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Tome 44 Nº 4 15 FÉVRIER 1983

LE JOURNAL DE PHYSIQUE - LETTRES

J. Physique — LETTRES **44** (1983) L-135 - L-141 15 FÉVRIER **1983**, PAGE L-135

Classification
Physics Abstracts
47.25

Pattern selection in a slowly varying environment

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(*Reçu le 18 octobre 1982, accepté le 23 décembre 1982*)

Résumé. — Habituellement, dans des conditions supercritiques, des structures cellulaires stationnaires, comme les rouleaux de Taylor-Couette, peuvent avoir n'importe quel nombre d'onde dans une bande finie, si la structure est illimitée. Si les conditions extérieures dépendent lentement de la position, la longueur d'onde devient fonction du paramètre de contrôle local. Si une région sous critique est reliée par une lente transition à une région supercritique, le nombre d'onde de la région supercritique est défini de façon unique, à des termes exponentiellement petits près.

Abstract. — Usually, in supercritical conditions, steady cellular structures (as rolls in Taylor-Couette experiments) may have any wavenumber in a finite band for an unbounded pattern. If the external conditions change slowly, the wavelength becomes a function of the local control parameter. If a subcritical region is smoothly connected to a supercritical one, the wavelength of steady rolls in the supercritical region is uniquely defined, up to exponentially small terms.

In a recent work [1] Kramer *et al.* have studied the problem of wavenumber selection in steady cellular structures. They reach the conclusion that a single wavenumber exists if the boundary pinning is eliminated, this wavenumber being the same as the one given by the condition found [2] by Paul Manneville and one of us [Y. P.], when the latter is applicable.

In this note, we analyse the wavelength selection in a slowly varying environment and propose an explanation to the computer findings of Kramer *et al.*

We shall first develop a formal approach and then apply it by using the amplitude equations, valid near the onset of emergence of cellular structures.

1. **General theory.** — A steady cellular structure is described by the solution of non linear (partial) differential equations. This solution is periodic with respect to one space variable, say x, the original equation being autonomous with respect to x. Furthermore, some external « control parameter » exists, as rotation speeds in Taylor-Couette experiments or a thickness for buckled plates, etc... We shall restrict ourselves to a single control parameter, denoted below as ε. The non linear equations of the problem are of the general form

$$\Lambda\{\varepsilon, A\} = 0, \tag{1}$$

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the quantity $A(x)$ is some function of x describing fluctuations around an uniform steady state and equation (1) is a non linear differential equation for $A(.)$. This has x-periodic solutions denoted below with the subscript zero. It expands in Fourier series as

$$A_0(x) = \sum_{n=1}^{\infty} a_n \sin(nqx), \tag{2}$$

q being the wavenumber of this periodic solution. As a general rule this wavenumber may vary in a whole band that depends on ε, so that the coefficients of the Fourier series (2) depend on ε and q. We shall introduce a phase $\psi(x)$ which, in the periodic case, is simply qx and equation (2) becomes :

$$A_0 = \sum_{n=1}^{\infty} a_n(\varepsilon, \psi_x) \sin(n\psi). \tag{3}$$

Now A_0 is formally a function of three variables $A_0[\varepsilon, \psi_x ; \psi]$. In the periodic case, it depends on x through ψ only. This function A_0 is such that

$$\Lambda \{ \varepsilon, A_0[\varepsilon, \psi_x ; \psi] \} = 0,$$

with the assumption $\psi = x\psi_x$, ψ_x constant wavenumber.

We shall assume now that ε depends slowly on x ($| \varepsilon/\varepsilon_x | \psi_x \gg 1$) and search an adiabatic solution in the vicinity of periodic solutions existing at ε constant. Equation (1) becomes

$$\Lambda \{ \varepsilon(x), A \} = 0, \tag{4}$$

where $\varepsilon(.)$ is slowly x-dependent. The adiabatic solution of (4) is

$$A(x) = A_0[\varepsilon(x), \psi_x(x) ; \psi] + A_1 + A_2 + \cdots, \tag{5}$$

where A_0 depends on $\varepsilon(x)$ and ψ_x, as specified by equation (3) and where $A_1 \sim \varepsilon_x$, $A_2 \sim (\varepsilon_x^2$ or $\varepsilon_{xx})$, etc...

