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► To cite this version:

A. Fogden. WEIERSTRASS REPRESENTATION OF PERIODIC MINIMAL SURFACES. Journal de Physique Colloques, 1990, 51 (C7), pp.C7-149-C7-156. 10.1051/jphyscol:1990714 . jpa-00231113

HAL Id: jpa-00231113

<https://hal.science/jpa-00231113>

Submitted on 1 Jan 1990

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WEIERSTRASS REPRESENTATION OF PERIODIC MINIMAL SURFACES

A. FOGDEN

Department of Applied Mathematics, Research School of Physical Sciences, Australian National University, GPO Box 4, Canberra 2601, Australia

Abstract - A general algorithm for the construction of infinite periodic minimal surfaces (IPMS) via their Weierstrass function is outlined, enabling parameterisation of known examples, and forming the bases of a systematic search for all such surfaces. For the 'regular' class of IPMS (including the low genus surfaces) the Weierstrass function has a simple form, permitting a complete listing of all possibilities. The method is then readily extended to parametrisation of the 'irregular' class of IPMS, and its generally illustrated with several previously unsolved examples.

1 - INTRODUCTION

Infinite periodic minimal surfaces (IPMS) have established an important place in a variety of areas ranging from crystalline structures to macromolecular assemblies. A complete theoretical treatment of phase behaviour in such systems necessitates analysis of all possible structures, thus relying on the existence of a full vocabulary of IPMS within each space group, and a precise mathematical description of each surface.

The union of differential geometry and complex function theory yields an analytic representation of minimal surfaces. The surface is transformed into the unit sphere via the Gauss map v , under which the image of a point (x, y, z) of a surface $F(x, y, z) = 0$ is the surface normal vector

$$(x', y', z') = \frac{\nabla F}{|\nabla F|},$$

then mapped via stereographic projection σ of this sphere coordinate to the point in the complex plane

$$\omega = \frac{x' + iy'}{1 - z'}.$$

Then the composition $\sigma \circ v$ conformally maps the neighbourhood of any point of non-zero Gaussian curvature on the surface to a simply connected region of $\mathbb{C}$, on which its inverse $\Phi = v^{-1} \circ \sigma^{-1}$ was found by Weierstrass to be of the general form

$$(x, y, z) = \text{Re} \int^{\omega} (1 - \omega'^2, i(1 + \omega'^2), 2\omega') R(\omega') d\omega' \quad (1)$$

for some complex function $R(\omega)$ analytic in this region/1/. The Weierstrass function $R(\omega)$ thus completely specifies the local differential geometry of the surface; in particular, the Gaussian curvature at the point corresponding to ω is $/2/$

$$K = -4(1 + |\omega|^2)^{-4} |R(\omega)|^2. \quad (2)$$

The domain of validity of the local parametrisation (1) for a particular minimal surface is determined by analytic continuation of the function $R(\omega)$. Now ω_i is a

singular point of the continuation if its image $\Phi(\omega_i)$ is a point of zero Gaussian curvature (called a flat point), in the vicinity of which the inverse Gauss map v^{-1} , and hence Φ and R , is in general multivalued. The natural domain of single-valued definition of such a multivalued function $R(\omega)$ is a Riemann surface over $\mathbb{C}$, branched at $\{\omega_i\}$, thus establishing the further connection between global representation of minimal surfaces and the theory of Riemann surfaces.

An IPMS is Gauss mapped to an infinity of superpositions of the image of the fundamental topological unit, from which the entire surface is generated by translation alone. This image is a compact covering of the sphere, corresponding to the Riemann surface of the Weierstrass function, so, by a general result /3/, this function is algebraic. Its form is then dictated by the singularity structure which is evident from simple consideration of the local differential geometry of the surface. In particular, consider a member of the finite set of flat points on the topological unit of the IPMS, and define the ratio of the angle of intersection of any two geodesics through this point with the Gauss map image of this angle to be the degree $b + 1$ of the Gauss map there. This corresponds to a branch point of order b on the Riemann surface, at which the $b + 1$ branches of the Weierstrass function joined there diverge as $a/b+1$ for some number a coprime to $b + 1$.

