of the finite sets of the family is provided by elimination of infinite quantifiers [BC02, Proposition 2.2]. \square

Lemma 3.25 *Let G be a locally^o solvable^o group of finite Morley rank. Then there exists a natural number m bounding uniformly the cardinals of finite exceptional subsets of G .*

Proof. Let m be as in Lemma 3.24. If X is a finite exceptional subset of G , then $C^\circ(X)$ is nonsolvable and $C(C^\circ(X))$ is finite by Lemma 3.20. As $C(C^\circ(X))$ is a finite centralizer, its cardinal is uniformly bounded by m . Now $X \subseteq C(C^\circ(X))$, and thus the cardinal of X is uniformly bounded by m . \square

If G is a locally^o solvable^o group of finite Morley rank, we call *exception index* and denote by $e(G)$ the maximal integer m such that G has an exceptional nonsolvable definable connected subgroup centralizing a subset X with m elements. Notice that X coincides with $\langle X \rangle$, so that $e(G)$ is the largest cardinal of an exceptional subgroup in \mathcal{E}_f .

Maximal exceptional subgroups of \mathcal{E}_f correspond to minimal exceptional subgroups of \mathcal{E}_s , and vice-versa. A case of particular interest is the following.

Lemma 3.26 *Minimal nontrivial exceptional subgroups are cyclic of prime order.*

Proof. Obvious. \square

One can clarify the structure of elements of \mathcal{E}_s as follows.

Lemma 3.27 *Let G be a locally^o solvable^o group of finite Morley rank and H an exceptional nonsolvable definable connected subgroup. Then H is semisimple^o, $R(H) = Z(H)$ is finite and $H/R(H)$ is semisimple.*

Proof. Lemma 3.15 (1). □

Of course, the sets of closed sets in \mathcal{E}_f and \mathcal{E}_s are at most reduced to $\{1\}$ and $\{G^\circ\}$ in the locally[◦] solvable case (in case G° is nonsolvable, and empty otherwise).

The next lemma seems to be the only way to get locally[◦] solvable groups out of locally[◦] solvable[◦] ones.

Lemma 3.28 *Let G be a locally[◦] solvable[◦] group of finite Morley rank and H a nonsolvable definable connected subgroup exceptional in G , which is minimal with respect to this property. Then $H/R(H)$ is locally[◦] solvable.*

Proof. The conclusions of Lemma 3.27 are valid in H and we will use them freely. Denote by $\bar{}$ the quotients by $R(H)$, and let A be the preimage in H of a nontrivial definable abelian subgroup \bar{A} of \bar{H} . Of course A is a definable solvable subgroup of H .

Let N be the preimage of $N_{\bar{H}}^\circ(\bar{A})$ in H . We have $N \leq N(A)$ and $\bar{N}^\circ = \bar{N}^\circ$, so that $N = N^\circ R(H)$.

If A is infinite modulo $R(H)$, then A is infinite as well, and as $N^\circ \leq N^\circ(A)$ we get N° solvable by local[◦] solvability[◦] of G and Lemma 3.2 (1). Then $N = N^\circ R(H)$ is solvable, as well as $N_{\bar{H}}^\circ(\bar{A})$.

If A is finite modulo $R(H)$, then A is finite as well as $R(H)$ is. Now N° acts on the nontrivial finite group A , and therefore centralizes it by Fact 1.2. The minimality of H yields N° solvable or $N^\circ = H$. In the first case one concludes that $N = N^\circ R(H)$ is solvable, as well as $N_{\bar{H}}^\circ(\bar{A})$. The second case implies that $A \leq Z(H) = R(H)$, and is thus impossible as A is nontrivial modulo $R(H)$. □

Lemma 3.28 seems to be a very rough indication that the new locally[◦] solvable[◦] groups which are not locally[◦] solvable are more or less as in Configuration 3.17.

We also note that exceptional nonsolvable definable connected subgroups attached to a nontrivial finite exceptional subgroup are of finite index in their normalizers.

Lemma 3.29 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. If X is an exceptional finite subset of G , then $N^\circ(C^\circ(X)) = C^\circ(X)$.*

Proof. $N^\circ(C^\circ(X))$ normalizes $C(C^\circ(X))$, which is finite and contains X . So it centralizes X by Fact 1.2, and we are done. □

A natural question is to know whether exceptional finite subsets X are contained in their attached exceptional nonsolvable definable connected subgroups, i.e. whether $X \subseteq C^\circ(X)$. This would follow from the more general, but similarly natural, question to know whether nonsolvable definable connected subgroups

are selfnormalizing. This is the kind of problem which seems optimistically trackable when $C^\circ(X)$ is generous in the ambient group, since the intensive experience on Weyl groups from [CJ04], and we will get positive answers in the most interesting situations in Section 4.2 below.

We are now going to look more closely at the interesting case in which an exceptional finite subgroup X of \mathcal{E}_f satisfies $X \leq C^\circ(X)$. In this case $X \leq Z(C^\circ(X))$, and X is in particular an abelian finite subgroup. Typical finite abelian groups belonging to the connected component of their centralizers are the finite subgroups of decent tori. (And this is in general not true around groups of bounded exponent, as noticed after Configuration 3.18.)

Lemma 3.30 *Let G be a locally[◦] solvable[◦] group of finite Morley rank and T a maximal definable decent torus of G . Then the union of elements of \mathcal{E}_f contained in T is finite and invariant under any automorphism of G leaving T invariant.*

Proof. For the finiteness we can use Lemma 3.25 to get a uniform bound, at most the exception index $e(G)$ of G , on the cardinals of the finite groups involved. Then, as Prüfer p -ranks are finite for any prime p in a decent torus, subgroups of order at most $e(G)$ must be contained in a finite subgroup of T .

The second point is obvious. \square

A question, which might be difficult, is to know whether the union in Lemma 3.30 is necessarily a (finite) subgroup of T , and is itself exceptional. If this were the case, then calling this group E , one would have a nonsolvable group $C^\circ(E)/R(C^\circ(E))$ where nontrivial toral elements are not exceptional anymore. This is a desirable property for certain questions such as bounding Prüfer ranks, as we will see later, in our treatment of odd type groups [DJ07]. This desirable property can however be obtained as follows.

Lemma 3.31 *Let G be a connected nonsolvable locally[◦] solvable[◦] group of finite Morley rank, and T a maximal definable decent torus of G . Then G has an exceptional nonsolvable definable connected subgroup H containing T and such that $C^\circ(\bar{t})$ is solvable for any nontrivial toral element \bar{t} of $H/R(H)$.*

Proof. Let X be a maximal exceptional finite subgroup of T . Then $H = C^\circ(X)$ is nonsolvable. As $X \leq T$ and T is abelian and connected, $X \leq T \leq H$.

Let now \bar{t} be a nontrivial toral element of $\bar{H} = H/R(H)$. By pullback of decent tori, Fact 2.4 (2) or rather [Fré06b, Lemma 3.1], and Fact 2.19, we may assume t in $TR(H)$, i.e. $t = t'r$ for some $t' \in T$ and some $r \in R(H)$. As in Lemma 3.28, one sees that the preimage of the centralizer[◦] of t modulo $R(H)$ cannot be nonsolvable: otherwise its connected component would centralize $t = t'r$, and as $r \in R(H) = Z(H)$ it would centralize t' , so that $X\langle t' \rangle$ would be an exceptional finite subgroup of T containing X properly, a contradiction. This finishes our proof. \square

Before moving ahead we close the present section by describing more precisely the set of exceptional subsets of a decent torus T as in Lemma 3.31, or more generally of an *arbitrary* subset T of a locally^o solvable^o group G of finite Morley rank.

First we naturally consider the notion of closure relative to T . For X an exceptional subset of T , we say that X is closed *in* T if $X = C_T(C^\circ(X))$. Of course the notion of relative closedness is robust.

Remark 3.32 *Any set of the form $C_T(C^\circ(X))$ is closed in T .*

Proof. As $X \subseteq C_T(C^\circ(X)) \subseteq \overline{X}$, $\overline{X} = C(C^\circ(C_T(C^\circ(X))))$ by taking the closure in G , and $C_T(C^\circ(C_T(C^\circ(X)))) = C_T(C^\circ(X))$ by taking the intersection with T . \square

The poset of exceptional subsets of T is best described as follows by the notion of minimal extensions of closed subsets. We say that (X_1, X_2) is a *minimal* extension of closed sets of T if $X_1 \subsetneq X_2$ are two exceptional subsets of T closed in T and any closed subset Y of T such that $X_1 \subseteq Y \subseteq X_2$ is either X_1 or X_2 . The relation “ (X_1, X_2) is a minimal extension of closed sets of T ” defines an oriented graph on the set of closed sets of T , which is clearly irreflexive, antisymmetric, and loop-free, that is without cycles preserving the orientation (but possibly with cycles not preserving the orientation). We call this graph the *graph of exceptional subsets of T* . Its main properties are the following.

Lemma 3.33 *Let G be a locally^o solvable^o group of finite Morley rank and T an arbitrary subset of G .*

- (1) *Assume (X_1, X_2) is a minimal extension in the graph of exceptional subsets of T and Y is a subset such that $X_1 \subsetneq Y \subseteq X_2$. Then $C_T(C^\circ(Y)) = X_2$. Moreover $C^\circ(X_2) < C^\circ(X_1)$.*
- (2) *Assume (X, X_1) and (X, X_2) are two minimal extensions in the graph of exceptional subsets of T . Then either $X_1 = X_2$ or $X_1 \cap X_2 = X$.*

Proof. (1). $X_1 \subsetneq Y \subseteq C_T(C^\circ(Y)) \subseteq X_2$ and as $C_T(C^\circ(Y))$ is closed in T by Remark 3.32 it must be X_2 by minimality of the extension (X_1, X_2) .

The claim that $C^\circ(X_2) < C^\circ(X_1)$ follows merely from the fact that $X_1 \neq X_2$ are closed in T .

(2). Let $Y = X_1 \cap X_2$. If $X \subsetneq Y$, then the first point implies that $X_1 = C_T(C^\circ(Y)) = X_2$. \square

Finally, we note that the graph of exceptional subsets of T as in Lemma 3.33 always has a “minimal” element, namely $T \cap Z(G)$, and “maximal” elements, corresponding to the maximal traces on T of exceptional sets in \mathcal{E}_f , which are of cardinal at most $e(G)$. We also note that the graph has a *finite height*: the length of a maximal chain of exceptional closed sets in T is at most $e(G)$.

When T is a nilpotent divisible subgroup of G (for example as in Lemmas 3.30 and 3.31), then exceptional subsets of T are necessarily in a same decent torus (the maximal decent torus of the definable hull of T) and by Lemma 3.30 applied in this decent torus the graph of exceptional subsets of T is finite.

3.5 Genericity

Fact 3.34 (Compare with [FJ08, Theorem 7.3]) *Let G be a locally solvable^o group of finite Morley rank with a nontrivial decent torus T , and Q a Carter subgroup of G containing T . Then Q is generous in G° , and $T \leq \tilde{T} \leq Q$ for some maximal definable decent torus \tilde{T} of G .*

Proof. The existence of Q is guaranteed by Fact 2.13, as decent tori are of minimal unipotence degree.

By Fact 2.19, $C^\circ(T)$ is generous in G° . Now $C^\circ(T)$ is solvable by local solvability^o of G , and the Carter subgroup Q is generous in $C^\circ(T)$ by Fact 2.26. It follows that Q is generous in G° by the transitivity of generosity provided in [Jal06, Lemma 3.9].

Doing the same argument as above for a maximal definable decent torus \tilde{T} containing T , one gets a generous Carter subgroup \tilde{Q} of G° containing \tilde{T} , and as generous Carter subgroups are conjugate by Fact 2.21 one gets that Q contains a maximal definable decent torus, which necessarily contains T . \square

We record here an application of Lemma 2.36 in the case of locally solvable groups of finite Morley rank. This will be the *clé de voûte* for a concentration argument in one of the most prominent theorem on odd type groups in [DJ07].

Lemma 3.35 (Compare with [Del07a, Corollaire 2.4]) *Let G be a group of finite Morley rank and X a nonempty definable G° -invariant subset of G° . Let M be a definable solvable subgroup of G° such that $X \cap M$ is generic in X .*

- (1) *If G is locally solvable and $X \neq \{1\}$, then G° is solvable.*
- (2) *If G is locally solvable^o and X is infinite, then G° is solvable.*

Proof. (1). Let Y be the definable G° -invariant subset of $X \cap M$ generic in X provided by Lemma 2.36. As X is nonempty, Y is also nonempty, and $G^\circ = N^\circ(\langle Y \rangle)$. Now $\langle Y \rangle$ is a subgroup of M , and hence is solvable. If it is nontrivial, then G° must be solvable by Lemma 3.4 (3). Otherwise, $\{1\}$ is a generic subset of X , and X must be finite. Hence X is a finite set of finite conjugacy classes, with one nontrivial by assumption. This nontrivial finite G° -conjugacy class must be central in G° by Fact 1.2, and as G° has then a nontrivial center it must again be solvable by local solvability.

(2). One argues in the same way. Now, as X is infinite, Y is also infinite by genericity. As $G^\circ = N^\circ(\langle Y \rangle)$ is $\langle Y \rangle$ -local^o with $\langle Y \rangle$ infinite and solvable, as contained in M , Lemma 3.4 (4) now gives the solvability of G° . \square

4 Local analysis

We now proceed to the *local analysis* of locally solvable groups of finite Morley rank, that is the analysis of intersections of their (most interesting) subgroups.

In Section 4.1 we deal with a series of results which correspond to the Bender method in finite group theory. In general these lemmas say in our context that sufficiently unipotent subgroups of locally solvable groups of finite Morley rank are disjoint, like unipotent subgroups in PSL_2 or SL_2 . They are the main tool for analyzing locally solvable groups, notably the only trick involving unipotence in the recognition of PSL_2 in the algebraic parts of our second paper [DJ07]. The original form was first proved in the context of minimal connected simple groups in the unpublished [Jal00]. It was in a form embryonal in characteristic 0 compared to the one provided later by the general abstract unipotence theory of Burdges as in Section 2.1, but both in positive and null characteristic. Then they appeared in the tame context in [CJ04, Section 3.4] where they were treated essentially as the positive characteristic case, i.e. involving no particular graduation in the unipotence theory. The positive characteristic case was recalled as the outline of [Bur07], and later the characteristic 0 case appeared in [Del07a, §3.2] for the recognition of PSL_2 in characteristic different from 2. We are going to give forms of these lemmas entirely uniform in the unipotence degrees, in particular independent of the characteristics, and in the most general context of locally[◦] solvable[◦] groups.

Section 4.3 will then concern the situation in which a particular consequence of such uniqueness theorems fails. This is a priori a possibility when the subgroups considered are not unipotent enough relative to the ambient group. The pathological situation appearing can be analyzed somehow by replacing the maximality in terms of unipotence degrees by a maximality for inclusion concerning a pair of Borel subgroups involved. The endless, but very precise, description resulting is the bulk of [Bur07], and in the context of locally[◦] solvable[◦] groups we will follow the exposition of this paper *verbatim*. This full description will be applied one time in a nonalgebraic situation in [DJ07] and that's why we need to restate, slightly more generally but in its full detail, this analysis from [Bur07].

4.1 Uniqueness Theorem

4.1.1 The main theorem

The following Uniqueness Theorem is our analog of the Bender method in groups of finite Morley rank and is the main tool for analyzing locally[◦] solvable[◦] groups of finite Morley rank. There are various forms of this theorem but the present one seems to be the most relevant, at least for our applications in [DJ07]. Its consequences on Borel subgroups in Section 4.1.2 below will be the closest analogs of the Uniqueness Theorem of Bender in finite group theory [Ben70a] [Ben70b] [Gag76, §5-7] [BG94, Chapter II]

Theorem 4.1 *Let G be a locally[◦] solvable[◦] group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and U a Sylow \tilde{p} -subgroup of G . Assume that U_1 is a nontrivial definable \tilde{p} -subgroup of U containing a nonempty (possibly trivial) subset X of G such that $d_p(C^\circ(X)) \leq r$. Then U is the unique Sylow \tilde{p} -subgroup of G containing U_1 , and in particular $N(U_1) \leq N(U)$.*

Before the proof, a few remarks.

- (1) If G° is solvable and $r = d_p(G) > 0$, then assuming that $F^\circ(G^\circ)$ has a nontrivial normal definable \tilde{p} -subgroup U_1 one gets with Theorem 4.1 (applied with $X = 1$ for example!) that G° has a unique Sylow \tilde{p} -subgroup, which is thus normal and contained in $F^\circ(G^\circ)$. In the event of the absence of such a subgroup U_1 one easily gets the same conclusion with Fact 2.12 (3) and (4). Hence in some sense Theorem 4.1 can be seen as a generalization from solvable groups to locally[◦] solvable[◦] groups of Fact 2.15.
- (2) The nontriviality of U_1 is needed in Theorem 4.1, as in a hypothetical \tilde{p} -homogeneous semisimple bad group the trivial subgroup would be contained in infinitely many conjugates of the Sylow \tilde{p} -subgroup.
- (3) Theorem 4.1 fails if $\tilde{p} = (\infty, 0)$. For example if G is of the form $T \times (U \rtimes T)$, with U p -unipotent for some prime p and T a good torus, whose second copy acts faithfully on U , then $d_\infty(G) = 0$, so that all assumptions of Theorem 4.1 are satisfied with U_1 the central copy of T , but the latter is contained in infinitely many conjugates of the maximal good torus $T \times T$. We will give in Lemma 4.2 below a version of Theorem 4.1 specific for the unipotence parameter $\tilde{p} = (\infty, 0)$, by replacing $d_p(C^\circ(X))$ by the absolute unipotence degree $d(C^\circ(X))$ but with no more local solvability assumption.

After these comments we pass to the proof of Theorem 4.1.

Proof. Assume V is a Sylow \tilde{p} -subgroup of G distinct from U and containing U_1 , and chosen so as to maximize the rank of $U_{\tilde{p}}(U \cap V)$. Let T denote $U \cap V$. As $U_1 \leq T$, the subgroup T is infinite. As T is nilpotent, $N := N^\circ(T)$ is solvable by local[◦] solvability[◦] of G and Lemma 3.4 (4). Notice that $T < U$, as otherwise $U = (U \cap V) \leq V$ and $U = V$ by maximality of U . Similarly $T < V$, as otherwise $V = (U \cap V) \leq U$ and $V = U$ by maximality of V . In particular by normalizer condition, Fact 2.7, $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_U(T))$ and $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_V(T))$.

We claim that $d_p(N) = r$. If $d_p(N) > r$, then $r < \infty$, $p = \infty$, and N contains a nontrivial Sylow (∞, r') -subgroup Σ with $r' > r$. Notice that $d_p(T) \leq r$ by Corollary 2.3 and Fact 2.5 and our assumption that the subset X of T satisfies $d_p(C(X)) \leq r$. Then $T \cdot \Sigma$ is nilpotent by Fact 2.12 (3) and (4), and T commutes with Σ by Fact 2.5. In particular Σ commutes with X and $d_p(C^\circ(X)) \geq r' > r$, a contradiction to our assumption. Hence $d_p(N) \leq r$, and as N contains $U_{\tilde{p}}(N_U(T))$ (or $U_{\tilde{p}}(N_V(T))$) which is nontrivial and of unipotence degree r we get $d_p(N) = r$.

