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Reconstruction of $\mathcal{N}=1$ Supersymmetry From Topological Symmetry

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Abstract

The scalar and vector topological Yang–Mills symmetries on Calabi–Yau manifolds geometrically define consistent sectors of Yang–Mills $D=4,6$ $\mathcal{N}=1$ supersymmetry, which fully determine the supersymmetric actions up to twist. For a CY_2 manifold, both $\mathcal{N}=1, D=4$ Wess and Zumino and superYang–Mills theory can be reconstructed in this way. A superpotential can be introduced for the matter sector, as well as the Fayet–Iliopoulos mechanism. For a CY_3 manifold, the $\mathcal{N}=1, D=6$ Yang–Mills theory is also obtained, in a twisted form. Putting these results together with those already known for the $D=4,8$ $\mathcal{N}=2$ cases, we conclude that all Yang–Mills supersymmetries with 4, 8 and 16 generators are determined from topological symmetry on special manifolds.

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1 Introduction

In a recent paper [1], it was shown that the scalar and vectorial topological Yang–Mills symmetries can be directly constructed, in four and eight dimensions, leading one to a geometrical definition of a closed off-shell twisted sector of Yang–Mills supersymmetric theories, with 8 and 16 generators, respectively. In fact, both scalar and vectorial topological symmetries completely determine the supersymmetric theory, (up to a twist that exists on special manifolds). Basically, the vector symmetry arises when one associates reparametrization symmetry and topological symmetry. It is important to work on manifolds that contain at least one covariantly constant vector. The $\mathcal{N}=2$ Poincaré supersymmetry is reached by untwisting the theory in the limit of flat manifolds.

The possibility of directly twisting the $\mathcal{N}=1$ super Yang–Mills theory in a "microscopic" TQFT was studied in [2, 3, 4, 5, 6]. In fact, the full topological symmetry $sA_\mu = \Psi_\mu + D_\mu c$ involves topological ghosts and antighosts, with twice as many degrees of freedom as there are in the gauge field, so it leads one to $\mathcal{N}=2$ supersymmetry. To get the $\mathcal{N}=1$ supersymmetry algebra in a twisted form, the number of independent topological transformations must be reduced by half. This leads one to build a TQFT on a Kähler manifold, such that the gauge field can be splitted in holomorphic and antiholomorphic components, $A_1 = A_{(0,1)} + A_{(1,0)}$. Only one of the components of A undergoes topological transformations, with [2, 3, 4, 5, 6]:

$$sA_m = \Psi_m + D_m c \qquad sA_{\bar{m}} = D_{\bar{m}} c \tag{1}$$

This holomorphic symmetry can be interpreted on a Calabi–Yau manifold as the symmetry of classical actions, which couple forms $B_{(0,n-2)}$, with only antiholomorphic components, to a Yang–Mills curvature $F = dA + A_\wedge A$ [7]:

$$I_{2n} = \int_{M_{2n}} \Omega_{(n,0)} \,{}_\wedge \mathrm{Tr}\; B_{(0,n-2)} \,{}_\wedge F_{(0,2)} \tag{2}$$

The BRST-invariant gauge-fixing of such actions provides in a twisted way the $\mathcal{N}=1$ Wess and Zumino and Yang–Mills models in 4 and 6 dimensions, on Calabi–Yau manifolds [7].

Here we show that both scalar and vector topological BRST symmetries can be also geometrically built for the $\mathcal{N}=1$ supersymmetry, in a way that justifies the choice of topological gauge functions of [7]. In fact, the various properties of $\mathcal{N}=1$ supersymmetry can be reformulated, in the context of topological symmetry.

Let us briefly summarize the situation for getting the scalar and vector topological symmetry, and, eventually $\mathcal{N}=2$ supersymmetry in twisted way. [1] shows that, for special manifolds with dimensions 4 or 8 that contain at least one constant vector κ, one can define an extended horizontality condition with its Bianchi identity, which involves the (twisted) fields of $\mathcal{N}=2$ supersymmetry in 4 and 8 dimensions. It reads:

$$
\begin{aligned}
(d+s+\delta-i_\kappa)\big(A+c+|\kappa|\bar{c}\big)+\big(A+c+|\kappa|\bar{c}\big)^2 &= F+\Psi+g(\kappa)\eta+i_\kappa\chi+\Phi+|\kappa|^2\bar{\Phi} \\
(d+s+\delta-i_\kappa)\big(F+\Psi+g(\kappa)\eta+i_\kappa\chi+\Phi+|\kappa|^2\bar{\Phi}\big)& \\
+[A+c+|\kappa|\bar{c}\,,\, F+\Psi+g(\kappa)\eta+i_\kappa\chi+\Phi+|\kappa|^2\bar{\Phi}] &= 0
\end{aligned}
\tag{3}
$$

By expansion in form degree and ghost number, both equations define the action of s and δ on all the fields, with the closure relations[1]:

$$
s^2=\delta^2=0 \qquad \{s,\delta\}=\mathcal{L}_\kappa+\delta_{\text{gauge}}(i_\kappa \overset{\circ}{A}) \tag{4}
$$

The property that s and δ close off-shell on a reparametrization, $\{s,\delta\}=\mathcal{L}_\kappa$, is at the heart of the property that the commutator of two supersymmetries is a translation in the super Poincaré algebra. In flat space, one defines the twisted scalar supersymmetric operator $Q=s_c$ and the vector supersymmetric operator Q_μ from the equivariant vector operator $\delta_{\bar{c}}=\kappa^\mu Q_\mu$, so one has $Q_\mu Q_\nu+Q_\nu Q_\mu=2g_{\mu\nu}\delta_{\text{gauge}}(\bar{\Phi})$ and $Q_\mu Q+QQ_\mu=D_\mu$. We refer to [1] for a detailed explanation of these formula and the twisted fields that they involve, with their relationship with $D=4,8$ $\mathcal{N}=2$ Yang–Mills supersymmetry, and the way reparametrization symmetry is encoded in s and δ.