2 - THE 'REGULAR' CLASS OF IPMS

It is convenient to distinguish the 'regular' class of IPMS for which each flat point has normal vector (Gauss map image) coincident only with those of other flat points exhibiting identical degeneracy on the surface, and hence for which the branch point structure characterising the Riemann surface is extremely simple. It may be readily proven that the Weierstrass function of such an IPMS must have the form

$$R(\omega) = e^{i\theta} \prod_{i=1}^n (\omega - \omega_i)^{-\frac{a_i}{b_i+1}} \quad (3)$$

where $\{\omega_i\}_{i=1}^n$ denote the distinct flat point images. Without loss of generality θ is assumed real and the isometric mapping of the minimal surface effected by the constant factor $e^{i\theta}$ is referred to as the Bonnet transformation /2/. In this study attention is restricted in the most part to the $\theta = 0$ surface in the family of Bonnet associates. The simplicity of the form (3) permits a systematic search for all IPMS within the regular class, via imposition of necessary conditions for the minimal surface specified by a choice of parameter set $\{\omega_i, a_i, b_i\}_{i=1}^n$ to be infinite periodic. In particular, a suitable coordinate rotation of the closed complex plane (if necessary) will ensure that the point at infinity (the north pole of the Riemann sphere) is not a flat point image. Then, from equation (2), $R(\omega) \sim \omega^{-4}$ as $\omega \rightarrow \infty$, yielding the constant

$$\sum_{i=1}^n \frac{a_i}{b_i + 1} = 4 \quad (4)$$

Additional conditions arise from the general geometrical and topological connections between the fundamental surface unit and its Gauss map image, the Riemann surface over the unit sphere of the Weierstrass function /4/. The Riemann-Hurwitz formula states that the Euler characteristic of a Riemann surface is simply specified by its number of sheets s and the total branch point order W as follows /5/

$$\chi = 2s - W \quad (5)$$

Note that χ is also the Euler characteristic of the fundamental unit of the IPMS since the Gauss map is a homeomorphism between the two surfaces. The Gauss-Bonnet theorem and the relation connecting Euler characteristic and total integral curvature, together demand that the Gauss map image satisfy /5/

$$\chi = -2s \quad (6)$$

(implying that the genus of an orientable IPMS is $g = -1/2\chi + 1 = s + 1$) and hence from equation (5)

$$W = 4s \quad (7)$$

By definition of regularity, above any flat point image ω_i there lie only branch points of order b_i on the Riemann surface, at which $b_i + 1$ sheets are pinned. Hence there are $s/(b_i + 1)$ such points above ω_i (implying that s is a multiple of each $b_i + 1$) and the total branch point order is

$$W = s \sum_{i=1}^n \frac{b_i}{b_i + 1}$$

Combining the above two equations yields the following constraint on the set of branch point orders:

$$\sum_{i=1}^n \frac{b_i}{b_i + 1} = 4 \quad (8)$$

and on subtracting equation (4), the result

$$\sum_{i=1}^n \frac{b_i - a_i}{b_i + 1} = 0.$$

If $a_i > b_i$ for some i then the singularity of $R(\omega)$ at ω_i is non-integrable and the surface diverges to infinity, so true IPMS, for which the fundamental unit is compact, must satisfy $a_i \leq b_i$ for each i . The above equation then implies $a_i = b_i$ for each i , thus the general form of the Weierstrass function for the regular class of IPMS is now

$$R(\omega) = e^{i\theta} \prod_{i=1}^n (\omega - \omega_i)^{\frac{-b_i}{b_i+1}} \quad (9)$$

subject to the constraint (8). In analysing the surface in the vicinity of a specific point, it is convenient to choose a coordinate orientation for which the point is mapped to the north pole of the unit sphere, projected to infinity in the closed complex plane. If the point under consideration is a flat point of the surface, say $\omega_n = \infty$, then the form (9) ceases to be appropriate and is replaced by

$$R(\omega) = \prod_{i=1}^{n-1} (\omega - \omega_i)^{\frac{-b_i}{b_i+1}} \quad (10)$$

subject to the previous constraint (8).