To understand better the origin of terms as $A_1, A_2, \ldots$, consider the quantity $\frac{d^2 A_0}{dx^2} [\varepsilon(x), \psi_x ; \psi]$, which will generally appear in the explicit form of $\Lambda \{ \varepsilon, A_0 \}$ which has by assumption, the reflection symmetry $(x) \to (-x)$.

$$\frac{d^2 A_0}{dx^2} = \psi_x^2 A_{0,\psi\psi} + 2 \varepsilon_x \psi_x A_{0,\varepsilon\psi} + \psi_{xx} A_{0,\psi} + 2 \psi_x \psi_{xx} A_{0,\psi\psi_x} + (\text{H.O.T.}), \tag{6}$$

where (H.O.T.) stands here and thereafter for higher order terms. In the present case, (H.O.T.) $\sim \varepsilon_x^2$, ε_{xx} or ψ_{xxx}. The first term on the right hand side of (6) is of zeroth order with respect to the small parameter ψ_x being the local wavenumber. Its eventual contribution to $\Lambda \{ \varepsilon, A_0 \}$ is cancelled because A_0 is a solution of equation (1). The next terms all involve a first derivative with respect to the slow variable : ε_x (resp. ψ_{xx}) is the small factor in $2 \varepsilon_x \psi_x A_{0,\psi\varepsilon}$ (resp. in $\psi_{xx} A_{0,\psi}$). Generalizing the expansion (6), one has :

$$\Lambda \{ \varepsilon, A_0[\varepsilon(x), \psi_x(x) ; \psi] \} = 0 + \varepsilon_x \Lambda_1 + \psi_{xx} \Lambda_2 + (\text{H.O.T.}). \tag{7}$$

The zero on the right hand side is to remind that the zeroth order contribution cancel. Furthermore both Λ_1 and Λ_2 are of order zero with respect to the small derivatives as we expect that the wavenumber and ε will be function of each other and that $\psi_{xx} \sim \varepsilon_x \ll \psi_x \sim 0(1)$. In the above example (Eq. (6)) :

$$A_1 \equiv 2\,\psi_x A_{0,\psi\varepsilon} \quad \text{and} \quad A_2 \equiv A_{0,\psi} + 2\,\psi_x\, A_{0,\psi\psi_x}\,.$$

The next (small) term in the expansion of the solution of (4), i.e. A_1 in (5), has to cancel $\varepsilon_x A_1 + \psi_{xx} A_2$ in $A\{\varepsilon(x), A_0\}$. It is given by

$$\frac{\delta A}{\delta A}\{\varepsilon, A\}\bigg|_{A=A_0} A_1 + \varepsilon_x A_1 + \psi_{xx} A_2 = 0 \tag{8}$$

$\frac{\delta A}{\delta A}\{\varepsilon, A\}\Big|_{A=A_0}$ being the linear operator obtained by linearizing A around $A = A_0$, this linearization is allowed because $A_1(\sim \varepsilon_x) \ll A_0$. The operator $\frac{\delta A}{\delta A}\Big|_{A_0}$ has $\frac{\mathrm{d}}{\mathrm{d}x} A_0$ as a non trivial kernel owing to the translation invariance of the equations. Whence equation (8) leads to a solvability condition. Let A^+ be the adjoint kernel of $\frac{\delta A}{\delta A}\Big|_{A_0}$ with respect to the some inner product (H, G) defined for periodic functions of x with the same period, λ_0, as A_0. Usually, but not necessarily, this scalar product is

$$(H, G) \equiv \int_x^{x+\lambda_0} \mathrm{d}y\; H(y)\, G(y)\,.$$

Thus, the solvability condition for (8) reads

$$\varepsilon_x(A^+, A_1) + \psi_{xx}(A^+, A_2) = 0\,. \tag{9}$$

In the limit of very slow variations of ε and ψ_x, the two inner products in (9) define two functions of ε and ψ_x :

$$F_{1,2}(\varepsilon, \psi_x) \equiv (A^+, A_{1,2})\,,$$

and equation (9) becomes a differential equation relating ε and ψ_x, i.e. the local values of the control parameter and of the wavenumber

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\psi_x} = -\frac{F_2(\varepsilon, \psi_x)}{F_1(\varepsilon, \psi_x)}\,. \tag{10}$$

The equation (10) does not imply directly that, for a given ε an unique ψ_x (= wavenumber) exists.

Equation (10) implies that, if $\varepsilon(x)$ has a slow transition from ε^a for $x \to -\infty$ to ε^b for $x \to +\infty$ and if the arbitrary wavenumber is ψ_x^a for $x \to -\infty$, thus the wavenumber ψ_x^b at $x \to +\infty$ is uniquely fixed for a stationary solution. Our theory could explain the computer results of reference [1] as follows. Consider a variation of the control parameter that is subcritical for $x \to -\infty$, and reaches a constant supercritical value for $x \to +\infty$. Following the idea presented before, the wavevector in the supercritical region must be in the (ε, wavenumber) Cartesian plane on the *unique* integral curve of the differential equation (10) starting from the onset of instability in this plane.