By Fact 2.15 and the assumption that $r \geq 1$ we get $U_{\tilde{p}}(N) \leq F^\circ(N)$. In particular $U_{\tilde{p}}(N)$ is nilpotent, and contained in a Sylow \tilde{p} -subgroup Γ of G . Now $U_1 \leq U_{\tilde{p}}(T) < U_{\tilde{p}}(N_U(T)) \leq \Gamma$, so our maximality assumption on V implies that $\Gamma = U$. In particular $U_{\tilde{p}}(N_V(T)) \leq \Gamma = U$. But then $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_V(T)) \leq U_{\tilde{p}}(U \cap V) = U_{\tilde{p}}(T)$, a contradiction which finishes the proof of our first statement.

The inclusion $N(U_1) \leq N(U)$ follows from the uniqueness. \square

We conclude with a version of Theorem 4.1 specific for the unipotence parameter $\tilde{p} = (\infty, 0)$, and which indeed does not rely on local solvability.

Lemma 4.2 *Let G be a group of finite Morley rank, T a maximal definable decent torus of G , and x an element of T such that $C^\circ(x)$ is solvable and $d(C^\circ(x)) \leq 0$. Then T is the unique maximal definable decent torus of G containing x , and in particular $N(\langle x \rangle) \leq N(T)$.*

Proof. By assumption and Lemma 2.11 (2), $C^\circ(x)$ is a good torus. As $x \in T$ and T is connected abelian, $T \leq C^\circ(x)$, and $T = C^\circ(x)$ by maximality of T . Now any maximal definable decent torus containing x must be in $C^\circ(x)$ for the same reason, hence in T , and hence equal to T by maximality of T . Furthermore, $N(\langle x \rangle) \leq N(C^\circ(x)) = N(T)$. \square

4.1.2 Consequences on Borel subgroups

Applied to the case of Borel subgroups Theorem 4.1 has the following corollaries. These can be seen as absolute approximations in the context of locally^o solvable^o groups of finite Morley rank of the fact that any unipotent subgroup of PSL_2 belongs to a unique Borel subgroup of the ambient group.

Corollary 4.3 *Let G be a locally^o solvable^o group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and B a Borel subgroup of G such that $d_p(B) = r$. Let U_1 be a nontrivial definable \tilde{p} -subgroup of $U_{\tilde{p}}(B)$ containing a nonempty subset X such that $d_p(C^\circ(X)) \leq r$. Then $U_{\tilde{p}}(B)$ is the unique Sylow \tilde{p} -subgroup of G containing U_1 , and in particular $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$. Furthermore, B is the unique Borel subgroup containing U_1 and admitting \tilde{p} as a unipotence parameter maximal in its characteristic.*

Proof. The fact that $U_{\tilde{p}}(B)$ is a Sylow \tilde{p} -subgroup of G is Lemma 3.10. The uniqueness of $U_{\tilde{p}}(B)$ among Sylow \tilde{p} -subgroups containing U_1 , as well as the inclusion $N(U_1) \leq N(U_{\tilde{p}}(B))$, is then Theorem 4.1.

Let now B_1 be a Borel subgroup of G containing U_1 and admitting $\tilde{p} = (p, r)$ as maximal in its characteristic. Notice that $U_{\tilde{p}}(B_1)$ is a Sylow \tilde{p} -subgroup of G by Lemma 3.10. As it contains U_1 , Theorem 4.1 now implies $U_{\tilde{p}}(B_1) = U_{\tilde{p}}(B)$. Now the normalizers^o of these (equal) groups are solvable by local^o solvability^o of G , contain B_1 and B respectively, hence are equal to B_1 and B respectively.

by maximality, and are equal. Hence $B_1 = B$, as desired for our second claim. \square

PSL_2 in positive characteristic offers a counterexample to Corollary 4.3 when $r = 0$. It suffices to consider for U_1 a maximal torus of the standard Borel subgroup B , so that $N(T) \not\leq N(B)$ and $T \leq B^w$ where w is a nontrivial Weyl group element associated to T . For the case $r = 0$ we refer to Lemma 4.2.

Corollary 4.3 takes the following form when (p, r) is maximal in its characteristic over the whole ambient group G .

Corollary 4.4 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$ such that $d_p(G) = r$. Let B be a Borel subgroup of G such that $d_p(B) = r$. Then $U_{\tilde{p}}(B)$ is a Sylow \tilde{p} -subgroup of G , and if U_1 is a nontrivial definable \tilde{p} -subgroup of B , then $U_{\tilde{p}}(B)$ is the unique Sylow \tilde{p} -subgroup of G containing U_1 , $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$, and B is the unique Borel subgroup of G containing U_1 .*

Proof. Let $X = 1$. Then $d_p(C^\circ(X)) = d_p(G) = r$, so Corollary 4.3 applies with $X = 1$. Notice that \tilde{p} is maximal in its characteristic for any Borel subgroup admitting it, and that when U_1 is a nontrivial definable \tilde{p} -group then any Borel subgroup containing it admits \tilde{p} . \square

As for Corollary 4.3, PSL_2 in positive characteristic offers a counterexample when $r = 0$ in Corollary 4.4.

The preceding uniqueness theorems are often used as follows to “fusion” Borel subgroups sharing too unipotent subgroups.

Lemma 4.5 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank. Assume that, for $i = 1$ and 2 , $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters with $r_i > 0$ and B_i are two distinct Borel subgroups of G such that $d_{p_i}(B_i) = r_i$. Then there is no Borel subgroup B_3 of G such that $d_{p_i}(B_i \cap B_3) = d_{p_i}(B_3) = r_i$ and $d_{p_i}(C^\circ(U_{\tilde{p}_i}(B_i \cap B_3))) \leq r_i$.*

Proof. Assume the contrary. Applying Corollary 4.3 with $U_1 = X = U_{\tilde{p}_i}(B_1 \cap B_3)$ implies that $B_1 = B_3$, and with $U_1 = X = U_{\tilde{p}_i}(B_2 \cap B_3)$ that $B_2 = B_3$. Hence $B_1 = B_2$, a contradiction. \square

We finish with a version of Lemma 4.5 concerning the case in which the unipotence degrees r_i ’s are maximized over the whole ambient group.

Lemma 4.6 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank. Assume that, for $i = 1$ and 2 , $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters with $r_i > 0$ and B_i are two distinct Borel subgroups of G such that $d_{p_i}(G) = d_{p_i}(B_i) = r_i$. Then there is no Borel subgroup B_3 of G such that $d_{p_i}(B_i \cap B_3) = r_i$.*

Proof. Under the stated assumptions $d_{p_i}(B_i) = r_i$. If there was a contradicting Borel subgroup B_3 , then $d_{p_i}(B_i \cap B_3) = r_i = d_{p_i}(B_i)$ and $d_{p_i}(C^\circ(U_{\tilde{p}_i}(B_i \cap B_3))) \leq r_i$, a contradiction to Lemma 4.5. \square

Again PSL_2 in positive characteristic offers counterexamples to Lemmas 4.5 and 4.6 when $r_i = 0$, as we may take for B_1 and B_2 two distinct conjugates of the standard Borel subgroup B and for B_3 any of these two.

4.1.3 Consequences on Fitting subgroups

The first paragraph of the proof of the following lemma appeared as [Bur07, Corollary 2.2].

Lemma 4.7 *Let G be a locally^o solvable^o group of finite Morley rank. If B_1 and B_2 are two distinct Borel subgroups and X denotes $F(B_1) \cap F(B_2)$, then X° is torsion free, $X = X^\circ \times S$ for a finite subgroup S , and for any subgroup S_1 of X $C^\circ(S_1)$ is nonsolvable if and only if $S_1 \leq S$.*

Proof. Assume X° not torsion free. Then it contains a nontrivial decent torus T or a nontrivial p -unipotent subgroup U . In the first case, $T \leq Z(B_1) \cap Z(B_2)$ by Fact 2.12 (1), $B_1 = N^\circ(T) = B_2$ by local^o solvability^o and Lemma 3.6, a contradiction. In the second case Corollary 4.4 with $\tilde{p} = (p, \infty)$ and $X = U$ yields $B_1 = B_2$, again a contradiction.

We have now $X = X^\circ \times S$ for some finite subgroup S of X by Fact 2.5.

Let S_1 be a subgroup of X . If $S_1 \not\leq S$, then S_1 contains an element of the form $s \cdot x$ for some s in S and some nontrivial element x in X° . As X° is torsion-free, x has infinite order, as well as $s \cdot x$, and $C^\circ(S_1) \leq C^\circ(H(s \cdot x)) \leq N^\circ(H^\circ(s \cdot x))$, which is solvable by local^o solvability^o of G . Hence $C^\circ(S_1)$ nonsolvable implies $S_1 \leq S$.

We now want to show that if $S_1 \leq S$, then $C^\circ(S_1)$ is nonsolvable. It suffices to do it for S , so we assume toward a contradiction $C^\circ(S)$ solvable. Let B_3 be a Borel subgroup of G containing $C^\circ(S)$. Notice that the finite nilpotent group S is the product of its Sylow p -subgroups. If p_1 and p_2 are two (not necessarily distinct) primes dividing the order of S , then we claim that one cannot have $U_{p_1}(B_1) \neq 1$ and $U_{p_2}(B_2) \neq 1$. Assume the contrary. As Sylow subgroups for primes different from p_1 in $F(B_1)$ commute with $U_{p_1}(B_1)$ (by Fact 2.5 (1)!), $U_{p_1}(B_1 \cap C^\circ(S))$ is nontrivial by Fact 2.24 (2). Similarly, $U_{p_2}(B_2 \cap C^\circ(S))$ is nontrivial. Now Lemma 4.6 gives a contradiction, which proves our claim. It follows that all nontrivial p -unipotent subgroups of B_1 or B_2 , for p dividing the order of S , are on one side, say they are all in B_1 . Notice then that all p -unipotence blocks of B_2 , for p dividing the order of S , are trivial. In particular $S \leq Z(B_2)$ by Lemma 2.25. Hence $B_2 \leq C^\circ(S) \leq B_3$, $B_3 = B_2$, and $C^\circ(S) = B_2$. Hence one cannot have $C^\circ(S) = B_1$, as $B_1 \neq B_2$. Hence S is not central in B_1 . By Lemma 2.25, there is a prime p dividing the order of S and such that $U_p(B_1) \neq 1$. As above, $U_p(C_{B_1}^\circ(S))$ is nontrivial by Fact 2.5 (1) and Fact 2.24 (2), and Corollary 4.4 gives then $B_1 = B_2$, a contradiction. \square

A subgroup S as in Lemma 4.7 could for example be the subgroup $Z(H)$ in the hypothetical Configuration 3.17.

We mention, parenthetically, that it is a version of the following lemma which has been baptized “Jaligot’s lemma” in [Bur07, §2] (see [CJ04, §3.4] and [Bur07, Lemma 2.1]).

Lemma 4.8 *Let G be a locally[◦] solvable[◦] group of finite Morley rank. Assume that, for $i = 1$ and 2 , $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters such that $d_{p_i}(G) = r_i$, and B_i are two distinct Borel subgroups such that $d_{p_i}(B_i) = r_i$. If X denotes $F(B_1) \cap F(B_2)$, then X is finite and $C^\circ(S_1)$ is nonsolvable for any subgroup S_1 of X .*

Proof. Assume X° non-trivial. By local[◦] solvability[◦] of G , $N^\circ(X)$ is solvable and hence contained in a Borel subgroup B_3 of G . As X° is torsion-free by Lemma 4.7, the assumption that $d_{p_i}(B_i) = r_i$ implies $r_i > 0$ for each i . By Fact 2.15, $U_{\tilde{p}_i}(B_i)$ is in $F^\circ(B_i)$, and by Fact 2.5, Γ_i , the last nontrivial iterated term of the descending central series of $U_{\tilde{p}_i}(B_i)$, is central in $F^\circ(B_i)$. Hence $\Gamma_i \leq N^\circ(X) \leq B_3 \cap B_i$. Now by assumption each Γ_i is nontrivial, and a \tilde{p}_i -group by Corollary 2.3. Corollary 4.4 implies that each Γ_i is contained in a unique Borel subgroup of G , which gives $B_3 = B_1$ and $B_3 = B_2$, contradicting the assumption that $B_1 \neq B_2$. Hence X is finite.

Our last claim is contained in Lemma 4.7. □

In absence of local[◦] solvability one might have $F(B_1) \cap F(B_2)$ (finite and nontrivial in Lemma 4.8, as for example in Configuration 3.17 again.

4.1.4 Consequences on soapy subgroups

We continue as in Sections 4.1.2 and 4.1.3 with consequences of the Uniqueness Theorem 4.1, now on soapy subgroups. All these properties make us think of a soap sliding between two hands, exactly like a unipotent subgroup which cannot be contained in two distinct Borel subgroups in PSL_2 . The following lemmas will be used in our most critical computations in [DJ07].

Lemma 4.9 *Let G be a locally[◦] solvable[◦] group of finite Morley rank, B_1 and B_2 two Borel subgroups each having a soapy subgroup U_1 and U_2 respectively. Then*

- (1) B_1 is unique among Borel subgroups of G containing U_1 and admitting the unipotence parameter of U_1 as maximal.
- (2) If $[U_1, U_2] = 1$, then $B_1 = B_2$.

Proof.

(1). By local[◦] solvability[◦] of G , $N^\circ(U_1)$ is solvable. As U_1 is normal in B_1 , the maximality of B_1 implies $N^\circ(U_1) = B_1$. If the unipotence parameter of U_1 is $(\infty, 0)$, then B_1 is a good torus, as well as any Borel subgroup admitting

$(\infty, 0)$ as maximal. So any such Borel subgroup is contained in $C^\circ(U_1) = B_1$, and thus equal to B_1 . Otherwise, as $C^\circ(U_1) \leq N^\circ(U_1)$, the first item is a mere application of Corollary 4.3.

(2). Again $N^\circ(U_1) = B_1$ and similarly $N^\circ(U_2) = B_2$. Hence $U_1, U_2 \leq B_1 \cap B_2$ under the assumption that U_1 and U_2 commute. If U_1 is a good torus, then as for the first item B_1 is a good torus as well, as well as its subgroup U_2 , and similarly B_2 also. We then get $B_2 \leq C^\circ(U_1) \leq N^\circ(U_1) = B_1$, and equality of B_1 and B_2 . One concludes symmetrically when U_2 is a good torus, so one can assume that both U_1 and U_2 are not good tori. As $U_1, U_2 \leq B_1 \cap B_2$, Corollary 4.4 gives $B_1 = B_2$ or $\max(d(U_1), d(U_2)) < \infty$. In any case Corollary 4.3 gives $B_1 = B_2$. \square

The following lemma allows one to build soapy subgroups in presence of two Borel subgroups.

Lemma 4.10 *Let G be a locally^o solvable^o group of finite Morley rank, B_1 and B_2 two Borel subgroups, and U_1 a soapy subgroup of B_1 . If $U_1 \leq B_2$, then B_2 contains a characteristically soapy subgroup.*

Proof. If $B_1 = B_2$, then U_1 is a soapy subgroup of B_2 and we may use Lemma 2.17.

Assume now $B_1 \neq B_2$, and let \tilde{q}_1 be the unipotence parameter attached to U_1 . Let \tilde{q}_2 be a unipotence parameter maximal for B_2 . If $\tilde{q}_2 = (\infty, 0)$, then B_2 is a good torus, as well as U_1 , as well as B_1 , and then one concludes as usual by local^o solvability^o that $B_1 = B_2$. Hence \tilde{q}_2 is not $(\infty, 0)$. If $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ is not central in B_2 , then we may apply Lemma 2.18.

So now assume toward a contradiction $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ central in B_2 . In particular $U_{\tilde{q}_2}(Z(F^\circ(B_2))) \leq C^\circ(U_1) \leq N^\circ(U_1) = B_1$ by local^o solvability^o of G . By Corollary 4.4, \tilde{q}_1 and \tilde{q}_2 do not represent subgroups of bounded exponent, as $B_1 \neq B_2$. The maximality of \tilde{q}_1 for B_1 and of \tilde{q}_2 for B_2 then yields $\tilde{q}_1 = \tilde{q}_2$. But Corollary 4.3 gives the uniqueness of B_2 among Borel subgroups containing $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ and admitting \tilde{q}_2 as maximal. Thus $B_1 = B_2$, a contradiction in the last case under consideration. \square

4.1.5 Consequences on Carter subgroups

Theorem 4.1 also gives information on Carter subgroups possessing a subgroup sufficiently unipotent relatively to the ambient group.

Lemma 4.11 *Let G be a locally^o solvable^o group of finite Morley rank, Q a Carter subgroup of G and $\tilde{p} = (p, r)$ a unipotence parameter admitted by Q . Assume Q contains a nontrivial definable central \tilde{p} -subgroup U_1 with a nonempty subset X such that $d_p(C(X)) \leq r$. Then exactly one of the following three cases occur.*

- (1) Q is a generous Carter subgroup.
- (2) For g generic in Q , $d_p(C^\circ(g)) > r$.
- (3) The generic element of Q is exceptional, and in particular any element of Q has order at most $e(G)$.

Proof. Notice that the assumption together with Corollary 2.3 and Fact 2.5 (2) implies that \tilde{p} is maximal in its characteristic for Q .

If Q is generous, then $C^\circ(g) \leq Q$ holds for g generic in Q by Fact 2.14 (see also [Jal06, Lemma 3.10]), so cases (2) and (3) cannot occur.

Assume Q not generous in G . By Fact 3.34, Q contains no nontrivial good torus, and thus $r > 0$ as \tilde{p} is maximal in its characteristic for Q . By Theorem 4.1, U_1 is contained in a unique Sylow \tilde{p} subgroup of G , say U , and $Q \leq N^\circ(U)$. Notice that $N^\circ(U)$ is solvable by local^o solvability^o of G . By condition (4) in Fact 2.14 a generic element g of Q is in infinitely many conjugates of Q .

Suppose toward a contradiction $d_p(C^\circ(g)) \leq r$ and $C^\circ(g)$ solvable. Then $\tilde{p} \neq (\infty, 0)$ is a unipotence parameter maximal in its characteristic for the definable connected solvable subgroup $C^\circ(g)$. It follows that $C^\circ(g)$ contains a unique Sylow \tilde{p} -subgroup by Fact 2.15, which is necessarily a \tilde{p} -subgroup of U as it contains U_1 . If γ is an element of G such that $g \in Q^\gamma$, then U_1 and U_1^γ are both contained in $U_{\tilde{p}}(C^\circ(g))$, and by uniqueness applied now to U_1^γ one gets $U = U^\gamma$. Hence all G -conjugates of Q containing g are actually $N^\circ(U)$ -conjugate. But now Q is generous in the definable connected solvable subgroup $N^\circ(U)$, and thus a generic element of Q is in a unique $N^\circ(U)$ -conjugate of Q by Fact 2.14. This is a contradiction. Hence when Q is not generous one of the two cases (2) or (3) must occur.

Notice that in case (3) a generic element of Q , being exceptional, has order at most $e(G)$, and then the exponent of Q is bounded by $e(G)$ by Fact 2.5 (2).

It just remains to show that cases (2) and (3) cannot occur simultaneously. But in case (2) r cannot be ∞ , and in case (3) it must. \square

Of course, by Corollary 2.3, Lemma 4.11 applies when $d_p(G) = d_p(Q) = r$. In particular a nongenerous Carter subgroup which is not divisible must be as in case (3) of Lemma 4.11.