To each IPMS there corresponds a Flächenstück (surface element) from which the entire surface is generated by repeated analytic continuation in its boundaries. These geodesic bounding arcs are lines of curvature and/or asymptotic curves of the surface. If the lines of curvature are planar and the asymptotic curves are linear then they define, respectively, mirror planes and two-fold axes. For the representation (1), the conditions

$$(x, y, z)(\bar{\omega}) = \pm (x, -y, z)(\omega) \quad (11)$$

for the real axis $\text{Im } \omega = 0$ to be the image of a line of curvature in the x - z plane (positive sign) or an asymptote along the y -axis (negative sign), reduce respectively to

$$R(\bar{\omega}) = \pm R(\omega) \quad (12)$$

In either case this functional constraint is satisfied by the form (9) (or its modified form (10) if a flat point image resides at infinity) only if the sets of branch points of equal order are conjugate invariant, that is,

$$\{\bar{\omega}_i\} = \{\omega_i\}, \quad b_i \text{ constant} \quad (13)$$

To analyse a general surface geodesic (with Gauss map image some great circle arc), transform the representation (1) by a rotation of the surface and unit sphere carrying the plane of interest into the x - z plane and an associated bilinear change of complex variable mapping the projected great circle image onto the real axis $\text{Im } \omega = 0$, then apply the criteria (11). In particular, consider a geodesic passing through a point imaged at the north pole of the unit sphere, and thus projected to a segment of a ray through the origin and infinity in the complex plane, at angle, say, ϕ to the positive real axis. Equation (11) then yields the generalisation of conditions (12):

$$e^{-i2\phi} R(e^{-i\phi}\bar{\omega}) = \pm e^{i2\phi} R(e^{-i\phi}\omega) \quad (14)$$

which are met by the appropriate form (9) or (10), with the constraint (8), only if the sets of rotated branch points of equal order are conjugate invariant:

$$\{e^{i\phi}\bar{\omega}_i\} = \{e^{i\phi}\omega_i\}, \quad b_i \text{ constant} \quad (15)$$

with ϕ assuming the particular values

$$\phi = \begin{cases} \left(\frac{b_n + 1}{b_n + 2} \right) m\pi & \text{for plane lines of curvature} \\ \left(\frac{b_n + 1}{b_n + 2} \right) \left(m + \frac{1}{2} \right) \pi & \text{for linear asymptotes} \end{cases} \quad (m \in \mathbb{Z}) \quad (16)$$

Thus there are a maximum of $b_n + 2$ plane lines of curvature (or linear asymptotes),

intersecting at angles of some multiple of $\frac{\pi}{(b_n + 2)}$ at a flat point of order $b_n \geq 0$ on the minimal surface - a general result of differential geometry.

The existence of perpendicular rotational symmetry about a particular point on the surface may be examined in a similar manner, by seeking angles ϕ' for which the rotation (about the z -axis in the coordinate system employed here) transforms the surface into itself. From the representation (1), this is equivalent to seeking solutions of

$$e^{-i2\phi'} R(e^{-i\phi'}\omega) = R(\omega) \quad (17)$$

which exist for the form (9) or (10), again subject to equation (8), only if the sets of branch points of equal order are invariant under the rotation, that is,

$$\{e^{i\phi}\omega_i\} = \{\omega_i\}, \quad b_i \text{ constant} \quad (18)$$

and ϕ' is specified by

$$\phi' = \left(\frac{b_n + 1}{b_n + 2} \right) 2m'\pi \quad (m' \in \mathbb{Z}) \quad (19)$$

Thus the minimal surface may exhibit perpendicular rotational symmetry at angles of

$\frac{2\pi}{b_n + 2}$ about a flat point of order $b_n \geq 0$, so the maximal symmetry number is $b_n + 2$, as expected. Hence if the surface possesses plane lines of curvature (and/or linear asymptotes) then the angle of rotational symmetry is twice that of the intersection of these curves - a geometrically obvious general result obtained by replacing ω by $e^{-i\phi}\omega$ in equation (14) and utilising condition (12). However the converse does not always apply. For the family of Bonnet associates, equation (14) is no longer satisfied while equation (17) is invariant, so rotational symmetries are preserved in the absence of plane lines of curvature and linear asymptotes.

