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FINITE-AMPLITUDE ACOUSTIC WAVE PROPAGATION THROUGH GAS-DROPLET MIXTURES : A BURGERS' EQUATION MODEL

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Shortened title for page heading : acoustics in gas ~ droplet mixtures.

Résumé.- Une équation modifiée de Burger est présentée, qui convient pour des milieux qui possèdent à la fois l'absorption thermovisqueuse acoustique ainsi que divers mécanismes d'absorption par relaxation. L'équation est appliquée au problème spécifique de propagation d'une onde plane à travers le brouillard air-eau, un milieu qui possède des sources de relaxation qui ont pour origine des mouvements de molécules de gaz et des interactions de transfert de masse, moment, et énergie entre le gaz et les gouttelettes d'eau suspendues. Cette méthode, utilisant l'équation de Burger, donne des renseignements sur l'atténuation et da dispersion d'ondes infinitesimales ainsi que sur les effects d'amplitude finie dans les brouillards du type air-eau.

Abstract.- An extended Burgers' equation, suitable for media which exhibit both thermoviscous acoustic absorption as well as various mechanisms of relaxational absorption, is presented. The equation is applied to the specific problem of plane wave propagation through an air-water fog, a medium which contains relaxation sources arising both from gas molecular motions and from mass, momentum, and energy transfer interactions between the gas and the suspended water droplets. The Burgers' equation approach provides information on infinitesimal wave attenuation and dispersion, and on finite-amplitude wave effects in air-water fogs.

1. List of symbols.

A	$1+C_V L(L-1)(\gamma_V-1)H_V$
B	$\lceil(\gamma-1(L-1)-1|\rceil(\gamma_V-1)(L-1)H_V-R_V/R_G|[\gamma A]^{-1}$
C_m	(droplet mass/cm^3) / (fog mass/cm^3)
C_V	(water vapour mass/cm^3)/(fog mass/cm^3)
c	speed of sound in fog
$\bar{c}$	specific heat of droplet
c_o	speed of sound in air = $(\gamma R_G T_o)^{1/2}$
c'	$1-c/c_o$
D	diffusivity of water vapour in air
f	sound frequency
g	dimensionless retarded time $=\omega(t-x/c)$
H_p	droplet specific heat/air specific heat at constant volume
H_V	ratio of water vapour to air specific heats at constant volume
h	droplet heat of vaporization
i	$(-1)^{1/2}$
K_j	relaxation strength of relaxation process j
k	velocity attenuation per wavelength
L	$h/R_V T_o$
ℓ	dimensionless distance = $\omega x/c_o$
n_o	droplet number density
P_r	Prandtl number
$R_{G,V}$	ideal gas constants of air and water vapour
r	water droplet radius
T_o	equilibrium temperature of fog
t	time
u	velocity perturbation due to acoustic field/c_o
x	distance
α	sound attenuation coefficient
γ	specific heat ratio of air
γ_V	specific heat ratio of water vapour
η	$(\mu/2\rho_o c_o{}^2)\ |4/3+(\gamma-1)/P_r|$
κ	thermal conductivity of air
μ	absolute viscosity of air
ρ_o	equilibrium density of air
$\bar{\rho}$	density of droplet material
τ_C	mass transfer relaxation time $=\bar{\rho}r^2/3D\rho_o$
τ_D	momentum relaxation time = $2\bar{\rho}r^2/9\mu$
τ_T	heat transfer relaxation time = $\rho\bar{c}r^2/3\kappa$
τ_j	relaxation time of relaxation process j
ω	sound frequency (radians/sec.)

2. Introduction.- Wave propagation processes have been successfully studied using single, relatively simple differential equations to model the full set of fluid mechanical conservation equation. Whitham |1|, for example, discusses the derivation of the wave equation, Korteweg - Devries equation, and Burgers'equation from more general conservation laws. These equations can then be used in the theoretical study of various types of wave problems. For finite-amplitude, plane wave propagation through gaseous media, such authors as Lighthill |2|, Blackstock |3|, and Lesser and Crighton |4| have demonstrated the utility of the Burgers' equation approach.

