Aubry sets vs Mather sets in two degrees of freedom

Daniel Massart

AUBRY SETS VS MATHER SETS IN TWO DEGREES OF FREEDOM

DANIEL MASSART

Abstract. Let L be an autonomous Tonelli Lagrangian on a closed manifold of dimension two. Let $\mathcal{C}$ be the set of cohomology classes whose Mather set consists of periodic orbits, none of which is a fixed point. Then for almost all c in $\mathcal{C}$, the Aubry set of c equals the Mather set of c.

1. Introduction

1.1. **Motivation.** We study Tonelli Lagrangian systems on closed manifolds, along the lines of [Mr91]. The Aubry set is a specific invariant set of the Euler-Lagrange flow, defined in [F]. Roughly speaking, it is the obstruction to push a Lagrangian submanifold inside a convex hypersurface of the cotangent bundle of a closed manifold without changing its cohomology class (see [PPS03]). Various nice results hold when the Aubry set is a finite union of hyperbolic, periodic orbits :

- asymptotic estimates for near-optimal periodic geodesics ([A03]), if the Lagrangian is a metric of negative curvature on a surface
- existence of 'physical' solutions of the Hamilton-Jacobi equation ([AIPS05])
- existence of C^∞ subsolutions of the Hamilton-Jacobi equation ([Be07]).

By [CI99] when there is a minimizing periodic orbit, a small perturbation makes it hyperbolic while still minimizing. The trouble is to find minimizing periodic orbits.

While this seems out of reach for the time being, there is a particular case when this difficulty is easily overcome : that is when the dimension of the configuration space is two, for then Proposition 2.1 of [CMP04] says that any minimizing measure with a rational homology class is supported on periodic orbits.

Even then, yet another problem arises : the Aubry set always contains the union of the supports of all minimizing measures (Mather set), but the inclusion may be proper. The purpose of this paper is to clarify the relationship between the Aubry set and the Mather set, when the latter consists of periodic orbits. In loose terms our main result says that in that case (and in two degrees of freedom) they almost always coincide. See the next paragraph.

1.2. **Definitions and precise statements.** After Fathi and Bernard we call autonomous Tonelli Lagrangian on a closed manifold M a C^2 function

Date: March 21, 2008.

L from TM to $\mathbb{R}$ which is fiberwise superlinear and such that $\partial^2 L/\partial v^2$ is positive definite everywhere. Let

- L be an autonomous Tonelli Lagrangian on a closed manifold M
- ϕ_t be the Euler-Lagrange flow of L
- p be the canonical projection $TM \longrightarrow M$.

The first object one encounters when using variational methods is Mañé's action potential : for each nonnegative t, and x, y in M, define

$$h_t(x,y) := \inf \int_0^t L\left(\gamma(s), \dot\gamma(s)\right) ds$$

over all absolutely continuous curves $\gamma : [0,t] \longrightarrow M$ such that $\gamma(0) = x$, $\gamma(t) = y$. The infimum is in fact a minimum due to the fiberwise strict convexity and superlinearity of L, and the curves achieving the minimum are pieces of orbits of ϕ_t.

Looking for orbits that realize the action potential between any two of their points, one is led to consider the Peierls barrier ([F])

$$h(x,y) := \liminf_{t\to\infty} h_t(x,y).$$

The projected Aubry set is then defined ([F]) as

$$\mathcal{A}(L) := \{x \in M \colon\ h(x,x) = 0\}.$$

Mather's Graph Theorem ([F], Theorem 5.2.8) then says that for any $x \in \mathcal{A}(L)$, there exists a unique $v \in T_xM$ such that $p \circ \phi_t(x,v) \in \mathcal{A}(L)$ for all $t \in \mathbb{R}$. The set

$$\tilde{\mathcal{A}}(L) := \{(x,v) \in M \colon\ p \circ \phi_t(x,v) \in \mathcal{A}(L)\ \forall t \in \mathbb{R}\}$$

is called the Aubry set of L, it is compact and ϕ_t-invariant.

As noticed by Mather, it is often convenient to deal with invariant measures rather than individual orbits. Define $\mathcal{M}_{inv}$ to be the set of Φ_t-invariant, compactly supported, Borel probability measures on TM. Mather showed that the function (called action of the Lagrangian on measures)

$$\begin{array}{ccc} \mathcal{M}_{inv} & \longrightarrow & \mathbb{R} \\ \mu & \longmapsto & \int_{TM} L d\mu \end{array}$$

is well defined and has a minimum. A measure achieving the minimum is called L-minimizing. The value of the minimum, times minus one, is called the critical value of L, and denoted $\alpha(L)$. The Mather set $\tilde{\mathcal{M}}(L)$ of L is then defined as the closure of the union of the supports of all minimizing measures. It is compact, ϕ_t-invariant, and contained in $\tilde{\mathcal{A}}(L)$.

The minimization procedure may be refined as follows. Mather observed that if ω is a closed one-form on M and $\mu \in \mathcal{M}_{inv}$ then the integral $\int_{TM} \omega d\mu$ is well defined, and only depends on the cohomology class of ω. By duality this endows μ with a homology class : $[\mu]$ is the unique $h \in H_1(M,\mathbb{R})$ such that

$$\langle h, [\omega] \rangle = \int_{TM} \omega d\mu$$

for any closed one-form ω on M. Besides, for any $h \in H_1(M,\mathbb{R})$, the set

$$\mathcal{M}_{h,inv} := \{\mu \in \mathcal{M}_{inv} \colon\ [\mu] = h\}$$

is not empty. Again the action of the Lagrangian on this smaller set of measures has a minimum, which is a function of h, called the β-function of the system. A measure achieving the minimum is called (L, h)-minimizing, or h-minimizing for short.

When the dimension of M is two, we get a bit of help from the topology. Let Γ be the quotient of $H_1(M, \mathbb{Z})$ by its torsion (we do not assume M to be orientable), Γ embeds as a lattice into $H_1(M, \mathbb{R})$. A homology class h is said to be 1-irrational if there exist $h_0 \in \Gamma$ and $r \in \mathbb{R}$ such that $h = rh_0$. Proposition 2.1 of [CMP04] (see also Proposition 5.6 of [BM]) reads :

Proposition 1. *Let M be a closed surface, possibly non-orientable, and let L be a Tonelli Lagrangian on M. If h is a 1-irrational homology class and μ is an h-minimizing measure, then the support of μ consists of periodic orbits, or fixed points.*

There is a dual construction : if ω is a closed one-form on M, then $L - \omega$ is a Tonelli Lagrangian, and furthermore $L - \omega$ has the same Euler-Lagrange flow as L. The Aubry set, Mather set, and critical value of $L - \omega$ are denoted $\tilde{\mathcal{A}}_L(c), \tilde{\mathcal{M}}_L(c), \alpha_L(c)$ respectively, or just $\tilde{\mathcal{A}}(c), \tilde{\mathcal{M}}(c), \alpha(c)$ when no ambiguity is possible. An $(L - \omega)$-minimizing measure is also called (L, ω)-minimizing, (L, c)-minimizing, or just c-minimizing for short if c is the cohomology of ω. In formal terms we have defined

$$
\begin{array}{rcl}
\beta_L\colon\ H_1(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\
h & \longmapsto & \min\left\{\int_{TM} L d\mu\colon\ [\mu]=h\right\} \\
\alpha_L\colon\ H^1(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\
c & \longmapsto & -\min\left\{\int_{TM} (L-\omega) d\mu\colon\ [\omega]=c\right\}.
\end{array}
$$

Mather proved that α_L and β_L are convex, superlinear, and Fenchel dual of one another. In particular $\min \alpha = -\beta(0)$, and we have the Fenchel inequality :

$$\alpha_L(c) + \beta_L(h) \geq \langle c, h \rangle \ \forall c \in H^1(M, \mathbb{R}),\ h \in H_1(M, \mathbb{R}).$$

Given $c \in H^1(M, \mathbb{R})$ (resp. $h \in H_1(M, \mathbb{R})$), the set of $h \in H_1(M, \mathbb{R})$ (resp. $c \in H^1(M, \mathbb{R})$) achieving equality in the Fenchel equality is called the Legendre transform of c (resp. h), and denoted $\mathcal{L}(c)$ (resp. $\mathcal{L}(h)$).

