The universal cover of a monomial triangular algebra without multiple arrows

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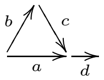
Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field k. Assuming that A is monomial and that the ordinary quiver Q of A has no oriented cycle and no multiple arrows, we prove that A admits a universal cover with group the fundamental group of the underlying space of Q.

Introduction

Let A be a finite dimensional k-algebra where k is an algebraically closed field. In order to study the category $mod(A)$ of (left) A-modules, one may assume that A is basic and connected. In [10] (see also [2]), C. Riedtmann has introduced the covering techniques which reduce the study of part of $mod(A)$ to the easier one of $mod(\mathcal{C})$, where $\mathcal{C} \to A$ is a Galois covering and $\mathcal{C}$ is locally bounded. These techniques are based on the coverings of translation quivers and their fundamental group, and therefore, have been particularly efficient for representation-finite and standard algebras A: In this case, P. Gabriel ([5]) has constructed a universal Galois covering of A, whose properties have led to a precise description of the standard form of a representation-finite algebra ([3]). Unfortunately, the above construction of [5] cannot be proceeded in the representation-infinite case precisely because the Auslander-Reiten quiver is no longer connected.

In [9], R. Martinez-Villa and J. A. de la Peña have constructed a Galois covering $k\widetilde{Q}/\widetilde{I} \to A$ associated with each presentation $kQ/I \simeq A$ (by quiver and admissible relations), for any algebra A. This Galois covering is induced by the universal cover $(\widetilde{Q}, \widetilde{I}) \to (Q, I)$ with fundamental group $\pi_1(Q, I)$ of the bound quiver (Q, I), as a generalisation of the universal cover of a translation quiver defined in [2] and [10]. Like in topology, the group $\pi_1(Q, I)$ is defined by means of a homotopy relation $\sim_I$ on the set of unoriented paths of Q. When A is representation-finite and standard, this Galois covering coincides with the one constructed by P. Gabriel. Therefore, it is a natural candidate for a universal cover of A in the general case. Alas, different presentations may have non-isomorphic fundamental groups. So there may exist many candidates for a universal cover. As an example, let $A = kQ/ < da >$, where Q is the quiver:



Then, $\pi_1(Q, < da >) \simeq \mathbb{Z}$. On the other hand, $A \simeq kQ/ < da - dcb >$, and $\pi_1(Q, < da - dcb >) = 1$. Notice that A is tilted of euclidean an type and therefore belongs to a quite well-understood class of algebras. This illustrates the fact that except for representation-finite algebras there are quite a few classes of algebras for which the existence of a universal cover is known.

In this text, we prove the existence of a universal cover for certain monomial algebras, that is, quotients of paths algebras of quivers by a monomial ideal (*i.e.* generated by a set of paths). More precisely, we prove the following main result.

Theorem 1. *Let $A = kQ/I_0$, where Q is a quiver without otiented cycle and without multiple arrows, and I_0 is a monomial admissible ideal of kQ. Let $\widehat{\mathcal{C}} \to kQ/I_0$ be the Galois covering with group $\pi_1(Q)$ defined by the presentation $kQ/I_0 \simeq A$ (see [9]). Then $\widehat{\mathcal{C}} \to A$ is a universal cover of A in the following sense. For any Galois covering $\mathcal{C} \to A$*

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with group G and with $\mathcal{C}$ connected and locally bounded, there exists a commutative "factorisation diagram":

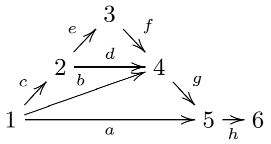
$$\begin{tikzcd}
\widehat{\mathcal{C}} \arrow[rd] \arrow[dd] & \\
& \mathcal{C} \arrow[d] \\
A \arrow[r, "\sim"] & A
\end{tikzcd}$$

where the bottom arrow is an isomorphism of k-algebras, extending the identity map on the set Q_0 of vertices, and where $\widehat{\mathcal{C}} \to \mathcal{C}$ is a Galois covering with group $N \lhd \pi_1(Q)$ such that there exists an exact sequence of groups: $1 \to N \to \pi_1(Q) \to G \to 1$.

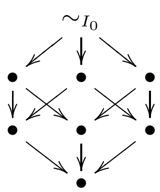
The author gratefully acknowledges an anonymous referee for pointing out the following example from [6, 3.2]. It shows that the assumption on multiple arrows cannot be removed: Let $A = k\left(\cdot \overset{x}{\underset{x'}{\rightrightarrows}} \cdot \overset{y}{\underset{y'}{\rightrightarrows}} \cdot\right)/I_0$, where $I_0 = < yx', y'x >$. Then, $\pi_1(Q, I_0) = \pi_1(Q) \simeq \mathbb{Z} \star \mathbb{Z}$, and if $char(k) \neq 2$, then $A \simeq kQ/I$ where $I = < yx - y'x', yx' - y'x >$, and $\pi_1(Q, I) \simeq \mathbb{Z}/2\mathbb{Z}$. Now, let $\mathcal{C} \to A$ be the Galois covering with group $\pi_1(Q, I)$ defined by the presentation $A \simeq kQ/I$. Then, with the notations of Theorem 1, it is easy to show that there is no k-linear functor $\widehat{\mathcal{C}} \to \mathcal{C}$. Thus, in this example, A admits no universal cover in the sense of Theorem 1. From the very definition of monomial algebras, one would expect that they all have a universal cover. The above counter-example shows that this is not always the case. It is all the more surprising as the involved algebra is gentle, so its representation theory its fairly well-known.

We now explain the strategy of the proof of Theorem 1. Our main tool is the quiver Γ of the homotopy relations $\sim_I$ of the presentations $kQ/I \simeq A$. It was introduced in [8] to prove the existence of a universal cover for algebras over a zero characteristic field, and whose ordinary quiver have no double bypass. In general, if $kQ/I \simeq A$ and $kQ/J \simeq A$, there is no simple relation between $\pi_1(Q, I)$ and $\pi_1(Q, J)$ (and therefore between the associated Galois coverings of A) unless A is representation-finite (in which case A is schurian, so that $\pi_1(Q, I) = \pi_1(Q, J)$). This is the main difficulty in proving the existence of a universal cover. Hopefully, such a relation exists when I and J are related by a transvection $\varphi_{\alpha,u,\tau}$, that is, $J = \varphi_{\alpha,u,\tau}(I)$, where (α, u) is a bypass (meaning that α is an arrow and u is a path parallel to and different from α), $\tau \in k$ and $\varphi_{\alpha,u,\tau} \in Aut(kQ)$ is the automorphism which maps α to $\alpha + \tau u$ and which fixes any other arrow. In such a case, there is a natural quotient relation between $\pi_1(Q, I)$ and $\pi_1(Q, J)$. Besides, the Galois coverings of A with groups $\pi_1(Q, J)$ and $\pi_1(Q, I)$ are the vertical arrows of a factorisation diagram like in Theorem 1, and the associated exact sequence of groups is given by the above quotient relation. The quiver Γ is then defined as follows. Its vertices are the homotopy relations $\sim_I$ of all the presentations $kQ/I \simeq A$, and there is an arrow $\sim \to \sim'$ if there exist presentations $kQ/I \simeq A$ and $kQ/J \simeq A$, and a transvection φ such that: $\sim = \sim_I$, $\sim' = \sim_J$, $J = \varphi(I)$, and $\pi_1(Q, J)$ is a strict quotient of $\pi_1(Q, I)$. The quiver Γ is then finite, connected, and it has no oriented cycle. Notice that Γ is reduced to a point (with no arrow) when A is schurian (and in particular when A is representation-finite). But, usually, Γ has many vertices and many arrows. We refer the reader to Section 1 for more details.

Roughly speaking, the existence of a universal cover is related to the existence of a unique source in Γ. More precisely, assume that there exists a presentation $kQ/I_0 \simeq A$ (which in our case will be the monomial presentation) such that for any other presentation $kQ/I \simeq A$, there exists a sequence of ideals $I_0, I_1 = \varphi_1(I_0), \ldots, I_n = \varphi_n(I_{n-1}) = I$, where $\varphi_1, \ldots, \varphi_n$ are transvections defining a path $\sim_{I_0} \to \sim_{I_1} \to \ldots \to \sim_{I_n} = \sim_I$ in Γ. Then, the Galois covering of A with group $\pi_1(Q, I_0)$ associated to the presentation $kQ/I_0 \simeq A$ is a universal cover of A. As an example, assume that $A = kQ/I_0$, where Q is the quiver



and $I_0 = < ha, gb, dc >$. Then Γ has the following shape:



We do not specify all the vertices, but one can check that for every $\sim_I \in \Gamma$, the group $\pi_1(Q, I)$ is free over $3 - l$ generators, where l is the length of any path from $\sim_{I_0}$ to $\sim_I$ (so two presentations may have distinct homotopy relations yet isomorphic fundamental groups). For the needs of the proof, we construct a specific total order on the set of bypasses of Q. In our example, this order is: $(d, fe) < (b, fec) < (b, dc) < (a, gfec) < (a, gdc) < (a, gb)$. Now, let $I =< ha + hgfec, gb + gfec, dc >$, then:

1. $I = \varphi_{a,gfec,1}\varphi_{b,fec,1}(I_0)$. Moreover, $\varphi_{a,gfec,1}\varphi_{b,fec,1}$ is the unique automorphism of kQ which transforms I_0 into I, and which maps every arrow α to the sum of α and a linear combination of paths of length greater than 1, none of which lying in I_0 (indeed: $gfec, fec \notin I_0$). For that reason, we set $\psi_I := \varphi_{a,gfec,1}\varphi_{b,fec,1}$.
2. The equality $\psi_I = \varphi_{a,gfec,1}\varphi_{b,fec,1}$ expresses ψ_I as a product of transvections $\varphi_{a,gfec,1}, \varphi_{b,fec,1}$. The associated sequence of bypasses is decreasing ($(a, gfec) > (b, fec)$). Actually, the sequence $\varphi_{a,gfec,1}, \varphi_{b,fec,1}$ is unique for this property.
3. We have a path $I_0 \to \varphi_{b,fec,1}(I_0) \to \varphi_{a,gfec,1} = \varphi_{b,fec,1}(I_0) = I$ in Γ.

In this example, it is easy to check that for any other presentation $A \simeq kQ/J$, the ideal J defines a unique automorphism ψ_J (as in 1.) which decomposes uniquely (as in 2.), giving rise to a path from $\sim_{I_0}$ to $\sim_J$ (as in 3.).

