Various Representations of Multi-Relaxation in Solids with Discrete Relaxation Spectra

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Abstract. A multilinear relaxation of any kind may have various representations which aid to comprehend natural phenomena concerning not only anelastic relaxation but also dielectric, magnetic and thermal relaxations. Five representations are given: representation 1 by $n+1$ non-equivalent conjugate variables composed of density and partial potential, representation 2 by a Voigt-type general linear solid of order n, representation 4 by a Maxwell-type general linear solid of order n, representation 4 by n different $\delta J^{(i)}\left(i=1,2,\cdots,n\right)$ or n different $\delta M^{(i)}\left(i=1,2,\cdots,n\right)$, and representation 5 by n Debye peaks of internal friction.

1. INTRODUCTION

The elasticity is a well-known property of solid since the work of Hooke [1] in the 17th century. Hooke's law is, however, not always valid. It occurs that strain is not uniquely determined even for a small stress, and strain lags behind a periodically variational harmonic-stress by a certain phase angle. Creep, elastic aftereffect and stress relaxation are typical quasi-static relaxations, while internal friction, frequency dependence of elastic modulus (or simply modulus) and compliance are dynamic phenomena different from the elasticity. Relaxations are also observed in ferromagnetic materials through magnetic field and magnetic flux density, and in ferroelectric materials through electric field and electric flux density. These relaxations are thermodynamically irreversible process ; that is , irreversible process represented phenomenologically by equations without terms of higher than and equal to the 2nd order of general force. Zener [2] defined the irreversible linear process on stress-strain as anelasticity, and stimulated researchers to lead to many theoretical and experimental works. This paper describes a general theory of relaxation with discrete relaxation spectra in solids, valid not only for mechanical but also for thermal, electric, magnetic and any other relaxations.

2. VARIOUS REPRESENTATIONS

A relaxation represented by a relaxation time is called a single relaxation and one represented by n different relaxation times, a multi-relaxation of order n or, simply, a relaxation of order n.

2.1 Representation by conjugate variables

Two physical quantities x, ξ, the product of which has the dimension of energy density, are called a conjugate variable and written as $\{x, \xi\}$, where x is called a partial potential and ξ a density. Two conjugate variables are equivalent, if their relaxation times are equal. A necessary and sufficient condition for the occurrence of a linear relaxation in solids is that there involves at least two non-equivalent pairs of conjugate variables in the relaxation, and a condition for the occurrence of a phenomenon yielding a multi-relaxation of order k is that there exist k non-equivalent pairs of conjugate variables other than a pair of conjugate variables for the use of observing the phenomenon. Further discussion gives two different expressions for the relaxation of the compliance in terms of the linear algebra. Although the following discussion is general enough, terminology for the anelasticity is used in this theory unless it leads to misunderstanding.

The elasticity is the property that the conjugate variable $\{x, \xi\}$ satisfies Hooke's law:

$$x = J\xi \quad \text{or} \quad \xi = Mx, \qquad M = 1/J,$$

where J and M are the compliance and modulus, respectively. As for the anelasticity, a conjugate variable is fixed for the use of observing a relaxation, called the observing conjugate variable and denoted by $\left\{x^{(0)}, \xi^{(0)}\right\}$. Furthermore, other n conjugate variables $\left\{x^{(i)}, \xi^{(i)}\right\}$ $(1, 2, \cdots, n)$ and k different relaxation times at constant $\xi^{(0)}$, $\tau_{\xi^{(0)}}^{(i)}$ $(i = 1, 2, \cdots, k)$, involve in the relaxation. The solid in an equilibrium state suffers a change in density $\xi^{(0)}$ from 0 (defined as 0 for the eqilibrium state) to a small non-zero constant value $\xi_0^{(0)}$. For each $\tau_{\xi^{(0)}}^{(i)}$ there are $n^{(i)}$ equivalent conjugate variables $\left\{x_j^{(i)}, \xi_j^{(i)}\right\}$ $\left(j = 1, 2, \cdots, n^{(i)}\right)$, so that $n = \sum_{i=1}^{k} n^{(i)}$. The observing potential is decomposed into two terms according to time dependence. The time-independent term is called the elastic partial potential written as $x_e^{(0)}$, and the time-dependent term the anelastic partial potential written as $x_a^{(0)}$, such that