The main geometric features of a convex function are its smoothness and strict convexity, or lack thereof. In general, the maps α_L and β_L are neither strictly convex, nor smooth ([Mt97]). The regions where either map is not strictly convex are called flats (see Section 2 for precise definitions). A flat is a convex subset of a linear space, hence it makes sense to speak of its relative interior, or interior for short. The sets $\mathcal{L}(c)$ for $c \in H^1(M, \mathbb{R})$ (resp. $\mathcal{L}(h)$ for $h \in H_1(M, \mathbb{R})$), if they contain more than one point, are non-trivial instances of flats ; conversely, by the Hahn-Banach Theorem, any flat is contained in the Legendre transform of some point.

Note that if two cohomology classes lie in the relative interior of a flat F of α_L, by [Mr91] their Mather sets coincide. We denote by $\tilde{\mathcal{M}}(F)$ the common Mather set to all the cohomologies in the relative interior of F. For any c in F, the Mather set of c contains the Mather set of F. We say a flat is rational if its Mather set consists of periodic orbits or fixed points. It is easy to see that any rational flat of α_L is contained in $\mathcal{L}(h)$ for some

1-irrational h. A partial converse is true when the dimension of M is two (see Lemma 22).

As to Aubry sets, Proposition 6 of [Mt03] reads :

Proposition 2. *If a cohomology class c_1 belongs to a flat F_c of α_L containing c in its interior, then $\mathcal{A}(c) \subset \mathcal{A}_{c_1}$. In particular, if c_1 lies in the interior of F_c, then $\mathcal{A}(c) = \mathcal{A}(c_1)$. Conversely, if two cohomology classes c and c_1 are such that $\tilde{\mathcal{A}}(c) \cap \tilde{\mathcal{A}}(c_1) \neq \emptyset$, then α_L has a flat containing c and c_1.*

So for any flat F of α and any c_1, c_2 in the interior of F, the Aubry sets $\tilde{\mathcal{A}}(c_1)$ and $\tilde{\mathcal{A}}(c_2)$ coincide. We denote by $\tilde{\mathcal{A}}(F)$ the common Aubry set to all the cohomologies in the interior of F.

A flat of α_L is called singular if its Mather set contains a fixed point of the Euler-Lagrange flow. A homology class h is called singular if its Legendre transform is a singular flat. So the set of singular classes is either empty, or it contains zero and is compact. When there are fixed points, we lose some of the perks of the low dimension, which explains why we have to exclude singular classes from our main result. The purpose of this paper is to prove that the Aubry set of a nonsingular rational flat equals its Mather set.

Theorem 3. *Assume*

- *M is a closed surface*
- *L is an autonomous Tonelli Lagrangian on M*
- *h is a 1-irrational, nonsingular homology class.*

Then $\tilde{\mathcal{A}}(\mathcal{L}(h)) = \tilde{\mathcal{M}}(\mathcal{L}(h))$, and $\tilde{\mathcal{A}}(\mathcal{L}(h))$ is a union of periodic orbits.

So in the interior of a nonsingular rational flat, the Aubry set is as small as possible since it must contain the Mather set. Note that the boundary of a convex set C is negligible in C, in any reasonable sense of negligible, which accounts for the phrase 'almost always coincide' we used in the 'Motivation' subsection.

Here is an outline of this paper. In Section 2 we have gathered some technical results about Fenchel dual pairs of convex functions. In Section 3 we prove the lemmas we need so as to include the case of non-orientable surfaces. In Section 4 we take a close look at the structure of the Aubry set in a neighborhood of a periodic orbit which is not a fixed point. In Section 5 we consider the flats of β_L. This is needed to prove that $\mathcal{L}(h)$ is a rational flat for any non-singular 1-irrational h. In the final section we put it all together and prove our main theorem.

2. Convex and superlinear functions

Let

- E be a finite dimensional Banach space
- $A\colon E \longrightarrow \mathbb{R}$ be a convex and superlinear map.

Then the Fenchel transform of A, defined by the formula

$$\begin{array}{rccl} B\colon & E^* & \longrightarrow & \mathbb{R} \\ & y & \longmapsto & \sup_{x\in E}\left(\langle y, x\rangle - A(x)\right) \end{array}$$

is well-defined, convex and superlinear. The Legendre transform (with respect to A) of a point x in E is the set $\mathcal{L}(x)$ of $y \in E^*$ which achieve the supremum above. Since B is convex and superlinear, there is a Legendre transform with respect to B as well. We call

- relative interior of a convex subset C of E, the interior of C in the affine subspace of E generated by C. For instance, the relative interior of the interval $[a, b]$ is $\{a\}$ if $a = b$, $]a, b[$ otherwise.
- supporting subspace to the graph of A any affine subspace of $E \times \mathbb{R}$ that meets the graph of A but not the open epigraph
$$\{(x, t) \in E \times \mathbb{R} \colon\ t > A(x)\}$$
- flat of A, the projection to E of the intersection of the graph of A with a supporting subspace
- dimension of a flat, the dimension of the affine subspace it generates in E
- interior of a flat, its relative interior as a convex set
- face of the graph of A, the projection to E of the intersection of the graph of A with a supporting *hyperplane*

Note that

- by the Hahn-Banach Theorem, any flat is contained in a face
- a face is not properly contained in any flat that is not a face
- for any $x \in E$ (resp. $x \in E^*$), the Legendre transform $\mathcal{L}(x)$ is a face of the graph of B (resp. A).

Conversely, by the Hahn-Banach Theorem, a face of A (resp. B) is the Legendre transform, with respect to B (resp. A), of a point of E^* (resp. E), that is, a subset F such that
$$\exists y_0 \in E^*,\ F = \{x \in E \colon\ A(x) + B(y_0) = \langle y_0, x\rangle\}\,.$$
Given a flat F of A, we denote
$$\mathcal{F}(F) := \{y \in E^* \colon\ \forall x \in F,\ A(x) + B(y) = \langle y, x\rangle\}\,,$$
that is, $\mathcal{F}(F)$ is the intersection of all Legendre transforms of points of F. Note that for any two flats F_1, F_2 of A, $F_1 \subset F_2$ is equivalent to $\mathcal{F}(F_2) \subset \mathcal{F}(F_1)$. In particular, if x, y are points of E^*,
$$(x \in \mathcal{F}(\mathcal{L}(y))) \iff (\mathcal{L}(y) \subset \mathcal{L}(x)) \iff (\mathcal{F}(\mathcal{L}(x))) \subset \mathcal{F}(\mathcal{L}(y)))\,.$$

Lemma 4. *Let F be a flat of A and let x_0 be a point in the relative interior of F. Then $\mathcal{F}(F)$ is the Legendre transform of x_0. In particular $\mathcal{F}(F)$ is a face of B.*

Proof. By definition of $\mathcal{F}(F)$, it is contained in the Legendre transform of x_0. Let us show the converse inclusion holds true. Take y such that $A(x_0) + B(y) = \langle y, x_0\rangle$. We want to show that $y \in \mathcal{F}(F)$, that is, $A(x) + B(y) = \langle y, x\rangle$ for all $x \in F$. Take $x \in F$. Since x_0 lies in the interior of F, there exists x' in F and $0 < t < 1$ such that $x_0 = tx + (1-t)x'$. Since any flat is contained in a face, there exists y_0 such that $F \subset \{x \in E \colon\ A(x) + B(y_0) = \langle y_0, x\rangle\}$, so
$$\begin{aligned} A(x) + B(y_0) &= \langle y_0, x\rangle \\ A(x') + B(y_0) &= \langle y_0, x'\rangle. \end{aligned}$$
Summing t times the first equation with $(1 - t)$ times the second equation yields $tA(x) + (1 - t)A(x') + B(y_0) = \langle y_0, x_0\rangle$, but since $x_0 \in F$, we have $A(x_0) + B(y_0) = \langle y_0, x_0\rangle$ whence $A(x_0) = tA(x) + (1 - t)A(x')$.

On the other hand by definition of B we have

$$\begin{aligned} A(x)+B(y) &\geq \langle y,x\rangle \\ A(x')+B(y) &\geq \langle y,x'\rangle. \end{aligned}$$

Summing t times the first inequality with $(1-t)$ times the second inequality yields the equality $A(x_0)+B(y)=\langle y,x_0\rangle$, thus both inequalities are equalities, which proves the lemma. □

Our next lemmas are Lemma 4.1 and 4.2 of [BM].