The proof of Theorem 1 mimickes the three steps we proceeded in that example. Indeed, we prove the three following technical points:

1. If $kQ/I \simeq kQ/I_0$, then there exists a unique product ψ_I of transvections such that $\psi_I(I_0) = I$, and such that ψ_I maps every arrow α to the sum of α and a linear combination of paths of length greater than 1, none of which lying in I_0.
2. There exists a suitable ordering on the set of bypasses such that if $\psi \in Aut(kQ)$ is a product of transvections, then ψ can be written uniquely as $\psi = \varphi_{\alpha_n,u_n\tau_n} \dots \varphi_{\alpha_1,u_1,\tau_1}$ with $\tau_1, \dots, \tau_n \in k^*$ and $(\alpha_n, u_n) > \dots > (\alpha_1, u_1)$.
3. If $kQ/I \simeq kQ/I_0$, the unique ordered sequence of transvections given by 1. and 2. yield a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_I$. Also, this sequence gives rise to the factorisation diagram of Theorem 1.

The text is therefore organised as follows. In Section 1 we recall all the notions that we need to prove Theorem 1. In Section 2, we prove some combinatorial facts on the paths in a quiver. These lead to the order and to the decomposition of the second point above. In Section 3 we prove the first point above. Finally, in Section 4 we prove the last point and Theorem 1.

1 Basic definitions

A **k-category** is a category $\mathcal{C}$ whose objects class $\mathcal{C}_0$ is a set, whose space of morphisms from x to y (denoted by ${}_y\mathcal{C}_x$) is a k-vector space for any $x, y \in \mathcal{C}_0$ and whose composition of morphisms is k-bilinear. All functors between k-categories will be assumed to be k-linear functors. In particular, $Aut(\mathcal{C})$ will denote the group of k-linear automorphism of $\mathcal{C}$, and $Aut_0(\mathcal{C})$ will denote by for the subgroup $\{\psi \in Aut(\mathcal{C}) \mid \psi(x) = x \text{ for any } x \in \mathcal{C}_0\}$ of $Aut(\mathcal{C})$. The k-category $\mathcal{C}$ is called **connected** if it cannot be written as the disjoint union of two full subcategories. An ideal I of $\mathcal{C}$ is the data of subspaces ${}_yI_x \subseteq {}_y\mathcal{C}_x$ (for any $x, y \in \mathcal{C}_0$) such that $fgh \in I$ whenever f, g, h are composable morphisms in $\mathcal{C}$ such that $g \in I$. The k-category $\mathcal{C}$ is called **locally bounded** provided that: 1) for any $x \in \mathcal{C}_0$, the vector spaces $\bigoplus\limits_{y\in\mathcal{C}_0} {}_y\mathcal{C}_x$ and $\bigoplus\limits_{y\in\mathcal{C}_0} {}_x\mathcal{C}_y$ are finite dimensional, 2) ${}_x\mathcal{C}_x$ is a local algebra for any $x \in \mathcal{C}_0$, 3) distinct objects are not isomorphic. Let A be a finite dimensional k-algebra and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. Then A is also a k-category: $A_0 := \{e_1, \dots, e_n\}$, ${}_{e_i}A_{e_i} := e_jAe_i$ and the product of A induces the composition of morphisms. Notice that different choices for the idempotents $e_1, \dots, e_n$ give rise to isomorphic k-categories. Also, A is connected (resp. basic) as a k-algebra if and only if it is connected (resp. locally bounded) as a k-category. In the sequel we shall make no distinction between a finite dimensional k-algebra and its associated k-category. If $\mathcal{C}$ is a locally bounded k-category, the radical of $\mathcal{C}$ is the ideal $\mathcal{RC}$ of $\mathcal{C}$ such that: ${}_y\mathcal{RC}_x$ is the space of non-isomorphisms $x \to y$ in $\mathcal{C}$, for any $x, y \in \mathcal{C}_0$. The ideal of $\mathcal{C}$ generated by compositions gf where f and g lie in $\mathcal{RC}$ will be denoted by $\mathcal{R}^2\mathcal{C}$.

A **Galois covering with group** G of $\mathcal{C}$ (by $\mathcal{C}'$) is a functor $F: \mathcal{C}' \to \mathcal{C}$ endowed with a group morphism $G \to Aut(\mathcal{C}')$ and such that: 1) the induced action of G on $\mathcal{C}'_0$ is free, 2) $F \circ g = F$ for any $g \in G$, 3) for any k-linear functor $F': \mathcal{C}' \to \mathcal{C}''$ such that $F' \circ g = F'$ for any $g \in G$, there exists a unique $\overline{F'}: \mathcal{C} \to \mathcal{C}''$ such that $\overline{F'} \circ F = F'$ (in other words, F is a quotient of $\mathcal{C}'$ by G in the category of k-categories). For short, the Galois covering F is called connected if $\mathcal{C}'$ is connected and locally bounded (this implies that $\mathcal{C}$ is connected and locally bounded). For more details on Galois coverings (in particular for the connections with representations theory), we refer the reader to [2].

Quivers, paths, bypasses. A quiver is a 4-tuple $Q = (Q_1, Q_0, s, t)$ where Q_1 and Q_0 are sets and $s, t: Q_1 \to Q_0$ are maps. The elements of Q_1 (resp. of Q_0) are called the arrows (resp. the vertices) of Q. If $\alpha \in Q_1$, the vertex $s(\alpha)$ (resp. $t(\alpha)$) is called the source (resp. the target) of α. The quiver Q is called **locally finite** if and only if any vertex is the source (resp. the target) of finitely many arrows. For example, if $\mathcal{C}$ is a locally bounded k-category, the **ordinary**

quiver of $\mathcal{C}$ is the locally finite quiver Q such that: $Q_0 := \mathcal{C}_0$ and for any $x, y \in \mathcal{C}_0$, the number of arrows starting at x and arriving at y is equal to $dim_k\ {}_y\mathcal{R}\mathcal{C}_x / {}_y\mathcal{R}^2\mathcal{C}_y$. A path in Q of length n ($n \geqslant 0$) with source $x \in Q_0$ (or starting at x) and target $y \in Q_0$ (or arriving at y) is a sequence of arrows $\alpha_1, \ldots, \alpha_n$ such that: $x = y$ if $n = 0$, $s(\alpha_1) = x$, $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i \in \{1, \ldots, n-1\}$ and $t(\alpha_n) = y$. If $n \geqslant 1$ this path will be written $\alpha_n \ldots \alpha_1$ and called non trivial. If $n = 0$ this path will be written e_x and called stationary at x. The length of this path is $|u| := n$. The mappings s, t are naturally extended to paths in Q. If u and v are paths, the concatenation vu is defined if and only if $t(u) = s(v)$ by the following rule: 1) $vu = v$ is u is stationary, 2) $vu = u$ is v is stationary, 3) $vu = \beta_m \ldots \beta_1 \alpha_n \ldots \alpha_1$ if $v = \beta_m \ldots \beta_1$ and $u = \alpha_n \ldots \alpha_1$ (with $\alpha_i, \beta_j \in Q_1$). Two paths in Q are called **parallel** whenever they have the same source and the same target. An **oriented cycle** in Q is a non trivial path whose source and target are equal. The quiver Q is said to have **multiple arrows** if and only if there exist in Q distinct parallel arrows. If Q has no oriented cycle and if (α, u, β, v) is a double bypass (see the introduction) there exists two unique paths u_1, u_2 such that $u = u_2\beta u_1$. In such a situation, the path $u_2 v u_1$ will be called obtained from $u = u_2\beta u_1$ after replacing β by v.

Admissible presentations (see [2, 2.1]). A quiver Q defines the **path category** kQ such that $(kQ)_0 = Q_0$, such that ${}_ykQ_x$ is the k-vector space with basis the family of paths starting at x and arriving at y, and the composition in kQ is induced by the concatenation of paths. For short, **a normal form for** $r \in\ {}_ykQ_x$ is an equality $r = \sum\limits_{i=1}^{n} t_i u_i$ where $t_1, \ldots, t_n \in k^*$ and $u_1, \ldots u_n$ are pairwise distinct paths in Q. With this notation, **the support** of r is the set $supp(r) := \{u_1, \ldots, u_n\}$ ($supp(0) = \emptyset$). A **subexpression of** r is a linear combination $\sum\limits_{i \in E} t_i u_i$ with $E \subseteq \{1, \ldots, n\}$. Later, we will need the following fact: if $r = r_1 + \ldots + r_n \in\ {}_ykQ_x$ is such that $supp(r_1), \ldots, supp(r_n)$ are pairwise disjoint, then $r_{i_1} + \ldots + r_{i_t}$ is a subsexpression of r, for any indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$. An ideal I of kQ is called **admissible** provided that: 1) any morphism in I is a linear combination of paths of length at least 2, 2) the factor category kQ/I is locally bounded. A morphism in I is called a **relation** (of I). In particular, **a minimal relation of** I (see [9]) is a non zero relation r of I such that 0 and r are the only subexpressions of r which are relations. With this definition, any relation of I is the sum of minimal relations with pairwise disjoint supports. A **monomial relation** is a path u lying in I and I is called monomial if it is generated by a set of monomial relations. A pair (Q, I) where Q is a locally finite and I is an admissible ideal of kQ is called a **bound quiver**. In such a case, kQ/I is locally bounded and it is connected if and only if Q is connected (i.e. the underlying graph of Q is connected). Conversely, if $\mathcal{C}$ is a locally bounded k-category, then there exists an isomorphism $kQ/I \xrightarrow{\sim} \mathcal{C}$ where (Q, I) is a bound quiver such that Q is the ordinary quiver of $\mathcal{C}$. Such an isomorphism is called **admissible presentation of** $\mathcal{C}$. If the ideal I is monomial, the admissible presentation and $\mathcal{C}$ are called monomial. Notice that $\mathcal{C}$ may have different admissible presentations.

Fundamental group of a presentation (see [9]). Let (Q, I) be a bound quiver and let $x_0 \in Q_0$. For every arrow $x \xrightarrow{a} y \in Q_1$ we define its formal inverse a^{-1} with source $s(a^{-1}) = y$ and target $t(a^{-1}) = x$. This defines a new quiver $\overline{Q} = (Q_0, Q_1 \cup \{a^{-1} \mid a \in Q_1\}, s, t)$. With these notations, **a walk in** Q is a path in $\overline{Q}$. The concatenation of walks in Q is by definition the concatenation of paths in $\overline{Q}$. The **homotopy relation** of (Q, I) is the equivalence relation on the set of walks in Q, denoted by $\sim_I$ and generated by the following properties:

1. $\alpha\alpha^{-1} \sim_I e_y$ and $\alpha^{-1}\alpha \sim_I e_x$ for any arrow $x \xrightarrow{\alpha} y$ in Q,
2. $u \sim_I v$ for any $u, v \in supp(r)$ where r is a minimal relation of I,
3. $wvu \sim_I wv'u$ for any walks w, v, v', u such that $v \sim_I v'$ and such that the concatenations wvu and $wv'u$ are well-defined (i.e. $\sim_I$ is compatible with the concatenation).