In previous papers, the Burgers' equation technique has been extended to describe wave propagation through aerosol media. Initially, aerosols consisting of monodisperse, solid particles suspended in an ideal gas were considered |5|. Subsequently, mass transfer effects were included for aerosols made up of liquid droplets suspended in an ideal gas mixture of liquid vapour and an inert gas |6|. Finally, the technique was extended to include droplet polydispersity and gas molecular relaxation, two effects which should be included in a realisic model for most fogs|7|.

This latter equation has so far only been applied to infinitesimal, plane wave propagation through air-water fogs|7|. The aim of this paper is to use the Burgers' equation approach to study finite-amplitude effects in such media.

3. Problem formulation.- A dimensionless Burgers' equation for air-water fogs may be shown to be |6,7|

$$u_\ell - c'u_g - \frac{1}{2}(\gamma + 1)uu_g = G(u) + D(u) \qquad (1)$$

Eq.1 describes the acoustic perturbation velocity u as a function of distance ℓ and retarded time g. Dispersion effects appear through the c' parameter.

The first term on the right side of Eq.1, which represents absorption due to gas molecular effects, may be written as

$$G(u)=\omega\eta u_{gg} + \sum_{j=1}^{4} \frac{K_j}{\omega\tau} \left[\frac{\exp(-g/\omega\tau_j)}{\omega\tau} \int_{-\infty}^{g} u\exp(g/\omega\tau_j)dg - u\right] \qquad (2)$$

Following|8|, four relaxational terms are included in this formulation, three arising from molecular vibration, and one from molecular rotation.For most frequencies of practical interest, however, rotational relaxation satisfies the restriction $\omega\tau_4 \ll 1$ For plane wave studies, introduction of this restriction into Eq. 2 allows the rotational relaxation term to be reduced to

$$\omega K_4 \tau_4 u_{gg} - K_4 u_g = \omega K_R' u_{gg} - K_4 u_g \qquad (3)$$

It can further be shown that the $K_4 u_g$ term represents a dispersion effect which is negligible except at extremely high frequencies. Accordingly, the rotational relaxation term may be simplified to the single term $\omega K_R' u_{gg}$.

A similar analysis allows the simplification of one of the three vibrational relaxation terms. The gas absorption terms of Eq. 1 therefore become

$$G(u) = \omega(\eta + K_R' + K'_v)u_{gg} + \sum_{j=1}^{2} \frac{K_j}{\omega\tau_j} \left[\quad \right] \qquad (4)$$

where the term in square brackets is defined in Eq.2. In this equation, the η parameter represents classical thermoviscous absorption, while τ_1 may be closely linked with vibration of nitrogen molecules and τ_2 with vibration of oxygen molecules |8|. Numerical values of these constants are listed in Table I for air at 100% relative humidity.

The second term on the right side of Eq.1, which represents absorption due to gas-droplet interactions, may also be written in a relaxational form formally identical to the molecular relaxation terms of Eq.4.

$$D(u) = \sum_{j=3}^{5} \frac{K_j}{\omega\tau_j} \left[\quad \right] \qquad (5)$$

The three terms in the summation of Eq. 5 describe respectively momentum, heat, and mass transfer between the gas and suspended water droplets. The relationship between the constants of Eq. 5 and aerosol properties is listed in Table II. Numerical values are also tabulated for a monodisperse fog containing water droplets of radius 5 microns and number density 250 droplets cm^{-3}.

Plane wave propagation through air-water fogs may therefore be described by the Burgers' equation

$$u_\ell - c'u_g - \frac{1}{2}(\gamma + 1)uu_g = A(u) \qquad (6)$$

with the absorption terms

$$A(u) = \omega(\eta + K_R' + K_V')u_{gg} + \sum_{j=1}^{5} \frac{K_j}{\omega\tau_j}\left[\frac{\exp(-g/\omega\tau_j)}{\omega\tau_j}\int_{-\infty}^{g} u\exp(g/\omega\tau_j)dg - u\right] \qquad (7)$$

3. Plane wave propagation.- To study plane wave propagation, Eq.6 is solved using a perturbation expansion in terms of the acoustic Mach number ε:

$$u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \ldots. \qquad (8)$$

The boundary conditions describing a sinusoidal excitation at the origin (ℓ=0) which dies out at infinity are approximately

$$u^{(1)}(0,g) = \varepsilon \sin g$$
$$u^{(m)}(0,g) = 0 \text{ for } m \geq 2 \qquad (9a)$$

$$u^{(m)}(\ell,g) \to 0 \quad \text{as } \ell \to \infty \qquad \text{for } m \geq 1 \qquad (9b)$$