The aim of this paper is to understand the way the extended horizontality condition (3) applies to the case of Calabi–Yau manifolds, for reconstructing $\mathcal{N}=1$ supersymmetry. In fact, by separation of holomorphic and antiholomorphic sectors, $\mathcal{N}=1$ supersymmetry will appear. The relevant information on the Wess and Zumino and Yang–Mills independent multiplets of $D=4$ or $D=6$ $\mathcal{N}=1$ will be obtained as off-shell closed sectors of the supersymmetry transformation laws. More precisely, the supersymmetry transformations will be encoded into scalar and vector topological BRST symmetries, corresponding to 3 (resp. 4) twisted generators in 4 (resp. 6) dimensions. The topological construction has the great advantage of purely geometrically determining a closed sector of the supersymmetric algebra, which is large enough to completely determine the theory. Furthermore, it determines the Faddeev–Popov ghosts for the supersymmetry algebra, in a way that is relevant for a control of the covariant gauge-fixing of the Yang–Mills symmetry.

[1] $\overset{\circ}{A}$ is a background connection that must be introduced for the sake of global consistency, but can be chosen equal to zero for trivial vacua.

2 Holomorphic vector symmetry in four dimensions

2.1 Pure Yang–Mills

We begin from the "semi-horizontality" condition for a Yang–Mills field A and its Faddeev–Popov ghost c, on a Kähler 2-fold:

$$(d+s)\big(A+c\big)+\big(A+c\big)^2=F+\Psi, \tag{5}$$

$\Psi=\Psi_m dz^m$ is the holomorphic 1-form topological ghost. If J is the complex structure on the manifold, one has $J\Psi=i\Psi$, Eq. (5) reproduces the "heterotic" BRST transformations, Eq. (1), and includes the ghost dependence, with:

$$\begin{aligned} sA_m&=\Psi_m+D_m c \qquad & sA_{\bar m}&=D_{\bar m}c\\ s\Psi_m&=-[c,\Psi_m] \qquad & sc&=-c^2 \end{aligned}$$

The Euclidean vector ghost $\Psi_{m(z^m,z^{\bar m})}$ must be considered as a complex field that counts for 2 real degrees of freedom in the quantum theory, as it will be explained in section 2.4. To introduce the vector symmetry, we suppose that the manifold contains at least a covariantly constant antiholomorphic vector $\kappa^{\bar m}$. κ defines the holomorphic 1-form $g(\kappa)=g_{m\bar m}\kappa^{\bar m}dz^m$. The norm of κ is $|\kappa|^2=\kappa^{\bar m}\bar\kappa_{\bar m}=i_{\bar\kappa}g(\kappa)$, where $\bar\kappa$ is the complex conjugate of κ. The differential δ will be geometrically constructed, and is somehow the mirror of s. In flat space, formula must be expanded as series in $\kappa^{\bar m}$. The vector symmetry operator $Q_{\bar m}$ is then defined by the identification $\delta+|\kappa|\delta_{\text{gauge}}(\bar c)=\kappa^{\bar m}Q_{\bar m}$, and determines, together with $s+\delta_{\text{gauge}}(c)$, the relevant closed sector of $\mathcal{N}=1$ supersymmetry.

The dual of the "holomorphic" 1-form topological ghost Ψ_m is made of a pair of a scalar η and an "antiholomorphic" 2-form $\chi_{\bar m\bar n}$, counting altogether for 2 real degrees of freedom. $\bar c$ is the Faddeev–Popov antighost. The ghost anti-ghost dependent horizontality condition that defines both s and δ symmetry is:

$$(d+s+\delta-i_\kappa)\big(A+c+|\kappa|\bar c\big)+\big(A+c+|\kappa|\bar c\big)^2=F+\Psi+g(\kappa)\eta+i_\kappa\chi \tag{6}$$

This gives

$$\begin{aligned} sA+d_Ac&=\Psi \qquad & \delta A+d_A|\kappa|\bar c&=g(\kappa)\eta+i_\kappa\chi\\ sc+c^2&=0 \qquad & \delta|\kappa|\bar c+\big(|\kappa|\bar c\big)^2&=0 \end{aligned}$$
$$\delta c+s|\kappa|\bar c+[c,|\kappa|\bar c]=i_\kappa\big(A-\overset{\circ}{A}\big) \tag{7}$$

One defines the "equivariant" differential $s_c\equiv s+\delta_{\text{gauge}}(c)$ and $\delta_{\bar c}\equiv\delta+|\kappa|\delta_{\text{gauge}}(\bar c)$.

The property $(d+s+\delta-i_\kappa)^2=0$, is equivalent to the Bianchi identity

$$(d_A+s_c+\delta_{\bar c}-i_\kappa)\big(F+\Psi+g(\kappa)\eta+i_\kappa\chi\big)=0 \tag{8}$$

The introduction of two scalar fields, b and h allows one to remove the indeterminacy that occurs in the determination of $s\bar c$ and $s\eta$. Eq. (6) and its Bianchi identity (8) provide by expansion in ghost number and form degree the following BRST transformations:

$$\begin{aligned}
s_c A_m &= \Psi_m & \delta_{\bar c} A_m &= \kappa_m \eta\\
s_c A_{\bar m} &= 0 & \delta_{\bar c} A_{\bar m} &= \kappa^{\bar n}\chi_{\bar n\bar m}\\
s_c \Psi_m &= 0 & \delta_{\bar c}\Psi_m &= \kappa^{\bar n}F_{\bar n m}-\kappa_m h\\
s_c\eta &= h & \delta_{\bar c}\eta &= 0\\
s_c h &= 0 & \delta_{\bar c} h &= \kappa^{\bar m}D_{\bar m}\eta\\
s_c\chi_{\bar m\bar n} &= F_{\bar m\bar n} & \delta_{\bar c}\chi_{\bar m\bar n} &= 0
\end{aligned} \tag{9}$$

$$\begin{aligned}
sc &= -c^2 & \delta c &= \kappa^{\bar m}\big(A_{\bar m}-\mathring{A}_{\bar m}\big)-|\kappa| b\\
s\bar c &= b-[c,\bar c] & \delta\bar c &= -|\kappa|\,\bar c^2\\
sb &= -[c,b] & \delta b &= \kappa^{\bar n}D_{\bar n}\bar c
\end{aligned} \tag{10}$$