$$x^{(0)} = x_e^{(0)} + x_a^{(0)}, \quad x_e^{(0)} = J_U^{(0)}\xi_0^{(0)}, \quad x_a^{(0)} = \sum_{i=1}^{k} x_{a,i}^{(0)}$$
$$x_{a,i}^{(0)} = \vec{\lambda}^{(i)} \bullet \vec{\xi}^{(i)} \tag{1}$$

where $J_U^{(0)}$ is the unrelaxed compliance for the observing conjugate variable, $x_{a,i}^{(0)}$'s are anelastic terms induced by the conjugate variables $\left\{\vec{x}^{(i)}, \vec{\xi}^{(i)}\right\}$ $\left('\vec{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \cdots, x_{n^{(i)}}^{(i)}\right), {}'\vec{\xi}^{(i)} = \left(\xi_1^{(i)}, \xi_2^{(i)}, \cdots, \xi_{n^{(i)}}^{(i)}\right)\right)$, and $\vec{\lambda}^{(i)} \bullet \vec{\xi}^{(i)}$ is the inner product of the two vectors. Each component of the density $\vec{\xi}^{(i)}$ in equilibrium, denoted by $\vec{\bar{\xi}}^{(i)}$, is proportional to $\xi_0^{(0)}$, namely, $\vec{\bar{\xi}}^{(i)} = \xi_0^{(0)}\vec{\mu}^{(i)}$, where $\vec{\mu}^{(i)}$ is the vector composed of the proportional constants.

2.1.1 Diffusion of Density

The diffusion of density is represented by the equation:

$$\left(\frac{d}{dt} + \frac{1}{\tau^{(i)}_{\xi^{(0)}}} \right) \bar{\xi}^{(i)} = \xi^{(0)}_0 \frac{\bar{\mu}^{(i)}}{\tau^{(i)}_{\xi^{(0)}}}$$

Operating $\left(\dfrac{d}{dt} + \dfrac{1}{\tau^{(i)}_{\xi^{(0)}}} \right)$ to both sides of eqn.(1) , is obtained

$$\dot{x}^{(0)}_{a,i} + \frac{x^{(0)}_{a,i}}{\tau^{(i)}_{\xi^{(0)}}} = \frac{\bar{\lambda}^{(i)} \bullet \bar{\mu}^{(i)}}{\tau^{(i)}_{\xi^{(0)}}} \xi^{(0)}_0 . \tag{2}$$

The partial potential $\bar{x}^{(i)}$ stands for the driving force of the density $\bar{\xi}^{(i)}$ for diffusion , bringing $x^{(0)}_{a,i}$ to an equilibrium. The differential of the compliance, denoted by $\delta J^{(0)}_i(t)$, is defined as

$$\delta J^{(0)}_i(t) \equiv x^{(0)}_{a,i}(t) \big/ \xi^{(0)}_0 ,$$

and, especially, the relaxation of the compliance is given by

$$\delta J^{(0)}_{R,i} \equiv x^{(0)}_{a,i}(\infty) \big/ \xi^{(0)}_0 .$$

Combined with eqn.(2), is derived

$$\delta J^{(0)}_{R,i} = \bar{\lambda}^{(i)} \bullet \bar{\mu}^{(i)}. \tag{3}$$

2.1.2 Condition of Equilibrium

The equilibrium state resulted from the onset of $\xi^{(0)}_0$ is attained by the condition that the driving force vanishes, i.e., $\bar{x}^{(i)} = \bar{0}$. Since $\bar{x}^{(i)}$ is given by

$$\bar{x}^{(i)} = -\left(\bar{\lambda}^{(i)} \right) \xi^{(0)} + B^{(i)}_{\xi^{(0)}} \bar{\xi}^{(i)}$$