Lemma 5. *Let F_1 and F_2 be two flats of A, both containing a point x_0, x_0 being an interior point of F_1. Then there exists a flat F containing $F_1 \cup F_2$.*

Proof. Let $y_i n E^*$ be such that for all x in F_2, we have $B(y)+A(x)=\langle y,x\rangle$. In particular we have $B(y)+A(x_0)=\langle y,x_0\rangle$ so y lies in the Legendre transform of x_0. Thus by Lemma 4 $y\in\mathcal{F}(F_1)$. This means that for all x in F_1, we have $B(y)+A(x)=\langle y,x\rangle$. Thus the face $\mathcal{L}(y)$ contains $F_1\cup F_2$. □

Lemma 6. *Let F_1 and F_2 be two flats of A, both containing a point x in their interiors. Then there exists a flat F containing $F_1 \cup F_2$ such that x is an interior point of F.*

3. What we need to know about non-orientable surfaces

Assume M is non-orientable. Let $\pi\colon M_o \longrightarrow M$ be the orientation cover of M. Then M_o is an orientable surface endowed with a fixed-point free, orientation-reversing involution I. Let I_* be the involution of $H_1(M_o,\mathbb{R})$ induced by I, and let E_1 (resp. E_{-1}) be the eigenspace of I_* for the eigenvalue 1 (resp.-1). Likewise, let I^* be the involution of $H^1(M_o,\mathbb{R})$ induced by I, and let E_1 (resp. E_{-1}) be the eigenspace of I^* for the eigenvalue 1 (resp.-1). We have ([BM], 2.2)

$$\ker\pi_* = E_{-1}\subset H_1(M_o,\mathbb{R}) \text{ and } \pi^*\left(H^1(M,\mathbb{R})\right) = E_1 \subset H^1(M_o,\mathbb{R})$$

Let

- $T\pi$ denote the tangent map of π
- L' denote the Lagrangian $L\circ T\pi$ on TM_o.

Likewise we denote with primes the Aubry and Mather sets of L'. Proposition 4 of [F98] says that

$$\mathcal{A}'_0 = \pi^{-1}(\mathcal{A}_0), \tilde{\mathcal{A}_0}' = T\pi^{-1}\left(\tilde{\mathcal{A}}_0\right).$$

Lemma 7. *We have*

$$\begin{aligned} \forall c\in H^1(M_o,\mathbb{R}),\ \alpha_o(I^*c) &= \alpha_o(c) \\ \forall h\in H_1(M_o,\mathbb{R}),\ \beta_o(I_*h) &= \beta_o(h) \end{aligned}$$

Proof. Take

- $c\in H^1(M_o,\mathbb{R})$
- a smooth one-form ω on M_o such that $[\omega]=c$
- a I^*c-minimizing measure μ.

We have

$$-\alpha_o(I^*c) = \int_{TM_o} \left(L' - I^*\omega\right) d\mu = \int_{TM_o} \left(L' - \omega\right) dI_*\mu \geq -\alpha_o(c)$$

where the second equality owes to the I-invariance of L'. So $\alpha_o(I^*c) \leq \alpha_o(c)$, whence $\alpha_o(c) = \alpha_o(I^*I^*c) \leq \alpha_o(I^*c)$, which proves the first statement of the lemma.

Now take $h \in H_1(M_o, \mathbb{R})$ and an h-minimizing measure μ. We have $[I_*\mu] = I_*h$ thus

$$\beta_o(I_*h) \leq \int_{TM_o} L' dI_*\mu = \int_{TM_o} L' d\mu = \beta_o(h)$$

whence $\beta_o(h) = \beta_o(I_*I_*h) \leq \beta_o(I_*h)$, which ends the proof of the lemma. □

Lemma 8. *For all $c \in H^1(M, \mathbb{R})$, $\alpha(c) = \alpha_o(\pi^*c)$.*

Proof. Take $c \in H^1(M, \mathbb{R})$ and a smooth one-form ω on M such that $[\omega] = c$. Then the lifted Lagrangian corresponding to $L - \omega$ is $L' - \pi^*\omega$. By [F98], Proposition 4, $L - \omega$ and $L' - \pi^*\omega$ share the same critical value, that is, $\alpha(c) = \alpha_o(\pi^*c)$.

□

Lemma 9. *For all $h \in E_1 \subset H_1(M_o, \mathbb{R})$, $\beta_o(h) = \beta(\pi_*h)$, and if μ is an h-minimizing measure, then $\pi_*\mu$ is π_*h-minimizing.*

Proof. Take

- $h \in E_1 \subset H_1(M_o, \mathbb{R})$
- an h-minimizing measure μ
- $c \in H^1(M_o, \mathbb{R})$ such that $\alpha_o(c) + \beta_o(h) = \langle c, h\rangle$.

Then by Lemma 7 $\alpha_o(I^*c) + \beta_o(I_*h) = \langle c, h\rangle$ and $\langle I^*c, I_*h\rangle = \langle c, h\rangle$ since I^* and I_* are dual of one another. Besides, $I_*h = h$ because $h \in E_1$. Setting $c_1 := 2^{-1}(c + I^*c)$, we have

$$\alpha_o(c_1) \leq \frac{1}{2}\left(\alpha_o(c) + \alpha_o(I^*c)\right) = \alpha_o(c)$$

by convexity of α, but on the other hand

$$\frac{1}{2}\left(\alpha_o(c) + \alpha_o(I^*c)\right) + \beta_o(h) = \langle c_1, h\rangle \leq \alpha_o(c_1) + \beta_o(h)$$

whence

$$\alpha_o(c_1) = \frac{1}{2}\left(\alpha_o(c) + \alpha_o(I^*c)\right) = \alpha_o(c)$$

and

$$\langle c_1, h\rangle = \alpha_o(c_1) + \beta_o(h).$$

Since $c_1 \in E_1 \subset H^1(M_o, \mathbb{R})$, there exists c_2 in $H_1(M, \mathbb{R})$ such that $\pi^*c_2 = c_1$. By lemma 8 $\alpha_o(c_1) = \alpha(c_2)$ so

$$\begin{aligned} \alpha(c_2) + \int_{TM_o} L' d\mu &= \langle \pi^*c_2, h\rangle \text{ that is,} \\ \alpha(c_2) + \int_{TM} L d\pi_*\mu &= \langle c_2, \pi_*h\rangle \end{aligned}$$

which proves that $I_*\mu$ is π_*h-minimizing and $\beta_o(h) = \beta(\pi_*h)$. □

Lemma 10. *Let h be an element of $H_1(M_o,\mathbb{R})$. We have*

$$\pi^*\left(\mathcal{L}(\pi_* h)\right) = \mathcal{L}(h)\cap E_1.$$

Proof. Take c in $\mathcal{L}(\pi_* h)$. We have

$$\alpha(c)+\beta(\pi_* h) = \langle c, \pi_* h\rangle$$

whence by lemmas 8, 9

$$\alpha_o(\pi^* c)+\beta_o(h) = \langle \pi^* c, h\rangle$$

that is, $\pi^* c \in \mathcal{L}(h)$. Therefore

$$\pi^*\left(\mathcal{L}(\pi_* h)\right) \subset \mathcal{L}(h)\cap E_1.$$

Now take $c\in\mathcal{L}(h)\cap E_1$. Since $c\in E_1$, there exists $c_1\in H^1(M,\mathbb{R})$ such that $\pi^* c_1 = c$. We have

$$\alpha_o(c)+\beta_o(h) = \langle c, h\rangle \text{ whence } \alpha(c_1)+\beta(\pi_* h) = \langle c_1, \pi_* h\rangle$$

so $c\in\pi^*\left(\mathcal{L}(\pi_* h)\right)$, which concludes the proof of the lemma. □

Lemma 11. *Let h be an element of $H_1(M,\mathbb{R})$, and h' be an element of $E_1\subset H_1(M_o,\mathbb{R})$ be such that $\pi_* h' = h$. We have*

$$\mathcal{F}(\mathcal{L}(h)) = \pi_*\left(\mathcal{F}(\mathcal{L}(h'))\cap E_1\right)$$

Proof. Let

- h_1 be an element of $\pi_*\left(\mathcal{F}(\mathcal{L}(h'))\cap E_1\right)$
- h_2 be an element of $\mathcal{F}(\mathcal{L}(h'))\cap E_1$ such that $\pi_*(h_2)=h_1$
- c be an element of $\mathcal{L}(h)$.

