INSTABILITIES, OSCILLATIONS, AND CHAOS

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Résumé. — Un certain nombre d'illustrations de début de turbulence sont commentées et opposées l'une à l'autre. Des modèles montrant différents types de comportement apériodique sont présentés et classés par catégories. Des études numériques basées sur ces illustrations sont présentées et comparées à des études expérimentales sur des fluides de faible turbulence dans la configuration de Rayleigh-Bénard.

Abstract. — A variety of pictures for the onset of turbulence are discussed and contrasted. Models that exhibit different types of aperiodic behaviour are exhibited and classified. Some numerical studies that bear on these pictures are presented and compared with experimental studies of weakly turbulent fluids in the Rayleigh-Bénard configuration.

There is an old story, that has many variations, and that my source attributes to Horace Lamb. At a meeting of the British Association in 1932 he is quoted as saying, *I am an old man and when I die and go to heaven, there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motions of fluids. About the former I am really rather optimistic.* There are many observations for which the quotation seems suited : that physicists have been concerned with both problems for many years ; that at least at the level of quantitative prediction, Lamb's assessment of the relative difficulty of the two problems was correct ; and even that there are close mathematical similarities between certain problems in strongly turbulent systems and problems that would be encountered in electrodynamics if the fine structure constant were large and perturbation theory could not be applied. I want to use his statement for a far less pretentious purpose, namely, to admit at the outset of this lecture that while the Dukes of Burgundy may be interred in this ancient town, it does not now qualify as heaven in Lamb's sense : you are not about to receive great enlightenment on turbulence. I hope that my talk will help to clarify a number of the problems and I think that some of the models of chaotic phenomena that I will discuss are illuminating and suggestive, but it is far too early to claim that any of them gives the essence of the phenomenon of turbulence — if indeed it is a single phenomenon and has an essence.

A joint session on phase transitions and hydrodynamic instabilities has been convened because the two phenomena seem to have many common features, and I shall discuss a few. There are, however, crucial differences, the most important of which is the fact that the latter typically involve only a finite number of degrees of freedom. As a result, the Landau theory seems to apply, at least at the first few thresholds. Less central, but not without effect, are the dissipative properties of the latter systems, and the fact that many of the broken symmetries in them involve time as well as space. While the experimental techniques that have been invaluable in understanding phase transitions promise to be very useful in the study of hydrodynamic phenomena, I suspect that the recent additions to our theoretical arsenal may be less effective than many had hoped.

The title of my talk is best explained by a picture. Figure 1 shows a cylinder moving through a fluid [1].

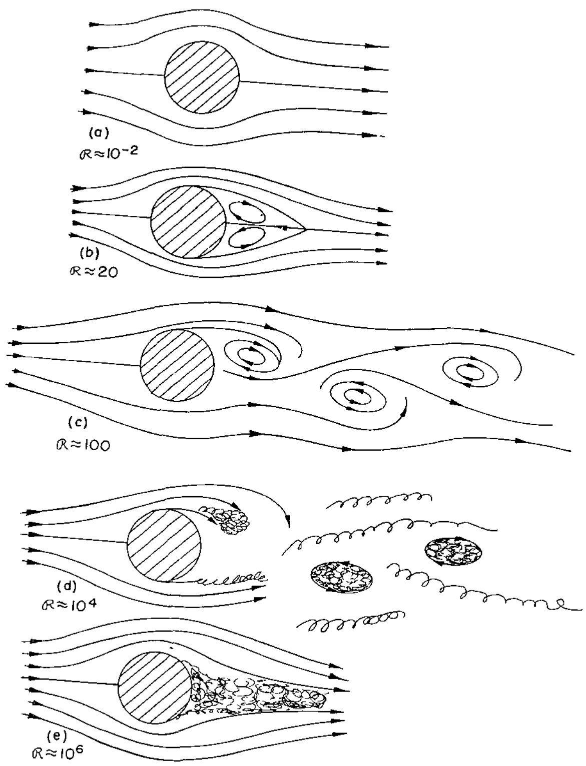

FIG. 1. — The behaviour of a fluid through which a cylinder moves uniformly for five different regimes of velocity.

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As its velocity, u, is increased, five regimes are encountered. As long as u is small compared to the sound velocity, the fluid may be treated as incompressible, and the behaviour discussed in terms of a single dimensionless parameter, the Reynolds number, R. R is equal to uL/v where L is the radius of the cylinder and v is the kinematic viscosity. As R increases, the following changes occur :

1. **New spatial symmetry.** — There is a transition to a new time-independent state with two vortices and a new spatial symmetry.