4.2 Uniqueness Theorem and cosets

In [CJ04] arguments pending on cosets and generosity were developed intensively for determining Weyl groups in groups of finite Morley rank in the specific case of minimal connected simple groups. This systematic approach was strongly inspired by the seminal work of Nesin in the context of bad groups [Nes89]. These arguments generally split into two parts. Cosets corresponding to an undesirable Weyl element are usually shown to be both generous *and* nongenerous in the ambient group, and then the coset as well as the unexpected Weyl element does not exist. Local properties of small groups often allow one to prove that

some cosets are generous, as this is done intensively in [CJ04], and normally this is contradictory by itself.

In the light of the fine analysis of generous sets of [Jal06] and in continuation of this work, these *coset arguments* have certain generalizations, and what follows is part of it. It is however worth recalling these arguments in the specific context of locally[◦] solvable[◦] groups. The interest is both to put in a uniform format this essential content of [CJ04] (here with the appeal to [Jal06]), and to see how the specific local analysis of small groups originates further such arguments. In the process we will also encounter an interesting pathological configuration.

As far as generosity is concerned, the fine analysis of conjugacy classes in [Jal06] definitively provided the right understanding concerning generosity.

Fact 4.12 [Jal06] *Let (G, Ω) be a permutation group of finite Morley rank in which the Morley rank is additive (or a ranked permutation group), H a definable subset of Ω , and assume that for r between 0 and $\text{rk}(G/N(H))$ the definable set H_r , consisting of those elements of H contained in a set of conjugates of H of rank exactly r , is nonempty. Then*

$$\text{rk}(H_r^G) = \text{rk}(G) + \text{rk}(H_r) - \text{rk}(N(H)) - r.$$

Proof. This is essentially the content of the fine analysis of conjugacy classes of [Jal06, §2.2]. Here the geometric proof for this mentioned later by Cherlin yields this equality exactly as in [Jal06, §2.3]. One uses the additivity of the Morley rank, or of the rank function if the structure is ranked as in the axioms of [BN94], for computing the ranks of the set of flags in the naturally associated geometry. \square

We say that a connected locally[◦] solvable[◦] group H of finite Morley rank is *sick* if it contains a generous Carter subgroup, H contains no nontrivial decent torus, the generous Carter subgroup has a nontrivial π -unipotent subgroup, H does not conjugate its maximal p -unipotent subgroups for any $p \in \pi$, and for any such p $N_H^\circ(U)$ is a Carter subgroup of bounded exponent of H for some maximal p -unipotent subgroup U of H .

Theorem 4.13 *Let G be a locally[◦] solvable[◦] group of finite Morley rank with a generous Carter subgroup. Let H be a definable connected subgroup of G and x an element of $N_{G^\circ}(H)$ not in H . Assume that H is solvable or that H contains a generous Carter subgroup of the ambient group and is not sick. Then xH is not generous in G .*

This essential content of [CJ04] was proved locally, usually for H a Carter subgroup or indeed the centralizer[◦] of a torus of the ambient group, and we reformat it in its natural form here, replacing the applications of [CJ04, Proposition 3.11] in that paper by Section 4.1 here and arguing directly for the production of bounded exponent torsion of [CJ04, §3.3].

Proof. Assume towards a contradiction xH is generous in G .

Notice that for the suitable r as in Fact 4.12 such that X_r^G is generic in G , where X denotes the coset xH , one has $\text{rk}(X_r) - \text{rk}(N(X)) = r \geq 0$, hence $\text{rk}(H) \leq \text{rk}(N(X)) \leq \text{rk}(X_r) \leq \text{rk}(xH) = \text{rk}(H)$, and thus $r = 0$, $\text{rk}(H) = \text{rk}(N(xH)) = \text{rk}([xH]_0)$, and a generic element of xH is in only finitely many conjugates of xH . Also $N^\circ(xH) = H$. (The argument of this paragraph is of course general.)

A generic element w of xH is also generic in G . By Fact 4.12, w is in only finitely many conjugates of xH , and this implies as in [Jal06, Fundamental Lemma 3.3] that $C^\circ(w) \leq N^\circ(xH) = H$. By Fact 2.14 (see also [Jal06, Lemma 3.10]), $C^\circ(w) \leq Q_w$, where Q_w denotes the unique conjugate of the generous Carter subgroup Q containing w .

(The next paragraph corresponds to the local applications of [CJ04, §3.3] in that paper, though things may be stated in somewhat reversed ways there. The paragraph following it will then concern the application of the uniqueness theorems of Section 4.1, and this was usually done first in the sequence of argumentations in [CJ04] via [CJ04, Proposition 3.11]. Actually [CJ04, Proposition 3.11] provided trivial intersections at the level of subgroups, and then cosets consisting generically of bounded exponent elements, and then [CJ04, §3.3] gave bounded exponent subgroups.)

Let n be the order of x modulo H . By assumption $n > 1$. As $w \in xH$ and x normalizes H , $H(w) \leq \langle x \rangle H$ and in the definable hull $H(w)$, w has order a nontrivial multiple of n modulo $H^\circ(w)$. This shows that the generic element of Q_w has the property of having order a nontrivial multiple of n modulo the connected component of its definable hull. Hence Q_w contains by Fact 2.5 a nontrivial definable connected subgroup of bounded exponent (whose elements are generically of order the above multiple of n).

The generic element w of Q_w centralizes a nontrivial definable connected abelian subgroup of exponent n by Facts 2.5 (2) and 2.24 (2). By Corollary 4.3, $C^\circ(w)$ is contained in a unique Borel subgroup, say B_w , and $C^\circ(w) \leq Q_w \leq B_w$.

Now wH is generous in any definable connected subgroup containing it. This is a general fact, for which one can proceed as in [Jal06, Lemma 3.9 b.]. Indeed the property of the generic element of xH of being contained in finitely many conjugates of xH is obviously preserved when passing to definable subgroup, and this suffices with Fact 4.12 and the fact that $\text{rk}(N(xH)) = \text{rk}(xH)$.

When H is solvable, we have as $C^\circ(w) \leq H$ also that $H \leq B_w$. In particular wH is generous in the connected solvable group B_w , and this is ridiculous. One can argue, being inside a connected solvable group. One can also argue noticing that $N_{Q_w}^\circ(\langle w \rangle(H \cap Q_w))$ normalizes $w(H \cap Q_w)$ by Fact 1.2, hence normalizes wH as in [Jal06, Fundamental Lemma 3.3], so it is in $N^\circ(wH) = H$, and the normalizer condition in connected nilpotent groups gives $w \in \langle w \rangle(H \cap Q_w) = Q_w$, a contradiction as Q_w is connected and w is not in $(H \cap Q_w)$.

This finishes our rearrangement of [CJ04] when H is solvable, and when H contains a generous Carter subgroup Q one can proceed as follows.

By the preceding case one may assume H nonsolvable. By a Frattini Argument following from the conjugacy of generous Carter subgroups, [Jal06, Corol-

lary 3.13], we may suppose that x normalizes the generous Carter subgroup Q of H .

Let π be the set of primes involved in the bounded exponent part of the generous Carter subgroup. Assume that H conjugates its maximal p -unipotent subgroups for some prime p in π , or that Q contains a nontrivial decent torus. In the first case one may assume after H -conjugacy, with Fact 2.15 and Lemma 4.6 that Q and Q_w are in a common Borel subgroup of G . Similarly, if Q and Q_w contain a nontrivial decent torus, one may assume after H -conjugacy that Q and Q_w are the centralizer^o of their common maximal decent torus, and hence that they are by local^o solvability^o in a same Borel subgroup of G . If we denote by B this Borel subgroup in both cases, then we get $Q \leq (H \cap B)^\circ < \langle w \rangle (H \cap B)^\circ \leq B$, and this is impossible by a Frattini argument as the Carter subgroup Q is selfnormalizing in B .

This leaves us with the case in which Q has a nontrivial π -unipotent subgroup, H does not conjugate its maximal p -unipotent subgroups for any $p \in \pi$, and H contains no nontrivial decent torus. Hence in this pathological situation H has all the symptoms of sickness, except maybe the last one. But this will be seen in Lemma 5.7 below (whose proof will be independent). \square

Theorem 4.13 represents coset arguments of [CJ04] for dealing with Weyl groups. We note that its proof actually provided the following much more general

Fact 4.14 [Jal08a] *Let G be a group of finite Morley rank in which the generic element of G° is in a connected nilpotent subgroup, and let H be a definable subgroup of G° . Then $H \setminus H^\circ$ is not generous in G .*

The main consequence of Fact 4.14 is the following, a general fact in which its conclusion is true, which also recasts some corresponding consequences as in [CJ04] somehow in their original content.

Fact 4.15 [Jal08a] *Let G be a group of finite Morley rank, n a natural number, H a definable connected generous subgroup with the property that, for h generic in H , h is in a connected nilpotent subgroup of H and h^n is also generic in H , and assume w is an element of G° of finite order n normalizing H without being inside. Then $C_H(w) < H$.*

Groups of finite Morley rank with a generous Carter subgroup not divisible or not abelian can be dealt with the Bender method, the results of Sections 4.1 and 4.3 here, and otherwise Fact 4.15 applies. In a locally^o solvable^o context and in presence of a generous Carter subgroup the situation will be considered in a separate paper with the results of Section 5.3 below. Here we merely mention the following basic commutation principle relevant for Weyl groups, and which builds upon [Del07a, Lemme 3.1].

Lemma 4.16 *Let $G = NQ$ be a group, with N and Q two subgroups and N normal. Assume σ is an automorphism of G normalizing Q and fixing N point-wise. Then*

- (1) N and $\langle[\sigma, Q]\rangle$ commute.
- (2) If $Q = \langle[\sigma, Q]\rangle C_Q(\sigma)$ and $N \leq N(C_Q(\sigma))$, then $N \leq N(Q)$.

Proof. For any element q in Q and h in N one has

$$h^{[\sigma, q]} = h^{q^{-1}\sigma q} = (h^{q^{-1}})^{\sigma q} = h^{q^{-1}q} = h$$

and thus $h \in C([\sigma, q])$. Hence $N \leq C(\langle[\sigma, Q]\rangle)$, the general commutation principle of [Del07a, Lemme 3.1]. The second item follows. \square

In particular, if in Lemma 4.16 (2) $G = NQ$ has finite Morley rank and Q is a Carter subgroup, then $N^\circ \leq C_Q^\circ(\sigma)$.

We finish this section with one word about centralizers of definable connected exceptional subgroups. If G , H , and n are as in Fact 4.15, with G locally^o solvable^o and H an exceptional definable connected nonsolvable subgroup of G , then $C(H)$ is finite by Lemma 3.25, and if x is an element of G° in this finite centralizer and of order n , then Fact 4.15 implies that x is in $Z(H)$.

4.3 Maximal pairs of Borel subgroups

When the absolute maximality assumptions concerning unipotence degrees fail in Lemma 4.8 one might have (or rather cannot exclude) pairs of Borel subgroups whose Fitting subgroups have an infinite intersection. This situation has been studied intensively in [Bur07]. In what follows, not only we claim no originality compared to this paper, but also we will tend to follow it word by word. The only differences will appear in the notation used for unipotence parameters and in a special care needed for dealing here with our weakest assumption of local^o solvability^o. Some additional results from [Del07a] will be mentioned in the process.

Definition 4.17 *Let G be a group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups. We say that (B_1, B_2) is a maximal pair (of Borel subgroups) if the definable connected subgroup $(B_1 \cap B_2)^\circ$ is maximal for inclusion among all definable connected subgroups of the form $(L_1 \cap L_2)^\circ$, with L_1 and L_2 two distinct Borel subgroups of G .*

Hypothesis 4.18 [Bur07, Hypothesis 3.2] *We assume the following configuration:*

- (1) G is a locally^o solvable^o group of finite Morley rank.
- (2) (B_1, B_2) is a maximal pair of Borel subgroups of G .

(3) $[F(B_1) \cap F(B_2)]^\circ$ is nontrivial.

Notation 4.19 [Bur07, Notation 3.3] *We let*

- (1) $H = (B_1 \cap B_2)^\circ$.
- (2) $X = F(B_1) \cap F(B_2)$.
- (3) $r' = d_\infty(X)$.

Recall that X° is torsion free by Lemma 4.7. In particular $0 < r' < \infty$. In particular $0 < d_\infty(B_1) < \infty$ and $0 < d_\infty(B_2) < \infty$.

Notice that by Lemma 4.8 one cannot have $d(B_1) = d(B_2) = \infty$. So at least one of the two Borel subgroups B_1 and B_2 , say B_i , has no bounded exponent subgroup. In particular $0 < d(B_i) < \infty$. The other Borel subgroup B_{i+1} might satisfy $0 < d(B_{i+1}) \leq \infty$ (this latter inequality will be shown to be also strict in the analysis below).

4.3.1 Homogeneity of X

We observe that $H' \leq X \trianglelefteq H$. We will show the asymmetry of the situation, i.e. $d_\infty(B_1) \neq d_\infty(B_2)$. We may assume in any case that

Hypothesis 4.20 $d_\infty(B_2) \leq d_\infty(B_1)$.

and we will indeed show that $d_\infty(B_2) < d_\infty(B_1)$. Notice that $d(H) = d_\infty(H)$.

Lemma 4.21 [Bur07, Lemma 3.5] $d_\infty(H) < d_\infty(B_1)$.

Proof. As there is no nontrivial p -unipotent subgroup in H , $d(H) < \infty$.

Suppose toward a contradiction $d_\infty(H) \geq d_\infty(B_1)$. As $H \leq B_1$, $d_\infty(H) \leq d_\infty(B_1)$ in any case, so our assumption becomes $d(H) = d(B_1)$. Since $d(H) \leq d_\infty(B_2) \leq d_\infty(B_1)$ by Hypothesis 4.21, all these unipotence degrees are equal to a certain d , and $U_{(\infty, d)}(H) \leq U_{(\infty, d)}(B_1) \cap U_{(\infty, d)}(B_2)$. As G is locally^o solvable^o, $N^\circ(U_{(\infty, d)}(H))$ is solvable, and thus contained in a Borel subgroup B_3 of G .

Now we contradict the fact that $B_1 \neq B_2$.

If $U_{(\infty, d)}(H) = U_{(\infty, d)}(B_i)$ for some $i = 1$ or 2 , then by local^o solvability^o and maximality of B_i , $B_i = N^\circ(U_{(\infty, d)}(H)) \leq B_3$, and $B_i = B_3$.

If $U_{(\infty, d)}(H) < U_{(\infty, d)}(B_i)$ for some $i = 1$ or 2 , then, as $d_\infty(B_i) = d \geq 1$, $U_{(\infty, d)}(B_i) \leq F^\circ(B_i)$ and is in particular nilpotent, Fact 2.7 gives

$$U_{(\infty, d)}(H) < U_{(\infty, d)}(N_{U_{(\infty, d)}(B_i)}(U_{(\infty, d)}(H))) \leq B_3.$$

Since $U_{(\infty, d)}(H) \trianglelefteq H$, we must get $H < (B_i \cap B_3)^\circ$. By maximality of H we get $B_i = B_3$.

As $B_1 \neq B_2$, $U_{(\infty, d)}(H)$ is proper in one of the two subgroups $U_{(\infty, d)}(B_i)$, and not in both. In any case we get $B_1 = B_3 = B_2$, a contradiction. \square

Lemma 4.22 [Bur07, Lemma 3.6] $d(H) = d_\infty(B_2)$.

Proof. Suppose toward a contradiction $d(H) < d_\infty(B_2)$.

By local^o solvability^o, $N^\circ(U_{(\infty, r')}(X))$ is solvable, and contained in a Borel subgroup B_3 of G . Since $U_{(\infty, r')}(X) \trianglelefteq H$, H is contained in B_3 . Since $d(H) < d_\infty(B_i)$ for $i = 1$ and 2 by Lemma 4.21 and Hypothesis 4.20, Fact 2.5 gives $U_{(\infty, d_\infty(B_i))}(B_i) \leq C^\circ(U_{(\infty, r')}(X)) \leq B_3$. Hence $H < (B_i \cap B_3)^\circ$ and $B_i = B_3$ by maximality of H for $i = 1$ and 2 , a contradiction to $B_1 \neq B_2$. \square

Corollary 4.23 $d(H) = d_\infty(H) = d_\infty(B_2) < d_\infty(B_1)$.

Proposition 4.24 [Bur07, Proposition 3.7] *If H is nonabelian, then B_1 and B_2 are the only Borel subgroup containing H .*

Proof. Suppose there is a Borel subgroup B_3 distinct from B_1 and B_2 and containing H . The maximality of H yields $H = (B_1 \cap B_3)^\circ = (B_2 \cap B_3)^\circ$. Since $1 \neq H' \leq F^\circ(B_3)$, the maximal pairs (B_1, B_3) and (B_2, B_3) satisfy Hypothesis 4.18. Since $d_\infty(H) < d_\infty(B_1)$ by Lemma 4.21, $d_\infty(H) = d_\infty(B_3)$ by Lemma 4.22 applied to the maximal pair (B_1, B_3) . But since $d_\infty(H) = d_\infty(B_2)$ by Lemma 4.22, $d_\infty(H) < d_\infty(B_3)$ by Lemma 4.21 applied to the maximal pair (B_2, B_3) . This is a contradiction. \square

As $d_\infty(B_2) < d_\infty(B_1)$, the Borel subgroups are not conjugate, a point we exploit in the next lemma.

Lemma 4.25 [Bur07, Lemma 3.8] $F^\circ(B_i) \not\leq H$ for $i = 1$ and 2 .

Proof. Since $d_\infty(H) < d_\infty(B_1)$ by Lemma 4.21, $F^\circ(B_1) \not\leq H$. Suppose toward a contradiction $F^\circ(B_2) \leq H$. Then $H \trianglelefteq B_2$ by Fact 2.22, and $H \leq B_1 \cap B_1^g$ for some $g \in B_2 \setminus N(B_1)$. By maximality of H , (B_1, B_1^g) is a maximal pair, and Corollary 4.23 applied to this maximal pair gives a contradiction. \square

Lemma 4.26 [Bur07, Lemma 3.9] *If X_1 is an infinite definable subgroup of X normal in H , then $N^\circ(X_1) \leq B_1$.*

Proof. By local^o solvability^o $N^\circ(X_1)$ is solvable, and hence contained in a Borel subgroup B_3 of G . By assumption $H \leq B_3$. Since $d_\infty(H) < d_\infty(B_1)$ by Lemma 4.21, Fact 2.5 yields $U_{(\infty, d_\infty(B_1))}(B_1) \leq C^\circ(X_1) \leq B_3$. Thus $H < (B_1 \cap B_3)^\circ$ and by maximality of H we get $B_1 = B_3$. In particular $N^\circ(X_1) \leq B_1$. \square

Corollary 4.27 $[X \cap Z(F(B_2))]^\circ = 1$.

Theorem 4.28 [Bur07, Theorem 3.10] *X° is a homogeneous (∞, r') -subgroup.*

Proof. Recall that X° is torsion-free. Suppose toward a contradiction that $U_{(\infty,r)}(X)$ is nontrivial for some $0 \leq r < r'$. By Fact 2.5 and Lemma 4.26

$$F^\circ(B_2) \leq C^\circ(U_{(\infty,r')}(X))C^\circ(U_{(\infty,r)}(X)) \leq B_1$$

and it follows that $F^\circ(B_2) \leq H$. This contradicts Lemma 4.25. Hence X° is homogeneous in the maximal unipotence parameter in its characteristic, that is (∞, r') . \square

4.3.2 Fitting subgroup of B_2

We delineate now $F^\circ(B_2)$, and in particular determine which of its factors are contained in H .