By Lemma 10 $\pi^* c\in\mathcal{L}(h')$ so

$$\alpha_o(\pi^* c)+\beta_o(h_2) = \langle \pi^* c, h_2\rangle$$

whence by Lemmas 8, 9

$$\alpha(c)+\beta(h_1) = \langle c, h_1\rangle$$

thus $h_1\in\mathcal{F}(\mathcal{L}(h))$, hence

$$\mathcal{F}(\mathcal{L}(h)) \supset \pi_*\left(\mathcal{F}(\mathcal{L}(h'))\cap E_1\right).$$

Conversely, let

- h_1 be an element of $\mathcal{F}(\mathcal{L}(h))$
- h_2 be an element of $E_1\subset H_1(M_o,\mathbb{R})$ such that $\pi_*(h_2)=h_1$
- c' be an element of $\mathcal{L}(h')$.

Setting $c_2 := 2^{-1}(c+I^* c)$, we see, as in the proof of Lemma 9, that $c_2\in\mathcal{L}(h')\cap E_1$. Since $c_2\in E_1\subset H^1(M_o,\mathbb{R})$, there exists c_1 in $H_1(M,\mathbb{R})$ such that $\pi^* c_1 = c_2$. Then by Lemmas 8, 9 $c_1\in\mathcal{L}(h)$. Since $h_1\in\mathcal{F}(\mathcal{L}(h))$ we have

$$\alpha(c_1)+\beta(h_1) = \langle c, h_1\rangle$$

thus

$$\alpha_o(c_2)+\beta_o(h_2) = \langle c_2, h_2\rangle$$

whence the two inequalities

$$\begin{aligned}\alpha_o(c')+\beta_o(h_2) &\geq \langle c_2, h_2\rangle\\ \alpha_o(I^* c')+\beta_o(h_2) &\geq \langle c_2, h_2\rangle\end{aligned}$$

sum to an equality, so both inequalities are equalities. Therefore $h_2 \in \mathcal{F}(\mathcal{L}(h'))$, so

$$\mathcal{F}(\mathcal{L}(h)) \subset \pi_* \left(\mathcal{F}(\mathcal{L}(h')) \cap E_1\right).$$

□

4. Local structure of the Aubry set at periodic orbits

We shall use Lemma 10 of [BM], and quote it below for the commodity of the reader. In [BM] only geodesic flows are considered but the proof extends without modification to the case of Lagrangian flows.

Let γ_0 be a closed, minimizing extremal on an oriented surface M, such that γ_0 is not a fixed point. Let U_0 be a neighborhood of γ_0 in M homeomorphic to an annulus.

Lemma 12. *There exists a neighborhood V_0 of $(\gamma_0, \dot{\gamma_0})$ in TM such that $p(V_0) = U_0$ and, for any simple extremal γ, if $(\gamma, \dot{\gamma})$ enters (resp. leaves) V_0 then either γ intersects transversally γ_0 or γ is forever trapped in $p(V_0)$ in the future (resp. past), that is*

$$\exists t_0 \in \mathbb{R},\ \forall t \geq (resp\ \ \leq) t_0,\ \gamma(t) \in p(V_0).$$

Besides, all intersections with γ_0, if any, have the same sign with respect to the chosen orientation.

4.1. Let γ_0 be a C^1 simple closed curve (not a fixed point) in an oriented surface M. We say a C^1 curve $\alpha\colon \mathbb{R} \longrightarrow M \setminus \gamma_0$ is positively asymptotic to γ_0 on the right if the ω-limit set of α is γ_0 and there exists some $t_0 \in \mathbb{R}$ such that, for any $t \geq t_0$, $\alpha(t)$ lies in the right-hand side (with respect to the chosen orientation of M) of a tubular neighborhood of γ_0. Similar definitions can be made replacing positively with negatively, and right with left. The lemma below will be used in the proof of Lemma 14.

Lemma 13. *Let γ_0 be a C^1 simple closed curve in an oriented surface M. Any extremal curve $\alpha\colon \mathbb{R} \longrightarrow M \setminus \gamma_0$ positively asymptotic to γ_0 on the right intersects transversally any extremal curve $\alpha\colon \mathbb{R} \longrightarrow M \setminus \gamma_0$ negatively asymptotic to γ_0 on the right.*

Proof. Let

- $\alpha_0\colon \mathbb{R} \longrightarrow M \setminus \gamma_0$ be a C^1 curve positively asymptotic to γ_0 on the right
- $\alpha_1\colon \mathbb{R} \longrightarrow M \setminus \gamma_0$ be a C^1 curve negatively asymptotic to γ_0 on the right
- δ be a C^1 transverse segment to γ_0, oriented so that its transverse intersection with γ_0 is positive.

Intersecting transversally with a given sign is an open property, so there exists a neighborhood U of $(\gamma_0, \dot{\gamma_0})$ in TM such that for any C^1 arc α in M, if $(\alpha(t), \dot{\alpha}(t))$ is contained in U for a sufficiently long time, then α intersects δ transversally with positive sign.

Since α_1 is negatively asymptotic to γ_0 on the right, there exists a tubular neighborhood V of γ_0 in M such that α_1 eventually leaves the right-hand side of V. Restricting U if necessary, we assume $p(U) \subset V$. Take t_1, t_2 two consecutive intersection points of α_0 with δ.

Consider the topological annulus A bounded by γ_0 on the left, and on the right, by $\alpha_0([t_1, t_2])$ glued to the segment of δ comprised between $\alpha_0(t_2)$ and $\alpha_0(t_1)$. Since α_1 eventually leaves the right-hand side of V, it must leave A. In so doing it cannot intersect δ for then the intersection of δ with α_1 would be negative. Therefore it must intersect α_0, which proves the lemma. □

4.2. **Periodic orbits which are not fixed points.** Besides the Aubry set, another set of note is the Mañé set $\tilde{\mathcal{N}}(L)$; all we need to know about it is that

- it is compact and ϕ_t-invariant
- it contains $\tilde{\mathcal{A}}(L)$
- no orbit contained in $\tilde{\mathcal{A}}(L)$ intersects tranversally an orbit contained in $\tilde{\mathcal{A}}(L)$ ([F], Theorem 5.2.4)
- it is lower semi-continuous with respect to the Lagrangian, that is, for any neighborhood U of $\tilde{\mathcal{N}}(L)$, for any sequence L_n of Tonelli Lagrangians converging to L in the C^2 topology, for n large enough $\tilde{\mathcal{N}}(L_n) \subset U$.

Lemma 14. *Let c be a cohomology class and let $\gamma_i, i \in I$ be a collection of c-minimizing periodic orbits which are not fixed points. Then there exists a face F of α_L containing c, and a neighborhood V of $\cup_{i\in I}\gamma_i$ in TM, such that $\tilde{\mathcal{A}}(F) \cap V$ consists of closed orbits whose projections to M are homologous to γ_i for some i.*

Proof. We may assume that $c = 0$. Note that by Mather's Graph Theorem, the γ_i have no self-intersections and are pairwise disjoint. Hence they can be partitioned into a finite number of homology classes $h_0, \dots h_k$. Let h_m be any barycenter with positive coefficients of $h_0, \dots h_k$. The face $F := \mathcal{L}(h_m)$ of α contains zero (not necessarily in its interior) because the curves γ_i are zero-minimizing. Choose one of the γ_i ; assume, without loss of generality, that it is γ_0 and its homology class is h_0.

Case 1 : M is orientable.

Denote by Int the symplectic intersection form induced on $H_1(M, \mathbb{R})$ by the algebraic intersection number of closed curves.

Case 1.1 : γ_0 does not separate M.

Pick $h_1 \in \Gamma$ such that $\mathrm{Int}(h_0, h_1) = 1$. Denote $h_n^{\pm} := nh_m \pm h_1 \in \Gamma, n \in \mathbb{N}$. Take, for each $n \in \mathbb{N}$,

- $n^{-1}h_n^{\pm}$-minimizing measures $\mu_n^{\pm}$
- $c_n^{\pm} \in H^1(M, \mathbb{R})$ such that μ_n^+ (resp. μ_n^-) is c_n^+-minimizing (resp. c_n^--minimizing).