2. **Oscillations.** — There is a transition to an oscillatory state. Even though the external conditions are time translationally invariant, there occurs a Karman vortex street, i. e., a flow with periodic oscillatory time dependence.

3. **Noise.** — Eventually the system develops a time dependence that appears random. In this state it seems possible to describe only statistical properties of the flow.

4. **Universal chaos.** — Finally, this chaotic behaviour involves fluctuations on so small a scale of time and space that it may not be unreasonable to hope that the details of the macroscopic motions that are indirectly responsible for the small fluctuations may be forgotten. It is in the last region that Richardson's poem,

Big whorls have little whorls,
Which feed on their velocity ;
And little whorls have lesser whorls
And so on to viscosity,

is expected to be relevant. Indeed, it is hoped that the smallest whorls will have no remembrance of the big whorls that feed them, as in the phase transition problem.

A more realistic picture [2] of these regimes is provided in figure 2 and a more schematic description is indicated in figure 3.

In this lecture we shall proceed along the path of increasing Reynolds number commenting on the relationship of the various changes to phase transition phenomena. Note incidentally that the path has a break at intermediate values of the Reynolds number. The reason is that the transition to turbulence is not very clear and is only now being subjected to detailed experimental study. Our own recent theoretical efforts [3] have been concerned with mathematical models of this transition. In these models, and possibly in different real physical systems, the omitted part of the path can take many different forms with different numbers and kinds of breaks.

Experiments and calculations are not usually performed for the conceptually simple and picturesque system described above, but for systems which exhibit greater spatial regularity : systems in which the fluid is driven by centrifugal forces (in the geometry of Couette flow) and systems in which the fluid is driven by thermally produced gravitational forces (in the geometry of Rayleigh-Bénard convection). In both cases the fluid appears to undergo the same sequence of transitions. The analogies between the regimes are indicated in table I.

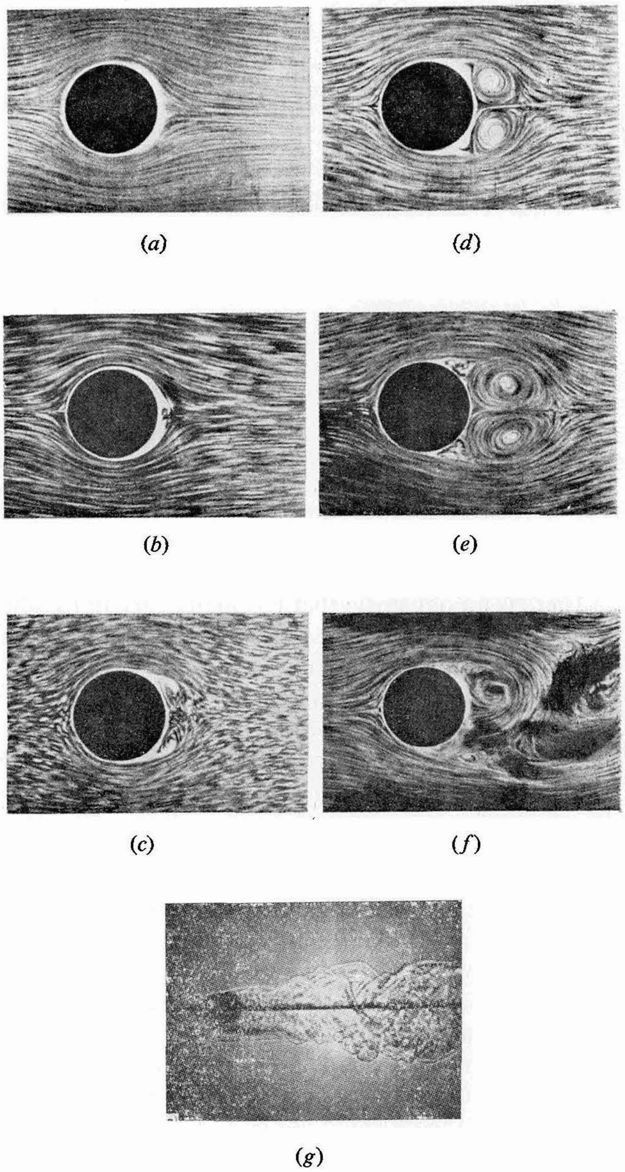


FIG. 2. — Actual photographs of a cylinder moving through a fluid at different velocities.