The $\sim_I$-equivalence class of a walk γ will be denoted by $[\gamma]_I$. Let $\pi_1(Q, I, x_0)$ be the set of equivalence classes of walks in Q with source and target equal to x_0. The concatenation of walks endows this set with a group structure (with unit e_{x_0}) and this group is called the **fundamental group of** (Q, I) **at** x_0. If Q is connected, the isomorphism class of this group does not depend on $x_0 \in Q_0$ and $\pi_1(Q, I, x_0)$ is denoted by $\pi_1(Q, I)$. If $\mathcal{C}$ is a connected locally bounded k-category and if $kQ/I \simeq \mathcal{C}$ is an admissible presentation, the fundamental group $\pi_1(Q, I)$ is called the fundamental group of this presentation.

Dilatations, transvections (see [8]). Let Q be a quiver. A **dilatation** of kQ is an automorphism $D \in Aut_0(kQ)$ such that $D(\alpha) \in k^*\alpha$ for any $\alpha \in Q_1$. The dilatations of kQ form a subgroup $\mathcal{D}$ of $Aut_0(kQ)$. Let (α, u) be a bypass in Q and let $\tau \in k$. This defines $\varphi_{\alpha,u,\tau} \in Aut_0(kQ)$ as follows: $\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau u$ and $\varphi_{\alpha,u,\tau}(\beta) = \beta$ for any arrow $\beta \neq \alpha$. The automorphism $\varphi_{\alpha,u,\tau}$ is called a **transvection**. The composition of transvections is ruled as follows. Let $\varphi_{\alpha,u,\tau}$ and $\varphi_{\alpha,u,\tau'}$, then $\varphi_{\alpha,u,\tau}\varphi_{\alpha,u,\tau'} = \varphi_{\alpha,u,\tau+\tau'}$ and $\varphi_{\alpha,u,\tau}^{-1} = \varphi_{\alpha,u,-\tau}$. If (α, u, β, v) and (β, v, α, u) are not a double bypasses, then $\varphi_{\alpha,u,\tau}\varphi_{\beta,v,\nu} = \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau},$. If (α, u, β, v) is a double bypass and if Q has no oriented cycle, then $\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} = \varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$, where w is the path obtained from u after replacing β by v. The subgroup of $Aut_0(kQ)$ generated by all the transvections is denoted by $\mathcal{T}$. The dilatations and the transvections are useful to compare admissible presentations of an algebra because of the following proposition:

Proposition 1.1. *(see [8, Prop. 2.1, Prop. 2.2]) Let $kQ/I \simeq A$ and $kQ/J \simeq A$ be admissible presentations of the basic finite dimensional algebra A. If Q has no oriented cycle, then there exists $\psi \in Aut_0(kQ)$ such that $\psi(I) = J$. Moreover, $\mathcal{T}$ is a normal subgroup of $Aut_0(kQ)$ and $Aut_0(kQ) = \mathcal{T}\mathcal{D} = \mathcal{D}\mathcal{T}$.*

The dilatations and the transvections were introduced because they allow comparisons between the fundamental groups of presentations of the same locally bounded k-category. Notice that if I, J are admissible ideals of kQ such that $\gamma \sim_I \gamma' \Rightarrow \gamma \sim_J \gamma'$ for any walks γ, γ', then the identity map on the set of walks induces a surjective group morphism $\pi_1(Q,I) \twoheadrightarrow \pi_1(Q,J)$. In particular, if $\sim_I$ and $\sim_J$ coincide, then $\pi_1(Q,I) = \pi_1(Q,J)$.

Proposition 1.2. *(see [8, Prop. 2.5]) Let I be an admissible ideal of kQ (with Q without oriented cycle), let $\varphi \in Aut_0(kQ)$ and set $J = \varphi(I)$. If φ is a dilatation, then $\sim_I$ and $\sim_J$ coincide. If $\varphi = \varphi_{\alpha,u,\tau}$ is a transvection, then:*

1. *if $\alpha \sim_I u$ and $\alpha \sim_J u$ then $\sim_I$ and $\sim_J$ coincide.*
2. *if $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then $\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$.*
3. *if $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $I = J$ and $\sim_I$ and $\sim_J$ coincide.*

If there exists a transvection φ such that $\varphi(I) = J$ and such the second point above occurs, then $\sim_J$ ***is called a direct successor of*** *$\sim_I$.*

Here the expression "$\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$" means that $\sim_J$ is the equivalence relation on the set of walks in Q, compatible with the concatenation and generated by the two following properties: 1) $\gamma \sim_I \gamma' \Rightarrow \gamma \sim_J \gamma'$, 2) $\alpha \sim_J u$. Following [8, Def. 2.7], if A is a basic connected finite dimensional algebra with ordinary quiver Q without oriented cycle, we define **the quiver Γ of the homotopy relations of** A to be the quiver such that $\Gamma_0 = \{\sim_I \ \mid kQ/I \simeq A\}$ and such that there exists arrow $\sim_I \to \sim_J$ if and only if $\sim_J$ is a direct successor of $\sim_I$. Recall ([8, Rem. 5, Prop. 2.8]) that Γ is finite, connected, without oriented cycle and such that for any (oriented) path with source $\sim_I$ and target $\sim_J$, the identity map on the set of walks in Q induces a surjective group morphism $\pi_1(Q,I) \twoheadrightarrow \pi_1(Q,J)$.

Gröbner bases Let E be a k-vector space with an ordered basis $(e_1,\ldots,e_n)$, let $(e_1^*,\ldots,e_n^*)$ be the associated dual basis of E^*, and let F be a subspace of E. A Gröbner basis (see [1] for the ususal definition) of F is a basis $(r_1,\ldots,r_d)$ such that:

1. $r_j \in e_{i_j} + Span(e_l \ ; \ l < i_j)$ for some i_j, for any $j \in \{1,\ldots,r\}$,
2. $i_1 < i_2 < \ldots < i_r$,
3. $e_{i_j}^*(r_{j'}) = 0$ for any $j \neq j'$.

It is well known that F admits a unique Gröbner basis. Also, $r \in F$ if and only if: $r = \sum\limits_{j=1}^{d} e_{i_j}^*(r) r_j$. In the sequel, we will use this notion in the following setting: E is the vector space with basis (for some order to be defined) the family of non trivial paths in a finite quiver Q without oriented cycles and F is the underlying subspace of E associated to an admissible ideal I of kQ. Notice that in this setting, the Gröbner basis of F is made of minimal relations of I. Also, if $r \in E$ and if u is a non trivial path, then: $u \in supp(r) \Leftrightarrow u^*(r) \neq 0$.

Until the end of the text, Q will denote a finite quiver without oriented cycle and without multiple arrows.

2 Combinatorics on the paths in a quiver

Recall from the previous section that if (α, u, β, v) is a double bypass and if τ, ν are scalars, then $\psi := \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}$ is equal to $\varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$ where w is the path obtained from u by replacing β by v. Remark that $\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau u + \tau\nu w$. Hence, the paths ($u$ and w) appearing in $\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}(\alpha) - \alpha$ are axactely those paths θ such that (α, θ) is a bypass appearing in one of the transvections of the product $\varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$. Moreover, the scalars (τ and $\tau\nu$) appearing with these paths are exactely the scalars of the corresponding transvections in this product. So, the computation of $\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}(\alpha)$ can be done just by looking for the occurences of α in the product $\varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$. From this point of view, the decomposition $\psi = \varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$ is more useful than the decomposition $\psi = \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}$. The aim of this section is to show that this phenomenon is a general one. In this purpose the useful notion of derivation of a path and a total order on the set of bypasses will be introduced.

2.1 Derivation of paths

Definition 2.1. *Let $u = \alpha_n \ldots \alpha_1$ and v be paths in Q. Then v is called derived of u (of order t) if there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ and bypasses $(\alpha_{i_1}, v_1), \ldots, (\alpha_{i_t}, v_t)$ such that v is obtained from u by replacing α_{i_l} by v_l, for each l:*

$$v = \alpha_n \ldots \alpha_{i_t+1} v_t \alpha_{i_t-1} \ldots \alpha_{i_l+1} v_l \alpha_{i_l-1} \ldots \alpha_{i_1+1} v_1 \alpha_{i_1-1} \ldots \alpha_1$$

Remark 2.2. *If $\alpha \in Q_1$, then u is derived of α if and only if (α, u) is a bypass.*

With the above definition, the following lemma is easily verified using the fact that Q has no multiple arrows and no oriented cycle.

Lemma 2.3.
1. *If v is derived of u with both orders t and t', then t=t'.*
2. *If v is derived of u of order t then there exists a sequence of paths $u_0 = u, u_1, \ldots, u_t = v$ such that u_i is derived of u_{i-1} of order 1 for any i.*
3. *If v is derived of u of order t, then $|v| \geqslant |u| + t$.*
4. *If v is derived of u of order t and if w is derived of v of order t', then w is derived of u of order at least t.*
5. *Let u, v, w be paths verifying:*
   - *v is derived of u,*
   - *w is derived of v,*
   - *w is derived of u of order 1,*

   then we have:

   $$u = u_2\alpha u_1, \quad v = u_2\theta u_1, \quad w = u_2\theta' u_1$$

   where u_1, u_2 are paths, (α, θ) is a bypass and θ' is derived of θ.
6. *If v (resp. v') is derived of u (resp. of u') of order t (resp. t'), then $v'v$ is derived of $u'u$ of order $t' + t$, whenever these compositions of paths are well defined.*

The following example shows that, in the preceding lemma, the inequality in the 4-th point may be an equality.

Example 2.4. *Let (α, u, β, v) be a double bypass. Let u_1, u_2 be the paths such that $u = u_2\beta u_1$. Then u is derived of α of order 1, $w := u_2vu_1$ is derived of u of order 1 and w is derived of u of order 1.*

2.2 Order between paths, order between bypasses

Now, we construct a total order on the set of non trivial paths in Q. This construction is a particular case of the one introduced in [4]. Also it depends on an arbitrary order $\lhd$ on Q_1. We assume that this order $\lhd$ is fixed for this subsection. We shall write $\lhd$ for the lexicographical order induced by $\lhd$ on the set of nontrivial paths in Q. For details on the correctness of the following definition we refer the reader to [4].

Definition 2.5. *For $\alpha \in Q_1$, set:*

$$W(\alpha) = Card(B(\alpha)) \quad where \quad B(\alpha) = \{(\alpha, u) \mid (\alpha, u) \text{ is a bypass in } Q\}$$

For $u = \alpha_n \ldots \alpha_1$ a path in Q (with $\alpha_i \in Q_1$), let us set:

$$W(u) = W(\alpha_n) + \ldots + W(\alpha_1)$$

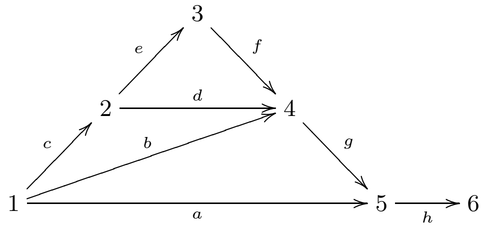
These data define a total order $<$ on the set of non trivial paths in Q as follows:

$$u < v \Leftrightarrow \begin{cases} & W(u) < W(v) \\ or & \\ & W(u) = W(v) \text{ and } u \lhd v \end{cases}$$

We shall write $<$ for the lexicographical order induced by $<$ on the set of couples of paths.