The homology classes $n^{-1}h_n^{\pm}$ remain in a compact subset of $H_1(M, \mathbb{R})$ so the cohomology classes $c_n^{\pm}$ remain in a compact subset of $H^1(M, \mathbb{R})$. Therefore the supports of the measures $\mu_n^{\pm}$, which lie in the energy levels $\alpha(c_n^{\pm})$ by [Ca95], remain in a compact subset of TM. Hence the sequences $\mu_n^{\pm}$, $n \in \mathbb{N}$, have weak* limit points $\mu^{\pm}$. Likewise, we may assume the sequences $c_n^{\pm}$ converge to $c^{\pm} \in H^1(M, \mathbb{R})$. Since the homology class is a continuous function of the measure, we have $[\mu^{\pm}] = h_m$. Besides, since $\mu_n^{\pm}$ is $c_n^{\pm}$-minimizing,

$$\begin{aligned} \langle c_n^+, h_n^+ \rangle &= \alpha(c_n^+) + \beta(h_n^+) \\ \langle c_n^-, h_n^- \rangle &= \alpha(c_n^-) + \beta(h_n^-) \end{aligned}$$

whence, taking limits,

$$\langle c^{\pm}, h_m \rangle = \alpha(c^{\pm}) + \beta(h_m)$$

that is, $c^{\pm} \in \mathcal{L}(h_m)$.

By the semi-continuity of the Mañé set with respect to the Hausdorff distance on compact sets,

$$\begin{aligned} \lim_{n\to+\infty} \text{supp}\mu_n^+ &\subset \tilde{\mathcal{N}}(c^+) \\ \lim_{n\to+\infty} \text{supp}\mu_n^- &\subset \tilde{\mathcal{N}}(c^-). \end{aligned}$$

On the other hand, by Proposition 2,

$$\begin{aligned} \mathcal{A}(F) \subset \mathcal{A}(c^+) \subset \ & \mathcal{N}(c^+) \\ \mathcal{A}(F) \subset \mathcal{A}(c^-) \subset \ & \mathcal{N}(c^-) \end{aligned}$$

and furthermore, no orbit contained in $\mathcal{N}(c^{\pm})$ intersects transversally an orbit in $\mathcal{A}(F)$. Now by Lemma 12 there exists a neighborhood V of γ_0 in TM, such that $\tilde{\mathcal{M}}(F) \cap V$ consists of closed orbits homotopic to γ_0. Therefore $\tilde{\mathcal{A}}(F) \cap V$ consists of closed orbits homotopic to γ_0 and orbits asymptotic to one of the latter.

Now since $\text{Int}(h_0, [\mu_n^+]) > 0$, the Hausdorff limit of $\text{supp}\mu_n^+$ must contain an orbit asymptotic to γ_0 positively on the right, and an orbit asymptotic to γ_0 negatively on the left. Likewise, since $\text{Int}(h_0, [\mu_n^-]) < 0$, the Hausdorff limit of $\text{supp}\mu_n^-$ must contain an orbit asymptotic to γ_0 positively on the left, and an orbit asymptotic to γ_0 negatively on the right. Thus by Lemma 13 any orbit asymptotic to a closed curve homotopic to γ_0 must intersect transversally either $\mathcal{N}(c^+)$ or $\mathcal{N}(c^-)$, and thus cannot be in $\mathcal{A}(F)$. So $\tilde{\mathcal{A}}(F) \cap V$ consists of closed orbits homotopic to γ_0.

Case 1.2 : γ_0 separates M.

Remark 15. *In that case the result is stronger : there exists a neighborhood of $(\gamma_0, \dot\gamma_0)$ in TM such that $\tilde{\mathcal{A}}_0 \cap V$ consists of closed orbits whose projection to M are homotopic to γ_0.*

Let U be a tubular neighborhood of γ_0 in M. Let V be the neighborhood of $(\gamma_0, \dot\gamma_0)$ in TM given by Lemma 12, such that $p(V) = U$. Let (x, v) be a point of $\tilde{\mathcal{A}}_0 \cap V$. Then by Lemma 12, $\Phi_t(x, v)$ is either a closed orbit homotopic to γ_0, or is asymptotic to a closed orbit homotopic to γ_0. In the former case, we are done, so assume the latter occurs, say, $\Phi_t(x, v)$ is positively asymptotic to a closed orbit γ_1 homotopic to γ_0. Let U_1 a tubular neighborhood of γ_1 in M such that U_1 is properly contained in U and $x \notin U_1$. Let V_1 be the neighborhood of $(\gamma_1, \dot\gamma_1)$ in TM given by Lemma 12, such that $p(V_1) = U_1$. In particular there exists t_1 such that for all $t \geq t_1$, $\Phi_t(x, v) \in V_1$.

Assume, by replacing L with $L-\alpha(0)$ if needed, that the critical value $\alpha(0)$ of L is zero. Then, by definition of $\tilde{\mathcal{A}}_0$ there exists a sequence of extremal curves

$$\alpha_n\colon\ [0,t_n]\longrightarrow M \text{ such that}$$

$$x=\alpha_n(0)=\alpha_n(t_n) \text{ and } \lim_{n\to+\infty}\int_0^{t_n} L(\alpha_n(t),\dot\alpha_n(t))dt=0.$$

Note that $\lim_{n\to+\infty}\dot\alpha_n(0)=v$. So for n large enough, there exists some t such that $(\alpha_n(t),\dot\alpha_n(t))\in V_1$. Then, by Lemma 12,

- either $(\alpha_n(t),\dot\alpha_n(t))$ is trapped in V_1
- or α_n intersects γ_0 with constant sign.

The latter case never occurs, for γ_0 separates M, so the algebraic intersection of γ_0 with any closed curve is zero. Neither does the former case, since α_n must return to its initial point, which lies outside of U_1. This proves that $\Phi_t(x,v)$ is a closed orbit homotopic to γ_0. The case when $\Phi_t(x,v)$ is negatively asymptotic to a closed orbit γ_1 homotopic to γ_0 is treated similarly.

Now let us finish the proof of the orientable case of the lemma. For every $i\in I$, we have found a neighborhood V_i of $(\gamma_i,\dot\gamma_i)$ in TM such that $\tilde{\mathcal{A}}\left(\mathcal{L}(h_m)\right)\cap V_i$ consists of periodic orbits homologous to γ_i. Define V to be the union over $i\in I$ of the V_i, then V is a neighborhood of $\cup_{i\in I}\left(\gamma_i,\dot\gamma_i\right)$ in TM, and $\tilde{\mathcal{A}}\left(\mathcal{L}(h_m)\right)\cap V$ consists of periodic orbits homologous to γ_i for some i.

Case 2 : M is not orientable. Let

- δ_j, $j\in J$ be the collection of all lifts to M_o of the γ_i
- $\{h'_1,\dots h'_l\}$ be the set of all homology classes of the δ_j, $j\in J$
- h'_m be $l^{-1}(h'_1+\ldots+h'_l)$
- h_m be $\pi_*(h'm)$.

The set $\{\delta_j\colon\ j\in J\}$ is invariant under I, thus the set $\{h'_1,\dots h'_l\}$ is invariant under I_*. Therefore $h'_m\in E_1\subset H_1(M_o,\mathbb{R})$. The δ_j, $j\in J$ are minimizing ([F98]) so by the orientable case of the lemma, there exists a neighborhood V of $\cup_{j\in J}(\delta_j,\dot\delta_j)$ in TM_o such that $\tilde{\mathcal{A}}\left(\mathcal{L}(h'_m)\right)\cap V$ consists of periodic orbits homologous to δ_j for some j. Taking a smaller V if we have to, we assume that V is invariant under I. Lemma 10 says

$$\pi^*\left(\mathcal{L}(h_m)\right)=\mathcal{L}(h'm)\cap E_1.$$

Now take c in the relative interior of $\mathcal{L}(h_m)$. Then, $\pi*$ being a linear isomorphism onto $E_1\subset H^1(M,\mathbb{R})$, π^*c lies in the relative interior of $\mathcal{L}(h'm)\cap E_1$. Since $h'_m\in E_1\subset H_1(M_o,\mathbb{R})$, it is easy to see that $\mathcal{L}(h'm)$ is I^*-invariant. Therefore π^*c lies in the relative interior of $\mathcal{L}(h'm)$, so

$$\tilde{\mathcal{A}}\left(\mathcal{L}(h'_m)\right)=\tilde{\mathcal{A}}(\pi^*c).$$

By [F98],

$$T\pi\left(\tilde{\mathcal{A}}(\pi^*c)\right)=\tilde{\mathcal{A}}(c)$$

that is

$$T\pi\left(\tilde{\mathcal{A}}\left(\mathcal{L}(h'_m)\right)\right)=\tilde{\mathcal{A}}\left(\mathcal{L}(h_m)\right).$$

Since V is invariant under I, we have

$$T\pi\left(\tilde{\mathcal{A}}\left(\mathcal{L}(h'_m)\right)\cap V\right) = T\pi\left(\tilde{\mathcal{A}}\left(\mathcal{L}(h'_m)\right)\right)\cap T\pi(V).$$

The projection π is a local diffeomorphism so $T\pi(V)$ is a neighborhood of $\cup_{i\in I}(\gamma_i,\dot{\gamma}_i)$ in TM. This finishes the proof of Lemma 14. □

The following corollary of Lemma 14 reduces the proof of our main result to proving that $\mathcal{L}(h)$ is a rational flat when h is 1-irrational and nonsingular.