By construction, one has the required relations:

$$s^2=0 \qquad \delta^2=0 \qquad \{s,\delta\}=\mathcal{L}_\kappa+\delta_{\rm gauge}(i_\kappa\mathring{A}) \tag{11}$$

One has also "equivariant" commutation relations for all fields, but c, $\bar c$ and b:

$$s_c^2=0 \qquad \delta_{\bar c}{}^2=0 \qquad \{s_c,\delta_{\bar c}\}=\mathcal{L}_\kappa+\delta_{\rm gauge}(i_\kappa A) \tag{12}$$

The most general δ-closed topological gauge function, which has ghost number -1, is gauge invariant and gives κ independent renormalizable terms, is:

$$\boldsymbol{\Psi}_{\rm YM}=\int_M d^4x\sqrt{g}\,{\rm Tr}\,\Big(\frac12\chi^{mn}F_{mn}+\eta\big(h+iJ^{m\bar n}F_{m\bar n}\big)\Big) \tag{13}$$

It defines the ungauge-fixed s_c and $\delta_{\bar c}$ invariant action $I_{\rm YM}=s\boldsymbol{\Psi}_{\rm YM}$. Integrating out the field h gives:

$$I_{\rm YM}=\approx\int_M d^4x\sqrt{g}\,{\rm Tr}\,\Big(\frac12F^{mn}F_{mn}+\frac14\big(J^{m\bar n}F_{m\bar n}\big)^2-\chi^{mn}D_m\Psi_n+\eta D^m\Psi_m\Big) \tag{14}$$

This is the twisted form of the $\mathcal{N}=1$ supersymmetric Yang–Mills action, as expressed in [2].

This invariant action is in fact $s\delta$-exact. One has indeed:

$$\boldsymbol{\Psi}_{\rm YM}=\tfrac{1}{(\kappa\cdot\bar\kappa)}\,\delta\int_M d^4x\sqrt{g}\,{\rm Tr}\,\Big(\ \eta\bar\kappa^m\Psi_m+\bar\kappa^{[m}g^{n]\bar m}\big(3\,A_{[\bar m}\partial_m A_{n]}+2A_{[\bar m}A_mA_{n]}\big)\Big) \tag{15}$$

The last term is nothing but the Chern–Simon term $g(\bar\kappa)\big(AdA+\frac23A^3\big)$

2.2 Wess and Zumino matter multiplet

The matter multiplet is defined from horizontality conditions, for both f extended fields $\bar\psi_{\bar m} dz^{\bar m} + \phi$ and $\frac{1}{2}\bar\chi_{mn} dz^m_{\wedge} dz^n + \kappa_m dz^m \bar\phi$. $\bar\chi$ is a "holomorphic" 2-form, $\bar\psi$ a "antiholomorphic" 1-form, and ϕ and $\bar\phi$ two scalars. These fields are valued in an arbitrarily given gauge group representation. The horizontality conditions and Bianchi identities are:

$$
\begin{aligned}
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\big(\bar\psi + \phi\big) &= \bar\partial_A \bar\psi + i_\kappa B \\
(d_A + s_c + \delta_{\bar c} - i_\kappa)\big(\bar\chi + g(\kappa)\bar\phi\big) &= \bar\partial_A \bar\chi + T - g(\kappa)\big(\bar\partial_A \bar\phi + \bar\eta\big) \\
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\big(\bar\partial_A \bar\psi + i_\kappa B\big) &= \big(F_{(0,2)} + i_\kappa \boldsymbol{\chi}\big)\big(\bar\psi + \phi\big) \\
(d_A + s_c + \delta_{\bar c} - i_\kappa)\big(\bar\partial_A \bar\chi + T - g(\kappa)\big(\bar\partial_A\bar\phi + \bar\eta\big)\big) &= \big(F + \Psi + g(\kappa)\boldsymbol{\eta} + i_\kappa \boldsymbol{\chi}\big)\big(\bar\chi + g(\kappa)\bar\phi\big)
\end{aligned}
\tag{16}
$$

The gauge field A obeys the same equations as defined in the previous section. This gives the following BRST transformations:

$$
\begin{aligned}
s_c \bar\psi_{\bar m} &= -D_{\bar m}\phi & \delta_{\bar c}\bar\psi_{\bar m} &= \kappa^{\bar n} B_{\bar n \bar m} \\
s_c \phi &= 0 & \delta_{\bar c}\phi &= -\kappa^{\bar m}\bar\psi_{\bar m} \\
s_c B_{\bar m \bar n} &= 2D_{[\bar m}\bar\psi_{\bar n]} + \boldsymbol{\chi}_{\bar m \bar n}\phi & \delta_{\bar c} B_{\bar m \bar n} &= 0 \\
s_c \bar\chi_{mn} &= T_{mn} & \delta_{\bar c}\bar\chi_{mn} &= 2\kappa_{[m} D_{n]}\bar\phi \\
s_c T_{mn} &= 0 & \delta_{\bar c} T_{mn} &= \kappa^{\bar p} D_{\bar p}\bar\chi_{mn} - 2\kappa_{[m} D_{n]}\bar\eta - 2\kappa_{[m}\Psi_{n]}\bar\phi \\
s_c \bar\phi &= \bar\eta & \delta_{\bar c}\bar\phi &= 0 \\
s_c \bar\eta &= 0 & \delta_{\bar c}\bar\eta &= \kappa^{\bar m} D_{\bar m}\bar\phi
\end{aligned}
\tag{17}
$$

They represent the twisted scalar and vector supersymmetric transformations for a Wess and Zumino multiplet.