The first-order problem, which describes infinitesimal plane waves, becomes

$$u_\ell^{(1)} - c'u_g^{(1)} = A(u^{(1)}) \qquad (10)$$

To conform with Eqs.9, wave solutions of the form

$$u^{(1)} = e^{-k^{(1)}\ell}\sin g^{(1)} = \mathrm{Im}\left\{\exp[i(g^{(1)} + ik^{(1)}\ell]\right\} \qquad (11)$$

are sought, where the $g^{(1)}$ notation is introduced since $c' = c^{(1)}$ appears in the definition of g. Substitution of Eq. 11 into Eq. 10 produces a complex algebraic equation which may be solved for the attenuation parameter $k^{(1)}$ and dispersion parameter $c^{(1)}$

$$k^{(1)} = \omega(\eta + K_R' + K_V') + \sum_{j=1}^{5} K_j \frac{\omega\tau_j}{1 + \omega^2\tau_j^2} \qquad (12)$$

$$c^{(1)} = \sum_{j=1}^{5} \frac{K_j}{1 + \omega^2\tau_j^2} \qquad (13)$$

The conventional sound attenuation coefficient α (units of decibels per unit length) may be obtained from Eq. 12 using

$$\alpha = 8.686\, k^{(1)}\, \omega / c_0 \qquad (14)$$

where c_0 is the speed of sound.Eqs. 13 and 14 are plotted as functions of frequency in Figures 1 and 2 using the numerical values of Tables I and II. Gas and droplet components are also identified. Good agreement has been found between plots of this kind and experimental data[7,9].

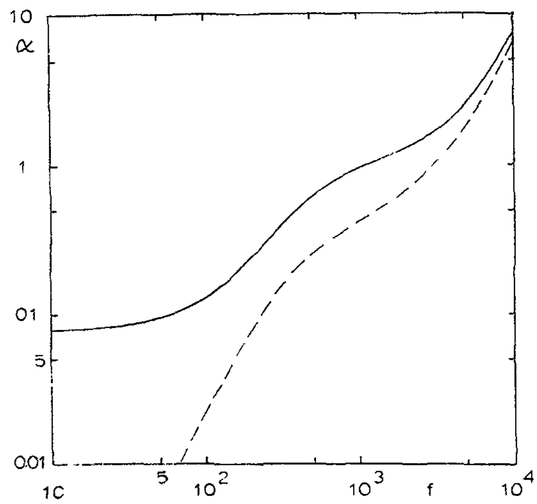

FIG.1 : The sound attenuation coefficient α, in units of 10^{-4}dB/cm, versus frequency f in hertz. The solid line indicates attenuation in fog, while the broken lines indicates attenuation in an air-water vapour mixture without droplets.

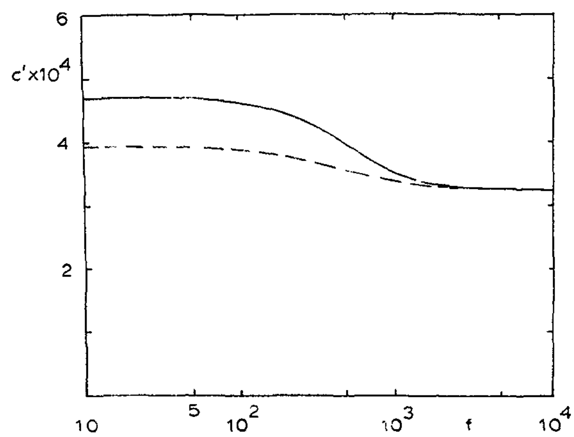

FIG.2 : The sound dispersion parameter c' versus frequency f in hertz. The solid line indicates dispersion in fog, while the broken line indicates dispersion in an air-water vapour mixture without droplets.