Below we show on examples that this integral curve has a finite slope near the instability threshold. This slope is the same as the one of the curve defining the optimal wavenumber [3], when this notion has a meaning. Before to study a « concrete example », we give the phase portrait of the integral curves of (10) in the neighbourhood of the onset of bifurcation by using the amplitude theory.

2. **Slowly varying environment in the amplitude theory.** — As it will appear below, we shall need to consider the amplitude equations at the next order after the dominant term for describing the kind of phenomena in which we are interested. Let $\chi(x)$ be the complex amplitude of the one dimensional structure. If, near the instability threshold ($\varepsilon \sim 0_+$) steady fluctuations with the wavenumber q_0 are linearly unstable around the homogeneous rest state, then one assumes that, in this weakly non linear domain these fluctuations depend on x as $\frac{1}{2}(\chi(x)\, e^{iq_0 x} + \chi^*(x)\, e^{-iq_0 x})$ where χ is the complex « amplitude ». It satisfies $|\chi| \ll 1$ and $|\chi_x/\chi| \ll 1$. If χ does not depend on x, the Landau theory gives $|\chi| \sim \varepsilon^{1/2}$, the coefficient in front of $\varepsilon^{1/2}$ being computed by standard methods. A small shift in the wavenumber from q_0 to $q_0 + \delta$ ($\delta \ll q_0$) is accounted by taking $\chi(x) \sim \hat{\chi}\, e^{i\delta x}$, $\hat{\chi}$ x-independent. The amplitude theory [4] extends the Landau theory to describe (among others) non linear saturation of the instabilities and various phenomena due to small changes in the wavenumber. If one accounts for the « next order terms » this amplitude equation reads :

$$\varepsilon\chi + \chi_{xx} - |\chi|^2 \chi = i(\alpha\chi_x |\chi|^2 + \beta\chi_x^* \chi^2) . \tag{11}$$

The left hand side of this equation is the standard form of the amplitude equation, the single dimensionless parameter left being ε. It is small and positive by assumption and measures the distance of the control parameter to its threshold value.

The left hand side of (11) vanishes for periodic solution in the form

$$\chi = (\varepsilon - \delta^2)^{1/2}\, e^{i\delta x}$$

which means that (at least near threshold) the wavenumber of the cellular structure is $q_0 + \delta$, δ arbitrary in $[-\varepsilon^{1/2}, \varepsilon^{1/2}]$, q_0 being the wavenumber of the unstable modulation at threshold ($\varepsilon = 0$).

In the range $\delta \sim \varepsilon$, the term χ_{xx} on the left hand side of (11) is of the same order ($\delta^2\, \varepsilon^{1/2} \sim \varepsilon^{5/2}$) as the terms on the right hand side of (11) ($\delta\varepsilon^{3/2} \sim \varepsilon^{5/2}$), which are usually considered as subdominant. This explains why we shall need this right hand side, although this is small in the range $\delta \sim \varepsilon^{1/2}$ (instead of $\delta \sim \varepsilon$).

Equation (11) has a variational structure if it can be put into the form $\mathrm{D}V[\chi, \chi^*]/\mathrm{D}\chi^* = 0$, where $\mathrm{D}/\mathrm{D}\chi^*$ is a Fréchet derivative and $V[\chi, \chi^*]$ a real functional of χ and χ^*. This is realized if $\beta = 0$ in equation (11) [compare to Eq. (2.6*b*) in Ref. 5] with

$$V[\chi, \chi^*] = \int \mathrm{d}x \left(\varepsilon |\chi|^2 - |\chi_x|^2 - \frac{1}{2}|\chi|^4 - \frac{i\alpha}{4}(\chi_x \chi^* |\chi|^2 - \chi_x^* \chi |\chi|^2) \right) .$$

Notice that α and β, as they occur in (11) must be real, due to the reflexion symmetry $x \to (-x)$, $\chi \to \chi^*$.

If one considers periodic solutions of (11) in the form

$$\chi = \hat{\chi}\, e^{i\delta x} ,$$

the optimal wavenumber gives the largest V per unit length. This potential per unit length is

$$(\varepsilon - \delta^2) |\hat{\chi}|^2 - \tfrac{1}{2} |\hat{\chi}|^4 (1 - \alpha\delta) . \tag{12}$$

It is stationary under amplitude variations if $|\hat{\chi}|^2 = \dfrac{\varepsilon - \delta^2}{1 - \alpha\delta}$, and its value is therefore $\dfrac{1}{2} \times \dfrac{(\varepsilon - \delta^2)^2}{1 - \alpha\delta}$. It has a maximum at $\delta = \alpha\varepsilon/4$ near $\varepsilon = 0$, with respect to variations of δ, and this defines the optimal wavenumber.