For a general simple non Abelian gauge group, the δ invariance uniquely determines the most general renormalizable and κ independent topological gauge function with ghost number -1 , $\boldsymbol{\Psi}_{\text{Matter}}$, as follows[2]:

$$
I_{\text{Matter}} = s\boldsymbol{\Psi}_{\text{Matter}}, \qquad \boldsymbol{\Psi}_{\text{Matter}} = \int_M d^4x\sqrt{g}\left(\frac{1}{2}\bar\chi^{\bar m \bar n} B_{\bar m \bar n} + \bar\phi D_m \bar\psi^m + \phi\boldsymbol{\eta}\bar\phi\right) \tag{18}
$$

The δ-closed gauge function $\boldsymbol{\Psi}_{\text{Matter}}$ turns out to be δ-exact:

$$
\boldsymbol{\Psi}_{\text{Matter}} = \tfrac{1}{(\bar\kappa\cdot\kappa)}\,\delta \int_M d^4x\sqrt{g}\left(\bar\kappa_{\bar m}\bar\psi_{\bar n}\bar\chi^{\bar m \bar n} + \bar\kappa^m \phi D_m \bar\phi\right) \tag{19}
$$

When on computes $I_{\text{Matter}} = s\boldsymbol{\Psi}_{\text{Matter}}$, one sees that T_{mn} and $B_{\bar m \bar n}$ identify thereself as both scalar auxiliary fields of the Wess and Zumino multiplet.

[2]We do not write explicitly the sum over the index of the matter representation, so that, one has for instance $\phi\bar\phi \equiv \sum_A \phi_A \bar\phi_A$

If one combines this result with that of section 2.1, one finds that the complete supersymmetric action for a matter field coupled to the Yang–Mills theory can be obtained by adding both gauge functions. Let t_α be the generators of the Lie algebra of the gauge group for the matter representation. The scale factor between the Yang–Mills and the matter gauge functions can be included in the definition of the trace. After integration of the fields h, T and B, the action is :

$$
\begin{aligned}
I_{\text{YM+Matter}} \approx \int_M d^4x\sqrt{g}\Bigg(\operatorname{Tr}\,\Big(\frac{1}{2}F^{mn}F_{mn} + \frac{1}{4}\big(J^{m\bar{n}}F_{m\bar{n}}\big)^2 - \chi^{mn}D_m\Psi_n + \eta D^m\Psi_m\Big) \\
+\Big(\bar{\eta}D_m\bar{\psi}^m - \bar{\chi}^{\bar{m}\bar{n}}D_{\bar{m}}\bar{\psi}_{\bar{n}} - \frac{1}{2}\bar{\phi}\big(D_mD^m + D^mD_m\big)\phi \\
-\frac{1}{2}\bar{\chi}_{mn}\chi^{mn}\phi + \bar{\phi}\Psi_m\bar{\psi}^m - \phi\eta\bar{\eta}\Big) - \sum_\alpha \frac{1}{4\operatorname{Tr}\,(t^\alpha t^\alpha)}\Big(\phi\, t^\alpha\bar{\phi}\Big)\Big(\phi\, t^\alpha\bar{\phi}\Big)\Bigg)
\end{aligned}
\tag{20}
$$

The $\mathcal{N}=1$ supersymmetric action for a Yang–Mills field and a scalar complex field can therefore be directly and uniquely constructed in a twisted form, as an $s\delta$-exact term.

2.3 Embedding of the $\mathcal{N}=1$ theory in the $\mathcal{N}=2$ theory

Consider the complexified twisted $\mathcal{N}=2$ theory on a Kähler manifold. The moduli space of instantons has a Kähler structure. The exterior differential on M can be decomposed into Dolbeault operators and a similar property exists for the BRST operator. [3] shows that the equivariant scalar BRST charge of the $\mathcal{N}=1$ theory can be identified as a component of the scalar BRST charge of the $\mathcal{N}=2$ theory. Here, we show that, starting from the $\mathcal{N}=2$ horizontality equation, the projection of a general constant vector field κ into antiholomorphic components gives the $\mathcal{N}=1$ vector symmetry for the Yang–Mills field and the matter field.

For a constant vector κ, with both holomorphic and antiholomorphic components, the expansion in ghost number of Eq.(3) determines the equivariant vector symmetry operator $\delta_{\bar{c}}$ for the $\mathcal{N}=2$ supersymmetry, that is [1]:

$$
\begin{aligned}
\delta_{\bar{c}} A_\mu &= -\kappa_\mu \eta + \kappa^\nu \chi_{\nu\mu} \\
\delta_{\bar{c}} \Psi_\mu &= \kappa^\nu \big(F_{\nu\mu} - T_{\nu\mu}\big) + \kappa_\mu [\Phi, \bar{\Phi}] \\
\delta_{\bar{c}} \Phi &= -\kappa^\mu \Psi_\mu \qquad\qquad\qquad\qquad \delta_{\bar{c}} \bar{\Phi} = 0 \\
\delta_{\bar{c}} \eta &= \kappa^\mu D_\mu \bar{\Phi} \\
\delta_{\bar{c}} \chi_{\mu\nu} &= -4\kappa_{[\mu} D_{\nu]^-} \bar{\Phi} \\
\delta_{\bar{c}} T_{\mu\nu} &= 4\kappa_{[\mu} \Big(D_{\nu]^-}\eta + \frac{1}{2} D^\sigma \chi_{\sigma|\nu]^-} - [\bar{\Phi}, \Psi_{\nu]^-}]\Big) + 2\kappa^\sigma D_{[\mu}\chi_{\sigma|\nu]^-}
\end{aligned}
\tag{21}
$$