Corollary 16. *Assume that for some c in $H^1(M,\mathbb{R})$, $\tilde{\mathcal{M}}(c)$ consists of periodic orbits which are not fixed points $\gamma_i, i\in I$. Let h be any barycenter with positive coefficients of the homology classes of $\gamma_i, i\in I$. Then*

$$\tilde{\mathcal{A}}\left(\mathcal{L}(h)\right) = \tilde{\mathcal{M}}\left(\mathcal{L}(h)\right) = \tilde{\mathcal{M}}(c).$$

Proof. By Lemma 14 there exists a neighborhood V of $\tilde{\mathcal{M}}(c)$, such that

$$\tilde{\mathcal{A}}\left(\mathcal{L}(h)\right)\cap V = \tilde{\mathcal{M}}(c). \tag{1}$$

Hence

$$\tilde{\mathcal{M}}(c)\subset\tilde{\mathcal{A}}\left(\mathcal{L}(h)\right),\ \text{so}\ \tilde{\mathcal{M}}(c)\subset\tilde{\mathcal{M}}\left(\mathcal{L}(h)\right).$$

On the other hand $c\in\mathcal{L}(h)$, thus

$$\tilde{\mathcal{M}}(c)\supset\tilde{\mathcal{M}}\left(\mathcal{L}(h)\right),\ \text{so}\ \tilde{\mathcal{M}}(c)=\tilde{\mathcal{M}}\left(\mathcal{L}(h)\right).$$

Now $\tilde{\mathcal{A}}(\mathcal{L}(h))$ consists of $\tilde{\mathcal{M}}(\mathcal{L}(h))$, and orbits homoclinic to $\tilde{\mathcal{M}}(\mathcal{L}(h))$. Such orbits enter any neighborhood of $\tilde{\mathcal{M}}(\mathcal{L}(h))$, so (1) implies

$$\tilde{\mathcal{A}}\left(\mathcal{L}(h)\right) = \tilde{\mathcal{M}}\left(\mathcal{L}(h)\right) = \tilde{\mathcal{M}}(c).$$

□

5. Faces and flats of β

5.1. **Notations.** Let M be a closed oriented surface and let L be a Tonelli Lagrangian on M. For $h\in H_1(M,\mathbb{R})\setminus\{0\}$, we define the maximal radial face R_h of β containing h as the largest subset of the half-line $\{th\colon\ t\in[0,+\infty[\}$ containing h (not necessarily in its relative interior) in restriction to which β is affine. Beware that R_h is a flat of β, but may not be a face of β. The possibility of radial flats is the most obvious difference between the β functions of Riemannian metrics ([Mt97], [BM]) and those of general Lagrangians. An instance of radial flat is found in [CL99]. We define the Mather set $\tilde{\mathcal{M}}(R_h)$ as the closure in TM of the union of the supports of all th-minimizing measures, for $th\in R_h$. The next lemma implies that if h is non-singular, then so is any element of R_h.

Lemma 17. *For any $h\in H_1(M,\mathbb{R})$, for any t such that $th\in R_h$, we have $\mathcal{L}(h)=\mathcal{L}(th)$. Consequently,*

$$R_h\subset\mathcal{F}(\mathcal{L}(h)).$$

Proof. Take $t\in\mathbb{R}$ such that $th\in R_h$. By definition of R_h there exists $c\in H^1(M,\mathbb{R})$ such that

$$\begin{aligned}\alpha(c)+\beta(h) &= \langle c,h\rangle\\ \alpha(c)+\beta(th) &= \langle c,th\rangle\,.\end{aligned}$$

The first equality says that $c \in \mathcal{L}(h)$. Take $c' \in \mathcal{L}(h)$. Let us show that $c' \in \mathcal{L}(th)$, which proves that $\mathcal{L}(h) \subset \mathcal{L}(th)$, whence $th \in \mathcal{F}(\mathcal{L}(h))$.

Since L is autonomous, by [Ca95] $\alpha(c') = \alpha(c)$ and $\langle c', h\rangle = \langle c, h\rangle$. So

$$\alpha(c') + \beta(th) = \alpha(c) + \beta(th) = \langle c, th\rangle = \left\langle c', th\right\rangle$$

that is, $c' \in \mathcal{L}th$.

Conversely, let us prove that $\mathcal{L}(h) \supset \mathcal{L}(th)$. Take $c' \in \mathcal{L}(th)$. Since L is autonomous, by [Ca95] $\alpha(c') = \alpha(c)$ and $\langle c', th\rangle = \langle c, th\rangle$. Therefore $\alpha(c') + \beta(h) = \langle c, h\rangle$, so $c' \in \mathcal{L}(h)$. □

Now we look at some consequences of Proposition 1. Assume h is 1-irrational. Then for all t such that $th \in R_h$, th is also 1-irrational. Furthermore R_h is contained in a face of β, so Mather's Graph Theorem and Proposition 1 combine to say that $\tilde{\mathcal{M}}(R_h)$ is a union of pairwise disjoint periodic orbits γ_i, $i \in I$ where I is some set, not necessarily finite. We denote by $V(R_h)$ the linear subspace of $H_1(M, \mathbb{R})$ generated by $[\gamma_i]$, $i \in I$. Since the γ_i are pairwise disjoint, there exist homology classes $h_1, \dots h_k$ with $k \leq 3/2(b_1(M) - 2)$, such that $\forall i \in I,\ \exists j = 1, \dots k, [\gamma_i] = h_j$.

Let T_i be the least period of γ_i. Then the invariant measure μ_i supported by γ_i has homology $T_i^{-1}[\gamma_i]$. By Lemma 6 the convex hull C of $T_i^{-1}[\gamma_i]$, $i \in I$, is a flat of β containing th in its interior whenever th is contained in the interior of R_h.

5.2. **Faces of β.** The following lemma is a rewriting of Lemma 12 of [Mt97].

Lemma 18. *Let F be a flat of β. Assume F contains a 1-irrational homology class h_0 in its interior. Then for all $h \in F$, for all h-minimizing measure μ, the support of μ consists of periodic orbits, or fixed points.*

Proof. Let c be a cohomology class such that for all $h \in F$, $\alpha(c) + \beta(h) = \langle c, h\rangle$. Take h in F. Since h_0 lies in the interior of F, there exist h' in F and $0 < t < 1$ such that $h_0 = th + (1-t)h'$. Take an h-minimizing (resp. h'-minimizing) measure μ (resp. μ'), so we have $\beta(h) = \int L d\mu$, $\beta(h') = \int L d\mu'$. Thus

$$\begin{aligned} \alpha(c) + \int L d\mu &= \langle c, h\rangle \\ \alpha(c) + \int L d\mu' &= \langle c, h'\rangle \end{aligned}$$

so

$$\alpha(c) + \int L d(t\mu + (1-t)\mu') = \langle c, th + (1-t)h'\rangle = \langle c, h_0\rangle$$

that is, the probability measure $t\mu + (1-t)\mu'$ is h_0-minimizing. Since h_0 is 1-irrational, Proposition 1 implies that the support of $t\mu + (1-t)\mu'$, hence that of μ, consists of periodic orbits, or fixed points. □

Here is a version of Theorem 6.1 of [BM] for general Lagrangians.