Finite-amplitude effects appear in the second-order problem :

$$u^{(2)} - c'u_g^{(2)} - A(u^{(2)}) = \frac{\gamma+1}{2} u^{(1)} u_g^{(1)} = \frac{\gamma+1}{4} \exp\left[2i(g^{(1)} + ik^{(1)}\ell)\right] \quad (15)$$

This equation is solved by first seeking a particular solution proportional to the inhomogeneous term. Eqs. 9 then lead to the appropriate homogeneous solution, and

$$u^{(2)} = p^{(2)} \left\{ \exp\left[2i(g^{(2)} + ik^{(2)}\ell)\right] - \exp\left[2i(g^{(1)} + ik^{(1)}\ell)\right] \right\} \quad (16)$$

where

$$p^{(2)} = \frac{\gamma+1}{8} \left[(k^{(1)} - k^{(2)}) + i(c^{(1)} - c^{(2)})\right]^{-1}$$

Parameters $k^{(2)}$ and $c^{(2)}$ may be obtained from Eqs. 12 and 13 by replacing ω by 2ω.

The $u^{(2)}$ solution describes nonlinear growth of the second harmonic and accompanying waveform distortion as propagation proceeds. The nature of the solution can be clarified by separating the imaginary part of Eq. 16 :

$$u^{(2)} = \sin 2g^{(1)} \left[P_R^{(2)} e^{-2k^{(2)}\ell} \cos 2\Omega\ell - P_R^{(2)} e^{-2k^{(1)}\ell} - P_I^{(2)} e^{-2k^{(2)}\ell} \sin 2\Omega\ell\right] + \cos 2g^{(1)} \left[P_I^{(2)} e^{-2k^{(2)}\ell} \cos 2\Omega\ell - P_I^{(2)} e^{-2k^{(1)}\ell} + P_R^{(2)} e^{-2k^{(2)}\ell} \sin 2\Omega\ell\right] \quad (17)$$

where

$$\Omega = c^{(1)} - c^{(2)} \quad (18)$$

$$p^{(2)} = P_R^{(2)} + iP_I^{(2)} \quad (19)$$

Eq. 17 can be further simplified by noting that, for the parameter values of Tables I and II, terms involving Ω are negligibly small. The second order solution thus may be reduced to

$$u^{(2)} = \frac{\gamma+1}{8} \frac{1}{k^{(1)} - k^{(2)}} \left(e^{-2k^{(2)}\ell} - e^{-2k^{(1)}\ell}\right) \sin 2g^{(1)} \quad (20)$$

It can be shown that the limiting forms of both Eq. 17 and Eq. 20 for the cases of no dissipation and no relaxation absorption agree with previously published results for lossless gases and for gases exhibiting only thermoviscous absorption /3,10/. In Figure 3, the growth of the second harmonic with propagation distance is illustrated, using Eq. 20 with the air-water fog constants of Tables I and II.

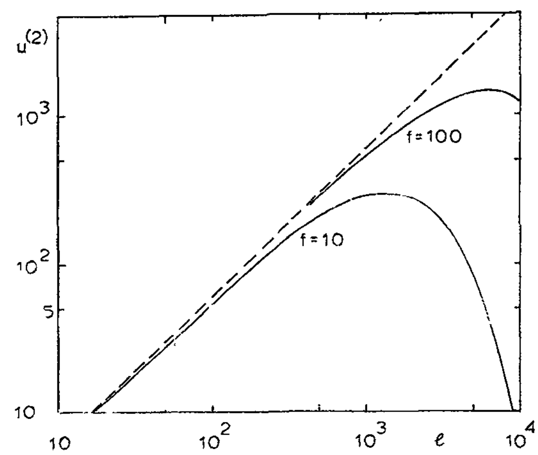

FIG. 3 : The amplitude of the dimensionless second harmonic solution versus dimensionless propagation distance, for excitation frequencies f = 10 hertz and f = 100 hertz. The broken line indicates the dissipationless gas limit $\frac{\gamma+1}{4}\ell$

An indication of when finite-amplitude effects are likely to be important may be obtained by examining the maximum value of the amplitude ratio of the first two terms of expansion (8) :

$$\varepsilon |u^{(2)}| / |u^{(1)}| = \varepsilon \frac{\gamma+1}{8} \frac{1}{k^{(1)} - k^{(2)}} \left[e^{(k^{(1)} - 2k^{(2)}\ell)} - e^{-k^{(1)}\ell}\right] \quad (21)$$