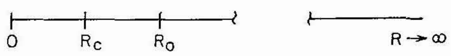
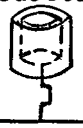
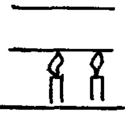


FIG. 3. — Schematic description of the special values of the Reynolds number for which the behavior of the fluid changes abruptly.

TABLE I

Flow 	$R = \frac{uL}{\nu}$	Laminar Flow	Vortex Pair	Karman Street	Turbulence
Couette 	$T = \frac{\Omega^2 L^4}{\nu^2}$	Laminar Flow	Taylor Cell	Oscillations	Turbulence
Bénard 	$R = \frac{g\beta L^3 \Delta T}{\nu \chi}$ $\sigma = \nu/\chi$	Thermal Conduction	Convective Rolls	Oscillations	Turbulence

The typical behaviour of the system at a point like R_c has been discussed at this meeting by Bergé. In the neighborhood of R_c the familiar Landau form can be introduced [4]. At R_c, the amplitude of a cellular fluctuation of the correct period is unstable. If one denotes the time dependent amplitude of such a fluctuation by $B(t)$, one may write

$$\frac{\mathrm{d}B}{\mathrm{d}t} = a_0 B + a_1 B^3$$

where a_1 is slowly varying and negative and a_0 is proportional to $R - R_c$. It follows that the steady state value of B will be given by $B_{eq} = 0$ for $a_0 < 0$ and by

$$B_{eq} = (a_0/-a_1)^{1/2}$$

for $a_0 > 0$. It also follows that if $B(t) = \frac{1}{2} B_{eq}$ at $t = 0$ for $a_0 > 0$, it will behave as

$$\frac{B^2(t)}{B_{eq}^2} = \frac{e^{2a_0 t}}{3 + e^{2a_0 t}}.$$

Thus the growth rate will be initially exponential but with a sluggish rate proportional to $R - R_c$ near R_c. In both of these respects, the behaviour near the instability is extremely similar to that of mean field theory. The mean-field-like static behaviour has been verified with considerable accuracy in the Bénard problem by Dubois and Bergé [5]. In the Couette problem, Donnelly and Schwarz [6] and Freilich and Gollub [7] have tested both the equilibrium properties and the time dependence.

Recently, higher harmonics on the static velocity field near R_c have been analyzed by both Freilich and Gollub, and Dubois and Bergé [8]. The latter are reporting for the first time at this conference, measurements that indicate that the $n = 2$ and $n = 3$ harmonics grow as predicted, as $(R - R_c)^{n/2}$, near threshold. Freilich and Gollub have found a different behaviour. If the discrepancies they report [9] are not due to some kind of azimuthal asymmetry, as suggested in a recent report, we will be confronted with serious problems. Since it can be almost rigorously demonstrated that near R_c, the harmonics must behave as $(R - R_c)^{n/2}$, we would have to conclude that the region over which this behaviour holds is too small to be experimentally accessible.

As discussed by Zaitsev and Shilomis [10], near the critical value of R, the correlation distance, ξ, and the correlation time, τ, grow according to the laws

$$\xi \sim L \frac{R_c}{|R_c - R|^{1/2}}, \qquad \tau \sim \frac{L^2}{\nu} \frac{R_c}{|R_c - R|}.$$

The slowing down has been observed in the experiments previously cited, but efforts to experimentally determine the growth of the correlation range in the mean field regime have not yet been successful.

In the absence of thermal fluctuations and external noise, there is no analog to the non-linear couplings, that occur in the phase transition problem, and that alter the mean field character of the transition in less than four dimensions and eliminate it entirely in one-dimensional systems. Even when these fluctuations are taken into account, their effect on the behaviour close to R_c is not likely to be experimentally accessible. Deviations from mean field behaviour near the transition normally result from slow spatial variations over distances much larger than the basic length scale but small compared to the size of the system so that boundary and size effects do not dominate and round the transition. These two conditions are difficult to meet in a macroscopic flow problem.