Theorem 19. *Let*

- *M be a closed oriented surface*
- *L be a Tonelli Lagrangian on M*
- *h_0 be a 1-irrational, nonsingular homology class of M*
- *V_0 be $V(R_{h_0})$*

- *h be an element of $V_0^\perp$.*

Then there exist $t(h_0,h) \in \mathbb{R}$ and $s(h_0,h) > 0$ such that the segment $[h_0, t(h_0,h)h_0 + s(h_0,h)h]$ is contained in a face of β.

Proof. We use the notation of Paragraph 5.1.

First case : $h \in V_0$. Take t_0 such that $t_0 h_0$ lies in the relative interior of R_{h_0}. Then $t_0 h_0$ lies in the relative interior of the convex hull C of $T_i^{-1}[\gamma_i]$, $i \in I$, so there exists a finite subset of I, say $\{1, \dots n\}$, and $\lambda_1, \dots \lambda_n$ in $]0,1[$ such that

- $\lambda_1 + \dots + \lambda_n = 1$
- $t_0 h_0 = \lambda_1 T_1^{-1}[\gamma_1] + \dots + \lambda_n T_n^{-1}[\gamma_n]$
- $[\gamma_1], \dots, [\gamma_n]$ generate V_0.

On the other hand, since $h \in V_0$, there exist real numbers $x_1, \dots x_n$ such that $h = x_1 T_1^{-1}[\gamma_1] + \dots + x_n T_n^{-1}[\gamma_n]$. Take $\epsilon > 0$ such that $\forall i = 1, \dots n,\ \epsilon x_i + \lambda_i > 0$. Then $(\epsilon x_1 + \lambda_1 + \dots + \epsilon x_k + \lambda_k)^{-1}(t_0 h_0 + \epsilon h)$ lies in the relative interior of C. Thus there exists a face of β containing h_0 and $th_0 + sh$, where

$$t := \frac{1}{\sum_1^k \epsilon x_i + \lambda_i},\ s := \frac{\epsilon}{t}.$$

Second case : $h \notin V_0$, that is, $h \in V_0^\perp \setminus V_0$.

Remark 20. *In that case the dimension of V_0 must be less than $b_1(M)/2$.*

For any $n \in \mathbb{N}^*$ let us denote $h_n := h_0 + n^{-1}h$. Let μ_n be an h_n-minimizing measure. For each $i \in I$ let V_i be the neighborhood of $(\gamma_i, \dot\gamma_i)$ given by Lemma 12. Let V be the union over $i \in I$ of the V_i. Be sure to take the V_i small enough so V is a disjoint union of annuli.

First let us prove that $V \cap \mathrm{supp}(\mu_n)$ is ϕ_t-invariant and consists of periodic orbits homotopic to some or all of the γ_i. Indeed by Lemma 12 a minimizing orbit γ that enters V is either

- asymptotic to one of the γ_i
- or homotopic to one of the γ_i
- or cuts one of the γ_i with constant sign.

The first case is impossible since γ is contained in the support of an invariant measure (see Lemma 5.5 of [BM]). The third case is impossible since it would imply $\mathrm{Int}([\mu_n], [\gamma_i]) \neq 0$, which contradicts $h \in V_0^\perp$. So $V \cap \mathrm{supp}(\mu_n)$ is ϕ_t-invariant and consists of periodic orbits homotopic to some γ_i.

Now let us show that for n large enough, $0 < \mu_n(V) < 1$. Note that any limit point, in the weak* topology, of the sequence μ_n is an h_0-minimizing measure, hence supported in V, so $\mu_n(V)$ tends to 1. On the other hand, if $\mu_n(V)$ were 1, then μ_n would be supported in V. By the Graph Theorem any minimizing measure supported inside V may be viewed as an invariant measure of a vector field defined in $p(V)$. But $p(V)$ is a union of annuli, hence by the Poincaré-Bendixon Theorem any minimizing measure supported inside V is supported on fixed points, or periodic orbits homotopic to some γ_i. Note that fixed points are ruled out by the nonsingularity of h, which implies that h_n is non-singular for n large enough. In particular if $\mu_n(V) = 1$, $[\mu_n] \in V_0$, which is a contradiction. So $0 < \mu_n(V) < 1$ and we may set, for any Borelian subset A of TM,

$$\begin{aligned}\mu_{n,1}(A) &:= \frac{\mu_n(A\cap V)}{\mu_n(V)}\\ \mu_{n,2}(A) &:= \frac{\mu_n(A\setminus V)}{\mu_n(TM\setminus V)}\\ \lambda_n &:= \mu_n(V).\end{aligned}$$

Then $\mu_{n,1}$ and $\mu_{n,2}$ are two probability measures on TM. They are invariant by the Lagrangian flow because $V\cap \operatorname{supp}(\mu_n)$, as well as its complement in $\operatorname{supp}(\mu_n)$, is ϕ_t-invariant. There exists a face of β containing $[\mu_{n,1}]$ and $[\mu_{n,2}]$ because

$$\mu_1 = \lambda_n\mu_{n,1} + (1-\lambda_n)\mu_{n,2}.$$

Let $\mu_{0,1}$ and $\mu_{0,2}$ be limit points, in the weak* topology, of the sequences $\mu_{n,1}$ and $\mu_{n,2}$. Then $\mu_{0,1}$ is an h_0-minimizing measure, and there exists a face of β containing $h_0 = [\mu_{0,1}]$ and $[\mu_{0,2}]$.

Now we prove that $[\mu_{0,2}] \notin V_0$. Assume to the contrary. Then, as in the first case, the face F_0 containing h_0 and $[\mu_{0,2}]$ contains th_0 in its interior for some t such that th_0 lies in R_0. Take $\lambda\in\,]0,1[$ and h' in F_0 such that

$$th_0 = \lambda\,[\mu_{0,2}] + (1-\lambda)h'.$$

Take an h'-minimizing measure μ'. Then $\lambda\mu_{0,2}+(1-\lambda)\mu'$ is a th_0-minimizing measure, hence is supported inside V, which is impossible since $\mu_{n,2}$ is supported outside V for all n.

Thus there exist $v\in V_0$ and $x\neq 0$ such that $[\mu_{0,2}] = v + xh$. Take t_0 such that t_0h_0 lies in the relative interior of the convex hull C of $T_i^{-1}[\gamma_i], i\in I$, so there exists a positive ϵ such that $t_0h_0-\epsilon v$ lies in the relative interior of C. Lemma 5 now says that there is a face of β containing h_0, $t_0h_0-\epsilon v$ and $[\mu_{0,2}] = v+xh$. Such a face must contain h_0 and

$$\frac{1}{1+\epsilon}\left(t_0h_0-\epsilon v\right) + \left(1-\frac{1}{1+\epsilon}\right)(v+xh) = \frac{t_0}{1+\epsilon}h_0 + \frac{x\epsilon}{1+\epsilon}h,$$

Now set

$$t(h_0,h) := \frac{t_0}{1+\epsilon},\ s(h_0,h) := \frac{x\epsilon}{1+\epsilon}$$

and the theorem is proved. □

6. Proof of the main theorem

Let h be a homology class. Recall that $\mathcal{L}(h)$ is the Legendre transform of h with respect to β, and that $\mathcal{F}(\mathcal{L}(h))$ is the set of homology classes h' that lie in $\mathcal{L}(c)$ for all $c\in\mathcal{L}(h)$. By Lemma 4, $\mathcal{F}(\mathcal{L}(h))$ is a face of β. It is clear that h lies in $\mathcal{F}(\mathcal{L}(h))$, although possibly on the boundary.

Lemma 21. *Let L be a Tonelli Lagrangian on a closed surface M. Let h be a 1-irrational, nonsingular homology class. Then the relative interior of R_h is contained in the relative interior of the face $\mathcal{F}(\mathcal{L}(h))$.*

Proof. **Orientable case**

Assume M is oriented. Take $h_i, i=1\dots k$ such that $h+h_i$ lies in $\mathcal{F}(\mathcal{L}(h))$ for all $i=1,\dots k$, and the convex hull of $h+h_i$, $i=1,\dots k$, contains an open subset of $\mathcal{F}(\mathcal{L}(h))$. Pick one of the h_i. Then the segment $[h, h+h_i]$ is contained in a face of β so by Mather's graph Theorem $h_i \in V_0^{\perp}$.