Nonlinear effects remain insignificant as long as this maximum is small relative to $\frac{1}{2}$, the value that the ratio would attain if nonlinearities were sufficiently strong to distort the initially sinusoidal wave into a sawtooh. Figure 4 has been constructed on this basis to show decibel-frequency combinations where finite-amplitude effects are likely to be significant in air-water fogs. The data of Tables I and II were used in these computations, along with the relationship between the acoustic Mach number and the sound pressure level in decibels /10/ :

$$SPL = 194 + 20\log_{10}(\gamma\varepsilon/\sqrt{2}) \qquad (22)$$

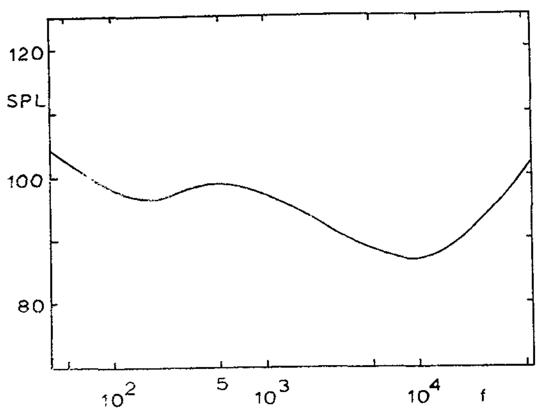

FIG. 4 : Regions where nonlinear effects will be significant in air-water fogs. Along the curve, the perturbation solution satisfies $[\varepsilon u^{(2)}/u^{(1)}]_{max} = 0.1$. Accordingly, infinitesimal acoustic results will be accurate in the lower part of the figure, while nonlinear corrections will become important in the upper part.

4. Concluding Remarks.- Plane wave propagation through air-water fog has been studied using a perturbation solution of an extended Burgers' equation. Acoustic parameters have been derived from the perturbation solution for constant values representative of real atmospheric fogs.

The Burgers's equation was developed from the full set of conservation equations for an ideal gas-monodisperse droplet fog, a set of ten coupled partial differential equations containing gas-droplet mass, momentum, and energy transfer interactions /6/. The simplification of dealing with one equation in place of the original ten not only allows analytical solutions to be found for finite-amplitude wave problems, but also allows the incorporation of such effects as gas molecular relaxation and droplet polydispersity /7/. The physical mechanisms of acoustic absorption, dispersion, and waveform distortion in fogs are thus clarified. The validity of the fog Burgers' equation is supported by good agreement found between theoretical predictions and experimental measurements of infinitesimal wave attenuation /7,9/.

The perturbation solution of the fog Burgers' equation is useful in identifying the limits of infinitesimal acoustics. However, it is unable to describe the actual waveform under those conditions when nonlinear distortion is significant.The Hopf-Cole transformation, which leads to an exact solutionof the perfect gas Burgers' equation /1,3/, does not enable the solution of the fog Burgers' equation. Accordingly, so far only approximate solutions for the distorted waveform in fogs have been found /6/.

5. Acknowledgments.- Financial support has been provided by the National Research Council of Canada and a University of Waterloo Research Grant. Thanks are also extended to Miss Colleen Male for preparation of the manuscript.

TABLE I Sound absorption constants for air at 100% relative humidity

τ_1	$3.825\text{x}10^{-4}$s	K_1	$6.087\ \text{x}10^{-5}$
τ_2	$1.348\text{x}10^{-6}$s	K_2	$3.158\text{x}10^{-4}$
		K'_R	$3.834\text{x}10^{-11}$
η	$1.214\text{x}10^{-10}$	K'_V	$1.724\text{x}10^{-12}$

TABLE II Sound absorption constants for water droplets in air ($r=5\mu, n_0=250$ droplets cm^{-3}.

$\tau_3 = \tau_D$	$3.036\text{x}10^{-4}$s
$\tau_4 = \tau_T/A$	$3.132\text{x}10^{-4}$s
$\tau_5 = C/C_m$	2.469s
$K_3 = \frac{1}{2} C_m$	$5.454\text{x}10^{-5}$
$K_4 = \frac{1}{2} C_m \frac{\gamma-1}{\gamma} \frac{H_p}{A}$	$2.144\text{x}10^{-5}$
$K_5 = \frac{1}{2} C_V B$	$7.617\text{x}10^{-2}$

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