with a symmetric $n^{(i)}$ square matrix $B^{(i)}_{\xi^{(0)}}$, the relaxation of the compliance is expressed in a form other than eqn.(3) as

$$\delta J^{(0)}_{R,i} = \left\| D^{(i)\,-1}_{\xi^{(0)}} {}^t P^{(i)}_{\xi^{(0)}} \bar{\lambda}^{(i)} \right\|^2 ,$$

where $D^{(i)}_{\xi^{(0)}}$ and $P^{(i)}_{\xi^{(0)}}$ are $n^{(i)}$ square diagonal and orthogonal matrices, respectively, such that

$$B^{(i)}_{\xi^{(0)}} = P^{(i)}_{\xi^{(0)}} D^{(i)\,2}_{\xi^{(0)}} {}^t P^{(i)}_{\xi^{(0)}}.$$

2.2 Representation by relaxation models

The multi-relaxation of order k may be represented by the relaxation models of the general linear solid of either Voigt-type or Maxwell-type [3]. Here, reciprocity and more strictly, equivalence between the two-type general linear solids are discussed. The permutation: $\xi \leftrightarrow x$, $J \leftrightarrow M$, $\delta J \leftrightarrow -\delta M$ enables one to exchange the stress-strain equation of the Voigt-type with that of the Maxwell-type and vice versa. From the equations:

$$\frac{\tau_{\xi^{(0)}}^{(i)}}{\tau_{x^{(0)}}^{(i)}} = 1 + \frac{\delta J^{(i)}}{J_U} = 1 + \frac{\delta M^{(i)}}{M_R} \ , \ \ \frac{\delta J^{(i)}}{J_U} = \frac{\delta M^{(i)}}{M_R} << 1 \ \ \left(1,2,\cdots,k\right)$$

and by neglecting terms of higher than and equal to the 2nd order of $\dfrac{\delta J^{(i)}}{J_U}$ and $\dfrac{\delta M^{(i)}}{M_R}$, the equivalence of the two-type general linear solids is proved.

2.3 Other representations

2.3.1 The relaxations of the compliance and the modulus

$$J_R = J_U + \sum_{i=1}^{k} n^{(i)} \delta J_R^{(i)} \ , \ \ M_U = M_R + \sum_{i=1}^{k} n^{(i)} \delta M_R^{(i)} \ .$$

2.3.2 Internal friction

The internal friction in the Voigt-type general linear solid of order k is expressed as

$$Q^{-1} = \frac{\displaystyle\sum_{v=1}^{k} \delta J^{(v)} \frac{\omega \tau_{\xi^{(0)}}^{(v)}}{1 + \left(\omega \tau_{\xi^{(0)}}^{(v)}\right)^2}}{J_U + \displaystyle\sum_{v=1}^{k} \delta J^{(v)} \frac{1}{1 + \left(\omega \tau_{\xi^{(0)}}^{(v)}\right)^2}} = \frac{\displaystyle\sum_{v=1}^{k} \Delta^{(v)} \frac{\omega \tau_{\xi^{(0)}}^{(v)}}{1 + \left(\omega \tau_{\xi^{(0)}}^{(v)}\right)^2}}{1 + \displaystyle\sum_{v=1}^{k} \Delta^{(v)} \frac{1}{1 + \left(\omega \tau_{\xi^{(0)}}^{(v)}\right)^2}} \approx \sum_{v=1}^{k} \Delta^{(v)} \frac{\omega \tau_{\xi^{(0)}}^{(v)}}{1 + \left(\omega \tau_{\xi^{(0)}}^{(v)}\right)^2}$$

and that in the Maxwell-type general linear solid of order k is expressed in a similar form.

References

[1] Hooke R., De Potentia Restitutiva (London,1678).

[2] Zener C., Elasticity and Anelasticity of Metals (The University of Chicago Press, Chicago, 1948).

[3] Iwasaki Y. and Fujimoto K., J.Phys. D: Appl. Phys., **13** (1980) 823-34.