Remark 2.6. *If u and v are (non trivial) paths such that vu is well defined, then $W(vu) = W(u) + W(v)$.*

Example 2.7. *Let Q be the following quiver without oriented cycle and without multiple arrows:*



and let $\lhd$ be any total order on Q_1. Then, $B(a) = \{(a, bg), (a, gdc), (a, gfe)\}$, $B(b) = \{(b, cd), (b, fec)\}$, $B(d) = \{(d, fe)\}$ and $B(x) = \emptyset$ for $x \in Q_1 \backslash \{a, b, d\}$. In particular, the paths with source 1 and target 5 are ordered as follows:

$$gfec < gdc < gb < a$$

Lemma 2.8. *1. If u, v, u', v' are paths such that $v < u$ and $v' < u'$ then $v'v < u'u$ whenever these compositions are well defined.*

2. If (α, u) is a bypass, then $W(u) < W(\alpha)$. So $u < \alpha$.

3. If v is derived of u, then $v < u$.

4. If (α, u, β, v) is a double bypass and if w is the path obtained from u after replacing β by v, then:

$$(\beta, v) < (\alpha, w) < (\alpha, u)$$

Proof: 1) is a direct consequence of Definition 2.5 and Remark 2.6.

2) Let us write $u = a_n \dots a_1$ with $a_i \in Q_1$ for each i (hence $a_i \neq a_j$ if $i \neq j$ because Q has no oriented cycle). Therefore:

. $B(a_1), \dots, B(a_n)$ are pairwise disjoint,

. $W(u) = W(a_1) + \dots + W(a_n)$

Notice that if $(a_i, v) \in B(a_i)$, then $(\alpha, a_n \dots a_{i+1} v a_{i-1} \dots a_1) \in B(\alpha)$. Thus, we have a well defined mapping:

$$\begin{array}{rrcl} \theta: & B(a_1) \sqcup \dots \sqcup B(a_n) & \longrightarrow & B(\alpha) \\ & (a_i, v) & \longmapsto & (\alpha, a_n \dots a_{i+1} v a_{i-1} \dots a_1) \end{array}$$

This mapping is one-to-one, indeed:

. if $\theta(a_i, v) = \theta(a_i, v')$ with $(a_i, v), (a_i, v') \in B(a_i)$ then:

$$a_n \dots a_{i+1} v a_{i-1} \dots a_1 = a_n \dots a_{i+1} v' a_{i-1} \dots a_1$$

and therefore $(a_i, v) = (a_i, v')$,

. if $\theta(a_i, v) = \theta(a_j, v')$ with $(a_i, v) \in B(a_i)$, $(a_j, v') \in B(a_j)$ and $j < i$, then:

$$a_n \dots a_{i+1} v a_{i-1} \dots a_1 = a_n \dots a_{j+1} v' a_{j-1} \dots a_1$$

So:

$$v a_{i-1} \dots a_1 = a_i \dots a_{j+1} v' a_{j-1} \dots a_1$$

Since v and a_i are parallel and since Q has no oriented cycle, we infer that $v = a_i$ which is impossible because $(a_i, v) \in B(a_i)$.

On the other hand, θ is not onto. Indeed, if there exists $(a_i, v) \in B(a_i)$ verifying $\theta(a_i, v) = (\alpha, u)$, then:

$$a_n \dots a_1 = u = a_n \dots a_{i+1} v a_{i-1} \dots a_1$$

which implies $a_i = v$, a contradiction. Since θ is one-to-one and not onto, we deduce that:

$$W(\alpha) = Card(B(\alpha)) > Card(B(a_1) \sqcup \dots \sqcup B(a_n)) = W(u)$$

This proves that $W(u) < W(\alpha)$ and that $u < \alpha$.

3) is a direct consequence of 1) and of 2).

4) Let us write $u = u_2 \beta u_1$ (with u_1, u_2 paths) so that $w = u_2 v u_1$. From 2), we have:

$$W(\alpha) > W(u) = W(u_1) + W(\beta) + W(u_2) \geqslant W(\beta)$$

So $\beta < \alpha$ and therefore $(\beta, v) < (\alpha, w)$. Using 2) again, we also have:

$$W(w) = W(u_2) + W(v) + W(u_1) < W(u_2) + W(\beta) + W(u_1) = W(u)$$

So $w < u$ and therefore $(\alpha, w) < (\alpha, u)$ ■

Unless otherwise specified, $<$ will always denote an order on the set of paths as in Definition 2.5.

2.3 Image of a path by a product of transvections

In this paragraph, we apply the previous constructions to find an easy way to compute $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path in Q. We begin with the following lemma on the description of $\psi(\alpha)$ when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$. Recall that Q has no multiple arrows and no oriented cycle.

Lemma 2.9. *Let $\psi \in \mathcal{T}$ and let $\alpha \in Q_1$. Then $\psi(\alpha) - \alpha$ is a linear combination of paths parallel to α and of length greater than or equal to 2. In particular, $\alpha \in supp(\psi(\alpha))$ and $\alpha^*(\psi(\alpha)) = 1$.*

Proof: The conclusion is immediate if ψ is a transvection because Q has no multiple arrows. The conclusion in the general case is obtained using an easy induction on the number of transvections whose product equal ψ. ■

The preceding lemma gives the following description of $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path. We omit the proof which is immediate thanks to Lemma 2.9 and to point 6) of Lemma 2.3.

Proposition 2.10. *Let $\psi \in \mathcal{T}$ and let $u = \alpha_n \dots \alpha_1$ be a path in Q (with $a_i \in Q_1$ for any i). For each i, let:*

$$\psi(\alpha_i) = \alpha_i + \sum_{j=1}^{m_i} \lambda_{i,j} u_{i_j}$$

be a normal form for $\psi(\alpha_i)$. Then $supp(\psi(u))$ is the set of the paths in Q described as follows. Let $r \in \{0, \dots, n\}$. Let $1 \leqslant i_1 < \dots < i_r \leqslant n$ be indices. For each $l \in \{1, \dots, r\}$, let $j_l \in \{1, \dots, m_{i_l}\}$. Then the following path obtained from u after replacing α_{i_l} by u_{j_l} for each l belongs to $supp(\psi(u))$:

$$\alpha_n \dots \alpha_{i_r+1} u_{j_r} \alpha_{i_r-1} \dots \alpha_{i_l+1} u_{j_l} \alpha_{i_l-1} \dots \alpha_{i_1+1} u_{j_1} \alpha_{i_1-1} \dots \alpha_1$$

Moreover, this path appears in $\psi(u)$ with coefficient:

$$\lambda_{i_1,j_1} \dots \lambda_{i_r,j_r}$$

As a consequence, $\psi(u) - u$ is a linear combination of paths derived of u.

Example 2.11. *The previous proposition does not hold if Q has multiple arrows. For example, if Q is the Kronecker quiver* $\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}$ *and if $\psi = \varphi_{a,b,1}\varphi_{b,a,-1}\varphi_{a,b,1}$, then $\psi(a) = b$ and $\psi(b) = -a$.*

Remark 2.12. *If (α, u) is a bypass and if $v \in supp(\psi(u) - u)$, then (α, v) is also a bypass and $(\alpha, v) < (\alpha, u)$.*

Now we are able to state the main result of this paragraph. It describes $\psi(\alpha)$ ($\alpha \in Q_1$) using a particular writing of ψ as a product of transvections. Notice that the following proposition formalises the phenomenon observed at the begining of the section.

Proposition 2.13. *Let $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \dots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n,u_n,\tau_n} \dots \varphi_{\alpha_1,u_1,\tau_1}$. For any $\alpha \in Q_1$, there is a normal form for $\psi(\alpha)$:*

$$\psi(\alpha) = \alpha + \sum_{i \text{ such that } \alpha = \alpha_i} \tau_i u_i$$

Proof: Let us prove that the conclusion of the proposition is true using an induction on $n \geqslant 1$. By definition of a transvection, the proposition holds of $n = 1$. Assume that $n \geqslant 2$ and that the conclusion of the proposition holds if we replace $\psi = \varphi_{\alpha_n,u_n,\tau_n} \dots \varphi_{\alpha_1,u_1,\tau_1}$ by $\varphi_{\alpha_{n-1},u_{n-1},\tau_{n-1}} \dots \varphi_{\alpha_1,u_1,\tau_1}$. Therefore, for $\alpha \in Q_1$, we have a normal form:

$$\varphi_{\alpha_{n-1},u_{n-1},\tau_{n-1}} \dots \varphi_{\alpha_1,u_1,\tau_1}(\alpha) = \alpha + \sum_{i \leqslant n-1,\ \alpha = \alpha_i} \tau_i u_i$$

So:

$$\psi(\alpha) = \varphi_{\alpha_n,u_n,\tau_n}(\alpha) + \sum_{i \leqslant n-1,\ \alpha = \alpha_i} \tau_i \varphi_{\alpha_n,u_n,\tau_n}(u_i) \qquad (i)$$

Let $i \in \{1, \dots, n-1\}$. Thanks to Lemma 2.8, the inequality $(\alpha_i, u_i) < (\alpha_n, u_n)$ implies that $(\alpha_i, u_i, \alpha_n, u_n)$ is not a double bypass. Thus, α_n does not appear in the path u_i. This proves that:

$$(\forall i \in \{1, \dots, n-1\}) \ \ \varphi_{\alpha_n,u_n,\tau_n}(u_i) = u_i \qquad (ii)$$

The definition of $\varphi_{\alpha_n,u_n,\tau_n}$, together with (i) and (ii), imply the equality:

$$\psi(\alpha) = \alpha + \sum_{\alpha = \alpha_i} \tau_i u_i \qquad (iii)$$

It only remains to prove that the equality *(iii)* is a normal form. Remark that all the scalars which appear in the right-hand side of *(iii)* are non zero. Moreover, if $i \in \{1, \ldots, n\}$ verifies $\alpha = \alpha_i$, then $\alpha \neq u_i$, because (α, u_i) is a bypass. Finally, if $1 \leqslant i < j \leqslant n$ verify $\alpha = \alpha_i = \alpha_j$, then $(\alpha, u_i) = (\alpha_i, u_i) < (\alpha_j, u_j) = (\alpha, u_j)$ so $u_i \neq u_j$. Therefore, *(iii)* is a normal form for $\psi(\alpha)$. ■

When $\psi \in \mathcal{T}$ is like in Proposition 2.13, we shall say that ψ is written as a decreasing product of transvections. Later we will prove that any $\psi \in \mathcal{T}$ can be written uniquely as a decreasing product of transvections. The description in Proposition 2.13 will be particularly useful in the sequel. We end this paragraph with two propositions concerning the description of $\psi(r)$ when $\psi \in \mathcal{T}$ and r is a linear combination of paths. The following proposition gives conditions for $\psi^{-1}(r')$ to be a subexpression of r when r' is a subexpression of $\psi(r)$.