If ε, as it occurs in equation (11) is now a slowly varying function of x, one may apply the method of construction of an adiabatic solution described before to find the relation between the slowly varying control parameter $\varepsilon(x)$ and the local wavenumber $q_0 + \delta(x)$. In the amplitude theory, translational invariance is equivalent to the phase invariance of equation (11) : if χ is any solution, $\chi\, e^{i\varphi}$, φ constant and real is also a solution. The infinitesimal translation ($\varphi \ll 1$) changes χ into $\chi - i\varphi\chi$, so that the kernel of the linearized operator is $(i\chi)$. The solvability condition equivalent to (9) is obtained by cancelling terms where the zeroth order solution is multiplied by i.

Thus our starting point is again equation (11) wherein ε is supposed to be x-dependent. The zeroth order solution is

$$\chi = \hat{\chi}(\varepsilon(x), \delta(x)) \exp i\psi(x)\,, \tag{13}$$

where $\psi_x = \delta(x)$, the local wavenumber being $q_0 + \delta(x)$. Furthermore, $\hat{\chi}(\varepsilon, \delta)$ is the function obtained by putting ε and δ constant into (11)

$$\hat{\chi}(\varepsilon, \delta) \equiv \left[\frac{\varepsilon - \delta^2}{1 + \delta(\beta - \alpha)}\right]^{1/2}.$$

Inserting now the zeroth order solution (13) into (11), one obtains first order quantities ($\sim \delta_x$ or ε_x) in the gradient expansion. The solvability condition follows and reads :

$$(\varepsilon_x\, \hat{\chi}_\varepsilon + \delta_x\, \hat{\chi}_\delta)\,(2\,\delta - (\alpha + \beta)\,|\,\hat{\chi}\,|^2) + \delta_x\, \hat{\chi} = 0\,, \tag{14}$$

which is the explicit form taken here by equation (9) [recall $\delta = \psi_x$].

Collecting all dominant terms in the domain $\delta \sim \varepsilon(\ll 1)$, one has

$$\varepsilon_x(2\,\delta - (\beta + \alpha)\,\varepsilon) + 2\,\delta_x\,\varepsilon = 0 \tag{15a}$$

or

$$\frac{d\phi(\varepsilon, \delta)}{dx} = 0 \tag{15b}$$

with

$$\phi \equiv 2\,\delta\varepsilon - (\alpha + \beta)\frac{\varepsilon^2}{2}.$$

Thus, at least near $(\varepsilon, \delta) = 0$, i.e. near the instability threshold, the integral curves of (15a) are the level curves of the adiabatic invariant $\phi(\varepsilon, \delta)$. These curves draw a system of hyperbolae with the common asymptotes $\varepsilon = 0$ and $\delta = \frac{(\alpha + \beta)\,\varepsilon}{4}$. The curve passing through the representative point of the onset of instability is the degenerate hyperbolae made of the two asymptotes. The line $\varepsilon = 0$ recalls that $\chi = 0$ is also a solution if ε depends on x, the other line is $\delta = (\alpha + \beta)\frac{\varepsilon}{4}$ and for $\beta = 0$ gives the wavenumber selected by the optimization principle.

3. **Application to the model of Kramer *et al.*** [1]. — To get analytical result from our consideration one must limit oneself to the vicinity of the instability threshold, where the weak amplitude approximation works. This explains why, although our general scheme of calculation can be applied to the strongly non linear case, the results presented below concern the weakly non linear domain only. In this domain it is only necessary to obtain α and β, as they appear in equation (11) to find the wavenumber selected in slowly varying external conditions. We outline this calculation

for the steady state reaction diffusion equations of reference [1],

$$D_1 u_{xx} + a_1 u(1 - u^2) - b_1 v = 0 \tag{16a}$$

$$D_2 v_{xx} + a_2 v(1 - v^2) + b_2 u = 0 . \tag{16b}$$

By simple substitutions, they become

$$\left(\varepsilon - \left(\frac{d^2}{dx^2} + q_0^2\right)^2\right) u - \lambda_1 u^3 - \lambda_2(u_{xx} u^2 + 2 u_x^2 u) - \lambda_3 u_{xx}^3 - \lambda_4 u u_{xx}^2 - - \lambda_5 u_x^2 u_{xx} + (\text{H.O.T.}) = 0 \tag{16c}$$