Lemma 4.29 [Bur07, Lemmas 3.11 and 3.12] *$F^\circ(B_2)$ is divisible (in particular $d(B_2) = d_\infty(B_2)$) and $U_{(\infty,r)}(F^\circ(B_2)) \leq Z(H)$ when $0 \leq r \leq d(B_2)$ and $r \neq r'$.*

Proof. If $d(B_2) = \infty$, then $U_p(B_2)$ is nontrivial for some prime p , and contained in $N^\circ(X^\circ) \leq B_1$ by Lemma 4.26. This is a contradiction to Lemma 4.7 or Lemma 4.8. Hence $d(B_2) = d_\infty(B_2)$ and $F^\circ(B_2)$ is divisible.

By Theorem 4.28, Fact 2.5, and Lemma 4.26, $U_{(\infty,r)}(F^\circ(B_2)) \leq C^\circ(X^\circ) \leq N^\circ(X^\circ) \leq B_1$, and hence each of these groups is contained in H . As each such group is nilpotent and normalized by the subgroup H of B_2 , each such group is in $F^\circ(H)$. Now Fact 2.8 and Theorem 4.28 give

$$[H, U_{(\infty,r)}(F^\circ(B_2))] \leq U_{(\infty,r)}(H') \leq U_{(\infty,r)}(X^\circ) = 1.$$

\square

Lemma 4.30 [Bur07, Lemma 3.13] *$U_{(\infty,r')}(F^\circ(B_2))$ is not contained in H and not abelian.*

Proof. By Fact 2.5 $F^\circ(B_2)$ is generated by its Sylow \tilde{p} -subgroups. But by Lemma 4.25 $F^\circ(B_2) \not\leq H$, so Lemma 4.29 implies $U_{(\infty,r')}(F^\circ(B_2)) \not\leq H$.

Since $N^\circ(X^\circ) \leq B_1$ by Lemma 4.26, $U_{(\infty,r')}(F^\circ(B_2))$ cannot be abelian. \square

Now we can deduce from the two preceding lemmas that the unipotence degree r' is uniquely determined by the structure of B_2 .

Corollary 4.31 [Bur07, Corollary 3.14] *$U_{(\infty,r)}(F^\circ(B_2))$ is not abelian if and only if $r = r'$.*

Lemma 4.32 [Bur07, Lemma 3.15] *$U_{(\infty,r)}(B_2) \leq F^\circ(B_2)$ for every $r > r'$.*

Proof. Let Q be a definable (∞, r) -subgroup of B_2 . By Fact 2.12 and our assumption $r > r'$, $U_{(\infty, r')}(F^\circ(B_2)) \cdot Q$ is nilpotent. It follows by Fact 2.5 that Q centralizes $U_{(\infty, r')}(F^\circ(B_2))$, and in particular Q centralizes its subgroup X° . Hence $Q \leq N^\circ(X^\circ) \leq B_1$ by Lemma 4.26, and $Q \leq H$. By Lemma 4.29, Q centralizes all factors of $F^\circ(B_2)$, except maybe the one of unipotence parameter (∞, r') . But as $U_{(\infty, r')}(F^\circ(B_2)) \cdot Q$ is nilpotent, $F^\circ(B_2) \cdot Q$ is nilpotent, and as it is normal in B_2 by Fact 2.22, we deduce that $Q \leq F^\circ(B_2)$, as desired. \square

4.3.3 Structure of H

Lemma 4.33 [Bur07, Lemma 3.16] $U_{(\infty, r')}(H) \leq F^\circ(B_2)$. In particular $U_{(\infty, r')}(H)$ is nilpotent and the unique Sylow (∞, r') -subgroup of H .

Proof. Let Q be any definable (∞, r') -subgroup of H . By Fact 2.12, the group $U_{(\infty, r')}(F^\circ(B_2)) \cdot Q$ is nilpotent. For any integer $r \neq r'$, Q centralizes $U_{(\infty, r)}(F^\circ(B_2))$ by Lemma 4.29. Hence $F^\circ(B_2) \cdot Q$ is nilpotent by Fact 2.5. By Fact 2.22, this product is normal in B_2 , and hence it must be contained in $F^\circ(B_2)$.

In particular $U_{(\infty, r')}(H)$ is nilpotent, and the unique Sylow (∞, r') -subgroup of H . \square

This has a consequence purely in B_2 , as seen in [Del07a].

Lemma 4.34 [Del07a, Lemme 3.11] *If a Carter subgroup of H is also a Carter subgroup of B_2 , then $U_{(\infty, r')}(B_2)$ is nilpotent, included in $F^\circ(B_2)$, and the unique Sylow (∞, r') -subgroup of B_2 .*

Proof. Let Q be a Carter subgroup of H , which is also a Carter subgroup of B_2 . Then $U_{(\infty, r')}(F^\circ(B_2)) \cdot U_{(\infty, r')}(Q)$ is a Sylow (∞, r') -subgroup of B_2 by Fact 2.30. By conjugacy of such subgroups in B_2 , Fact 2.29, it suffices to show that it is contained in $F^\circ(B_2)$. But the first factor clearly is, and the second also by Lemma 4.33. \square

We return to the structure of H .

Notation 4.35 *Let $Y = U_{(\infty, r')}(H)$ be the unique Sylow (∞, r') -subgroup of H . It is normal in H , and in $F^\circ(H)$.*

We find that Y has properties antisymmetric to those of X .

Lemma 4.36 [Bur07, Lemma 3.17] $N^\circ(Y) \leq B_2$ and $X^\circ < Y$. In addition $U_{(\infty, r')}(N_{F(B_2)}(Y)) \not\leq H$.

Proof. Let $P = U_{(\infty, r')}(F^\circ(B_2))$. Then $Y \leq P$ by Lemma 4.33. By Lemma 4.30 $P \not\leq H$, so $Y < P$. By Normalizer Condition, Fact 2.7, $Y < U_{(\infty, r')}(N_{\tilde{P}}^\circ(Y))$. Now $X^\circ < Y$ by Lemma 4.26, and $N^\circ(Y) \leq B_2$ by maximality of H . \square

Theorem 4.37 [Bur07, Theorem 3.18] *Every definable connected nilpotent subgroup of H is abelian.*

Proof. By Fact 2.5 it suffices to show that any Sylow \tilde{p} -subgroup of H is abelian. As this is true for decent tori and there is no nontrivial p -unipotent subgroup in H , it suffices to show this when $\tilde{p} = (p, r)$, with $1 \leq r < \infty$. For $r \neq r'$, $U_{(\infty, r)}(H') = 1$ by Theorem 4.28, so a Sylow (∞, r) -subgroup of H must be abelian by Fact 2.8. It remains to show that the unique Sylow (∞, r') -subgroup of H , Lemma 4.33, is also abelian. For if Y' is not trivial, then $N_{B_2}^\circ(Y) \leq N_{B_2}^\circ(Y') \leq B_1$ by Lemma 4.26, contradicting Lemma 4.36. This completes the proof. \square

Lemma 4.38 [Bur07, Lemma 3.19] *If H is not abelian, then $N^\circ(H) = H$.*

Proof. Lemma 4.36 implies that $N^\circ(H) \leq N^\circ(Y) \leq B_2$. When $H' \neq 1$, Lemma 4.26 implies also that $N^\circ(H) \leq N^\circ(H') \leq B_1$. \square

4.3.4 Structure of B_1

Lemma 4.39 [Bur07, Lemma 3.20] *$F^\circ(B_1)$ is divisible (and in particular $d(B_1) = d_\infty(B_1)$) and $U_{(\infty, 0)}(F^\circ(B_1)) \leq Z^\circ(H)$.*

Proof. Y is an (∞, r') -group, with $0 < r' < \infty$, normalizing an hypothetic p -unipotent subgroup of B_1 . By Fact 2.12 (4), it centralizes all of them. As $N^\circ(Y)$ is solvable by local^o solvability^o of G , one gets if $U_p(B_1) \neq 1$ for some prime p that $N^\circ(Y) \leq B_1$ by Lemma 4.4. But $N^\circ(Y) \leq B_2$ also by Lemma 4.36, which gives a nontrivial p -unipotent subgroup in H , a contradiction to Lemma 4.4 or Lemma 4.7. Hence $F^\circ(B_1)$ is divisible and $d(B_1) = d_\infty(B_1)$.

As the maximal definable decent torus of $F^\circ(B_1)$ is central in B_1 , it centralizes Y . Hence Lemma 4.36 gives $U_{(\infty, 0)}(F^\circ(B_1)) \leq N^\circ(Y) \leq B_2$, and hence $U_{(\infty, 0)}(F^\circ(B_1))$ is contained in H . But the latter normalizes the former, and hence centralizes it by Fact 2.12 (1). \square

Lemma 4.40 [Bur07, Lemma 3.21] *$X^\circ = U_{(\infty, r')}(F^\circ(B_1))$, and also $B_1 = N^\circ(X^\circ)$.*

Proof. By Fact 2.12, $U_{(\infty, r')}(F^\circ(B_1)) \cdot Y$ is nilpotent. Lemma 4.36 gives the inclusion $N_{U_{(\infty, r')}(F^\circ(B_1)) \cdot Y}^\circ(Y) \leq H$. So $U_{(\infty, r')}(F^\circ(B_1)) \leq Y$ by Normalizer Condition, Fact 2.7. By Lemma 4.33, $Y \leq F(B_2)$, so $U_{(\infty, r')}(F^\circ(B_1)) \leq X^\circ$. But the converse to the latter inclusion holds by Theorem 4.28.

Our last claim follows by local^o solvability^o of G and maximality of B_1 . \square

Corollary 4.41 [Bur07, Corollary 3.22] $U_{(\infty, r')}(F^\circ(B_1))$ is abelian, and $F^\circ(B_1) \leq C^\circ(X^\circ)$.

Proof. By Lemma 4.40 and Theorem 4.37 $U_{(\infty, r')}(F^\circ(B_1))$ is abelian, and contained in $C^\circ(X^\circ)$. For an integer $r \neq r'$, $U_{(\infty, r)}(F^\circ(B_1)) \leq C^\circ(X^\circ)$ by Fact 2.5. So our last claim follows. \square

Notation 4.42 We let Q denote a Carter subgroup of H .

Lemma 4.43 [Bur07, Lemma 3.23] $U_{(\infty, r')}(Q) = U_{(\infty, r')}(Z(H))$, and this group is not trivial.

Proof. By Lemma 4.36, $U_{(\infty, r')}(H/H')$ is not trivial. So $U_{(\infty, r')}(Q)$ is not trivial by Facts 2.22 and 2.28.

By Theorem 4.37, Q and Y are abelian. By Lemma 4.33, $U_{(\infty, r')}(Q) \leq Y$. So $U_{(\infty, r')}(Q)$ centralizes both Q and the subgroup H' of Y . So $U_{(\infty, r')}(Q) \leq Z(H)$ by Fact 2.28. Conversely, $Z^\circ(H) \leq Q$. \square

Theorem 4.44 [Bur07, Theorem 3.24] $N^\circ(U_{(\infty, r')}(Q)) \leq B_2$. So $N^\circ(Q) \leq B_2$, and Q is a Carter subgroup of B_1 .

Proof. We first show that $N^\circ(U_{(\infty, r')}(Q)) \leq B_2$. By Lemma 4.36, $N^\circ(Y) \leq B_2$. So we may assume that $U_{(\infty, r')}(Q) < Y$, and hence H is not abelian by Lemma 4.43. So B_1 and B_2 are the only Borel subgroups of G containing H by Proposition 4.24. By Lemma 4.43, $H \leq N^\circ(U_{(\infty, r')}(Q))$. By local^o solvability^o of G the latter group is solvable. If it contains H properly, then it can grow only in one Borel B_1 or B_2 , and must agree with H on the other. By Lemma 4.40, $N^\circ(X^\circ) = B_1$. Since $Y = X^\circ \cdot U_{(\infty, r')}(Q)$ by Fact 2.30, $N_{B_1}^\circ(U_{(\infty, r')}(Q)) \leq N^\circ(Y) \leq B_2$ by Lemma 4.36. So $N^\circ(U_{(\infty, r')}(Q)) \leq B_2$.

It follows that $N^\circ(Q) \leq N^\circ(U_{(\infty, r')}(Q)) \leq B_2$, and $N_{B_1}^\circ(Q) \leq N_H^\circ(Q) = Q$, so that Q is a Carter subgroup of B_1 . \square

We show now that r' is the only unipotence degree ≥ 1 (and in fact ≥ 0 as well) appearing in both $F(B_1)$ and $F(B_2)$.

Lemma 4.45 [Bur07, Lemma 3.25] $U_{(\infty, r)}(F^\circ(B_1)) = 1$ for any $r \neq r'$ with $1 \leq r \leq d(B_2)$.

Proof. Let $T = U_{(\infty, r)}(F^\circ(B_1))$. We claim that $T \leq H$. First suppose that $d(B_2) = r'$. Then $T \cdot Y$ is nilpotent by Fact 2.12, and Y centralizes T by Fact 2.5. So $T \leq N^\circ(Y) \leq B_2$ by Lemma 4.36, and $T \leq H$. Next, suppose that $d(B_2) > r'$. Then $U_{(\infty, d(B_2))}(B_2) \leq Z(H)$ by Lemma 4.29. By Fact 2.5, $U := T \cdot U_{(\infty, d(B_2))}(B_2)$ is nilpotent. If $r \neq d(B_2)$, then $T \leq C^\circ(U_{(\infty, d(B_2))}(B_2))$ by Fact 2.5, and $T \leq H$. So we may assume that $r = d(B_2)$. If $T \not\leq B_2$, then by Normalizer Condition, Fact 2.7, $U_{(\infty, r)}(B_2) < U_{(\infty, r)}(N_U^\circ(U_{(\infty, r)}(B_2)))$, a

contradiction to the fact that $B_2 = N^\circ(U_{(\infty, r)}(B_2))$ by local solvability of G . Thus $T \leq H$.

Since $T \leq H$, and $U_{(\infty, r)}(H') = 1$ by Theorem 4.28, T is contained in a Carter subgroup of H by Fact 2.30. Now $T \leq Q$ because $T \trianglelefteq H$ and Carter subgroups are conjugate in H . Clearly $T \leq F^\circ(H)$ too. By Fact 2.28 and Theorem 4.37 $H = F^\circ(H)Q \leq C^\circ(T)$, and hence $T \leq Z(H)$.

Now consider the case where $r > r'$. Then $U_{(\infty, r')}(F^\circ(B_2)) \cdot T$ is nilpotent by Fact 2.12, and both factors commute by Fact 2.5. If $T \neq 1$, then $B_1 = N^\circ(T)$ by local solvability of G and $U_{(\infty, r')}(F^\circ(B_2)) \leq N^\circ(T) \leq B_1$, a contradiction to Lemma 4.30. Thus $T = 1$.

Finally consider the case where $r < r'$. Since $T \leq Z(H)$, TY is abelian by Theorem 4.37. Recall that $Y \leq F(B_2)$ by Lemma 4.33. Let P denote the group $U_{(\infty, r')}(N_{F(B_2)}(Y))$. Then $[x, h] \in Y$ for any $x \in X^\circ$ and any $h \in P$, and hence $[x, h] = [x, h]^t = [x, h^t]$ for any $t \in T$. So $[h^{-1}, t] = hh^{-t} \in C(X^\circ)$. Now $[P, T] \leq Y$ by Lemma 4.26 and Fact 2.8. Since P is nilpotent, and T commutes with Y , the product TP is nilpotent. By Fact 2.5, $P \leq N^\circ(T)$, which is equal to B_1 by local solvability of G if $T \neq 1$. This contradiction to Lemma 4.36 shows that $T = 1$. \square

As a result, r' is also uniquely determined by B_1 .

Corollary 4.46 [Bur07, Corollary 3.26] *r' is the minimal unipotence degree $1 \leq r < \infty$ such that $F(B_1)$ admits the unipotence parameter (∞, r) .*

Corollary 4.47 [Bur07, Corollary 3.27] *For $1 \leq r \leq d(B_2)$, a Sylow (∞, r) -subgroup of H is a Sylow (∞, r) -subgroup of B_1 .*

Proof. By Lemmas 4.45 and 4.40, $U_{(\infty, r)}(F^\circ(B_1)) \leq H$. Since Q is a Carter subgroup of B_1 by Theorem 4.44, the subgroup

$$U_{(\infty, r)}(F^\circ(B_1)) \cdot U_{(\infty, r)}(Q)$$

of H is a Sylow (∞, r) -subgroup of H and of B_1 by Theorem 2.30. One concludes then by conjugacy of Sylow (∞, r) -subgroups. \square

4.3.5 Nonabelian intersections

Remark 4.48 *Tor (X) is toral and in $Z(B_1) \cap Z(B_2)$, and $C^\circ(X) = C^\circ(X^\circ)$.*

Proof. Let S be the (finite) torsion subgroup of X , as in Lemma 4.7. As $d(B_1) < \infty$ and $d(B_2) < \infty$, S is a toral subgroup of B_1 and B_2 , and in $Z(B_1) \cap Z(B_2)$ by Lemma 2.25.

By Lemma 4.26, $C^\circ(X) \leq C^\circ(X^\circ) \leq B_1$, and as $X = X^\circ \times S$ with $S \leq Z(B_1)$, $C^\circ(X) = C^\circ(X^\circ)$. \square

Lemma 4.49 [Bur07, Lemma 3.28] *The subgroup $C^\circ(X^\circ)$ is not nilpotent. If H is not abelian, then B_1 is the unique Borel subgroup of G containing $C^\circ(X^\circ)$.*

Proof. By Lemma 4.26, $C^\circ(X^\circ) \leq B_1$. By Lemma 4.33 and Theorem 4.37, $U_{(\infty, r')}(Q) \leq C^\circ(X^\circ)$. By Fact 2.5 and the fact that $d(B_1) \neq r'$ (Corollary 4.23), $U_{(\infty, d(B_1))}(B_1) \leq C^\circ(X^\circ)$ too. By Theorem 4.44 and Fact 2.5, $U_{(\infty, d(B_1))}(B_1) \cdot U_{(\infty, r')}(Q)$ is not nilpotent. So $C^\circ(X^\circ)$ is not nilpotent.

Suppose now H not abelian. Suppose then toward a contradiction that a Borel subgroup of G distinct from B_1 contains $C^\circ(X^\circ)$. So there is a maximal pair (B_3, B_4) which contains $C^\circ(X^\circ)$. We may assume $d(B_3) \geq d(B_4)$. Let $K = [C^\circ(X^\circ)]'$. By Corollary 4.41, $F^\circ(B_1) \leq C^\circ(X^\circ)$. So $C^\circ(X^\circ) \trianglelefteq B_1$ by Fact 2.22. Now $N^\circ(K) = B_1$ by local^o solvability^o of G and maximality of B_1 . Since $K \leq [F(B_3) \cap F(B_4)]^\circ$, we have by, Corollary 4.41 applied to the pair (B_3, B_4) , $F^\circ(B_3) \leq C^\circ([F(B_3) \cap F(B_4)]^\circ) \leq C^\circ(K)$. Thus $d(B_1) \geq d(B_3) > d(B_4)$ by Lemma 4.23 applied again to the pair (B_3, B_4) . But as $F^\circ(B_1) \leq B_4$ also, $d(B_4) \geq d(B_1)$, a contradiction. Hence when H is not abelian B_1 is the unique Borel subgroup of G containing $C^\circ(X^\circ)$. \square

Corollary 4.50 [Bur07, Corollary 3.29] *Suppose H not abelian. Then, for any infinite definable subgroup $X_1 \leq X$, B_1 is the unique Borel subgroup of G containing $C^\circ(X_1)$.*

Proof. Recall that $C^\circ(X) = C^\circ(X^\circ)$. $C^\circ(X) \leq C^\circ(X_1)$, the latter being solvable by local^o solvability^o of G , so the preceding lemma gives the desired result. \square

Corollary 4.51 *If H is nonabelian, then $C^\circ(Y) \leq C^\circ(X^\circ) \leq B_1$.*

Proof. $X^\circ \leq Y$. \square

Lemma 4.52 (Compare with [Del07a, Lemma 3.10]) *If H is nonabelian, then any Sylow (∞, r') -subgroup of G containing Y is contained in B_2 .*

Proof. We want to show that $\Sigma \leq B_2$ for any Sylow (∞, r') -subgroup Σ of G containing Y . One can assume $Y < \Sigma$, and then $Y < U_{(\infty, r')}(N_\Sigma^\circ(Y))$ by normalizer condition, Fact 2.7. By Lemma 4.36, $N^\circ(Y) \leq B_2$, and thus $U_{(\infty, r')}(N_\Sigma^\circ(Y)) \leq B_2$.