Proposition 2.14. *Let $\psi \in \mathcal{T}$, let $r \in {}_y kQ_x$ and let r' be a subexpression of $\psi(r)$. Let $\simeq$ be the equivalence relation on the set of paths in Q generated by:*

$$v \in supp(\psi(u)) \Rightarrow u \simeq v$$

Assume that for any $u, v \in supp(\psi(r))$ verifying $u \simeq v$ we have:

$$u \in supp(r') \Leftrightarrow v \in supp(r')$$

Then $\psi^{-1}(r')$ is a subexpression of r.

Proof: Let $\simeq'$ be the trace of $\simeq$ on $supp(r)$ and let us write $supp(r) = c_1 \sqcup \ldots \sqcup c_n$ as a disjoint union of its $\simeq'$-classes. This partition of $supp(r)$ defines a decomposition of $r = r_1 + \ldots + r_n$ where r_i is the subexpression of r verifying $supp(r_i) = c_i$. For each i, let us fix a normal form:

$$r_i = \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

so that we have the following normal form for r:

$$r = \sum_{i=1}^{n} \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

Let us set $r'_i := \psi(r_i)$. In order to prove that $\psi^{-1}(r')$ is a subexpression of r, we will prove that there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ verifying $r' = r'_{i_1} + \ldots + r'_{i_t}$ (so that $\psi^{-1}(r') = r'_{i_1} + \ldots + r'_{i_t}$). In this purpose, we will successively prove the following facts:

1) $u, v \in supp(r'_i) \Rightarrow u \simeq v$, for any i,
2) $supp(r'_1), \ldots, supp(r'_n)$ are pairwise disjoint,
3) for each i, r'_i is a subexpression of $\psi(r)$,
4) if $i \in \{1, \ldots, n\}$ verifies $supp(r') \cap supp(r'_i) \neq \emptyset$, then $supp(r'_i) \subseteq supp(r')$,

1) Let $i \in \{1, \ldots, n\}$ and let $u, v \in supp(r'_i)$. So there exist $u', v' \in supp(r_i)$ such that $u \in supp(\psi(u'))$ and $v \in supp(\psi(v'))$. By definition of $\simeq$ and of r_i, we deduce that:

$$u, v \in supp(r'_i) \Rightarrow u \simeq v \tag{i}$$

2) Let $i, j \in \{1, \ldots, n\}$ be such that there exists $v \in supp(r'_i) \cap supp(r'_j)$. So there exist $u \in supp(r_i)$ and $u' \in supp(r_j)$ such that $v \in supp(\psi(u))$ and $v \in supp(\psi(u'))$. This implies that $u \simeq v \simeq u'$. Since $u \in c_i = supp(r_i)$ and $u' \in c_j = supp(r_j)$, we deduce that $c_i = c_j$ and therefore $i = j$. So:

$$i \neq j \Rightarrow supp(r'_i) \cap supp(r'_j) = \emptyset \tag{ii}$$

3) We have $\psi(r) = r'_1 + \ldots + r'_n$ so *(ii)* implies that:

$$r'_i \text{ is a subexpression of } \psi(r) \text{ for any } i \tag{iii}$$

4) Let $i \in \{1, \ldots, n\}$ and assume that there exists $u \in supp(r'_i) \cap supp(r')$. If $v \in supp(r'_i)$ then $u \simeq v$ thanks to *(i)*. So, by assumption on r', we have $v \in supp(r')$. This proves that:

$$supp(r'_i) \cap supp(r') \neq \emptyset \Rightarrow supp(r'_i) \subseteq supp(r') \tag{iv}$$

Now, we can prove that $\psi^{-1}(r')$ is a subexpression of r. Thanks to *(iii)*, the elements $r', r'_1, \ldots, r'_n$ are subexpressions of $\psi(r)$. So *(iv)* and the equality $\psi(r) = r'_1 + \ldots + r'_n$ imply that there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ such that $r' = r'_{i_1} + \ldots + r'_{i_t}$. So $\psi^{-1}(r') = r_{i_1} + \ldots + r_{i_n}$. This proves that $\psi^{-1}(r')$ is a subexpression of r. ■

The last proposition of this subsection gives a sufficient condition on $u \in supp(r)$ to verify $u \in supp(\psi(r))$.

Proposition 2.15. *Let $\psi \in \mathcal{T}$, let $r \in {}_ykQ_x$ and let $u \in supp(r)$. Then, at least one of the two following facts is verified:*

1. *$u \in supp(\psi(r))$,*
2. *there exists $v \in supp(r)$ such that $u \neq v$ and such that $u \in supp(\psi(v))$.*

As a consequence, if u is not derived of v for any $v \in supp(r)$, then:

$$u \in supp(\psi(r)) \ \ and \ \ u^*(\psi(r)) = u^*(r)$$

Proof: Let us fix a normal form $r = \sum\limits_{i=1}^{n} t_i u_i$ where we may assume that $u = u_1$. Let us assume that $u \notin supp(\psi(r))$, i.e. $u^*(\psi(r)) = 0$. Recall from Proposition 2.10 that $u^*(\psi(u)) = 1$, so:

$$0 = u^*(\psi(r)) = t_1 + \sum_{i=2}^{n} t_i u^*(\psi(u_i)) \tag{i}$$

Therefore, there exists $i_0 \in \{2, \ldots, n\}$ such that $u^*(\psi(u_{i_0})) \neq 0$. So:

$$u_{i_0} \in supp(r), \ \ u_{i_0} \neq u_1 = u \ \text{ and } \ u_1^*(\psi(u_{i_0})) \neq 0$$

This proves the first assertion of the proposition. Now let us assume that u is not derived of v for any $v \in supp(r)$. Let $i \in \{2, \ldots, n\}$. Since $u = u_1 \neq u_i$, Proposition 2.10 gives the following implications:

$$u \in supp(\psi(u_i)) \Rightarrow u \in supp(\psi(u_i) - u_i) \Rightarrow u \text{ is derived of } u_i$$

By assumption on u, this implies that $u^*(\psi(u_i)) = 0$ for any $i \geqslant 2$. Using (i), we deduce the announced conclusion: $u^*(\psi(r)) = t_1 = u^*(r) \neq 0$ ■

2.4 Ordering products of transvections

In Proposition 2.13 we have seen that $\psi(\alpha)$ may be computed easily when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$ provided that ψ is written as a decreasing product of transvections. The main result of this subsection proves that any $\psi \in \mathcal{T}$ can be uniquely written that way. Recall that $<$ is an order on the set of non trivial paths in Q defined in Definition 2.5. The following notations will be useful.

Definition 2.16. *Let (α, u) be a bypass. We set $\mathcal{T}_{<(\alpha,u)}$ and $\mathcal{T}_{\leqslant(\alpha,u)}$ to be the subgroups of $\mathcal{T}$ generated by the following sets of transvections:*

$$\begin{array}{l} \{\varphi_{\beta,v,\tau} \mid (\beta, v) < (\alpha, u) \ and\ \tau \in k\} \ \ for\ \mathcal{T}_{<(\alpha,u)} \\ \{\varphi_{\beta,v,\tau} \mid (\beta, v) \leqslant (\alpha, u) \ and\ \tau \in k\} \ \ for\ \mathcal{T}_{\leqslant(\alpha,u)} \end{array}$$

Also, we define $\mathcal{T}_{(\alpha,u)}$ to be the following subgroup of $\mathcal{T}$:

$$\mathcal{T}_{(\alpha,u)} = \{\varphi_{\alpha,u,\tau} \mid \tau \in k\}$$

Remark 2.17.

- *$\mathcal{T}_{(\alpha,u)}$ is indeed a subgroup of $\mathcal{T}$ because $\varphi_{\alpha,u,\tau}\varphi_{\alpha,u,\tau'} = \varphi_{\alpha,u,\tau+\tau'}$ for any $\tau, \tau' \in k$. Actually, the following mapping is an isomorphism of abelian groups:*

$$\begin{array}{ccc} k & \longrightarrow & \mathcal{T}_{(\alpha,u)} \\ \tau & \longmapsto & \varphi_{\alpha,u,\tau} \end{array}$$

- *$\mathcal{T}_{\leqslant(\alpha,u)}$ is generated by $\mathcal{T}_{<(\alpha,u)} \cup \mathcal{T}_{(\alpha,u)}$.*
- *If $(\alpha, u) < (\beta, v)$, then $\mathcal{T}_{\leqslant(\alpha,u)} \subseteq \mathcal{T}_{\leqslant(\beta,v)}$ and $T_{<(\alpha,u)} \subseteq T_{<(\beta,v)}$.*
- *$\mathcal{T} = \bigcup\limits_{(\alpha,u)} \mathcal{T}_{\leqslant(\alpha,u)}$ and if (α_m, u_m) is the greatest bypass in Q, then $\mathcal{T} = \mathcal{T}_{\leqslant(\alpha_m,u_m)}$ (recall that Q has finitely many bypasses because it has no oriented cycle).*

The following lemma proves that any $\psi \in \mathcal{T}$ is a decreasing product of transvections.

Lemma 2.18.

- *$\mathcal{T}_{<(\alpha,u)}$ is a normal subgroup of $\mathcal{T}_{\leqslant(\alpha,u)}$, for any bypass (α, u).*
- *Let $(a_1, v_1) < \ldots < (a_N, v_N)$ be the (finite) increasing sequence of all the bypasses in Q. Then:*
  - *$\mathcal{T}_{<(a_i,v_i)} = \mathcal{T}_{\leqslant(a_{i-1},v_{i-1})}$ if $i \geqslant 1$,*
  - *$\mathcal{T}_{<(a_1,v_1)} = 1$,*
  - *$\mathcal{T}_{\leqslant(a_i,v_i)} = \mathcal{T}_{(a_i,v_i)}\mathcal{T}_{(a_{i-1},v_{i-1})} \cdots \mathcal{T}_{(a_1,v_1)}$.*

Proof: Thanks to Remark 2.17, it is sufficient to prove that if $\tau, \nu \in k$ and if $(\beta, v), (\alpha, u)$ are bypasses such that $(\beta, v) < (\alpha, u)$, then:

$$\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} \in \varphi_{\alpha,u,\tau}\mathcal{T}_{<(\alpha,u)} \tag{$\star$}$$

There are two situations wether (α, u, β, v) is a double bypass or not. If (α, u, β, v) is a double bypass, then Section 1 gives:

$$\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} = \varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$$

where w is the path obtained from u after replacing β by v. Moreover, Lemma 2.8 implies that $(\beta, v) < (\alpha, w) < (\alpha, u)$. Therefore, $(\star)$ is satisfied when (α, u, β, v) is a double bypass. If (α, u, β, v) is not a double bypass, then Section 1 gives (notice that thanks to Lemma 2.8 and to the inequality $(\beta, v) < (\alpha, u)$ we know that (β, v, α, u) is not a double bypass):

$$\varphi_{\alpha,u,\tau}\varphi_{\beta,v,\nu} = \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}$$

So $(\star)$ is also satisfied when (α, u, β, v) is not a double bypass. ■

Using the preceding lemma and Proposition 2.13, it is now possible to prove that any $\psi \in \mathcal{T}$ is uniquely a decreasing product of transvections.