Notice that, in holomorphic coordinates, the antiselfduality condition is $\chi_{m\bar{n}} = \frac{1}{2} J_{m\bar{n}} J^{n\bar{m}} \chi_{n\bar{m}}$. Therefore, the Kähler metric allows one to define scalar fields χ and t, with:

$$
\chi_{m\bar{n}} = g_{m\bar{n}}\chi \qquad \chi_{\bar{m}n} = -g_{\bar{m}n}\chi \qquad T_{m\bar{n}} = g_{m\bar{n}} t \qquad T_{\bar{m}n} = -g_{\bar{m}n} t \tag{22}
$$

If one chooses a constant antiholomorphic vector $\kappa^{\bar{m}}$, Eq.(21) is projected into:

$$
\begin{aligned}
&\delta_{\bar{c}} A_m = -\kappa_m \big(\eta + \chi\big) && \delta_{\bar{c}} A_{\bar{m}} = \kappa^{\bar{n}} \chi_{\bar{m}\bar{n}} \\
&\delta_{\bar{c}} \Psi_m = \kappa^{\bar{n}} F_{\bar{n}m} + \kappa_m \big(t + [\Phi, \bar{\Phi}]\big) && \delta_{\bar{c}} \Psi_{\bar{m}} = \kappa^{\bar{n}} \big(F_{\bar{n}\bar{m}} - T_{\bar{n}\bar{m}}\big) \\
&\delta_{\bar{c}} \Phi = -\kappa^{\bar{m}} \Psi_{\bar{m}} && \delta_{\bar{c}} \bar{\Phi} = 0 \\
&\delta_{\bar{c}} \eta = \kappa^{\bar{m}} D_{\bar{m}} \bar{\phi} \\
&\delta_{\bar{c}} \chi_{mn} = -4\kappa_{[m} D_{n]} \bar{\Phi} \qquad \delta_{\bar{c}} \chi = -\kappa^{\bar{m}} D_{\bar{m}} \bar{\Phi} && \delta_{\bar{c}} \chi_{\bar{m}\bar{n}} = 0 \\
&\delta_{\bar{c}} T_{mn} = \kappa^{\bar{m}} D_{\bar{m}} \chi_{mn} + 4\kappa_{[m} \big(D_{n]}\eta - [\bar{\Phi}, \Psi_{n]}]\big) \\
&\delta_{\bar{c}} t = \kappa^{\bar{m}} D_{\bar{m}} \big(\eta + \chi\big) - [\bar{\Phi}, \kappa^{\bar{m}} \Psi_{\bar{m}}] && \delta_{\bar{c}} T_{\bar{m}\bar{n}} = 2\kappa^{\bar{p}} D_{[\bar{m}} \chi_{\bar{p}|\bar{n}]}
\end{aligned}
\tag{23}
$$

By comparison with Eq. (9), one sees that, up to field redefinitions, the antiholomorphic component of the vector BRST symmetry of the twisted $\mathcal{N}=2$ theory is nothing but the vector symmetry of the $\mathcal{N}=1$ twisted theory, as directly constructed in the last section.

As for the holomorphic component of the scalar BRST operator s of $\mathcal{N}=2$, it can be obtained by looking for an operator that act in a nilpotent way on the multiplet and verifies $\{s_c, \delta_{\bar{c}}\} = \mathcal{L}_\kappa + \delta_{\text{gauge}}(i_\kappa A)$. This defines the same operator s as in Eq. (9).

Looking for a Lagrangian $I = s_c \mathbf{\Psi}$, which is invariant under antiholomorphic s_c and $\delta_{\bar{c}}$ transformations, one finds both independent $\delta_{\bar{c}}$-exact topological gauge functions $\mathbf{\Psi}_{\text{YM}}$ and $\mathbf{\Psi}_{\text{Matter}}$ of the previous section. Relaxing the antiholomorphicity condition of κ, only a special combination of both gauge functions is $\delta_{\bar{c}}$ invariant, which turns out to be that of the $\mathcal{N}=2$ theory.

2.4 Matching with the untwisted theory

To twist the anticommuting fields $(\Psi_m, \eta, \chi_{\bar{m}\bar{n}})$ of a topological multiplet into a Dirac spinor, a pair of covariantly constant antichiral spinors $\zeta_\pm$, with $iJ^{m\bar{n}}\sigma_{m\bar{n}}\zeta_\pm = \pm\zeta_\pm$ is needed. This implies that the manifold must be hyperKähler. We can normalize $\zeta_\pm$ with $\left(\zeta_{-\dot{\alpha}}\zeta_+^{\dot{\alpha}}\right) = 1$. Then, the Euclidean twist formula are [2, 3, 6]:

$$\lambda_\alpha = \Psi_m \sigma^m{}_{\alpha\dot{\alpha}}\zeta_-^{\dot{\alpha}} \qquad \lambda^{\dot{\alpha}} = \eta\zeta_+^{\dot{\alpha}} + \chi_{\bar{m}\bar{n}}\sigma^{\bar{m}\bar{n}\dot{\alpha}}{}_{\dot{\beta}}\zeta_+^{\dot{\beta}} \tag{24}$$

Performing these changes of variables in the topological actions found in the previous section, one finds the Euclidean "Majorana action" for Dirac spinors $(\lambda_\alpha, \lambda^{\dot{\alpha}})$ described by Nicolai [8], that is, the analytic continuation of the Minkowski $\mathcal{N} = 1$ superYang–Mills theory in Euclidean space. (The untwisted action is independent on $\zeta_\pm$, as a result of the change of variables). Both twisted and untwisted actions do not depend on the complex conjugates of the complex fields, i.e, λ in the untwisted theory and η, χ, Ψ in the twisted theory. In both cases, the path integral is formally understood as counting four real degrees of freedom. The Euclidean prescription must be considered as justified by the analytic continuation of the Minkowski case, where real representations do exist (Majorana condition) [8].