Then $-h_i$ lies in $V_0^{\perp}$ so by Theorem 19 there exist $t_i \in \mathbb{R}$, $s_i := s(h, h_i) > 0$ such that the segment $[h, t_i h - s_i h_i]$ is contained in a face of β. So there exists $c \in H^1(M, \mathbb{R})$ such that

$$\begin{aligned} \alpha(c) + \beta(h) &= \langle c, h \rangle \\ \alpha(c) + \beta(t_i h - s_i h_i) &= \langle c, t_i h - s_i h_i \rangle. \end{aligned}$$

The first equality above says that $c \in \mathcal{L}(h)$. Since $h + h_i$ lies in $\mathcal{F}(\mathcal{L}(h))$, we also have

$$\alpha(c) + \beta(h + h_i) = \langle c, h + h_i \rangle.$$

Therefore $\mathcal{F}(\mathcal{L}(h))$ contains the convex hull C_i of h, $h + h_i$, $t_i h - s_i h_i$ for $i = 1, \dots k$.

We claim that for each $i = 1, \dots k$, C_i contains some th in its interior. Indeed, the following three cases may occur :

- $h_i \notin \mathbb{R}h,\ t_i \neq 1$: then C_i is a 2-simplex containing th in its relative interior, for some $t \in]1, t_i[$
- $h_i \notin \mathbb{R}h,\ t_i = 1$: then C_i is the segment $[h + h_i, h - s_i h_i]$, which contains h in its relative interior
- $h_i \in \mathbb{R}h$: then C_i is a segment of the straight line $\mathbb{R}h$, hence it contains some th in its relative interior.

Now the convex hull C of $\cup_{i=1}^{k} C_i$ is contained in $\mathcal{F}(\mathcal{L}(h))$ by convexity of the latter. The relative interior of C is open in $\mathcal{F}(\mathcal{L}(h))$ by our hypothesis on $h_i, i = 1 \dots k$. On the other hand C contains an interior point of R_h in its relative interior. So the intersection of the relative interiors of R_h and $\mathcal{F}(\mathcal{L}(h))$ is non-empty, convex, and contains h in its closure. Hence its closure is the whole of R_h.

Non-orientable case

Take

- h a 1-irrational, nonsingular homology class of M
- $h' \in E_1 \subset H_1(M_o, \mathbb{R})$ such that $\pi_* h' = h$
- $c \in H^1(M, \mathbb{R})$ such that $\alpha(c) + \beta(h) = \langle c, h \rangle$.

Since π_* sends an integer class to an integer class, and is one-to-one on E_1, h' is 1-irrational.

Any support of an h'-minimizing measure is the lift to M_o of the support of an h-minimizing measure by [F98], hence h' is nonsingular.

The orientable case of the lemma then says that for any t such that th is in the relative interior of $R_{h'}$, th' lies in the relative interior of $\mathcal{F}(\mathcal{L}(h'))$. Furthermore $h' \in E_1$ so th' lies in the relative interior of $\mathcal{F}(\mathcal{L}(h')) \cap E_1$. Thus, π_* being linear, $\pi_*(th') = th$ lies in the relative interior of $\pi_*\left(\mathcal{F}(\mathcal{L}(h')) \cap E_1\right)$. This combines with Lemma 11 to end the proof. $\square$

A consequence of our last lemma is that for any 1-irrational, nonsingular homology class h, $\mathcal{L}(h)$ is a nonsingular rational flat of α_L :

Lemma 22. *Let h be a 1-irrational, nonsingular homology class. We have*

$$\tilde{\mathcal{M}}(\mathcal{L}(h)) = \tilde{\mathcal{M}}(R_h).$$

Proof. Let μ be a minimizing measure supported in $\tilde{\mathcal{M}}(\mathcal{L}(h))$. Then for all $c \in \mathcal{L}(h)$ we have

$$\alpha(c) + \beta([\mu]) = \langle c, [\mu] \rangle$$

whence $[\mu] \in \mathcal{F}(\mathcal{L}(h))$. Take t such that th lies in the interior of $\mathcal{F}(\mathcal{L}(h))$. Take $\lambda \in]0,1[$ and h' in $\mathcal{F}(\mathcal{L}(h))$ such that

$$th = \lambda\,[\mu] + (1-\lambda)h'.$$

Take an h'-minimizing measure μ'. Then $\lambda\mu + (1-\lambda)\mu'$ is a th-minimizing measure, hence its support is contained in $\tilde{\mathcal{M}}(R_h)$. □

6.1. **Proof of Theorem 3.** Take a nonsingular, 1-irrational homology class h. Note that th is 1-irrational for any t. For any measure μ supported in $\tilde{\mathcal{M}}(\mathcal{L}(h))$, the homology class of μ lies in $\mathcal{F}(\mathcal{L}(h))$. Thus by Lemma 21 and Lemma 18, the support of μ consists of periodic orbits and fixed points. The latter are ruled out by the nonsingularity of th. Thus $\tilde{\mathcal{M}}(\mathcal{L}(h))$ is a union of non-trivial minimizing periodic orbits. Note that by Lemmas 21 and 22, if th lies in the relative interior of R_h, then th is a barycenter with positive coefficients of the homology classes of the periodic orbits contained in $\tilde{\mathcal{M}}(\mathcal{L}(h))$. So by Corollary 16 and Lemma 17,

$$\tilde{\mathcal{M}}(\mathcal{L}(h)) = \tilde{\mathcal{M}}(\mathcal{L}(th)) = \tilde{\mathcal{A}}(\mathcal{L}(th)) = \tilde{\mathcal{A}}(\mathcal{L}(h))$$

which finishes the proof of Theorem 3. □

References

[A03] N. Anantharaman *Counting geodesics which are optimal in homology* Ergodic Theory Dynam. Systems 23 (2003), no. 2, 353–388.

[AIPS05] N. Anantharaman ; R. Iturriaga ; P. Padilla ; H. Sánchez-Morgado *Physical solutions of the Hamilton-Jacobi equation* Discrete Contin. Dyn. Syst. Ser. B 5 (2005), no. 3, 513–528.

[BM] F. Balacheff, D. Massart *Stable norms of non-orientable surfaces*, to appear, Annales de l'Institut Fourier

[Be07] P. Bernard *Smooth critical sub-solutions of the Hamilton-Jacobi equation* Math. Res. Lett. 14 (2007), no. 3, 503–511.

[Ca95] M.J. Carneiro *On minimizing measures of the action of autonomous Lagrangians* Nonlinearity 8 (1995), no. 6, 1077–1085

[CL99] M.J. Carneiro,A. Lopes *On the minimal action function of autonomous Lagrangians associated to magnetic fields* Ann. Inst. H. Poincar Anal. Non Linaire 16 (1999), no. 6, 667–690.

[CI99] G. Contreras, R. Iturriaga *Convex Hamiltonians without conjugate points* Ergodic Theory Dynam. Systems 19 (1999), no. 4, 901–952.

[CMP04] G. Contreras, L. Macarini, G. Paternain, *Periodic orbits for exact magnetic flows on surfaces* Int. Math. Res. Not. 2004, no. 8, 361–387.

[F98] A. Fathi, *Orbites hétéroclines et ensemble de Peierls* C. R. Acad. Sci. Paris Sr. I Math. 326 (1998), no. 10, 1213–1216.

[F] A. Fathi, *Weak KAM theorem in Lagrangian dynamics* to appear, Cambridge University Press.

[Mn92] R. Mañé *On the minimizing measures of Lagrangian dynamical systems* Nonlinearity 5 (1992), no. 3, 623–638.

[Mt97] D. Massart *Stable norms of surfaces: local structure of the unit ball at rational directions* Geom. Funct. Anal. 7 (1997), **6**, 996–1010.

[Mt03] D. Massart *On Aubry sets and Mather's action functional* Israël Journal of Mathematics **134** (2003), 157-171.

[Mr91] J. N. Mather *Action minimizing invariant measures for positive definite Lagrangian systems* Math. Z. **207**, 169-207 (1991).

[PPS03] G. Paternain, L. Polterovich, K. F. Siburg, *Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory* Mosc. Math. J. 3 (2003), no. 2, 593–619, 745.

Département de Mathématiques, Université Montpellier II, France
e-mail : massart@math.univ-montp2.fr