Proposition 2.19. *Let (α, u) be a bypass and let $\psi \in \mathcal{T}_{\leqslant(\alpha,u)}$. Then, there exist a non negative integer n, a sequence of bypasses $(\alpha_1, u_1), \ldots, (\alpha_n, u_n)$ and non zero scalars $\tau_1, \ldots, \tau_n \in k^*$ verifying:*

(i) $\psi = \varphi_{\alpha_n,u_n,\tau_n} \cdots \varphi_{\alpha_1,u_1,\tau_1}$,

(ii) $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n) \leqslant (\alpha, u)$.

Moreover, the integer n and the sequence $(\alpha_1, u_1, \tau_1), \ldots, (\alpha_n, u_n, \tau_n)$ are unique for these properties.

Proof: The existence is given by Lemma 2.18. So it suffices to characterise the triples (α_i, u_i, τ_i) using ψ only. Let A, B and T be the following sets:

$$\begin{aligned}
&A := \{\alpha \in Q_1 \mid \psi(\alpha) \neq \alpha\}\\
&B := \{(\alpha, u) \mid (\alpha, u) \text{ is a bypass, } \alpha \in A \text{ and } u \in \mathit{supp}(\psi(\alpha))\}\\
&T := \{(\alpha, u, \tau) \mid (\alpha, u) \in B \text{ and } \tau = u^*(\psi(\alpha))\}
\end{aligned}$$

Notice that the definition of A, B, T depend on ψ only (and not on the triples (α_i, u_i, τ_i)). Let $\beta \in Q_1$. Then Proposition 2.13 gives a normal form:

$$\psi(\beta) = \beta + \sum_{i \text{ such that } \beta = \alpha_i} \tau_i u_i$$

By definition of a normal form and because of (i) and (ii), the following equalities hold:

$$\begin{aligned}
&A = \{\alpha_1, \ldots, \alpha_n\}\\
&B = \{(\alpha_1, u_1), \ldots, (\alpha_n, u_n)\}\\
&T = \{(\alpha_1, u_1, \tau_1), \ldots, (\alpha_n, u_n, \tau_1)\}
\end{aligned}$$

This proves that n and $(\alpha_1, u_1, \tau_1), \ldots, (\alpha_n, u_n, \tau_n)$ are uniquely determined by the sets A, B, T (which depend on ψ only) and by the total order $<$. ■

3 Comparison of the presentations of a monomial algebra

Let $A = kQ/I_0$ with I_0 a monomial admissible ideal of kQ and let $kQ/I \simeq A$ be an admissible presentation of A. Thanks to Proposition 1.1, there exists ψ a product of transvections and of a dilatation such that $\psi(I_0) = I$. The aim of this section is to exhibit ψ_I the "simplest" possible among all the ψ's verifying $\psi(I_0) = I$. It will appear that ψ_I verifies a property which makes it unique. The construction of ψ_I will use specific properties of the Gröbner basis of I, due to the fact that I_0 is monomial. So, throughout the section, $<$ will denote a total order on the set of non trivial paths in Q, as in Definition 2.5. Before studying the Gröbner basis of I, it is useful to give some properties on the automorphisms $\psi \in AUt_0(kQ)$ verifying $\psi(I_0) = I_0$.

Lemma 3.1. *Let $D \in \mathcal{D}$ be a dilatation. Then $D(I_0) = I_0$. As a consequence, if $kQ/I \simeq A$ is an admissible presentation, then there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$.*

Proof: The first assertion is due to the fact that $D(u) \in k^*u$ for any path u and to the fact that I_0 is monomial. The second one is a consequence of the first one and of Proposition 1.1. ■

Lemma 3.2. *Let (α, u) be a bypass in Q. Then exactly one of the two following assertions is satisfied:*

- . *$\varphi_{\alpha,u,\tau}(I_0) = I_0$ for any $\tau \in k$.*
- . *$\varphi_{\alpha,u,\tau}(I_0) \neq I_0$ for any $\tau \in k^*$.*

Proof: Assume that $\tau \in k^*$ verifies $\varphi_{\alpha,u,\tau}(I_0) = I_0$ and let $\mu \in k$. Let $v \in I_0$ be a path. If α does not appear in v, then $\varphi_{\alpha,u,\nu}(v) = v \in I_0$. Assume that α appears in v, i.e. $v = v_2\alpha v_1$ with v_1, v_2 paths in which α does not appear (because Q has no oriented cycle). Therefore, $\varphi_{\alpha,u,\tau}(v) = v + \tau v_2 u v_1 \in I_0$. Thus, $v_2 u v_1 \in I_0$. This implies that $\varphi_{\alpha,u,\nu}(v) = v + \nu v_2 u v_1 \in I_0$. Since I_0 is monomial, $\varphi_{\alpha,u,\nu}(I_0) = I_0$. ■

Lemma 3.3. *Let $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \ldots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n,u_n,\tau_n} \ldots \varphi_{\alpha_1,u_1,\tau_1}$. Then:*

$$\psi(I_0) = I_0 \quad \Leftrightarrow \quad \varphi_{\alpha_i,u_i,\tau_i}(I_0) = I_0 \text{ for any } i$$

Proof: Assume that $\psi(I_0) = I_0$. Let $i \in \{1, \ldots, n\}$, let $u = a_r \ldots a_1 \in I_0$ be a path (with $a_i \in Q_1$) and fix $i \in \{1, \ldots, n\}$. If $a_j \neq \alpha_i$ for any $j \in \{1, \ldots, r\}$ then $\varphi_{\alpha_i,u_i,\tau_i}(u) = u \in I_0$. Now assume that there exists $j \in \{1, \ldots, r\}$ such that $a_j = \alpha_i$ (j is necessarily unique because Q has no oriented cycle). Therefore:

$$\varphi_{\alpha_i,u_i,\tau_i}(u) = u + \tau_i a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \qquad (i)$$

On the other hand, Proposition 2.10 and Proposition 2.13 imply that $a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \in supp(\psi(u))$. Thus (recall that $\psi(u) \in I_0$ and that I_0 is monomial):

$$a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \in I_0 \qquad (ii)$$

From (i) and (ii) we deduce that $\varphi_{\alpha_i,u_i,\tau_i}(u) \in I_0$ for any path $u \in I_0$. So $\varphi_{\alpha_i,u_i,\tau_i}(I_0) = I_0$ for any i. The remaining implication is immediate. ■

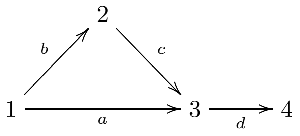
Remark 3.4. *The three preceding lemmas imply that the group $Aut_0(kQ, I_0)$ defined as follows:*

$$Aut_0(kQ, I_0) := \{\psi \in Aut(kQ) \mid \psi(x) = x \text{ for any } x \in Q_0, \text{ and } \psi(I_0) = I_0\}$$

is generated by the dilatations and by all the transvections preserving I_0:

$$Aut_0(kQ, I_0) = < \mathcal{D} \cup \{\varphi \mid \varphi \text{ is a transvection such that } \varphi(I_0) = I_0\} >$$

Example 3.5. *The preceding remark does not hold for any ideal I, even if kQ/I is monomial. For example, let Q be the quiver:*



and let $I = < da - dcb >$. Notice that $kQ/I \simeq kQ/I_0$ where $I_0 = \varphi_{a,cb,1}(I) = < da >$. On the other hand:

1. *$Id = \varphi_{a,cb,0}$ is the only transvection lying in $Aut_0(kQ, I)$,*
2. *for $t \in k\backslash\{0, 1\}$, the dilatation D_t such that $D_t(a) = ta$ and $D_t(x) = x$ for any other arrow x does not belong to $Aut_0(kQ, I)$,*
3. *$D_t\varphi_{a,cb,t} \in Aut_0(kQ, I)$ for any $t \in k^*$.*

So $Aut_0(kQ, I)$ is not generated by $\mathcal{D}$ and by the transvections it contains.

The following proposition gives the announced properties on the Gröbner bases of the admissible ideals I of kQ such that $kQ/I \simeq A$. Recall that for such an I, there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$ (see Lemma 3.1).

Proposition 3.6. *Let $\psi \in \mathcal{T}$ and let let $I = \psi(I_0)$. Let B_0 (resp. B) be the Groebner basis of I_0 (resp. of I). Then B_0 is made of all the paths in Q which belong to I_0. Moreover, the mapping:*

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ r & \longmapsto & max(supp(r)) \end{array} \qquad (\star)$$

is well defined and bijective. For $u \in B_0$, let $r_u \in B$ be the inverse image of u under $(\star)$. Then $supp(r_u - u)$ is a set of paths derived of u.

Proof: Let $u_1 < \ldots < u_n$ be the increasing sequence of all the non trivial paths in Q. Let $(r_1, \ldots, r_d)$ be the Gröbner basis of I and for each $j \in \{1, \ldots, d\}$, let $i_j \in \{1, \ldots, n\}$ be such that:

$$r_j \in u_{i_j} + Span(u_l \ ; \ l < i_j)$$

Since I_0 is monomial, B_0 is made of all the paths in Q belonging to I_0.

Let $j \in \{1, \ldots, d\}$. Since $u_{i_j} = max(supp(r_j))$, the path u_{i_j} is not derived of u for any $u \in supp(r_j)$ (thanks to Lemma 2.8). So Proposition 2.15 implies that $u_{i_j} \in supp(\psi^{-1}(r_j)) \in I_0$. Because I_0 is monomial, this proves that $u_{i_j} \in I_0$. Therefore, the mapping $(\star)$ is well defined. It is also one-to-one because of the definition of the Groebner basis of I. Let $u \in B_0$. Proposition 2.10 implies that $u = max(supp(\psi(u))$. Since $\psi(u) \in I$, there exists $j \in \{1, \ldots, d\}$ such that $u = u_{i_j} = max(supp(r_j))$. This proves that $(\star)$ is onto and therefore bijective.