If $U_{(\infty, r')}(N_\Sigma^\circ(Y))$ is abelian, then it centralizes Y . But $C^\circ(Y) \leq C^\circ(X^\circ) \leq B_1$ by Lemma 4.49. Hence $U_{(\infty, r')}(N_\Sigma^\circ(Y)) \leq (B_1 \cap B_2)^\circ = H$ and then $U_{(\infty, r')}(N_\Sigma^\circ(Y)) = Y$, a contradiction.

Hence $U_{(\infty, r')}(N_\Sigma^\circ(Y))$ is nonabelian. Now it follows from the result obtained in Theorem 4.37 that in a locally^o solvable^o group of finite Morley rank, a nonabelian definable connected nilpotent subgroup is contained in a unique Borel subgroup. As $U_{(\infty, r')}(N_\Sigma^\circ(Y))$ is in B_2 and in Σ , this gives $\Sigma \leq B_2$. \square

Lemma 4.53 [Bur07, Lemma 3.30] *Let B be a Borel subgroup of G , distinct from B_1 . Suppose that (B, B_1) is a maximal pair, that $H_1 = (B \cap B_1)^\circ$ is not abelian, and that $d(B_1) \geq d(B)$. Then B is $F^\circ(B_1)$ -conjugate to B_2 .*

Proof. We can apply the results of the above analysis to the maximal pair (B_1, B) . We observe that $H'_1 \leq F(B_1) \cap F(B)$. By Corollary 4.46 and Theorem 4.28, $r' = d(H'_1)$, and both H' and H'_1 are contained in $U_{(\infty, r')}(F^\circ(B_1))$. By Lemma 4.40, $U_{(\infty, r')}(F^\circ(B_1))$ is contained in both H and H_1 . Let Q and Q_1 be Carter subgroups of H and H_1 respectively. By Theorem 4.44, Q and Q_1 are Carter subgroups of B_1 . By conjugacy of Carter subgroups in connected solvable groups, $Q_1 = Q^h$ for some $h \in B_1$, and we may assume $h \in F^\circ(B_1)$ by Facts 2.22 and 2.28. By Facts 2.22 and 2.28, Q and Q_1 cover H/H' and H_1/H'_1 respectively. By Lemma 4.40,

$$H^h = U_{(\infty, r')}(F^\circ(B_1)) \cdot Q^h = U_{(\infty, r')}(F^\circ(B_1)) \cdot Q_1 = H_1.$$

Since H_1 is not abelian, $B_2^h = B$ by Proposition 4.24. □

4.3.6 Conclusions

Proposition 4.54 [Bur07, Proposition 4.1] *Let G be a locally[◦] solvable[◦] group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G , and H a nontrivial definable connected subgroup of $B_1 \cap B_2$. Then the following hold:*

- (1) H' is a homogeneous (∞, r') -group for some $1 \leq r' < \infty$ (or trivial).
- (2) Every definable connected nilpotent subgroup of H is abelian.
- (3) $U_{(\infty, r')}(F^\circ(H)) = U_{(\infty, r')}(H)$ is the unique Sylow (∞, r') -subgroup of H .
- (4) $U_{\tilde{q}}(F^\circ(H)) \leq Z(H)$ for any $\tilde{q} \neq (\infty, r')$.
- (5) $0 \leq d_\infty(H) = d(H) \leq d(C(H')) \leq d(N(H')) \leq \infty$, all inequalities, except maybe the third one, being strict when H is not abelian.

Proof. We may assume H not abelian, as otherwise all statements are trivially true once one has noticed that $d_\infty(H) = d(H)$ by Corollary 4.4.

Let (B_3, B_4) be a maximal pair containing H , with $d(B_3) \geq d(B_4)$. The first two conclusions follow immediately from Theorems 4.28 and 4.37. The third conclusion follows from Lemma 4.33. For the fourth conclusion, if $\tilde{q} = (\infty, r)$ with $r \neq r'$, then $U_{\tilde{q}}(F^\circ(H))$ lies in a Carter subgroup Q of H by Fact 2.30, and $H \leq QH'$ (Facts 2.22 and 2.28) $\leq C^\circ(U_{\tilde{q}}(F^\circ(H)))$ (by the second point), which shows the fourth point.

By Corollary 4.23, $\infty > d(B_3) > d(B_4) \geq d(H) = d_\infty(H) > 0$ (be careful, this is not the same H , and one uses also the divisibility of $F^\circ(B_3)$ and of $F^\circ(B_4)$). By Fact 2.5, $U_{(\infty, d_\infty(B_3))}(B_3) \leq C(H')$, thus $d_\infty(C(H')) > d_\infty(H) =$

$d(H)$. Hence for the last point it suffices to show that $d(N(H')) < \infty$. Otherwise, $U_p(N(H'))$ is nontrivial for some prime p ; now the nontrivial group $U_{(\infty, d_\infty(B_3))}(B_3)$, which is also in $N(H')$, normalizes $U_p(N(H'))$, and centralizes it by Fact 2.12 (4), so that $U_p(N(H')) \leq N^\circ(U_{(\infty, d_\infty(B_3))}(B_3)) = B_3$ (by local^o solvability^o and Lemma 3.6), a contradiction to the divisibility of $F^\circ(B_3)$. Hence $d(N(H')) < \infty$ and this completes the proof of the fifth point. \square

Corollary 4.55 [Bur07, Corollary 4.2] *Let G be a locally^o solvable^o group of finite Morley rank. Then a definable connected nonabelian nilpotent subgroup is contained in exactly one Borel subgroup of G .*

Corollary 4.56 [Bur07, Corollary 4.2'] *Let G be a locally^o solvable^o group of finite Morley rank. If Q is a Carter subgroup of a Borel subgroup B , and if Q is not abelian, then Q is a Carter subgroup of G .*

Proof. $N^\circ(Q)$ is contained in a Borel subgroup B_1 of G by local^o solvability^o. As $Q \leq B \cap B_1$, $B = B_1$ by Corollary 4.55, and $N_G^\circ(Q) \leq N_{B_1}^\circ(Q) = N_B^\circ(Q) = Q$ \square

Lemma 4.57 [Bur07, Lemma 4.4] *Let G be a locally^o solvable^o group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G . Suppose that $H = (B_1 \cap B_2)^\circ$ is not abelian, and that $C^\circ(H') \leq B_1$. Then B_1 and B_2 are the only Borel subgroups containing H .*

Proof. Suppose toward a contradiction that G contains a Borel subgroup B distinct from both B_1 and B_2 and which contains H . We may choose B such that $H_2 = (B \cap B_2)^\circ$ is maximal subject to $B \neq B_1$ and $H \leq B, B_2$. Consider a maximal pair (B_3, B_4) containing H_2 and such that $d_\infty(B_3) \geq d_\infty(B_4)$. Corollary 4.50 applied to (B_3, B_4) implies that B_3 is the unique Borel subgroup of G containing $C^\circ(H')$. So $B_1 = B_3$. Thus $H = H_2$. By Proposition 4.24, $B_1 = B_3$ and B_4 are the unique Borel subgroups containing the connected component of their intersection. So we may assume $B_4 \neq B_2$, as otherwise we are done. Therefore we may also assume that $B = B_4$. So $H_1 = (B_1 \cap B)^\circ = (B_3 \cap B_4)^\circ$ corresponds to the intersection^o of maximal pairs, and we can apply the previous results to this intersection. We observe that

$$r' := d_\infty(H') = d_\infty(F(B_1) \cap F(B))$$

by Theorem 4.28.

Consider first the case $U_{(\infty, r')}(F^\circ(B_2)) \leq B_1$. Since H' is (∞, r') -homogeneous, $U_{\tilde{q}}(F^\circ(B_2)) \leq C^\circ(H') \leq B_1$ for every $\tilde{q} \neq (\infty, r')$. Hence $F^\circ(B_2) \leq H$, and $H \trianglelefteq B_2$. By local^o solvability^o of G and maximality of B_2 , $N^\circ(H') = B_2$. But Corollary 4.41 applied to (B, B_1) yields $F^\circ(B_1) \leq C^\circ([F(B) \cap F(B_1)]^\circ) \leq C^\circ(H') \leq N^\circ(H') = B_2$. Then $U_{(\infty, d_\infty(B_1))}(B_1) \leq H$ by Fact 2.15, and $d_\infty(B_1) \leq d_\infty(H_1)$. This contradicts Lemma 4.21 applied with (B, B_1) .

Consider next the case $U_{(\infty, r')}(F^\circ(B_2)) \not\leq B_1$. Let $P = U_{(\infty, r')}(H)$, the unique Sylow (∞, r') -subgroup of H by Proposition 4.54 (3), and $M = N^\circ(P)$, a solvable group by local^o solvability^o. Since P normalizes $U_{(\infty, r')}(F^\circ(B_2))$, $U_{(\infty, r')}(F^\circ(B_2)) \cdot P$ is nilpotent by Fact 2.12. By Normalizer Condition, Fact 2.7, $P < U_{(\infty, r')}(N_{U_{(\infty, r')}(F^\circ(B_2))} \cdot P(P))$. Since $H = (B_1 \cap B_2)^\circ$ and $P = U_{(\infty, r')}(H)$, it follows that $H \leq (M \cap B_2)^\circ \not\leq B_1$ and that $H < (M \cap B_2)^\circ$. Hence $M \leq B_2$ by maximality of $H_2 (= H)$. By Lemma 4.30, $U_{(\infty, r')}(F^\circ(B)) \not\leq H_1$. Since P normalizes $U_{(\infty, r')}(F^\circ(B))$, $U_{(\infty, r')}(F^\circ(B)) \cdot P$ is nilpotent by Fact 2.12. By Normalizer Condition, Fact 2.7, and using $M \leq B_2$, $P < U_{(\infty, r')}(N_{U_{(\infty, r')}(F^\circ(B))} \cdot P(P)) \leq (B \cap B_2)^\circ = H_2 = H$, a contradiction to $P = U_{(\infty, r')}(H)$. \square

We can now characterize maximal pairs with nonabelian intersections.

Theorem 4.58 [Bur07, Theorem 4.3] *Let G be a locally^o solvable^o group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G . Suppose $H = (B_1 \cap B_2)^\circ$ nonabelian. Then the following are equivalent:*

- (1) B_1 and B_2 are the only Borel subgroups containing H .
- (2) (B_1, B_2) is a maximal pair.
- (3) If $B_3 \neq B_1$ is a Borel subgroup containing H , then $(B_1 \cap B_3)^\circ = H$.
- (4) $C^\circ(H')$ is contained in B_1 or B_2 .
- (5) B_1 and B_2 are not conjugate under the action of $C^\circ(H')$.
- (6) $d_\infty(B_1) \neq d_\infty(B_2)$.

Proof. Clearly (1) implies (2), (2) implies (3), and (4) implies (5). By Lemmas 4.21 and 4.22, (2) implies (6). Clearly (6) implies (5). By local^o solvability^o of G , there exists a Borel subgroup B_c of G containing $N^\circ(H')$.

We show now that (3) implies (4). Let B_x denotes B_1 , unless $B_c = B_1$, in which case we let B_x denote B_2 . By (3), $H = (B_c \cap B_x)^\circ$. By Lemma 4.57 applied to the pair (B_c, B_x) , $B_c \geq C^\circ(H')$ must be one of B_1 or B_2 , so (4) holds.

We show now that (5) implies (1). Assume (1) fails. Then, for $i = 1$ and 2 , $C^\circ(H') \not\leq B_i$ by Lemma 4.57. But (B_c, B_1) and (B_c, B_2) are maximal pairs, by Lemma 4.57 again. So $d(B_c) \geq d(B_1), d(B_2)$, by Lemma 4.49. By Lemma 4.53, B_1 is $F^\circ(B_c)$ -conjugate to B_2 . By Corollary 4.41, $F^\circ(B_c) \leq C^\circ(H')$, so (5) fails. \square

We can now describe the maximal pairs having a nonabelian intersection^o, collecting the results from [Bur07] with the additional results from [Del07a]. We slightly change the presentation in comparison to [Bur07, Theorem 4.5], as we prefer to distinguish between a symmetric version and an asymmetric one. We start with the symmetric version.

Theorem 4.59 *Let G be a locally^o solvable^o group of finite Morley rank, and (B_1, B_2) a maximal pair of Borel subgroups such that $H := (B_1 \cap B_2)^{\circ}$ is non-abelian. Let $r' = d_{\infty}(H')$.*

- (1) $0 < d(B_1) < \infty$ and $0 < d(B_2) < \infty$.
- (2) $N^{\circ}(H) = H$.
- (3) $[F(B_1) \cap F(B_2)]^{\circ}$ is (∞, r') -homogeneous, and $r' > 0$.

Furthermore, if Q denotes a Carter subgroup of H and $Q_{r'}$ denotes $U_{(\infty, r')}(Q)$, then

- (4) $Q_{r'} \neq 1$,

and exactly one of the following cases occur:

- (4.a) $N^{\circ}(Q_{r'}) = H$.
- (4.b) $H < N_{B_1}^{\circ}(Q_{r'})$; furthermore $N_{B_2}^{\circ}(Q_{r'}) = H$ and B_1 is the unique Borel subgroup containing $N^{\circ}(Q_{r'})$.
- (4.c) $H < N_{B_2}^{\circ}(Q_{r'})$; furthermore $N_{B_1}^{\circ}(Q_{r'}) = H$ and B_2 is the unique Borel subgroup containing $N^{\circ}(Q_{r'})$.

Proof.

- (1): 4.7, 4.39, 4.29.
- (2): 4.38.
- (3): 4.7, 4.28.
- (4): 4.43, and proof of [Del07a, Lemme 3.9] for the trichotomy. □

We finish with the description once the asymmetry is fixed.

Theorem 4.60 *Assume in addition to Theorem 4.59 that $d(B_1) \geq d(B_2)$. Then*

- (1) $0 < d(B_2) < d(H) = d(B_1) < \infty$.
- (2) Q is a Carter subgroup of B_1 .
- (3) $U_{(\infty, r')}(F(B_1)) = [F(B_1) \cap F(B_2)]^{\circ}$.
- (4) B_1 is the unique Borel subgroup containing $C^{\circ}(U_{(\infty, r')}(F(B_1)))$.
- (5) $N^{\circ}(Q) \leq B_2$.
- (6) $U_{(\infty, r')}(H) \leq F^{\circ}(B_2)$, and $N^{\circ}(U_{(\infty, r')}(H)) \leq B_2$.
- (7) $U_{\tilde{q}}(F(B_2)) \leq Z(H)$ for any $\tilde{q} \neq (\infty, r')$, and $U_{(\infty, r')}(F(B_2))$ is nonabelian (in particular $U_{\tilde{q}}(F(B_2))$ is nonabelian iff $\tilde{q} = (\infty, r')$).

- (8) Any Sylow (∞, r') -subgroup of G containing $U_{(\infty, r')}(H)$ is contained in B_2 .
- (9) If Q is a Carter subgroup of B_2 , then $U_{(\infty, r')}(F(B_2))$ is the unique Sylow (∞, r') -subgroup of B_2 , and in particular the unique Sylow (∞, r') -subgroup of G containing $U_{(\infty, r')}(H)$.

Proof.

- (1): 4.59 (1), 4.23.
(2): 4.44.
(3): 4.40.
(4): 4.40, 4.49.
(5): 4.44
(6): 4.33, 4.36.
(7): 4.29, 4.30.
(8): 4.52.
(9): 4.34, 4.52. □

Finally, we record a point about exceptional elements, which applies in particular in Theorems 4.59 and 4.60.

Theorem 4.61 *Let G be a locally[◦] solvable[◦] group of finite Morley rank and (B_1, B_2) a maximal pair of Borel subgroups such that $[F(B_1) \cap F(B_2)]^\circ$ is non-trivial. Then the finite subgroup S of $F(B_1) \cap F(B_2)$ as in Lemma 4.7 is toral and central, both in B_1 and B_2 .*

Proof. $F^\circ(B_1)$ and $F^\circ(B_2)$ are divisible by Lemmas 4.29 and 4.39, and Remark 4.48 applies. □

4.4 An extra homogeneity result

The following extra homogeneity result proved for the purpose of [Del07a] is essentially a corollary of Corollary 4.55.

Lemma 4.62 (Compare with [Del07a, Lemme 3.8]) *Let G be a locally[◦] solvable[◦] group of finite Morley rank, B and B^g two distinct conjugates of a same Borel subgroup B . If $[F(B) \cap F(B^g)]^\circ$ is not homogeneous, then $F^\circ(B)$ is abelian.*

Proof. By assumption $[F(B) \cap F(B^g)]^\circ$ contains two nontrivial Sylow subgroups U_1 and U_2 with two distinct unipotence parameters, say \tilde{p} for U_1 and \tilde{q} for U_2 .

By local[◦] solvability[◦] of G , $N^\circ(U_1)$ is contained in a Borel subgroup B_1 . If $B_1 \neq B$, then by Fact 2.5 (2) Corollary 4.55 implies that Sylow subgroups of $F^\circ(B)$ of unipotence parameters different from \tilde{p} are abelian. If $B_1 = B$, then $B_1 \neq B^g$ and one sees similarly that Sylow subgroups of unipotence parameters different from \tilde{p} of $F^\circ(B^g)$, and thus also of $F^\circ(B)$, are also abelian.

Considering a Borel subgroup B_2 containing $N^\circ(U_2)$, one sees similarly that Sylow subgroups of $F^\circ(B)$ of unipotence parameters different from \tilde{q} are abelian. Now $F^\circ(B)$ is abelian by Fact 2.5 (2). \square

4.5 Exceptional connected subgroups

Section 4.3 concerned the analysis of intersections of maximal pairs of Borel subgroups. In the present section we continue a little bit in this vein when one of the two subgroups involved is not necessarily solvable, a possibility in the context of locally $^\circ$ solvable $^\circ$ groups of finite Morley rank in comparison to the context of minimal connected simple groups.

Definition 4.63 *Let G be a group of finite Morley rank and K a definable connected subgroup of G . We say that a Borel subgroup B of G has maximal intersection with K if $B \not\leq K$ and $(K \cap B)^\circ$ is maximal for inclusion among groups of the form $(K \cap B_1)^\circ$ for some Borel subgroup B_1 of G such that $B_1 \not\leq K$.*

We note in Definition 4.63 that if K is solvable and not a Borel subgroup, then it has a maximal intersection with any Borel subgroup containing it. If K is a Borel subgroup of G , and a Borel subgroup B has a maximal intersection with K , then G° is not solvable.