The lines of curvature and/or asymptotic curves bounding the Flächenstück of an IPMS are constrained to lie in the faces of a cell which partitions three-dimensional Euclidean space by congruence alone. The Gauss map image of the Flächenstück is a spherical geodesic polygon, that is, its edges are arcs of great circles. Reflection (or rotation) of the cell, and hence the Flächenstück, in a face (or about an edge) is then equivalent to reflection of the polygon on the unit sphere in the corresponding edge, and thus to analytic continuation of the Weierstrass function to this region. Repetition of this leads to construction of the fundamental topological unit of the IPMS, and in the process the polygon tessellates the Riemann surface of the Weierstrass function over the unit sphere. The branch point structure is thus obtained from propagation of the flat point images in the polygon unit by the repeated reflections of the tessellation.

Thus in order to generate the regular class of IPMS via their common Weierstrass functional form (9) or (10), consider the spherical geodesic polygons tessellating a finite number of copies of the unit sphere and assigned a branch point set which, in the tessellation, yields a distribution $\{\omega_i, b_i\}_{i=1}^n$ satisfying equation (9), with the property that only branch points of the same order are superposed. In seeking such tessellating polygons, a natural starting point is the geodesic triangles which tile the sphere in a finite number of coverings. An exhaustive list of the fifteen basic cases of these (one of which permits an infinity of subdivisions) was compiled last century by Schwarz /6/. The vocabulary of possible Flächenstück images is thus defined, for each case, as the polygons, with assigned branch points, which comprise a finite union of tiles and tessellate some number of copies of this underlying tiling. On repeated reflection of the polygonal unit, the resulting branch point distribution is then identical for each sheet only if it is identical for each underlying tile. In this manner the problem reduces to consideration of the permissible allocations of branch points to the single tile representing each Schwarz case. Further, these points must then be propagated from one tile to its neighbours by symmetry operations of the tessellation - namely edge reflection or composition of this with reflection in any existing internal symmetry axis of the tile.

With the requirement that the Schwarz triangle tessellations be consistent with the regularity of the Gauss map covering, it is readily shown that it suffices to consider only cases 1 (with polar angle π/n , $n \geq 2$), 2, 4 and 6. Addressing the remaining condition (8), the number of distinct images, under the reflection operations of the tessellation, of a branch point on a single Schwarz triangle (defined by vertex angles

$(\lambda_1, \lambda_2, \lambda_3)\pi$) is given by the Gauss-Bonnet theorem. With $\{b_j\}_{j=1}^3$, $\{b'_k\}_{k=1}^{N'}$, $\{b''_l\}_{l=1}^{N''}$ denoting the set of non-negative orders of the branch points distributed at the vertices and on the edges and face, respectively, of the single tile, equation (8) becomes

$$\sum_{j=1}^3 \lambda_j \frac{b_j}{b_j + 1} + \sum_{k=1}^{N'} \frac{b'_k}{b'_k + 1} + 2 \sum_{l=1}^{N''} \frac{b''_l}{b''_l + 1} = 2 \left(\sum_{j=1}^3 \lambda_j - 1 \right) \quad (20)$$

For each possible Schwarz case listed above, the non-negative integer sets satisfying this equation are given in Table 1 - in particular, cases 2 and 6 yield no non-trivial solutions.

Table 1 - List of branch point set solutions (ϕ denotes an empty subset).