It follows that both supersymmetric $\mathcal{N} = 1$ Yang–Mills and Wess and Zumino actions are truly determined by a subsector of supersymmetry algebra with three independent generators, corresponding to both scalar and antiholomorphic vector symmetries, s_c and $\delta_{\bar{c}}$. The closure of $s_c = Q$ and $\delta_{\bar{c}} = \kappa^{\bar{m}}Q_{\bar{m}}$, and thus of the 3 generators Q and $Q_{\bar{m}}$ under the form $Q^2 = 0, Q_{\bar{m}}Q_{\bar{n}} + Q_{\bar{n}}Q_{\bar{m}} = 0$ and $Q_{\bar{m}}Q + QQ_{\bar{m}} = D_{\bar{m}}$, stems from the property $(s + d + \delta - i_\kappa)^2 = 0$. The above change of variables actually maps the four supersymmetry generators Q^α and $Q_{\dot{\alpha}}$ on the four twisted generators $(Q, Q_{\bar{m}}, Q_{mn})$. The symmetry of the action under the action of the fourth twisted generator Q_{mn} appears as an additional symmetry, which is not needed, geometrically. Moreover, it is not needed for enforcing supersymmetry, for instance in technical proofs, such as those concerning renormalization.

2.5 Formal reality condition for the action

The Euclidean action that is the analytic continuation of the Minkowski $\mathcal{N} = 1$ superYang–Mills action is not Hermitian in the usual sense. However, as suggested in [8], a "formal complex conjugation" can be defined, for which the action is Hermitian.

Such a "formal complex conjugation" can be extended to the twisted case. It is the composition of a Wick rotation, an ordinary complex conjugation, and an inverse Wick rotation. The complex conjugation takes into account the fact that, in Minkowski space one has the Majorana condition for the spinors. This "formal complex conjugation", which we define as $*$, breaks the Lorentz invariance, since we must define which direction defines the imaginary time one. However, the operation $*$ can be covariantly defined, by defining the temporal direction of the Minkowski space to be the covariantly constant vector field κ of the Euclidean manifold. The action of $*$ is the ordinary complex conjugation on c-numbers and the following transformations on the fields of the theory[3]:

$$
\begin{aligned}
&*\partial_A* = \bar\partial_A - g(\bar\kappa)\mathscr{L}_{\kappa-\bar\kappa} && *\bar\partial_A* = \partial_A + g(\kappa)\mathscr{L}_{\kappa-\bar\kappa} && \\
&\big(*\Psi\big)_{\bar m} = \bar\kappa_{\bar m}\boldsymbol{\eta} + \kappa^{\bar n}\boldsymbol{\chi}_{\bar n\bar m} && *\boldsymbol{\eta} = \bar\kappa^{m}\Psi_m && \big(*\boldsymbol{\chi}\big)_{mn} = 2\kappa_{[m}\Psi_{n]} \\
&*h = -h - \bar\kappa^{m}\kappa^{\bar n}F_{m\bar n} && && \\
&\big(*\bar\psi\big)_{m} = \kappa_{m}\bar\eta + \bar\kappa^{n}\bar\chi_{nm} && *\bar\eta = \kappa^{\bar m}\bar\psi_{\bar m} && \big(*\bar\chi\big)_{\bar m\bar n} = 2\bar\kappa_{[\bar m}\bar\psi_{\bar n]} \\
&*\phi = -\bar\phi && *\bar\phi = -\phi && \\
&\big(*T\big)_{\bar m\bar n} = B_{\bar m\bar n} && \big(*B\big)_{mn} = T_{mn} &&
\end{aligned}
\tag{25}
$$

This $*$ operation interchanges s_c and $\delta_{\bar c}$:

$$
s_c = \delta_{\bar c} \qquad *\,\delta_{\bar c}* = s_c \tag{26}
$$

The "reality" condition of the action means that, after integration of auxiliary fields, one has:

$$
*I_{\text{YM}} = I_{\text{YM}} \qquad *\,I_{\text{Matter}} = I_{\text{Matter}} \tag{27}
$$

modulo the addition of the topological term $\int_M \text{Tr}\ F_\wedge F$. In fact, this topological term is not invariant under the $*$ operation, only the Yang–Mills action $\int_M \text{Tr}\ F \star F$ is.

2.6 Introduction of the WZ superpotential and Fayet–Iliopoulos term in the twisted formalism

The introduction of a s and δ invariant superpotential for a Calabi–Yau manifold involves terms that have zero ghost number only modulo 2. This parallels the breaking of chirality induced by a superpotential in the untwisted theory.

Consider a scalar field φ valued in a certain representation of the gauge group. Let $f(\varphi)$ be a superpotential, which is an analytical function of φ. Looking for an action I_{SP},

[3]To simplify the notations, we normalize $|\kappa|$ to 1, and define $\mathscr{L}_\xi \equiv \{i_\xi, d_A\}$.

which is s, δ and $*$ invariant and has ghost number zero modulo 2, gives:

$$I_{\rm SP} = \int_M d^4x\sqrt{g}\Bigg(\overline{\Omega}^{mn}\Big(T^A_{mn}f_A(\bar\phi) - \bar\chi^A_{mn}\bar\eta^B f_{AB}(\bar\phi)\Big) + \Omega^{\bar m\bar n}\Big(B^A_{\bar m\bar n}f_A(-\phi) + \bar\psi^A_{\bar m}\bar\psi^B_{\bar n}f_{AB}(-\phi)\Big)\Bigg) \tag{28}$$

Ω and $\overline{\Omega}$ are respectively the holomorphic and the antiholomorphic 2-form in the Calabi–Yau 2-fold, and f_A and f_{AB} stand for the first and second derivatives of the superpotential[4].