Lemma 4.64 *Let G be a group of finite Morley rank, K a definable connected subgroup of G , and B a Borel subgroup of G having maximal intersection with K . Then any Borel subgroup B_1 of G such that $(K \cap B)^\circ < (K \cap B_1)^\circ$ is in K .*

Proof. This is immediate by definition. \square

It follows that if K is a Borel subgroup of a locally $^\circ$ solvable $^\circ$ group G and B is a Borel subgroup of G having maximal intersection with K , then if $(K \cap B)^\circ$ is nonabelian any Borel subgroup B_3 of G containing $(K \cap B)^\circ$ such that $(K \cap B)^\circ < (K \cap B_3)^\circ$ must be K , and hence (K, B) is a maximal pair of Borel subgroups of G by the equivalence provided in Theorem 4.58 (3).

In the general case of a locally $^\circ$ solvable $^\circ$ group G a proper definable connected subgroup K can be nonsolvable, and we slightly clarify the situation in this general case.

Lemma 4.65 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank, K a nontrivial definable connected subgroup of G , B a Borel subgroup of G having maximal intersection with K , and let $H = (K \cap B)^\circ$. Then assuming H nontrivial exactly one of the following cases occurs.*

- (1) H is an abelian Carter subgroup of K and of B .
- (2) H is an abelian Carter subgroup of K and $H < N_B^\circ(H) \leq B$.

- (3) H is an abelian Carter subgroup of B , and $H < N^\circ(H) \leq K$. In this case any Borel subgroup of K containing H is a Borel subgroup of G .
- (4) H is a nonabelian Borel subgroup of K .
- (5) H is nonabelian and not a Borel subgroup of K . In this case any Borel subgroup of K containing H is a Borel subgroup of G .

Proof. Notice that $N^\circ(H)$ is solvable by local^o solvability^o of G .

Assume first H abelian. If H has finite index in its normalizer in K and in B then we are in case (1).

Assume now $H < N_B^\circ(H)$. Then the maximality of the intersection forces $N_K^\circ(H) = H$, and H is an abelian Carter subgroup of K . Hence we are in case (2).

Assume now $H < N_K^\circ(H)$. Then the maximality of the intersection forces $N^\circ(H) \leq K$ with Lemma 4.64. Now $N_B^\circ(H) \leq (K \cap B)^\circ = H$, and H is an abelian Carter subgroup of B . Hence we are in case (3) by Lemma 4.64.

This treats all cases corresponding to the case H abelian, so we may now assume H nonabelian. If H is a Borel subgroup of K , then we are in case (4).

It remains only to consider the case in which H is not abelian and not a Borel subgroup of K . By Lemma 4.64, any Borel subgroup of K containing H is a Borel subgroup of G . We are in case (5). \square

5 Homogeneous cases and torsion

In this final section we collect various additional results of specialized nature about locally^o solvable^o groups of finite Morley rank, generally pending on the uniqueness theorems of Section 4.1.

The first type of results concerns the homogeneous cases. Recall from [FJ08] or Section 2.1 that a group of finite Morley rank is homogeneous if it is \tilde{p} -homogeneous for some unipotence parameter \tilde{p} , that is every definable connected nilpotent subgroup is a \tilde{p} -group. (This is weaker than the definition in [Fré06a], which requires to consider all definable connected subgroups, not only the nilpotent ones.) In a \tilde{p} -homogeneous group one sees easily with Lemma 2.11 and Fact 2.15 that any Borel subgroup is a (homogeneous) \tilde{p} -group, and in particular nilpotent. Hence we will more generally consider the case in which all Borel subgroups are nilpotent, and look at the homogeneous cases at various levels of generality.

The torsion-free case will be fairly well understood in this context, and with torsion this connects to a bit of Sylow theory. As far as torsion is concerned, there is in general no Sylow theory as in Fact 2.24 available in an arbitrary group of finite Morley rank. The following fact shows however similarities with Fact 2.24 in the general case.

Fact 5.1 [BC07, Theorem 3 and Corollary 3.1] *Let G be a connected group of finite Morley rank, t a π -element of G for some set π of primes p . If*

$U_p(C(t)) = 1$ for every p in π , then t belongs to a , and in fact to any, maximal π -torus of G and of $C^\circ(t)$.

Notice that the second statement is a mere corollary of the first, together with the fact that toral elements belong to the connected components of their centralizers and Fact 2.19.

5.1 Nilpotent Borel subgroups

In this section we consider locally^o solvable^o groups in which each Borel subgroup is nilpotent. We start with a lemma concerning abelian Borel subgroups.

Lemma 5.2 *Let G be a locally^o solvable^o group of finite Morley rank with an abelian Borel subgroup B . Let B_u denote the maximal definable connected subgroup of B of bounded exponent. Then B has a subgroup E finite modulo B_u such that $B \cap B^g \leq E$ for any element g of G not in $N(B)$, and one of the following two cases occurs.*

- (1) B is a generous abelian Carter subgroup.
- (2) B is an abelian Carter subgroup of bounded exponent.

Proof. For any g in $G \setminus N(B)$, $N^\circ(B \cap B^g)$ contains B and B^g , and if $B \cap B^g$ is infinite then $N^\circ(B \cap B^g)$ is solvable by local^o solvability^o of G and one gets $B, B^g \leq N^\circ(B \cap B^g)$ and $B = B^g$ by maximality, a contradiction. Hence $B \cap B^g$ is finite for every $g \in G \setminus N(B)$.

The uniformly definable family of finite subgroups $B \cap B^g$, for $g \in B \setminus N(B)$, consists of subgroups of uniformly bounded cardinals by elimination of infinite quantifiers. As Prüfer p -ranks are finite for any prime p , all these subgroups must be contained modulo B_u in a finite subgroup of the maximal definable decent torus of B . Calling E the preimage in B of this group, this proves our first statement.

If $B_u < B$, then E is not generic in B and one can conclude that the Carter subgroup B of G is generous by the equivalence given in Fact 2.14 (3). This proves our alternative. \square

We note that the two cases in Lemma 5.2 are a priori not necessarily mutually exclusive. In the locally^o solvable context E is necessarily trivial, and B is then necessarily generous in any case.

We pass now to nilpotent Borel subgroups, replacing the commutativity assumption by a nilpotence assumption on all Borel subgroups of the ambient group. The first lemma is essentially the content of the first part of the proof of Lemma 5.2 and typical of earlier work on bad groups [BN94, Chapter 13].

Lemma 5.3 *Let G be a locally^o solvable^o group of finite Morley rank in which all Borel subgroups are nilpotent. Then any two distinct Borel subgroups have a finite intersection.*

Proof. Assume the contrary. Let B_1 and B_2 contradict our claim, with $[B_1 \cap B_2]^\circ$ of maximal rank. Call the latter group H , and notice that $H < B_1$ and $H < B_2$. By normalizer condition in nilpotent groups, [BN94, Lemma 6.3], $H < N_{B_1}^\circ(H)$ and $H < N_{B_2}^\circ(H)$. Now $N^\circ(H)$ is solvable by local^o solvability^o of G , and contained in a Borel subgroup B_3 . As $H < (B_1 \cap B_3)^\circ$ and $H < (B_2 \cap B_3)^\circ$, our maximality assumption forces $B_1 = B_3 = B_2$, a contradiction. \square

We get in any case conclusions similar to those of Lemma 5.2.

Lemma 5.4 *Let G be a locally^o solvable^o group of finite Morley rank in which all Borel subgroups are nilpotent. Let B be a Borel subgroup of G . Then B has a definable subgroup E , finite modulo the bounded exponent part of B , such that $B \cap B^g \leq E$ for any intersection $B \cap B^g$ with $g \in G \setminus N(B)$. Moreover one of the following two cases occurs.*

- (1) B is a generous Carter subgroup.
- (2) B is a Carter subgroup of bounded exponent.

Proof. With Lemma 5.3 applied to distinct conjugates of B , the existence of E follows as in the proof of Lemma 5.2. The alternative proposed follows similarly as well. \square

As for Lemma 5.2, the two cases in Lemma 5.4 are a priori not mutually exclusive, and if the ambient group G is locally^o solvable then distinct conjugates of B are necessarily pairwise disjoint by the same proof as in Lemma 5.3, and B is always generous.

5.2 The torsion-free homogeneous case

We shall now evacuate, or rather collect in Pandora's box of bad groups, \tilde{p} -homogeneous locally^o solvable^o groups of finite Morley rank, with \tilde{p} not of the form $(\infty, 0)$ or (p, ∞) for p a prime. In this case Borel subgroups are nilpotent and torsion-free by Facts 2.1 and 2.6. More generally, we have the following result for such groups.

Theorem 5.5 *Let G be a torsion-free locally^o solvable^o group of finite Morley rank in which Borel subgroups are all nilpotent. Then Borel subgroups are conjugate and either*

- (1) G is nilpotent, or
- (2) $B < G$ is a full Frobenius group for some Borel subgroup B of G .

As far as torsion is concerned there is a classical lifting result.

Fact 5.6 [BN92] *Let G be a group of finite Morley rank, H a definable normal subgroup, and x a p -element modulo H , for some prime p . Then the definable hull $H(x)$ of x contains a p -element.*

Proof. Notice that G is connected by absence of torsion and Fact 5.6.

By Lemmas 5.3 and 5.4, distinct Borel subgroups have trivial intersections, and each Borel subgroup is generous. As G is connected it cannot have two disjoint generic subsets. If B_1 and B_2 are two Borel subgroups, then two conjugates of B_1 and B_2 must have a nontrivial intersection by generosity, and then are equal. This shows that Borel subgroups are conjugate.

If G is not nilpotent, then $B < G$ for some Borel subgroup B of G . By Fact 5.6, $N(B) = B$, and B is malnormal in G by disjointness of distinct Borel subgroups. As B^G is generic, any element g of G has an infinite centralizer (this is also an easy consequence of the main result of [BBC07] in arbitrary connected groups), and in particular normalizes a Borel subgroup by Lemma 2.11 (1) and the disjointness of Borel subgroups. Hence $G = B^G$, and $B < G$ is a full Frobenius group. \square

We note that a connected (∞, r) -homogeneous group of finite Morley rank, with $0 < r < \infty$, is torsion-free by Facts 2.6 and 5.1, and in particular Theorem 5.5 applies to such homogeneous connected locally^o solvable^o groups.

Otherwise in the torsion free case all results of Section 4 still apply, where all definable subgroups are connected. In this case Carter subgroups are conjugate by the same proof as in [Fre07].

5.3 The bounded exponent case

In presence of bounded exponent torsion the uniqueness theorems of Section 4.1 can be applied in their most straightforward forms for dealing with generosity, as seen in Section 4.2 already.

Lemma 5.7 *Let G be a locally^o solvable^o group of finite Morley rank such that $U_p(G)$ is nontrivial for some prime p . Then one of the following three cases occur.*

- (1) *Maximal p -unipotent subgroups are conjugate in G^o and $N^o(U)$ is a generous Borel subgroup of unbounded exponent for any maximal p -unipotent subgroup U (and in fact one may assume also $N^o(U) = UC^o(U)$).*
- (2) *There is a maximal p -unipotent subgroup U normalized but not centralized by a nontrivial q -torus T for some (and in fact infinitely many) prime(s) $q \neq p$. Moreover T is contained in a generous Carter subgroup of G .*
- (3) *$N^o(U)$ is a Carter subgroup of bounded exponent for some maximal p -unipotent subgroup U .*

Proof. First recall that $N^\circ(U)$ is a Borel subgroup of G for any maximal p -unipotent subgroup U of G by Lemma 3.9.

Assume case (3) does not occur. This means that for any maximal p -unipotent subgroup U , $N^\circ(U)$ is not nilpotent of bounded exponent. By Fact 2.15, this simply means that any such group $N^\circ(U)$ has unbounded exponent.

If $UC^\circ(U) < N^\circ(U)$ for some maximal p -unipotent subgroup U , then Wagner's theorem [Wag01, Corollary 8] gives a nontrivial q -torus in $N^\circ(U)$, for some prime $q \neq p$, acting nontrivially on U (see for example [FJ05, Fact 2.5] and Zilber's field theorem [BN94, §9.1]). The fact that there are infinitely many primes q occurring in the definable subgroup of the multiplicative group of the field of characteristic p is due to [Wag03]. Then Fact 3.34 shows that we are in case (2).

This leaves us with the case in which $N^\circ(U) = UC^\circ(U)$ is a Borel subgroup of unbounded exponent for any maximal p -unipotent subgroup U .

If $N^\circ(U) \cap N^\circ(U^g)$ has a nontrivial connected component X for some $g \in G$, then $N^\circ(X)$ is solvable by local solvability of G . As $N^\circ(U) = UC^\circ(U)$, X centralizes a nontrivial p -unipotent subgroup of U by Fact 2.24 (2), and similarly a nontrivial p -unipotent subgroup of $N^\circ(U^g)$. Now, as $N^\circ(X)$ is contained in a Borel subgroup, Lemma 4.6 implies $N^\circ(U) = N^\circ(U^g)$. Hence distinct conjugates of $N^\circ(U)$ have finite intersections.

As $N^\circ(U)$ has unbounded exponent, these finite intersections cannot cover $N^\circ(U)$ generically by Lemma 2.35. In particular they land in a (definable) non-generic subset of $N^\circ(U)$, and one concludes easily that $N^\circ(U)$ is generous in G° (see for instance [CJ04, Lemma 3.3], bearing in mind that $N^\circ(U)$ is of finite index in its normalizer, as a Borel subgroup).

We have thus $N^\circ(U) = UC^\circ(U)$ a generous Borel subgroup of unbounded exponent for any maximal p -unipotent subgroup U .

Now let U_1 and U_2 be two maximal p -unipotent subgroups of G . By generosity of $N^\circ(U_1)$ and [Jal06, Proposition 2.1], a generic element g of G° is in a conjugate of $N^\circ(U_1)$, and in finitely many such. Similarly, g is in a conjugate of $N^\circ(U_2)$, say $N^\circ(U_2)$ after conjugacy, and in finitely many such. Now $Z^\circ(U_2)$ centralizes g as $N^\circ(U_2) = U_2C^\circ(U_2)$. So it permutes naturally by conjugation the finitely many conjugates of $N^\circ(U_1)$ containing g , and one can argue as in [Jal06, Fundamental Lemma 3.3]. By Fact 1.2, it fixes each of them, and in particular it normalizes a conjugate of U_1 , say U_1 up to conjugacy. Hence $Z^\circ(U_2) \leq N^\circ(U_1)$, $Z^\circ(U_2) \leq U_p(N^\circ(U_1)) = U_1$, and $U_1 = U_2$ by Theorem 4.1. This shows that U_1 and U_2 are conjugate and completes our proof. \square

First we note that this completes the proof of Theorem 4.13.

The fact that $N^\circ(U) = UC^\circ(U)$ is stated between parentheses in case (1) of Lemma 5.7 is to depreciate this aspect not true in the algebraic case. A conclusion closer to the algebraic case would be case (2) combined with case (1) without this aspect. But even in the well described context of [CJ04] there are potentially Borel subgroups as in case (2) but not as in case (1) without this aspect (in sets of Borel subgroups usually denoted by \mathfrak{B} in [CJ04]).

If the ambient group G is locally[◦] solvable in Lemma 5.7, then one sees by the same argument, and using the results of Section 4.1 adapted to the locally[◦] solvable case, that Borel subgroups as in cases (1) and (3) have trivial intersections indeed, and are all generous. In particular a maximal p -unipotent subgroup U as in case (3) must satisfy $N^\circ(U)$ generous, and must be conjugate to one as in case (1) if it exists. But in this case one also has $N^\circ(U)$ of unbounded exponent, and thus cases (1) and (3) are mutually exclusive. It follows also that cases (2) and (3) are mutually exclusive, and as cases (1) and (2) are obviously mutually exclusive all cases are pairwise mutually exclusive, and with a generous Carter subgroup in any case. One can summarize this as follows.

Lemma 5.8 *Let G be a locally[◦] solvable group of finite Morley rank such that $U_p(G)$ is nontrivial for some prime p . Then exactly one of the following two cases occur.*

- (1) *Maximal p -unipotent subgroups are conjugate in G° and Borel subgroups of the form $N^\circ(U)$, for U a maximal p -unipotent subgroup, are pairwise disjoint, generous, of the form $UC^\circ(U)$, and either of unbounded exponent or nilpotent of bounded exponent.*
- (2) *There is a maximal p -unipotent subgroup U normalized but not centralized by a nontrivial q -torus T for some (and in fact infinitely many) prime(s) $q \neq p$. Moreover T is contained in a generous Carter subgroup of G .*

As in Section 5.2 one may wish to consider the (p, ∞) -homogeneous case for some prime p , or more generally the case in which all Borel subgroups are nilpotent but now of bounded exponent. In this case any Borel subgroup is a Carter subgroup of bounded exponent, and cases (1) and (2) of Lemma 5.7 cannot occur (recall that in case (1) $N^\circ(U)$ has unbounded exponent). One can also see in this case that any two distinct Borel subgroups have a finite intersection, by using Corollary 4.3.

We continue with the mere presence of a nontrivial p -unipotent subgroup for some prime p .

Lemma 5.9 *Let G be a locally[◦] solvable[◦] group of finite Morley rank, p and q two primes (possibly the same). Assume a nontrivial p -unipotent subgroup U of G commutes with a nontrivial q -torus T of G . Then there is a Borel subgroup B of G containing T , U , a Carter subgroup of G (and B) containing T and generous in G , and a maximal p -unipotent subgroup of G .*

Proof. Let Q be a Carter subgroup of G containing T , which exists and is generous in G by Fact 3.34. We have Q and U in $N^\circ(T)$, and $N^\circ(T) \leq B$ for some Borel subgroup by local[◦] solvability[◦] of G . Now B is the unique Borel subgroup of G containing U by the Uniqueness Theorem, here Corollary 4.3 or Corollary 4.4, and our claim follows. \square

Definition 5.10 *If M is a proper definable subgroup of a group G of finite Morley rank and p a prime, we say that*

- (1) *M is p -weakly embedded in G if M has infinite p -subgroups and $M \cap M^g$ has no infinite p -subgroups for any g in $G \setminus M$.*
- (2) *M is p -strongly embedded in G if M has nontrivial p -subgroups and $M \cap M^g$ has no nontrivial p -subgroups for any g in $G \setminus M$.*

Again the following remarks were obviously made around [CJ04], but they were not explicitly stated there to keep that paper not too long.

Lemma 5.11 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank, p a prime, U_1 and U_2 two distinct maximal p -unipotent subgroups of G (which are then necessarily nontrivial). Then p -subgroups of $N(U_1) \cap N(U_2)$ are exceptional and have order at most $e(G)$.*

Proof. By assumption, $N^\circ(U_1)$ and $N^\circ(U_2)$ are two distinct Borel subgroups of G .