case no.	$(\lambda_1, \lambda_2, \lambda_3)$		$\{\{b_j\}_{j=1}^3, \{b'_k\}_{k=1}^{N'}, \{b''_l\}_{l=1}^{N''}\}$
1	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n}\right)$	n	
		2	$\{\{1, 2, 5\}, \phi, \phi\}, \{\phi, \{1, 1\}, \phi\}, \{\{0, 1, 1\}, \{1\}, \phi\}, \{\{0, 0, 2\}, \{2\}, \phi\}, \{\{0, 0, 1\}, \{3\}, \phi\}, \{\phi, \phi, \{1\}\}$
		3	$\{\{0, 3, 7\}, \phi, \phi\}, \{\{0, 4, 4\}, \phi, \phi\}, \{\{0, 5, 3\}, \phi, \phi\}, \{\{0, 8, 2\}, \phi, \phi\}, \{\{1, 5, 0\}, \phi, \phi\}, \{\phi, \{2\}, \phi\}, \{\{0, 0, 1\}, \{1\}, \phi\}$
		4	$\{\{0, 3, 1\}, \phi, \phi\}, \{\phi, \{1\}, \phi\}$
		5	$\{\{0, 1, 3\}, \phi, \phi\}, \{\{0, 4, 0\}, \phi, \phi\}$
		6	$\{\{0, 1, 1\}, \phi, \phi\}, \{\{0, 2, 0\}, \phi, \phi\}$
		8	$\{\{0, 1, 0\}, \phi, \phi\}$
4	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$		$\{\{0, 1, 0\}, \phi, \phi\}, \{\{0, 0, 2\}, \phi, \phi\}$

In general each of the above solutions yields a multiplicity of possible classes given by the set of distinct allocations of the edge branch points and the choices of the operations defining the branch point propagation over the three edges. For each possibility, the family of edges assigned the reflection operation and common to a particular vertex and/or branch point on the tiled Riemann surface is then identified as the images of plane lines of curvature or linear asymptotes in accordance with the local condition (16). If continuation of this procedure leads to a consistent global identification on the entire Riemann surface then the Flächenstück image is the minimal polygon on this surface with all edges the image of such curves. Further, if the cell defined by the Flächenstück is space-filling then this case corresponds to an IPMS (which may possibly be self-intersecting), and its adjoint IPMS (Bonnet associate for $\theta=\pi/2$) obtained by interchanging plane lines of curvature and linear asymptotes.

Applying this procedure to the solution in Table 1, the complete categorisation is not detailed here⁽¹⁾; only specific illustrative cases are considered. First analysing Schwarz case 4, generating IPMS of cubic symmetry, the solution $\{\{0, 0, 2\}, \phi, \phi\}$ gives the I-WP surface /7/, while all other distributions of six second order flat point images: $\{\{0, 0, 2\}, \{2\}, \phi\}$, $\{\phi, \{2\}, \phi\}$, $\{\{0, 2, 0\}, \phi, \phi\}$ in Schwarz 1 subcases $n = 2, 3$ and 6 , respectively, yield no IPMS, so in particular there are no reduced symmetry relatives of the I-WP surface within the regular class. The other solution $\{\{0, 1, 0\}, \phi, \phi\}$ generates the D (and the adjoint P) surface /2/, and of its lower symmetry Schwarz 1 derivatives $\{\phi, \{1, 1\}, \phi\}$, $\{\{0, 1, 1\}, \{1\}, \phi\}$, $\{\phi, \phi, \{1\}\}$ for $n = 2$, $\{\{0, 0, 1\}, \{1\}, \phi\}$ and $\{\phi, \{1\}, \phi\}$ for $n = 3$ and 4 , respectively, IPMS corresponding to rhombohedral, tetragonal, and a pair of orthorhombic, distortions of the D and P surfaces are found to exist. In addition, the $n = 3$ set $\{\{0, 0, 1\}, \{1\}, \phi\}$ also gives the H and H-CLP surfaces /8/, the self-adjoint special case of the latter being the $n = 6$ solution $\{\{0, 1, 1\}, \phi, \phi\}$. The CLP family, exhausting all remaining IPMS derived via Schwarz triangles from a regular distribution of eight first order branch points, pertains to the particular solutions of $\{\phi, \{1, 1\}, \phi\}$, $\{\phi, \{1\}, \phi\}$ and $\{\{0, 1, 0\}, \phi, \phi\}$ for $n = 2, 4$ and 8 , in which all branch points reside on a common great circle. Figure 1 gives the tetragonal CLP surface /9/ (with its self adjoint special case for $n = 8$) together with the IPMS resulting from its orthorhombic distortion; further it illustrates that admission of the degenerate tessellation $n = 1$ (in which the triangle is replaced by a sphere quadrant) introduces a monoclinically distorted IPMS, as it does in other cases above.

(1) Fogden, A. and Hyde, S.T., in preparation.