with

$$q_0^2 \equiv \frac{a_1 D_2 - a_2 D_1}{2 D_1 D_2} (> 0 \text{ by assumption}), \quad \varepsilon \equiv \frac{(b_1 b_2 - a_1 a_2) D_1 D_2 - (a_1 D_2 - a_2 D_1)^2}{4 D_1^2 D_2^2},$$

$$\lambda_1 \equiv \frac{a_1 a_2(b_1^2 + a_1^2)}{b_1^2 D_1 D_2}, \quad \lambda_2 \equiv -\frac{3 a_1}{D_1}, \quad \lambda_3 \equiv \frac{a_2 D_1^2}{b_1^2 D_2},$$

$$\lambda_4 \equiv \frac{3 a_1 a_2 D_1}{D_2 b_1^2} \quad \text{and} \quad \lambda_5 \equiv \frac{3 a_1^2 a_2}{b_1^2 D_2}.$$

The amplitude equation can be derived from (16*c*) in a form equivalent to equation (11), and one obtains :

$$\varepsilon\chi + 4 q_0^2 \chi_{xx} - \frac{|\chi|^2 \chi}{\chi_0^2} = i(\tilde{\alpha} |\chi|^2 \chi_x + \tilde{\beta}\chi^2 \chi_x^*),$$

with

$$\frac{1}{\chi_0^2} \equiv \frac{1}{4}(3 \lambda_1 - \lambda_2 q_0^2 - 3 \lambda_3 q_0^6 + 3 \lambda_4 q_0^4 - \lambda_5 q_0^4) \quad (> 0 \text{ by assumption})$$

$$\tilde{\alpha} \equiv q_0 \lambda_2 + 3 \lambda_3 q_0^5 - 2 \lambda_4 q_0^3 + \frac{3}{4} q_0^3 \lambda_5$$

and

$$\tilde{\beta} \equiv q_0 \frac{\lambda_2}{2} - \frac{3 \lambda_3 q_0^5}{2} + \lambda_4 q_0^3 - \frac{\lambda_5 q_0^3}{4}.$$

With the above defined parameters, the wavenumber selected by non pinning boundary conditions (= for a structure at $1 \gg \varepsilon > 0$ joined smoothly to a subcritical region) is

$$\frac{\delta}{q_0} \simeq \frac{\varepsilon\chi_0^2(\tilde{\alpha} + \tilde{\beta})}{16 q_0^3}.$$

4. **Conclusion.** — We have shown that for slowly varying external conditions, a differential equation relates the wavenumber of a steady structure and the control parameter. If this parameter varies in space from sub- to supercritical values, an unique wavenumber exists for a steady state in the supercritical domain.

This contrasts with the « pinning » boundary conditions (b.c.), as the one generated by the condition $\varepsilon > 0$ for $x \geqslant 0$ and $\varepsilon = -\infty$ for $x < 0$. In this case a whole band of selected wavenumber is known to exist [6] for steady structures in the supercritical domain. It is easy to think of a continuum of b.c. interpolating between pinning and non pinning b.c. For instance $\varepsilon_\mu(x) =$

$\varepsilon_0(1 - e^{-\mu x})$, $\varepsilon_0 > 0$. For $\mu \to 0\,|_+$ this tends to a « non pinning » b.c., although for $\mu \to +\infty$ this becomes a « pinning » b.c. By continuity the band width of selected wavenumber that is finite for $\mu = +\infty$ shrinks to zero for non pinning b.c. ($\mu = 0_+$). One may conjecture that this width decreases as $\exp(-c/\mu)$ $c > 0$ as $\mu \to 0_+$. Actually, as quoted by Landau [7], the adiabatic Ehrenfest invariants, as the function $\phi(\varepsilon, \delta)$ introduced in equation (16*b*) are truly constants up to exponentially small terms. This explains why the band width selected for non pinning b.c. is not accessible to our gradient expansion, because it involves quantities of order $\exp(-c/|\varepsilon_x|)$, clearly outside of the scope of the perturbation calculation in powers in ε_x. Nevertheless, as noted by Landau too, this order of magnitude estimate fails if the control parameter $\varepsilon(x)$ is not an analytic function of x, which is the case in the computer experiments of reference [1]. More precisely the order of magnitude of the non constant part of the adiabatic invariant is fixed by the distance to the real axis (measured in units defined by the fast modulation) of the singularity of the complex extension of $\varepsilon(.)$ closest to this real axis.

Acknowledgments. — Y. Pomeau thanks very much L. Kramer for sending him the preprint in reference [1] and discussing various questions related to this.

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