Assume toward a contradiction $N(U_1) \cap N(U_2)$ contains a p -subgroup X with $C^\circ(X)$ solvable. We have then $C^\circ(X) \leq B$ for some Borel subgroup B . Notice that $C_{U_1}^\circ(X)$ and $C_{U_2}^\circ(X)$ are both nontrivial by Fact 2.24 (2). Now Lemma 4.6 implies $N^\circ(U_1) = N^\circ(U_2)$, a contradiction. \square

Corollary 5.12 *Let G be a locally $^\circ$ solvable $^\circ$ group of finite Morley rank with G° nonsolvable. Assume that for some prime p maximal p -unipotent subgroups of G are nontrivial. Then $N(U)$ is p -weakly embedded in G for any such maximal p -unipotent subgroup U of G , and p -strongly embedded whenever G is locally $^\circ$ solvable.*

Proof. Assume $N(U) \cap N(U^g)$ has an infinite p -subgroup S for some g in G . We have $S^\circ \leq N^\circ(U) \cap N^\circ(U^g)$, and as S is infinite S° is infinite as well. By Fact 2.24 (1), S° is a central product of a p -unipotent subgroup V and a p -torus T , and one of the two factors is nontrivial by assumption. Now Lemma 4.6 or Lemma 5.11 gives in any case $N^\circ(U) = N^\circ(U^g)$. Thus $g \in N(U)$.

When G is locally $^\circ$ solvable one proceeds similarly, but now the only exceptional p -element is the identity. \square

We also observe that when a nontrivial p -unipotent subgroup commutes with a nontrivial p -torus, then a maximal p -torus commutes with a maximal p -unipotent subgroup by Lemma 5.9. One can then build a p -weakly embedded subgroup as for the elimination of 2-mixed type simple groups [ABC08]. If U is a definable p -unipotent subgroup of G , we denote by U^\perp the definable connected subgroup $T_p(C(U))$, the subgroup of $C(U)$ generated by the definable hulls of its p -tori. By local $^\circ$ solvability $^\circ$, this group is solvable (for U nontrivial). One sees easily that if $[U_1, U_2] = 1$, then $U_1^\perp = U_2^\perp$. Then one observes that

the graph on the set of nontrivial p -unipotent subgroups, where adjacency is commutation, is not connected, as otherwise U^\perp is independent of the choice of U , hence normal in G , and as it is nontrivial connected and solvable, G° is solvable by local^o solvability^o, a contradiction to the assumption. Let \mathcal{C} be a connected component of the graph. The group G acts naturally on this graph. Let M be the normalizer in G of a connected component of \mathcal{C} . If $U \in \mathcal{C}$, then $M \leq N(U^\perp)$. In particular M° is solvable, and M has a unique maximal p -unipotent subgroup U . $M = N(U) = N(U^\perp)$. Notice that $B = M^\circ$ is a Borel subgroup of G . But in any case one shows that M is p -weakly embedded in G .

With $p = 2$ these notions will suffice to eliminate connected non-solvable mixed type locally^o solvable^o groups in [DJ07], by methods and/or results from the simple case. For $p \neq 2$ Configuration 3.17 stands around. Prüfer ranks will be controlled with the notion of strongly embedded subgroup.

If one is not interested in conjugacy in Lemma 5.7 but merely in genericity, then one can notice that a connected locally^o solvable^o group with $U_p(G)$ nontrivial with no generous Borel subgroup must satisfy that $N^\circ(U)$ is a Carter subgroup of bounded exponent for each maximal p -unipotent subgroup U ; otherwise $N^\circ(U)$ has unbounded exponent and one gets as in the proof of Lemma 5.7 either a nontrivial decent torus or $N^\circ(U)$ generous, a contradiction to the assumption.

In particular, if the generic element of a connected locally^o solvable^o group G of finite Morley rank is not in a connected nilpotent subgroup, then G contains no decent tori (Fact 3.34), contains nontrivial p -unipotent subgroups (Facts 5.1 and 5.6), and $N^\circ(U)$ is a Carter subgroup of bounded exponent for each such maximal p -unipotent subgroup U , generically composed of exceptional elements by Lemma 4.11.

5.4 The toral homogeneous case

We shall now consider the case in which there is no bounded exponent subgroup, and more specifically the toral homogeneous case. Before studying this specific case precisely, we note that Carter subgroups are conjugate in any locally^o solvable^o group G of finite Morley rank such that $d(G) < \infty$, by the same proof as in [Fre07].

Theorem 5.13 *Let G be a locally^o solvable^o group of finite Morley rank in which Borel subgroups are divisible abelian. Assume furthermore that nontrivial toral elements are not exceptional and that G contains no involution. Then, either*

- (1) G° is abelian, or
- (2) $T < G^\circ$ is a full Frobenius group for some (any) Borel subgroup T .

We will use the following fact.

Fact 5.14 (Ali's Lemma) *Let G be a group, T_1 and T_2 two disjoint subgroups, $x_1 \in T_1 \cap (N(T_2) \setminus T_2)$ and $x_2 \in T_2 \cap (N(T_1) \setminus T_1)$ satisfying $x_1 T_2 = x_1^{T_2}$,*

$x_2T_1 = x_2^{T_1}$, and $(x_1^2)T_2 = (x_1^2)^{T_2}$. Then x_1 and x_1^2 are conjugate in G . In particular, if x_1 has prime order $p \neq 2$ and is central in T_1 , $N(T_1)$ controls fusion in T_1 , and $N(T_1)/T_1$ is finite, then some nontrivial prime divisor of $N(T_1)/T_1$ divides $p - 1$.

Proof. This is one of the essential contents of [Nes89], already re-employed through the scope of [CJ04, Lemma 7.23]. By the fusion assumptions one can conjugate x_1 to x_1x_2 in x_1T_2 , x_1x_2 to $x_1^2x_2$ in x_2T_1 , and $x_1^2x_2$ to x_1^2 in $x_1^2T_2$, which yields the G -conjugacy of x_1 and x_1^2 .

For the second point we have now a nontrivial induced automorphism of $\langle x_1 \rangle$ in $N(T_1)/T_1$, and the cyclic group $\langle x_1 \rangle$ of prime order p has an automorphism group of order $p - 1$. \square

We now proceed to the proof of Theorem 5.13.

Proof. In order to prove Theorem 5.13 we consider now G a connected locally^o solvable^o group, and fix a Borel subgroup T , which is divisible abelian by assumption. If G is solvable, then $G = T$ and we are in case (1). So we may assume G not solvable.

As in Lemma 5.2, any two Borel subgroups T_1 and T_2 must have a finite intersection E , and being a finite subgroup of a divisible abelian group E must be toral if it is nontrivial. If $E \neq 1$, then $T_1, T_2 \leq C^\circ(E)$, and one gets either $T_1 = T_2$ when $C^\circ(E)$ is solvable, or a nontrivial exceptional toral subgroup otherwise, which is excluded by assumption. Hence distinct Borel subgroups are pairwise disjoint. As usual, each is generous, and they are all conjugate.

As any element of G also has an infinite centralizer, any such element must centralize an infinite abelian subgroup by Lemma 2.11 (1), and in particular normalizes the unique conjugate of T containing it.

This shows that $G = N(T)^G$. If $N(T) = T$, then T is malnormal in G by disjointness of pairwise distinct Borel subgroups, and $G = T^G$, and thus $T < G$ is a full Frobenius group as desired.

Hence the analysis boils down to showing that T is selfnormalizing. Assume on the contrary $T < N(T)$, and let x be an element of order p modulo T for some prime p , which may be assumed to be itself a p -element of G by Fact 5.6, and in fact inside a p -torus.

By conjugacy, one concludes that T contains a maximal p -torus T_p which is nontrivial. Now x is in a conjugate T_p^g of T_p and $x^p \in T_p$. As $T^g \cap T = 1$, as otherwise $T = T^g$ and $x \in T^g = T$, $x^p \in T \cap T^g = 1$. For any element y in xT , the definable hull $H(y)$ of y contains also a p -element y_1 by Fact 5.6, which similarly belongs to a maximal torus T_1 distinct from T . Now $C_T^\circ(y) \leq C_T^\circ(y_1) \leq T \cap T_1$, and thus any element y in xT has a finite centralizer in T . Hence y^T is generic in xT for any y in xT , and as the Morley degree is one one gets $xT = x^T$. Now x normalizes T and centralizes a nontrivial element z in the elementary abelian p -subgroup of T . We have z normalizing T_x , the torus containing x , without being inside, and similarly $zT_x = z^{T_x}$ (this is typical

of [Nes89]. See also [CJ04, Lemma 7.19]). We are now in situation to apply Fact 5.14. Noticing that $N(T)$ controls fusion in the torsion subgroup of T by Corollary 2.20, this gives a contradiction by choosing for p the smallest prime divisor $N(T)/T$. \square

We note similarities between groups as in Theorem 5.13 (2) with those of [JOH04]. These are far from being stable by [JMN08], but there are some hints for the existence of (at least partially) stable such groups, as envisioned in [Jal08b, §1].

We also note that a reduction to Fact 5.14 yields involutions or triviality of Weyl groups in general in groups of finite Morley rank without non-trivial p -unipotent subgroups [BC07]. In particular for the last paragraph of the proof of Theorem 5.13 we could have referred to this.

With this triviality of Weyl groups in connected groups without involutions and without p -unipotent subgroups, one can give a general decomposition as in Theorem 5.13 without non-exceptionality assumption, with the Galois connection of Section 3.4.

Theorem 5.15 *Let G be a locally^o solvable^o group of finite Morley rank in which Borel subgroups are divisible abelian, and without involutions. Then G^o has an abelian generous selfnormalizing Carter subgroup T such that $G^o = T^{G^o}$.*

Proof. This is similar to the proof of Theorem 5.13, using Lemma 5.2 for the generosity of (divisible abelian) Borel subgroups T . Notice also that such (conjugate) Borel subgroups T are selfnormalizing by the above mentioned result of [BC07], or a more direct reduction to Fact 5.14 here, and cover G^o by the same argument as in the proof of Theorem 5.13 again. \square

In general one cannot say much more in Theorem 5.15, except describing the full group G^o by the graph of finite exceptional closed subsets of the divisible abelian Borel subgroup T introduced at the end of Section 3.4 and delineated in Lemma 3.33. In fact, exceptional subsets of T are in the divisible torsion subgroup of T , and in a finite subset of it by Lemma 3.30. One sees easily that closed exceptional subsets of T correspond exactly to intersections of T with distinct conjugates of T . If (X_0, \dots, X_k) is a maximal chain of exceptional closed subsets of T in the graph of exceptional closed subsets of T (i.e. with (X_i, X_{i+1}) a minimal extension for each i), then $X_0 = Z(G^o)$, $C^o(X_i) = C_{G^o}(X_i)$ for each i (by a Frattini Argument following the conjugacy of generous Carter subgroups and triviality of the Weyl group $N(T)/T$ in G^o) and the center of this group is X_i , and each group

$$C^o(X_i)/X_i$$

also satisfies the assumptions of Theorem 5.15, with exception indices decreasing as i increases. The last factor $C^o(X_k)/X_k$ is as in Theorem 5.13 by Lemma 3.31.

Hence any group as in Theorem 5.15 is entirely described as above by the finite graph of exceptional closed subsets of T . In particular the picture in Theorem 5.15 looks like Configuration 3.17, where all Borel subgroups involved are decent tori but potentially with more complexity involved in the finite graph of exceptional subsets of T . As for Theorem 5.13, constructions of such abstract groups can be obtained as in [JOH04] with *any* finite graph for T (compatible with the conditions of Lemma 3.33), similarly with a bad control on the complexity of their model theory by the general construction but perhaps with some stability if more care is taken.

The case of groups as in Theorems 5.13 and 5.15 but with involutions will be considered in [DJ07], and eventually disappear by the analysis of this paper and the contents of [Nes89].

5.5 Prüfer ranks and strong embedding

In this final section we recast the dichotomy represented by Sections 6 and 7 of [CJ04] in its actual content. If S is an abelian p -group for some prime p and n a natural number, then we denote by $\Omega_n(S)$ the subgroup of S generated by all elements of order p^n .

Theorem 5.16 *Let G be a connected nonsolvable locally[◦] solvable[◦] group of finite Morley rank of Prüfer p -rank at least 2 for some prime p , and fix a maximal p -torus S of G . Assume that every proper definable connected subgroup containing S is solvable, that elements of S of order p are not exceptional, and let*

$$B = \langle C^\circ(s) \mid s \in \Omega_1(S) \setminus \{1\} \rangle.$$

Then either

- (1) *$B < G$, in which case B is a Borel subgroup of G , and moreover $N(B)$ is p -strongly embedded in G assuming additionally that $U_p(C(s)) = 1$ for every element s of order p of S and that S is a Sylow p -subgroup of $N_{N(B)}(S)$, or*
- (2) *$B = G$, in which case S has Prüfer p -rank 2.*

Let $M = N(B)$. As B contains a Carter subgroup Q of G containing S , Facts 2.21 and 3.34 and a Frattini Argument give $M = N(B) \subseteq BN(Q)$, and as Q is almost selfnormalizing $B = M^\circ$. (With the same notation, this holds of course for an arbitrary p -torus S in an arbitrary group G of finite Morley rank.)

Assume first

- (1) $B < G$

By assumption $B \leq B_1$ for some Borel subgroup B_1 of G . As $S \leq B \leq B_1$, Corollary 2.34 implies that $B = B_1$, and thus B is a Borel subgroup of G .

We adopt now the extra assumptions that $U_p(C(s)) = 1$ for every element s of order p of S and that S is a Sylow p -subgroup of $N_{N(B)}(S)$. We claim that

$M = N(B)$ is p -strongly embedded in G in this case by using a “black hole” principle (a term going back to Harada) similar to the one used in [BCJ07, §2.2], and already contained in [CJ04, Lemma 7.3]. We note that our additional assumptions imply that S is a Sylow p -subgroup of B (Facts 2.23 and 2.24), and of M as well, as $M = BN_{N(B)}(S)$ by a Frattini Argument. In particular M/B has trivial Sylow p -subgroups by Fact 5.6.

Assume that $M \cap M^g$ contains an element s of order p for some g in G . Notice that s is actually in $B \cap B^g$, and p -toral. By connectedness and conjugacy of Sylow p -subgroups in connected solvable groups, the definition of B implies that $C^\circ(s') \leq B$ for any element s' of order p of B . Similarly, $C^\circ(s') \leq B^g$ whenever s' has order p and is in B^g . By conjugacy in B we may assume s in S , and if Q denotes a Carter subgroup of G containing S then $\Omega_1(S) \leq S \leq Q \leq C^\circ(s) \leq B \cap B^g$. By Lemma 2.33 or Corollary 2.34 applied in B and in B^g we get $B^g = \langle C^\circ(s) \mid s \in \Omega_1(S) \setminus \{1\} \rangle = B$. Thus g normalizes B , and is in M .

Hence $M = N(B)$ is p -strongly embedded in G under the two extra assumptions, and this proves clause (1) of Theorem 5.16.

Now we pass to the second case

$$(2) \quad B = G$$

We will eventually show that clause (2) of Theorem 5.16 holds by reworking the beginning of Section 6 of [CJ04]. We first put aside p -unipotent subgroups.

Lemma 5.17 *Any Borel subgroup containing a toral element of order p has trivial p -unipotent subgroups.*

Proof. Assuming the contrary, we may assume after conjugacy of decent tori that a Borel subgroup L with $U_p(L)$ nontrivial contains an element s of S of order p . Then $U_p(C(s))$ is nontrivial by Fact 2.24, contained in a unique Borel subgroup B_1 of G . (Actually $B_1 = L$.) By Corollary 4.4, B_1 is the unique Borel subgroup containing any given nontrivial p -unipotent subgroup of $U_p(C(s))$. Now any element s' of order p of S normalizes $U_p(C(s))$, and thus $U_p(C(s, s')) \neq 1$ by Fact 2.24, and as B_1 is the unique Borel subgroup containing the latter group we get $C^\circ(s') \leq B_1$. This shows that $B \leq B_1$, a contradiction as $B = G$ is nonsolvable under the current assumption. \square

In other words, nontrivial p -toral elements commute with no nontrivial p -unipotent subgroups. This can be stated more carefully as follows.

Corollary 5.18 *Any connected solvable subgroup $\langle s \rangle$ -invariant for some p -toral element s of order p has trivial p -unipotent subgroups.*

Proof. Otherwise s would normalize a nontrivial p -unipotent subgroup, and by Fact 2.24 it would centralize a nontrivial p -unipotent subgroup. \square

Our assumption (2) on B yields similarly a property antisymmetric to the black hole principle implied by assumption (1). Let E denote the elementary abelian p -group $\Omega_1(S)$.

Lemma 5.19 *Let E_1 be a subgroup of E of order at least p^2 . Then for any proper definable connected subgroup L there exists an element s of order p of E_1 such that $C^\circ(s) \not\leq L$. In particular $G = \langle C^\circ(s) \mid s \in E_1 \setminus \{1\} \rangle$.*

Proof. Assume on the contrary $C^\circ(s) \leq L$ for any element s of order p of E_1 .

We claim that $C^\circ(t) \leq L$ for any element t of order p of E . In fact, as $E_1 \leq S \leq C^\circ(t)$, $C^\circ(t)$ is by Lemma 2.33 generated by its subgroups of the form $C^\circ(t, s)$, with s of order p in E_1 . As these groups are all contained in L by assumption, our claim follows.

Hence we have $B \leq L < G$. But under our current assumption $B = G$, and this is a contradiction.

Our last claim follows, as proper definable connected subgroups of G containing S are solvable by assumption. \square

Corollary 5.20 *There exists an element s of order p of E such that $C^\circ(S) < C^\circ(s)$.*

Proof. $C^\circ(S)$ is S -local $^\circ$, and thus solvable by local $^\circ$ solvability $^\circ$ of G and Lemma 3.4. As $C^\circ(S) \leq C^\circ(s)$ for any element s of order p of S , it suffices to apply Lemma 5.19. \square

Lemma 5.21 *There exists an element s of order p of E such that*

$$d(O_{p'}(C^\circ(s))) \geq 1.$$

Proof. Assume the contrary, and let s be an arbitrary element of order p of E . By our assumption that $d(O_{p'}(C^\circ(s))) \leq 0$, $O_{p'}(C^\circ(s))$ is trivial or a good torus by Lemma 2.11, and central in $C^\circ(s)$ by Fact 2.12 (1). Notice that $U_p(C^\circ(s)) = 1$ by Lemma 5.17. As $C^\circ(s)/O_{p'}(C^\circ(s))$ is abelian by Fact 2.31, $C^\circ(s)$ is nilpotent. Now S is central in $C^\circ(s)$ by Fact 2.5 (2). In particular $C^\circ(S) = C^\circ(s)$, and this holds for any element s of order p of E . We get a contradiction to Corollary 5.20. \square

It follows in particular from Lemma 5.21 that there exist definable connected subgroups L containing $C^\circ(s)$ for some element s of order p of E and such that $O_{p'}(L)$ is not a good torus. Choose then a unipotence parameter $\tilde{q} = (q, r)$ different from $(\infty, 0)$ such that r is maximal in the set of all $d_q(O_{p'}(L))$, where L varies in the set of all definable connected solvable subgroups with the above property.

Notice that there might exist several such maximal unipotence parameters \tilde{q} , maybe one for $q = \infty$ and several ones for q prime, except for $q = p$ by Corollary 5.18.

It will also be shortly and clearly visible below that the notion of maximality for \tilde{q} is the same when L varies in two smaller subsets of definable connected solvable subgroups containing $C^\circ(s)$ for some s of order p of E : the set of Borel subgroups with this property on the one hand, and exactly the finite set of subgroups of the form $C^\circ(s)$ on the other.