Fig. 1 - Stereographically projected Gauss map image, comprising one of the pair of Riemann surface sheets (with p and l denoting plane line of curvature and linear asymptote images), and corresponding Flächenstück, of the (a) tetragonal, (b) orthorhombic and (c) monoclinic IPMS of the CLP family.

As a final case, the solution $\{(0,1,3), \phi, \phi\}$ for $n = 5$ permits a consistent identification of plane line of curvature and linear asymptote images on the tiled Riemann surface, giving the Flächenstück illustrated in Figure 2.

Fig. 2 - Surface element given by the $n = 5$ solution $\{(0,1,3), \phi, \phi\}$.

As this pentagonal analogue of the CLP Flächenstück possesses flat points (order three) of five-fold symmetry on the surface, the bounding cell is not associated with a space group, so repeated reflection and rotation of the surface element results in self-intersections of arbitrary density in $\mathbb{R}^3$. However introduction of a suitable set of defects could yield an infinite surface displaying local five-fold symmetry, thus offering a description of the atomic positions of quasicrystalline alloy phases.

3 - THE 'IRREGULAR' CLASS OF IPMS

The generalisation to the 'irregular' class admits IPMS for which the set of points on the fundamental unit with common normal vector may contain points of different Gauss map degree (for example, points of zero and non-zero Gaussian curvature). Recalling our previous observation that the Weierstrass function of a general IPMS is algebraic since its Riemann surface has a finite number s of sheets, this function is then defined by a s^{th} degree polynomial equation /3/

$$\sum_{m=0}^s a_m(\omega) R^m = 0 \quad (21)$$

for some set of polynomials $\{a_m(\omega)\}_{m=0}^s$. The regular class are then naturally recovered as the special case $a_m(\omega) = 0$ for $m = 1, \dots, s-1$.

The method of constructing polynomial sets corresponding to IPMS is illustrated here for the known irregular surface discovered by Neovius, for which the Gauss map image of the Flächenstück is the geodesic polygon comprising four triangles of the underlying Schwarz case 4 tiling, and assigned branch points, as given in Figure 3.

Fig. 3 - Stereographically projected Schwarz case 4 tessellation with Neovius surface Gauss map image (p and l as for Fig. 1).

The topological unit genus $g=9$, implying from above that $s=8$, consistent with the tessellation of eight copies of the unit sphere by the geodesic polygon. The branch point structure of the Riemann surface generated by this tessellation consists of two second order branch points (both pinning three sheets) and two regular points under each of the six $\pi/4$ vertices of the Schwarz tiling, together with one first order branch point (pinning a pair of sheets) and six regular points under each of the eight $\pi/3$ vertices, thus satisfying the general condition (7). Considering the two 'regular' class IPMS likewise derived from the Schwarz case 4 tiling - the D and I-WP surfaces discussed previously, the Gauss map of the Neovius surface is locally identical to that of the D surface at the first order branch point over the $\pi/3$ vertex and the regular points over the $\pi/4$ vertex, and to that of the I-WP surface at the second order branch points over the $\pi/4$ vertex and the regular points over the $\pi/3$ vertex. The constraint that the form (21) asymptotically recovers the appropriate 'regular' class form in

these limits then specifies the polynomial set $\{a_m(\omega)\}_{m=0}^8$ up to multiplicative factors. With these factors determined by the symmetry conditions that the discriminant of equation (21) vanish only at the vertices of the underlying tiling, the Neovius surface Weierstrass function is the solution $R(\omega)$ of the equation

$$\frac{1}{4} p_D p_I^4 R^8 \mp p_I^4 R^6 \pm 2 p_D R^2 + 1 = 0 \quad , \quad (22)$$

where

$$p_D = \omega^8 + \frac{28}{3} \omega^6 - \frac{14}{3} \omega^4 + \frac{28}{3} \omega^2 + 1 \quad , \quad p_I = \omega^6 - 5\omega^4 - 5\omega^2 + 1 \quad (23)$$

are the Weierstrass polynomials of the D and I-WP surfaces in the chosen orientation, and the $\pm$ sign denotes the surface and its adjoint. The solution is equivalent to that derived in the original Neovius parametrisation /10/.

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