This term is neither s- nor δ-exact. However, it can be written as follows:

$$I_{\rm SP} = (s+\delta)\int_M d^4x\sqrt{g}\Big(\overline{\Omega}^{mn}\bar\chi^A_{mn}f_A(\bar\phi) + 2\Omega^{\bar m\bar n}\bar\kappa_{\bar m}\bar\psi^A_{\bar n}f_A(-\phi)\Big) \tag{29}$$

After expansion, one recovers, up to twist, the Wess and Zumino formula for a superpotential. The superpotential is at most a cubic function, to ensure renormalizability.

In order to make the distinction between the topological action, which has ghost number zero, and the superpotential, which has non vanishing even ghost number, we may interpret the later as the insertion of an operator in the path integral.

The s and δ symmetry actually does not constrain the potential of ϕ (which we will name holomorphic) to be equal to that of $\bar\phi$ (antiholomorphic): we could have chosen independent functions f and $\bar f$. However, the "formal reality condition" $*I_{\rm SP} = I_{\rm SP}$, constrains the holomorphic and the antiholomorphic superpotential to be related. This condition determines a real action when one goes to Minkowski space.

Finally, if the group has a $U(1)$ sector, one can add the Fayet–Iliopoulos term, under the simplest invariant form:

$$I_{FI} = \int_M d^4x\sqrt{g}s\,\delta\,\left(\bar\kappa^m A^{U(1)}_m\right) = \int_M d^4x\sqrt{g}h^{U(1)} \tag{30}$$

The integration of the auxiliary field h gives then a mass term for the Abelian component of the scalar fields, plus a topological term

$$\int_M J_\wedge F^{U(1)} \tag{31}$$

This parallels the property that the full superYang–Mills action is BRST exact, modulo a topological term. $J_\wedge F^{U(1)}$ is not invariant under the formal complex conjugation $*$.

[4]The index A denotes the group representation of the matter.

Thus, the formal reality condition of the twisted action implies that this topological term must be subtracted from the action.

So, we conclude that most of the features of the supersymmetric theory in four dimensions are captured in the TQFT formalism, from the principle of s and δ invariance.

3 Yang–Mills $\mathcal{N}=1$ on Calabi–Yau 3-fold

By analogy with the four dimensional case, we might tentatively define the horizontality condition in six dimensions, (with 8 spinorial generators in the untwisted formalism), as:

$$(d+s+\delta-i_\kappa)\big(A+c+|\kappa|\bar{c}\big)+\big(A+c+|\kappa|\bar{c}\big)^2=F+\Psi_{(1,0)}+g(\kappa)\eta_{(0,0)}+i_\kappa\chi_{(0,2)} \quad (32)$$

However, something more elaborated must be done, because the counting of degrees of freedom would not be quite right. In four dimensions there is as much degrees of freedom in the connexion, modulo gauge transformations, as in a selfdual curvature, whereas in the six dimensional case the counting is more subtle and involves a scalar field [6] .

In order to solve the question, we must refine the Kähler decomposition of extended differentials, which include the BRST scalar and vector operators s and δ. We define:

$$\tilde{d}=\tilde{\partial}+\tilde{\bar{\partial}} \quad (33)$$

These extended "heterotic" Dolbeault operators are defined as:

$$\tilde{\partial}\equiv\partial \qquad \tilde{\bar{\partial}}\equiv\bar{\partial}+s+\delta-i_\kappa \quad (34)$$

The twisted $\mathcal{N}=1$ algebra takes into account this asymmetry between holomorphic and antiholomorphic sectors, in a way that generalizes the four dimensional case. In fact, for a Kähler manifold, the Bianchi identity of the curvature F can be decomposed as follows:

$$\partial_A F_{(2,0)}=0 \qquad \partial_A F_{(1,1)}+\bar{\partial}_A F_{(2,0)}=0 \qquad \bar{\partial}_A F_{(1,1)}+\partial_A F_{(0,2)}=0 \qquad \bar{\partial}_A F_{(0,2)}=0 \quad (35)$$

Eq. (32) decomposes into two horizontality conditions:

$$\begin{aligned}(\bar{\partial}+s+\delta-i_\kappa)\big(A_{(0,1)}+c+|\kappa|\bar{c}\big)+\big(A_{(0,1)}+c+|\kappa|\bar{c}\big)^2&=F_{(0,2)}+i_\kappa\chi\\ \big\{\bar{\partial}_A+s_c+\delta_{\bar{c}}-i_\kappa,\partial_A\big\}&=F_{(1,1)}+\Psi+g(\kappa)\eta\end{aligned} \quad (36)$$

with their Bianchi identities

$$
\begin{aligned}
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\big(F_{(0,2)} + i_\kappa\chi\big) = 0 \\
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\big(F_{(1,1)} + \Psi + g(\kappa)\eta\big) + \partial_A\big(F_{(0,2)} + i_\kappa\chi\big) = 0
\end{aligned}
\tag{37}
$$

These equations only determine part of the BRST algebra. Indeed, they miss a dependence on the holomorphic curvature $F_{(2,0)}$. One must therefore introduce a third horizontality condition, which completes Eq. (32), and enforces the definition of $F_{(2,0)}$ as a curvature. In order to have the necessary balance between the ghosts and antighosts degrees of freedom, we understand that one further degree of freedom with ghost number one must be introduced. It can be represented as a holomorphic 3-form ς. The third horizontality condition and Bianchi identity are:

$$
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\,\varsigma_{(3,0)} = \bar\partial_A\varsigma_{(3,0)} + \Phi_{(3,0)} + g(\kappa)F_{(2,0)} \tag{38}
$$

$$
(\bar\partial_A + s_c + \delta_{\bar c} - i_\kappa)\big(\bar\partial_A\varsigma + \Phi + g(\kappa)F_{(2,0)}\big) = [F_{(0,2)} + i_\kappa\chi,\, \varsigma\,] \tag{39}
$$