Lemma 5.22 *Let L be any definable connected solvable subgroup containing $C^\circ(s)$ for some element s of order p of E . Then $U_{\tilde{q}}(O_{p'}(L))$ is a normal definable connected nilpotent subgroup of L .*

Proof. As $O_{p'}(L)$ is normal in L , it suffices to show that its definably characteristic subgroup $U_{\tilde{q}}(O_{p'}(L))$ is nilpotent. But the latter is in $F(O_{p'}(L))$ by Fact 2.15 and the maximality of r . \square

Corollary 5.23 *Let L be any definable connected solvable subgroup containing $C^\circ(s)$ for some element s of order p of E . Then any definable \tilde{q} -subgroup of L without elements of order p is in $U_{\tilde{q}}(F(O_{p'}(L)))$.*

Proof. Let U be such a subgroup. As $U_p(L) = 1$ by Lemma 5.17, $L/O_{p'}(L)$ is (divisible) abelian by Fact 2.31, and thus $U \leq O_{p'}(L)$, and $U \leq U_{\tilde{q}}(O_{p'}(L))$. Now it suffices to apply the normality and the nilpotence of the latter. \square

We now prove a version of the Uniqueness Theorem 4.1 with a combined action, more precisely where the assumption on unipotence degrees of centralizers is replaced by an assumption of invariance by a sufficiently “large” p -toral subgroup. For this purpose we first note the following.

Lemma 5.24 *Let E_1 be a subgroup of order at least p^2 of E , and H a definable connected solvable E_1 -invariant subgroup. Then $d_q(O_{p'}(H)) \leq r$.*

Proof. Assume toward a contradiction $r' > r$, where r' denotes $d_q(O_{p'}(H))$. In this case r is necessarily finite, and $q = \infty$. By Fact 2.15, $U_{(\infty, r')}(O_{p'}(H)) \leq F^\circ(O_{p'}(H))$, and this nontrivial definable (∞, r') -subgroup is E_1 -invariant. Fact 2.32 gives an element s of order p in E_1 such that

$$C_{U_{(\infty, r')}(O_{p'}(H))}(s) \neq 1.$$

But the latter is an (∞, r') -group by Fact 2.9. Now considering the definable connected solvable subgroup $C^\circ(s)$ gives a contradiction to the maximality of r , as $C^\circ(s)/O_{p'}(C^\circ(s))$ is (divisible) abelian as usual and the centralizer above is connected without elements of order p , and thus contained in $O_{p'}(C^\circ(s))$. \square

As mentioned already around the definition of maximal parameters \tilde{q} , the same argument shows that r is also exactly the maximum of the $d_q(O_{p'}(L))$ different from 0, with L varying in the set of *Borel* subgroups containing $C^\circ(s)$ for some element s of order p of E (instead of all definable connected solvable subgroups L with the same property), and similarly with L varying in the set of subgroups $C^\circ(s)$ for some element s of order p of E .

We now prove the specific version of the Uniqueness Theorem 4.1.

Theorem 5.25 *Let E_1 be a subgroup of order at least p^2 of E . Then any E_1 -invariant nontrivial definable \tilde{q} -subgroup without elements of order p is contained in a unique maximal such.*

Proof. Let U_1 be the \tilde{q} -subgroup under consideration. Fix U a maximal E_1 -invariant definable \tilde{q} -subgroup without elements of order p containing U_1 .

Assume V is another such subgroup, distinct from U , and chosen so as to maximize the rank of $U_2 = U_{\tilde{q}}(U \cap V)$. As $1 < U_1 \leq U_2$, the subgroup U_2 is nontrivial. As U_2 is nilpotent, $N := N^\circ(U_2)$ is solvable by local^o solvability^o of G . Note that $U_2 < U$, as otherwise $U = U_2 \leq V$ and $U = V$ by maximality of U . Similarly $U_2 < V$, as otherwise $V = U_2 \leq U$ and $V = U$ by maximality of V . In particular by normalizer condition, Fact 2.7, $U_2 < U_{\tilde{q}}(N_U(U_2))$ and $U_2 < U_{\tilde{q}}(N_V(U_2))$.

We claim that $d_q(O_{p'}(N)) = r$. Actually $d_q(O_{p'}(N)) \leq r$ by Lemma 5.24, and as $O_{p'}(N)$ contains U_2 which is nontrivial and of unipotence degree r in characteristic q we get $d_q(O_{p'}(N)) = r$.

By Fact 2.15 and the fact that $r \geq 1$ we get $U_{\tilde{q}}(O_{p'}(N)) \leq F^\circ(O_{p'}(N))$. In particular $U_{\tilde{q}}(O_{p'}(N))$ is nilpotent, and contained in a maximal definable E_1 -invariant \tilde{q} -subgroup without elements of order p , say Γ . Notice that N , being E_1 -invariant, satisfies $U_p(N) = 1$, and $N/O_{p'}(N)$ is abelian as usual. Now $U_1 \leq U_2 < U_{\tilde{q}}(N_U(U_2)) \leq \Gamma$, so our maximality assumption implies that $\Gamma = U$. In particular $U_{\tilde{q}}(N_V(U_2)) \leq \Gamma = U$. But then $U_2 < U_{\tilde{q}}(N_V(U_2)) \leq U_{\tilde{q}}(U \cap V) = U_2$, a contradiction. \square

Corollary 5.26 *Let E_1 be a subgroup of order at least p^2 of E .*

- (1) *If U_1 is a nontrivial E_1 -invariant definable \tilde{q} -subgroup without elements of order p , then U_1 is contained in a unique maximal E_1 -invariant definable connected solvable subgroup B . Furthermore $U_{\tilde{q}}(O_{p'}(B))$ is the unique maximal E_1 -invariant definable \tilde{q} -subgroup without elements of order p containing U_1 , and, for any element s of order p of E_1 with a nontrivial centralizer in U_1 , $C^\circ(s) \leq B$ and B is a Borel subgroup of G .*
- (2) *$U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$ is trivial.*

Proof. (1). Assume B_1 and B_2 are two maximal E_1 -invariant definable connected solvable subgroups containing U_1 . We have $U_p(B_1) = U_p(B_2) = 1$. Hence B_1 and B_2 are both abelian modulo their $O_{p'}$ subgroups.

Let $U = U_{\tilde{q}}(O_{p'}(B_1 \cap B_2))$. This group contains U_1 and is in particular nontrivial, and is E_1 -invariant, as well as $U_{\tilde{q}}(O_{p'}(B_1))$ and $U_{\tilde{q}}(O_{p'}(B_2))$. Now all these three subgroups are contained in a (unique) common maximal E_1 -invariant definable \tilde{q} -subgroup without elements of order p by the Uniqueness Theorem 5.25, say \tilde{U} . Notice that $B_1 = N^\circ(U_{\tilde{q}}(O_{p'}(B_1)))$ and $B_2 = N^\circ(U_{\tilde{q}}(O_{p'}(B_2)))$ by maximality of B_1 and B_2 . Now applying the normalizer condition, Fact 2.7, in the subgroup \tilde{U} without elements of order p yields easily $U_{\tilde{q}}(O_{p'}(B_1)) = \tilde{U} = U_{\tilde{q}}(O_{p'}(B_2))$. Now taking their common normalizers^o yields $B_1 = B_2$.

Our next claim follows from the same argument.

For the last claim, we note that there exists an element s in E_1 of order p such that $C_{U_1}(s)$ is nontrivial. By Fact 2.9 the latter is a \tilde{q} -group, and of course it is E_1 -invariant. So the preceding uniqueness applies to $C_{U_1}(s)$, and as $C_{U_1}(s) \leq U_1 \leq B$ we get that B is the unique maximal E_1 -invariant definable connected solvable subgroup containing $C_{U_1}(s)$. But $C_{U_1}(s) \leq C^\circ(s) \leq B_s$ for some Borel subgroup B_s and $E_1 \leq B_s$, so B_s satisfies the same conditions as B , so $B_s \leq B$ and $B = B_s$ is a Borel subgroup of G .

(2). Suppose toward a contradiction $U := U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$ nontrivial. It is of course E_1 -invariant. Recall that Q is a fixed Carter subgroup of G containing the maximal p -torus S . As $Q \leq C^\circ(E_1)$, Q normalizes the subgroup U . Now for any element s of order p or E_1 we have $UQ \leq C^\circ(s)$.

As $E_1 \leq Q$, any Borel subgroup containing UQ is E_1 -invariant, and by the first point there is a *unique* Borel subgroup containing UQ . Now $C^\circ(s)$ is necessarily contained in this unique Borel subgroup containing UQ , and this holds for any element s of order p of E_1 . We get a contradiction to Lemma 5.19. \square

We note that the proof of the second point in Corollary 5.26 actually shows that any definable connected subgroup containing E_1 and U_1 for some nontrivial E_1 -invariant definable \tilde{q} -subgroup U_1 without elements of order p is contained in a *unique* Borel subgroup of G . Furthermore with the notation of Corollary 5.26 (1) we have in any case $N(U_1) \cap N(E_1) \leq N(U_{\tilde{q}}(O_{p'}(B))) = N(B)$.

There are two possible ways to prove that the Prüfer p -rank is 2. One may use the Uniqueness Theorem 5.25 provided by the local^o solvability^o of the ambient group, or use the general signalizer functor theory, which gives similar consequences in more general contexts. We explain now how to use the signalizer functor theory to get the bound on the Prüfer p -rank, but we will rather continue the analysis with the Uniqueness Theorem 5.25 which is closer in spirit to [CJ04, Lemma 6.1], and our original proof anyway. It also gives much more information in the specific context under consideration, including when the Prüfer p -rank is 2, while the general signalizer functor theory just provides the bound.

For s a nontrivial element of E we let

$$\theta(s) = U_{\tilde{q}}(O_{p'}(C(s))).$$

If t is another nontrivial element of E , then it normalizes the connected nilpotent \tilde{q} -group without elements of order p $\theta(s)$, and by Facts 2.9 and 2.31 $C_{\theta(s)}(t) \leq U_{\tilde{q}}(O_{p'}(C(t))) = \theta(t)$. Hence one has the two following properties:

- (1) $\theta(s)^g = \theta(s^g)$ for any s in $E \setminus \{1\}$ and any g in G .
- (2) $\theta(s) \cap C_G(t) \leq \theta(t)$ for any s and t in $E \setminus \{1\}$.

In the parlance of finite group theory one says that θ is an *E-signalizer functor* on G . In groups of finite Morley rank one says that θ is a *connected nilpotent E-signalizer functor*, as any $\theta(s)$ is connected (by definition) and nilpotent, which follows from Corollary 5.23. When E_1 is a subgroup of E one defines

$$\theta(E_1) = \langle \theta(s) \mid s \in E_1 \setminus \{1\} \rangle$$

In groups of finite Morley rank there is no Solvable Signalizer Functor Theorem available as in the finite case [Asc93, Chapter 15] (see [Gol72a, Gol72b, Gla76, Ben75] for the story in the finite case). However Borovik imported from finite group theory a Nilpotent Signalizer Functor Theorem for groups of finite Morley rank [Bor90, Bor95] [BN94, Theorem B.30], stated as follows in [Bur04b, Theorem A.2] (and which suffices by the unipotence theory of [Bur04b] for which it has been designed originally).

Fact 5.27 (Nilpotent Signalizer Functor Theorem) *Let G be a group of finite Morley rank, p a prime, and $E \leq G$ a finite elementary abelian p -group of order at least p^3 . Let θ be a connected nilpotent E -signalizer functor. Then $\theta(E)$ is nilpotent. Furthermore $\theta(E) = O_{p'}(\theta(E))$ and $\theta(s) = C_{\theta(E)}(s)$ for any s in $E \setminus \{1\}$.*

(From the finite group theory terminology one says that θ is *complete* when it satisfies the two properties of the last statement.)

In our situation one has thus, assuming toward a contradiction the Prüfer p -rank is at least 3, that $\theta(E)$ is nilpotent. Notice that the definable connected subgroup $\theta(E)$ is nontrivial, as $\theta(s)$ is nontrivial at least for some s by Facts 2.9 and 2.32. In particular $N^\circ(\theta(E))$ is solvable by local $^\circ$ solvability $^\circ$ of G .

From this point on one can use arguments formally identical to those of [Bor95, §6.2-6.3] used there for dealing with “proper 2-generated cores”.

If E_1 and E_2 are two subgroups of E of order at least p^2 , then for any s in $E_1 \setminus \{1\}$ one has $\theta(s) \leq \langle C_{\theta(s)}(t) \mid t \in E_2 \setminus \{1\} \rangle \leq \theta(E_2)$ and thus $\theta(E_1) = \theta(E_2)$.

In particular $\theta(E) = \theta(E_1)$ for any subgroup E_1 of E of order at least p^2 .

Now if g in G normalizes such a subgroup E_1 , then $\theta(E)^g = \theta(E_1)^g = \theta(E_1^g) = \theta(E_1) = \theta(E)$ and thus $g \in N(\theta(E))$.

Take now as in Lemma 5.19 an element s of order p in E such that $C^\circ(s) \not\leq N^\circ(\theta(E))$.

Then, still assuming E of order at least p^3 , there exists a subgroup E_2 of E of order at least p^2 and disjoint from $\langle s \rangle$. By Lemma 2.33,

$$C^\circ(s) = \langle C_{C^\circ(s)}(t) \mid t \in E_2 \setminus \{1\} \rangle.$$

But now if t is in E_2 as in the above equality, then $E_1 := \langle s, t \rangle$ has order p^2 as E_2 is disjoint from $\langle s \rangle$, hence $C_{C^\circ(s)}(t) \leq C(s, t) \leq N(\langle s, t \rangle) = N(E_1) \leq N(\theta(E))$, and this shows that $C^\circ(s) \leq N^\circ(\theta(E))$. This is a contradiction, and as our only extra assumption was that the Prüfer p -rank was at least 3, it must be 2.

Anyway, we can get the bound similarly, by using more directly the Uniqueness Theorem 5.25 here instead of the axiomatized signalizer functor machinery.

Actually the proof below is the core of the proof of the Nilpotent Signalizer Functor Theorem, and the Uniqueness Theorem here gives a shortcut to the passage to a quotient for the induction in the general case.

Theorem 5.28 *S has Prüfer p -rank 2.*

Proof. Assume towards a contradiction E has order at least p^3 .

We then claim that there exists a *unique* maximal nontrivial E -invariant definable \tilde{q} -subgroup without elements of order p . Let U_1 and U_2 be two such subgroups. Then by Facts 2.9 and 2.32 $C_{U_1}(E_1)$ and $C_{U_2}(E_2)$ are nontrivial \tilde{q} -subgroups for some subgroups E_1 and E_2 of E , each of index p in E . Assuming $|E| \geq p^3$ gives then an element s of order p in $E_1 \cap E_2$. Now $C_{U_1}(s)$ and $C_{U_2}(s)$ are both nontrivial, and these are both \tilde{q} -subgroups by Fact 2.9. Clearly they are both E -invariant, as E centralizes s , and in $U_{\tilde{q}}(O_{p'}(C^\circ(s)))$ as usual, which is also E -invariant. Now the Uniqueness Theorem 5.25 gives $U_1 = U_2$, as desired.

Hence there is a unique maximal E -invariant definable \tilde{q} -subgroup without elements of order p , say “ $\theta(E)$ ” in the notation of the signalizer functor theory. For the same reasons as mentioned above, Facts 2.9 and 2.32, it is nontrivial.

Now by Facts 2.9 and 2.32 again, $C_{\theta(E)}(E_1)$ is a nontrivial definable \tilde{q} -subgroup of $\theta(E)$ for some subgroup E_1 of E of index p . As $U_p(C^\circ(E_1)) = 1$, $C^\circ(E_1)/O_{p'}(C^\circ(E_1))$ is abelian as usual, and the definable connected subgroup $C_{\theta(E)}(E_1)$ is in $O_{p'}(C^\circ(E_1))$, and in $U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$.

But as $|E| \geq p^3$, $|E_1| \geq p^2$, and we get a contradiction to Corollary 5.26 (2). \square

This proves clause (2) of Theorem 5.16 and completes the proof of Theorem 5.16. \square

We can also record informally some information gained along the proof of case (2) of Theorem 5.16, which can be compared to [CJ04, 6.1-6.6]. We let G and S be as in case (2) of Theorem 5.16, and Q be a Carter subgroup of G containing S . Then Q is contained in at least two distinct Borel subgroups of G by Lemma 5.19, and in particular Q is divisible abelian by Corollary 4.4 and Proposition 4.54. Now there are unipotence parameters $\tilde{q} \neq (\infty, 0)$ as in the proof of case (2) of Theorem 5.16 (maybe one for $q = \infty$, several for q prime, but none for $q = p$ by Lemma 5.17). All the results of the above analysis apply, now with $|\Omega_1(S)| = p^2$ necessarily.

By Corollary 5.26,

$$U_{\tilde{q}}(O_{p'}(C^\circ(\Omega_1(S)))) = 1.$$

As $\Omega_1(S)$ has order p^2 , it contains in particular

$$\frac{p^2 - 1}{p - 1} = p + 1$$

pairwise noncolinear elements. It follows that there are at most $p + 1$ nontrivial subgroups of the form $U_{\tilde{q}}(O_{p'}(C^\circ(s)))$ for some nontrivial element s of order p of

S , and at most $p+1$ Borel subgroups B containing Q (actually $\Omega_1(S)$ -invariant suffices as noticed after Corollary 5.26) and such that $U_{\bar{q}}(O_{p'}(B)) \neq 1$. By Corollary 5.26, any such Borel subgroup would contain $C^\circ(s)$ for any element s of order p of S having a nontrivial centralizer in $U_{\bar{q}}(O_{p'}(B))$, and $\Omega_1(S)$ has a trivial centralizer in $U_{\bar{q}}(O_{p'}(B))$.

The following corollary of Theorem 5.16 will be of crucial use in [DJ07] to get a bound on Prüfer ranks.

Corollary 5.29 *Let G be a connected nonsolvable locally^o solvable^o group of finite Morley rank and of Prüfer p -rank at least 2 for some prime p , and fix a maximal p -torus S of G . Let X be a maximal exceptional (finite) subgroup of S (as in Lemma 3.31), $\overline{H} = C^\circ(X)/X$, \overline{K} a minimal definable connected nonsolvable subgroup of \overline{H} containing \overline{S} , and let*

$$\overline{B} = \langle C_{\overline{K}}^\circ(\overline{s}) \mid \overline{s} \in \Omega_1(\overline{S}) \setminus \{1\} \rangle.$$

Then either

- (1) $\overline{B} < \overline{K}$, in which case \overline{B} is a Borel subgroup of \overline{K} , and moreover $N_{\overline{K}}(\overline{B})$ is p -strongly embedded in \overline{K} assuming additionally that $U_p(C_{\overline{K}}(\overline{s})) = 1$ for every element \overline{s} of order p of \overline{S} and that \overline{S} is a Sylow p -subgroup of $N_{N_{\overline{K}}(\overline{B})}(\overline{S})$, or
- (2) $\overline{B} = \overline{K}$, in which case \overline{S} , as well as S , has Prüfer p -rank 2.

Proof. It suffices to apply Theorem 5.16 in \overline{K} . We note that \overline{S} and S have the same Prüfer p -rank, as X is finite by Lemma 3.20. \square

Cases (1) and (2) of Theorem 5.16 and Corollary 5.29 correspond respectively to Sections 7 and 6 of [CJ04] in presence of divisible torsion. The remaining analysis of both of these sections, as well as the treatment without the extra assumption for p -strong embedding in case (1), will be considered in our separate paper on Weyl groups, mentioned already in Section 4.2.

For $p = 2$ case (1) will entirely disappear in [DJ07] by an argument similar to the one used in [BCJ07, Case I].

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