It remains to prove the last assertion of the proposition. This will be done by proving by induction on $j \in \{1, \ldots, d\}$ that the following assertion is true:

$$H_j : \text{"}supp(r_j - u_{i_j}) \text{ is a set of paths derived of } u_{i_j}\text{"}$$

Remark that Proposition 2.10 implies that for any j:

$$u_{i_j} = max(supp(\psi(u_{i_j}))) \ \text{ and } u^*_{i_j}(\psi(u_{i_j})) = 1 \tag{i}$$

Moreover, $\psi(u_{i_j}) \in I$ because $(\star)$ is well defined and because $\psi(I_0) = I$. Now begins the induction. Both r_1 and $\psi(u_{i_1})$ lie in I. Moreover, $u_{i_1} = max(supp(r_1))$ by definition of u_{i_1} and $u_{i_1} = max(supp(\psi(u_{i_1})))$ because of Proposition 2.10. So H_1 is true. Assume that $j \geqslant 2$ and that $H_1, \ldots, H_{j-1}$ are true. Since $\psi(u_{i_j}) \in I$ and because of (i), the following holds:

$$\psi(u_{i_j}) = r_j + \sum_{\substack{j' < j, \\ u_{i_{j'}} \in supp(\psi(u_{i_j}))}} u^*_{i_{j'}}(\psi(u_{i_j}))r_{j'}$$

So:

$$r_j - u_{i_j} = \psi(u_{i_j}) - u_{i_j} - \sum_{\substack{j' < j, \\ u_{i_{j'}} \in supp(\psi(u_{i_j}))}} u^*_{i_{j'}}(\psi(u_{i_j})) \left[(r_{j'} - u_{i_{j'}}) + u_{i'_j}\right] \tag{ii}$$

Notice that in the above equality:

- (iii) $supp(\psi(u_{i_j}) - u_{i_j})$ is a set of paths derived of u_{i_j} (thanks to Proposition 2.10),
- (iv) if $j' < j$ verifies $u_{i_{j'}} \in supp(\psi(u_{i_j}))$, then:
  - (v) $u_{i_{j'}}$ is derived of u_{i_j} (see (iii) above),
  - (vi) $supp(r_{j'} - u_{i_{j'}})$ is a set of paths derived of $u_{i_{j'}}$ (because $H_{j'}$ is true) and therefore derived of u_{i_j} (thanks to (v) and to Lemma 2.3).

The points $(ii) - (vi)$ prove that H_j is true. Hence, H_j is true for any $j \in \{1, \ldots, d\}$. This finishes the proof of the proposition. ■

Now it is possible to define precisely the automorphism ψ_I mentionned at the beginning of the section.

Proposition 3.7. *Let $kQ/I \simeq A$ be an admissible presentation. Then there exists a unique $\psi_I \in \mathcal{T}$ verifying the following conditions:*

1) $\psi_I(I_0) = I$,
2) *if (α, u) is a bypass such that $u \in supp(\psi_I(\alpha))$ then $\varphi_{\alpha,u,\tau}(I_0) \neq I_0$ for any $\tau \in k^*$ (see Lemma 3.2).*

Proof: • First, the existence of ψ_I. Thanks to Lemma 3.1, there exists $\psi \in \mathcal{T}$ verifying 1). Set:

$$\mathcal{A} := \{\psi \in \mathcal{T} \mid \psi(I_0) = I\}$$

and assume that for any $\psi \in \mathcal{A}$, the condition 2) is not verified. So, for any $\psi \in \mathcal{A}$, there is a finite (recall that Q has no oriented cycle) and non empty set of bypasses (see Lemma 3.2):

$$B_\psi = \left\{ (\alpha, u) \;\middle|\; \begin{array}{l} (\alpha, u) \text{ is a bypass} \\ u \in supp(\psi(\alpha)) \\ \varphi_{\alpha,u,\tau}(I_0) = I_0 \text{ for any } \tau \in k \end{array} \right\}$$

For each $\psi \in \mathcal{A}$, let $(\alpha_\psi, u_\psi) = max\ B_\psi$ and let $\psi \in A$ be such that:

$$(\alpha_\psi, u_\psi) = min\ \{(\alpha_{\psi'}, u_{\psi'}) \mid \psi' \in \mathcal{A}\}$$

For simplicity, set $(\alpha, u) := (\alpha_\psi, u_\psi)$, $\tau := u^*(\psi(\alpha))$ $\psi' := \psi\varphi_{\alpha,u,-\tau}$. Notice that $\psi' \in \mathcal{A}$ because $(\alpha, u) \in B_\psi$. In order to get a contradiction, let us prove that $(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u)$. To do this, let us prove first that $(\alpha, u) \notin B_{\psi'}$. Thanks to Proposition 2.19, the following equality holds:

$$\psi = \varphi_{\alpha_n,u_n,\tau_n} \cdots \varphi_{\alpha_1,u_1,\tau_1}$$

where $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ and where $\tau_1, \ldots, \tau_n \in k^*$. On the other hand, since $u^*(\psi(\alpha)) = \tau \neq 0$, Proposition 2.13 gives:

$$(\exists! i \in \{1, \ldots, n\}) \;\; (\alpha_i, u_i, \tau_i) = (\alpha, u, \tau)$$

Let us set:

$$\psi_1 := \varphi_{\alpha_{i-1},u_{i-1},\tau_{i-1}} \cdots \varphi_{\alpha_1,u_1,\tau_1} \in \mathcal{T}_{<(\alpha,u)}$$

Hence, the following equality holds:

$$\psi' = \varphi_{\alpha_n,u_n,\tau_n} \cdots \varphi_{\alpha_{i+1},u_{i+1},\tau_{i+1}} \varphi_{\alpha,u,\tau} \psi_1 \varphi_{\alpha,u,\tau}^{-1}$$

Since $\psi_1 \in \mathcal{T}_{<(\alpha,u)}$, Lemma 2.18 implies that $\varphi_{\alpha,u,\tau}\psi_1\varphi_{\alpha,u,\tau}^{-1} \in \mathcal{T}_{<(\alpha,u)}$. Therefore, Proposition 2.19 gives the equality:

$$\varphi_{\alpha,u,\tau}\psi_1\varphi_{\alpha,u,\tau}^{-1} = \varphi_{\beta_m,v_m,\nu_m} \cdots \varphi_{\beta_1,v_1,\nu_1}$$

where $(\beta_1, v_1) < \ldots < (\beta_m, v_m) < (\alpha, u)$ and $\nu_1, \ldots, \nu_m \in k^*$. As a consequence:

$$\psi' = \varphi_{\alpha_n,u_n,\tau_n} \cdots \varphi_{\alpha_{i+1},u_{i+1},\tau_{i+1}} \varphi_{\beta_m,v_m,\nu_m} \cdots \varphi_{\beta_1,v_1,\nu_1}$$

where $(\beta_1, v_1) < \ldots < (\beta_m, v_m) < (\alpha, u) < (\alpha_{i+1}, u_{i+1}) < \ldots < (\alpha_n, u_n)$ and where $\tau_{i+1}, \ldots, \tau_n, \nu_1, \ldots, \nu_m \in k^*$. In particular, Proposition 2.13 implies that $u \notin supp(\psi'(\alpha))$. Therefore, $(\alpha, u) \notin B_{\psi'}$ and in particular, $(\alpha, u) \neq (\alpha_{\psi'}, u_{\psi'}) = max\ B_{\psi'}$. Thus, in order to prove that $(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u)$, it suffices to pove that the following implication holds for any bypass (β, v):

$$v \in supp(\psi'(\beta)) \text{ and } (\alpha, u) < (\beta, v) \;\Rightarrow \varphi_{\beta,v,t}(I_0) \neq I_0 \text{ for any } \tau \in k^* \tag{i}$$

Let (β, v) be a bypass such that $v \in supp(\psi'(\beta))$ and such that $(\alpha, u) < (\beta, v)$. Since $\psi' = \psi\varphi_{\alpha,u,-\tau}$, the following holds:

$$\psi'(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \neq \alpha \\ \psi(\beta) - \tau\psi(u) & \text{if } \beta = \alpha \end{cases}$$

Therefore, $v \in supp(\psi'(\beta)) \subseteq supp(\psi(\beta)) \cup supp(\psi(u))$. Remark that if $v \in supp(\psi(u)) \backslash supp(\psi(\beta))$, then $\alpha = \beta$ and Proposition 2.10 implies that v is derived of u (we have $u \neq v$ because $\beta = \alpha$ and $(\alpha, u) < (\beta, v)$) and therefore $(\alpha, u) > (\alpha, v) = (\beta, v)$ whereas we assumed that $(\alpha, u) < (\beta, v)$. This proves that $v \in supp(\psi(\beta))$. Since $(\beta, v) > (\alpha, u) = (\alpha_\psi, u_\psi) = max\ B_\psi$ we deduce that $\varphi_{\beta,v,\tau}(I_0) = I_0$ for any $\tau \in k$. This proves that the implication (i) is satisfied. Thus:

$$(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u) = (\alpha_\psi, u_\psi)$$

This contradicts the minimality of (α_ψ, u_ψ) and proves the existence of ψ.

- It remains to prove the uniqueness of ψ_I. Assume that $\psi, \psi' \in \mathcal{T}$ verify the conditions 1) and 2). In order to prove that $\psi = \psi'$, it is sufficient to prove that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$ for any bypass (α, θ). Let $\alpha \in Q_1$ and assume that there exists a minimal path θ such that (α, θ) is bypass and such that $\theta^*(\psi(\alpha)) \neq \theta^*(\psi'(\alpha))$. We may assume that $\theta^*(\psi(\alpha)) \neq 0$, i.e. $\theta \in supp(\psi(\alpha))$. Since ψ verifies 2), we deduce that there exist paths u and v such that:

$$u \in I_0,\; v \notin I_0 \text{ and } \varphi_{\alpha,\theta,1}(u) = u + v \notin I_0$$

Notice that Proposition 2.10 gives:

$$\begin{cases} v^*(\psi(u)) = \theta^*(\psi(\alpha)) \text{ and } u^*(\psi(u)) = 1 \\ v^*(\psi'(u)) = \theta^*(\psi'(\alpha)) \text{ and } u^*(\psi'(u)) = 1 \end{cases} \tag{ii}$$

Moreover, $\psi(u), \psi'(u) \in I_0$ because $u \in I$. Therefore, Proposition 3.6 gives, the same notations concerning the Groebner bases, we have:

$$\begin{cases} \psi(u) = r_u + \sum\limits_{w \in \mathcal{A}_\psi} w^*(\psi(u)) r_w \\ \psi'(u) = r_u + \sum\limits_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u)) r_w \end{cases}$$

where $\mathcal{A}_\psi$ is equal to:

$$\mathcal{A}_\psi := \{w \in supp(\psi(u)) \mid w \neq u \text{ and } w \in I_0\}$$

So:

$$\begin{cases} v^*(\psi(u)) = v^*(r_u) + \sum\limits_{w \in \mathcal{A}_\psi} w^*(\psi(u)) v^*(r_w) \\ v^*(\psi'(u)) = v^*(r_u) + \sum\limits_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u)) v^*(r_w) \end{cases} \qquad (iii)$$

Let $w \in \mathcal{A}_\psi$ be such that $v^*(r_w) \neq 0$, i.e. $v \in supp(r_w)$. Remark that $v \in supp(r_w - w)$ because $v \notin I_0$ and $w \in I_0$. So:

. v is derived of w (thanks to Proposition 3.6 and because $v \in supp(r_w - w)$).