Expending the equations in form degree and ghost number determines the action of s_c and $\delta_{\bar c}$:

$$
\begin{aligned}
s_c A_m &= \Psi_m & \delta_{\bar c} A_m &= \kappa_m \eta \\
s_c A_{\bar m} &= 0 & \delta_{\bar c} A_{\bar m} &= \kappa^{\bar n}\chi_{\bar n\bar m} \\
s_c \Psi_m &= 0 & \delta_{\bar c}\Psi_m &= \kappa^{\bar n}F_{\bar n m} - \kappa_m h \\
s_c \eta &= h & \delta_{\bar c}\eta &= 0 \\
s_c \chi_{\bar m\bar n} &= F_{\bar m\bar n} & \delta_{\bar c}\chi_{\bar m\bar n} &= \kappa^{\bar p}\bar\Phi_{\bar p\bar m\bar n} \\
s_c h &= 0 & \delta_{\bar c} h &= \kappa^{\bar m} D_{\bar m}\eta \\
s_c \bar\Phi_{\bar p\bar m\bar n} &= 3D_{[\bar p}\chi_{\bar m\bar n]} & \delta_{\bar c}\bar\Phi_{\bar p\bar m\bar n} &= 0 \\
s_c \varsigma_{pmn} &= \Phi_{pmn} & \delta_{\bar c}\varsigma_{pmn} &= 3\kappa_{[p}F_{mn]} \\
s_c \Phi_{pmn} &= 0 & \delta_{\bar c}\Phi_{pmn} &= \kappa^{\bar q}D_{\bar q}\varsigma_{pmn} - 6\kappa_{[p}D_m\Psi_{n]}
\end{aligned}
$$

The δ invariance determines the following topological gauge function:

$$
\mathbf{\Psi} = \int_M d^6x\sqrt{g}\,\mathrm{Tr}\,\left(\frac{1}{2}\chi^{mn}F_{mn} + \eta\big(h + iJ^{m\bar n}F_{m\bar n}\big) - \frac{1}{6}\bar\Phi^{mnp}\varsigma_{mnp}\right) \tag{40}
$$

Integrating out h, Φ, $\bar\Phi$ gives the following s and δ invariant action:

$$
I = s\mathbf{\Psi} = \int_M d^6x\sqrt{g}\,\mathrm{Tr}\,\left(\frac{1}{2}F^{mn}F_{mn} + \frac{1}{4}\big(J^{m\bar n}F_{m\bar n}\big)^2 - \chi^{mn}D_m\Psi_n + \eta D_{\bar m}\Psi^{\bar m} + \frac{1}{2}\varsigma^{\bar p\bar m\bar n}D_{\bar p}\chi_{\bar m\bar n}\right) \tag{41}
$$

This action is the twisted, $\mathcal{N}=1, D=6$ supersymmetric action, up to a topological term $\int_M iJ_\wedge \mathrm{Tr}\ F_\wedge F$, as in [6]. As in the four dimensional case, this action is $s\delta$-exact,

$$I = s\,\delta_{\frac{1}{(\kappa\cdot\bar\kappa)}} \int_M d^4x\sqrt{g}\,\mathrm{Tr}\left(\eta\bar\kappa^m\Psi_m + \frac{1}{2}\varsigma_{pmn}\bar\kappa^p\chi^{mn} + \bar\kappa^{[m}g^{n]\bar m}\big(3A_{[\bar m}\partial_m A_{n]} + 2A_{[\bar m}A_m A_{n]}\big)\right) \tag{42}$$

The last term is just $ig(\bar\kappa)_\wedge J_\wedge \mathrm{Tr}\ \left(AdA + \frac{2}{3}A^3\right)$.

4 Conclusion

Putting together the results of this paper and of [1], we reach an interesting conclusion for the super Yang–Mills symmetries with 4, 8 and 16 generators, which can be represented as $\mathcal{N}=1$ theories in 4 and 6 dimensions and $\mathcal{N}=2$ theories in 4 and 8 dimensions.

On the one hand, one can directly see that the known spinorial generators of the superPoincaré "on-shell algebra" can be mapped on tensor operators with either Lorentz indices or holomorphic indices, as follows:

$$\begin{array}{lll} \mathcal{N}=2 & D=4,8 & (Q,\ Q_\mu, Q_{\mu\nu^-}) \\ \mathcal{N}=1 & D=4 & (Q, Q_{\bar m},\ Q_{mn}) \\ \mathcal{N}=1 & D=6 & (Q, Q_{\bar m},\ Q_{mn},\ Q_{\bar m\bar n\bar p}) \end{array}$$

In some cases, auxiliary fields exist, giving "off-shell" closed transformations.

On the other hand, we found a reverse construction, that clarifies the structure of supersymmetry. In all cases, the set of both scalar and vector generators (Q, Q_μ) or $(Q, Q_{\bar m})$ are determined by horizontality conditions, which only involve fields related to the geometry of the Yang–Mills fields. The theory is in fact defined by these equivariant scalar and vector BRST differential operators. This reduced set of generators builds a closed "off-shell" algebra, which is large enough to determine the supersymmetric actions. It allows one to reconstruct by an untwisting procedure the complete structure of Poincaré supersymmetry. Moreover, the dependence on Faddeev–Popov ghosts can be computed, in a way that clearly allows a consistent and convenient covariant Yang–Mills gauge-fixing of the supersymmetric actions. Tensor generators, such as $Q_{\mu\nu^-}$, Q_{mn}, $Q_{\bar m\bar n\bar p}$, are decoupled sets of generators. They appear as additional symmetries of the actions that are defined by invariance under scalar and vector topological symmetries, but are not needed, neither for geometrical reasons, nor for the sake of defining the theories. However, by completion, they allow the construction of the irreducible set of on-shell supersymmetry spinorial generators in flat space.

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