. v is derived of u of order 1 (because $\varphi_{\alpha,\theta,1}(u) = u + v$).

. w is derived of u (because $w \in \mathcal{A}_\psi$ and thanks to Proposition 2.10).

Using Lemma 2.3, these three facts imply that:

$$u = u_2 \alpha u_1,\ v = u_2 \theta u_1 \text{ and } w = u_2 \theta' u_1 \qquad (iv)$$

where u_1, u_2 are paths and where θ' is a path derived of θ. In particular, (α, θ') is a bypass such that $\theta' < \theta$ (see Lemma 2.8). Therefore, the minimality of θ forces $\theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha)))$. Moreover, (iv) and Proposition 2.10 imply that

$$w^*(\psi(u)) = \theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha)) = w^*(\psi'(u))$$

Therefore we have proved the following implication:

$$w \in \mathcal{A}_\psi \text{ and } v^*(r_w) \neq 0 \ \Rightarrow\ w^*(\psi(u)) v^*(r_w) = w^*(\psi'(u)) v^*(r_w) \qquad (v)$$

After exchangeing the roles of ψ and ψ', the arguments used to prove (v) also give the following implication:

$$w \in \mathcal{A}_{\psi'} \text{ and } v^*(r_w) \neq 0 \ \Rightarrow\ w^*(\psi(u)) v^*(r_w) = w^*(\psi'(u)) v^*(r_w) \qquad (vi)$$

Then, (iii), (v) and (vi) give $v^*(\psi(u)) = v^*(\psi'(u))$. This and (ii) imply that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$, a contradiction. This proves that $\psi = \psi'$. ■

4 Proof of the main theorem

Let $A = kQ/I_0$ where I_0 is a monomial admissible ideal of kQ. The aim of this section is to prove that the quiver Γ of the homotopy relations of the admissible presentations of A admits $\sim_{I_0}$ as unique source. This fact will be used in order to the existence of the universal cover of A. Notice that $\sim_{I_0}$ is a source of Γ. Indeed, all minimal relations in I_0 are monomial relations so, for any $\sim_I \in \Gamma_0$ we have $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$. In order to prove that $\sim_{I_0}$ is the unique source in Γ it will be proved that for any admissible presentation $kQ/I \simeq A$, the decomposition of ψ_I (given by Proposition 3.7) into a decreasing product of transvections (see Proposition 2.19) defines a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_I$. In this purpose, the following proposition will be useful.

Proposition 4.1. *Let $kQ/I \simeq A$ be an admissible presentation. Then, for any bypass (α, u):*

$$u \in supp(\psi_I(\alpha)) \Rightarrow u \sim_I \alpha$$

Proof: For simplicity, set $\psi := \psi_I$. Thanks to Proposition 2.19 there is an equality:

$$\psi = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$$

with $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ and $\tau_1, \dots, \tau_n \in k^*$. Thanks Proposition 2.13 it suffices to prove that $\alpha_i \sim_I u_i$ for any i. This will be done using a decreasing induction on $m \in \{1, \dots, n\}$. Let H_m be the assertion:

$$H_m : \text{"}\alpha_i \sim_I u_i \text{ for any } i \in \{m, m+1, \dots, n\}\text{"}$$

H_{n+1} is true because $\{i \mid n+1 \leqslant i \leqslant n\}$ is empty. So assume that H_{m+1} is true ($m \in \{1, \dots, n\}$). In order to prove that H_m is true, it thus suffices to prove that $\alpha_m \sim_I u_m$. From Proposition 2.13, the path u_m lies in $supp(\psi(\alpha_m))$. Hence, Proposition 3.7 provides a path $u \in I_0$ such that $\varphi_{\alpha_m, u_m, 1}(u) \notin I_0$. Therefore, there exist paths v_1, v_2 such that:

$$u = v_2 \alpha_m v_1,\quad v := v_2 u_m v_1 \notin I_0 \text{ and } \varphi_{\alpha_m, u_m, 1}(u) = u + v \qquad (i)$$

Since $\psi(u) \in I$, there exists a decomposition:

$$\psi(u) = r_1 + \dots + r_N$$

where $r_1, \ldots, r_N$ are minimal relations in I with pairwise disjoint supports. Remark that $u, v \in supp(\psi(u))$ thanks to Proposition 2.10 and to Proposition 2.13. Without loss of generality, it may be assumed that $v \in supp(r_1)$. Let $i \in \{1, \ldots, N\}$ be such that $u \in supp(r_i)$. If $i = 1$ then $u \sim_I v$ and (i) gives $\alpha_m \sim_I u_m$. So assume that $i \neq 1$. Remark that $\psi^{-1}(r_1) \in I_0$ because $r_1 \in I$. Since I_0 is monomial, this also implies that $v \notin supp(\psi^{-1}(r_1))$. And thanks to Proposition 2.15, this proves that:

$$\text{there exists } w \in supp(r_1) \text{ such that } v \text{ is derived from } w \qquad (ii)$$

Therefore:

. w is derived of u since $w \in supp(r_1) \subseteq supp(\psi(u))$ (see Proposition 2.10, notice that $u \neq w$ because $u \notin supp(r_1)$),

. v is derived of w (see (ii)),

. v is derived of u of order 1 (because of (i)).

Thanks to Lemma 2.3, these three points imply that:

$$w = v_2 \theta v_1 \text{ and } u_m \text{ is derived of } \theta \qquad (iii)$$

Since $w \in supp(\psi(u))$, the equalities $w = v_2\theta v_1$, $u = v_2\alpha_m v_1$ and Proposition 2.10 imply that $\theta \in supp(\psi(\alpha_m))$. Hence, there exists $j \in \{1, \ldots, n\}$ such that:

$$(\alpha_m, \theta) = (\alpha_j, u_j)$$

Since u_m is derived of θ (see (iii)), this last equality gives $u_j = \theta > u_m$ (see Lemma 2.8) and therefore $j > m$. On the other hand, H_{m+1} is true, so:

$$\alpha_m = \alpha_j \sim_I u_j = \theta \qquad (iv)$$

Finally, $v \sim_I w$, because r_1 is a minimal relation in I such that $v, w \in supp(r_1)$. This together with (i), (iii) and (iv) imply that $\alpha_m \sim_I u_m$. So H_m is true and the induction is finished. ■

Remark 4.2. *The preceding proposition proves that $\alpha \sim_I u$ for any $u \in supp(\psi(\alpha))$. On the other hand, $\sim_{I_0}$ is weaker than $\sim_I$ (i.e. $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$). These two properties are linked in general. Indeed, in [7, Prop. 4.2.35, Prop. 42.36] the author has proved that if I is an admissible ideal (non necessarily monomial) of kQ and if $\psi \in \mathcal{T}$ is such that $\alpha \sim_{\psi(I)} u$ for any bypass (α, u) such that $u \in supp(\psi(\alpha))$, then $\sim_I$ is weaker than $\sim_{\psi(I)}$.*

Now it is possible to provethe existence of a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_I$, whenever $kQ/I \simeq A$.

Proposition 4.3. *Let $kQ/I \simeq A$ be an admissible presentation. Let $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ be the bypasses and $\tau_1, \ldots, \tau_n \in k^*$ the scalars such that $\psi_I = \varphi_{\alpha_n, u_n, \tau_n} \ldots \varphi_{\alpha_1, u_1, \tau_1}$ (see Proposition 2.19). For each $i \in \{1, \ldots, n\}$, set:*

$$I_i := \varphi_{\alpha_i, u_i, \tau_i} \ldots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$$

then, for each i, exactly one of the two following situations occurs:

. *$\sim_{I_{i-1}}$ and $\sim_{I_i}$ coincide,*

. *$\varphi_{\alpha_i, u_i, \tau_i}$ induces an arrow $\sim_{I_{i-1}} \to \sim_{I_i}$ in Γ.*

In particular, there exists a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_{I_n} = \sim_I$.

Proof: Let $i \in \{1, \ldots, n\}$ and set $\psi_i := \varphi_{\alpha_i, u_i, \tau_i} \ldots \varphi_{\alpha_1, u_1, \tau_1}$. Thus $I_i = \psi_i(I_0)$. Using Proposition 2.13 and Proposition 3.7 it is easily verified that $\psi_i = \psi_{I_i}$. Therefore, Proposition 4.1 applied to I_i gives $\alpha_i \sim_{I_i} u_i$. Since $I_i = \varphi_{\alpha_i, u_i, \tau_i}(I_{i-1})$, this proves that (see Proposition 1.2) either $\sim_{I_{i-1}}$ and $\sim_{I_i}$ coincide or $\varphi_{\alpha_i, u_i, \tau_i}$ induces an arrow $\sim_{I_{i-1}} \to \sim_{I_i}$ in Γ. Thus, the vertices $\sim_{I_0}, \sim_{I_1}, \ldots, \sim_{I_n} = \sim_I$ of Γ are the vertices of a path in Γ (maybe with repetitions) starting at $\sim_{I_0}$ and ending at $\sim_I$. ■

The preceding proposition and the fact that Γ has no oriented cycle gives immediately the following corollary which was proved by the author in [8] in the case of algebras without double bypass over an algebraically closed field of characteristic zero.

Corollary 4.4. *Let Q be a quiver without oriented cycle and without multiple arrows. Let I_0 be an admissible and monomial ideal of kQ and let $A = kQ/I_0$. Then the quiver Γ of the homotopy relations of the admissible presentations of A admits $\sim_{I_0}$ as unique source.*

The following example shows that the preceding corollary does not hold if Q has multiple arrows.

Example 4.5. *Let* $A = kQ/I_0$ *where* Q *is the quiver* $1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \xrightarrow{c} 3$ *and* $I_0 =< ca >$*. Then* Γ *is equal to:*

$$\begin{tikzcd}
\sim_{I_0} \arrow[dr] & & \sim_{I_1} \arrow[dl] \\
& \sim_{I_2} &
\end{tikzcd}$$

where $I_1 =< cb >$ *and* $I_2 =< ca - cb >$*. In particular,* Γ *has two distinct sources. Notice however, that the mapping* $a \mapsto b$*,* $b \mapsto a$*,* $c \mapsto c$ *defines a group isomorphism* $\pi_1(Q, I_0) \simeq \pi_1(Q, I_1)$*. One has* $\pi_1(Q, I_0) \simeq \pi_1(Q, I_1) \simeq \mathbb{Z}$ *and* $\pi_1(Q, I_2) = 1$*.*

Proposition 4.3 also allows one to prove Theorem 1. It extends [8, Thm. 2] to monomial triangular algebras without multiple arrows. Notice that Theorem 1 makes no assumption on the characteristic of k. Also recall that $\pi_1(Q, I_0) = \pi_1(Q)$.

Proof of Theorem 1: The proof is identical to the proof of [8, Thm. 2] except that one uses Proposition 4.3 instead of [8, Lem. 4.3]. ■

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