For a Rayleigh-Bénard configuration in which the fluid is contained between two infinite plates, Graham and Pleiner [11] have estimated that at room temperature the size of the critical region would be

$$(R - R_c)/R_c \ll 10^{-6} .$$

Their estimate is based on an approximate analysis which has many points in common with the analysis by the spherical model of a transition in a two dimensional system with long but finite-ranged forces. In a two-dimensional system, an analysis of this type gives the size of the critical region but the analysis incorrectly predicts that there is no phase transition. The arguments of Graham and Pleiner on the character of the transition are also open to some question [12], but the estimate of the size of the critical region is probably reliable. If it is correct the critical properties would only be apparent when the correlation range exceeded 10^3 L and this large distance was negligible compared to the distance over which the vertical boundaries had an effect. The area does not seem to be a promising one for experimental study.

The phenomenon that occurs as R increases through R_c is a bifurcation. Below R_c all orbits in phase space are attracted towards a single point. Above R_c they are attracted towards one of either two points which describe the rolls going either clockwise or counterclockwise. The phenomenon of a continuous transition from decay to growth, of a motion, as a parameter increases, often occurs for oscillatory rather than steady state motion. In that case, above the transitional value R_0, orbits in phase space are attracted not to a fixed point but to a limit cycle. In the example of flow past a cylinder, the time at which a vortex will split off from the top of the cylinder depends on details of the initial state, but for all initial values the system will settle down to the oscillatory state.

From the point of view of Landau [4] and Hopf [13], the phenomenon of turbulence is nothing more than the effect of a sequence of bifurcations as different oscillatory modes with incommensurate periods of oscillation become unstable. Above each new instability a new cyclical component is introduced, and since the relative phase of the cyclical components is undetermined, the behaviour

$$u(rt) = \sum B_n \cos (\omega_n t + \varphi_n) u_n(r)$$

becomes more complicated and more irreproducible since each mode enters with a new arbitrary phase φ_n relative to the others.

The Landau-Hopf picture of turbulence may be likened to walking down a path with many bifurcations until, eventually one is lost. At no point along the way can it be said that this is where one became lost (or turbulent) ; it just gradually happened. In the soluble mathematical model that was invented by Hopf, exactly this kind of behaviour occurs. But it is not the only possibility.

In the past decade, several alternatives to this experimentally unverified picture have received considerable attention. Some of these possibilities, which fill in the missing part of figure 3, are illustrated in figure 4. They share in common the prediction that there is an abrupt transition, at a well-defined value of R_t, from periodic behavior to *turbulent* behaviour. The first of these alternative pictures is closer to Landau's, and shares with it, the assumption that each of the sequence of modes which becomes unstable tends to stabilize with a small non-vanishing amplitude as the Reynolds number increases. This alternative, which was advocated by Ruelle and Takens [14] a few years ago, is accompanied by the assertion that the Hopf model is untypical (non-generic) and that interactions between the modes would tend to make the system with two independent incommensurate periods periodic. On the other hand, Ruelle and Takens assert that after the third or fourth bifurcation, the generic behaviour will be neither periodic nor quasi-periodic. Instead the orbits in phase space will be attracted to more complex subsets of the configuration space which they call strange attractors. Ruelle and Takens propose that these attractors are characteristic of turbulent behaviour. The basis for their proposal seems somewhat arbitrary to me but that may reflect my ignorance. Their predictions seem to agree with calculations [3] by John McLaughlin and myself, for a problem of the Bénard type. While our calculations seem to be in semi-quantitative accord with the experiments and to exhibit features of the Ruelle-Takens picture, the agreement could be fortuitous. For example, our calculations have not been critically examined for spurious computer effects.

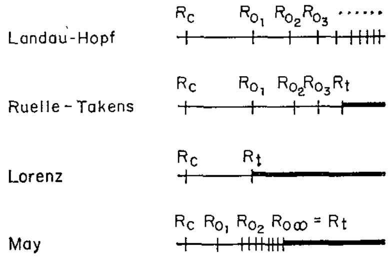


FIG. 4. — Schematic description of different proposals for the singular points that precede the onset of turbulence or chaos.

Before discussing the Ruelle-Takens picture which seems to describe several recent experiments, let me stress that it is not inevitable. There are natural models that contain a few degrees of freedom from the hydrodynamic equations that describe the Bénard problem, and exhibit, contrary to the Ruelle-Takens proposal, a transition to *turbulent* behaviour at a specific value R_t, without any intermediate oscillatory regime. For detailed reasons these models may not apply to actual fluids that conduct heat poorly, when they become turbulent in the Bénard configuration. But at the level at which the previous two pictures have been proposed, there seems to be nothing to rule out these models. We should therefore view cautiously the assumptions on which the pictures of Landau-Hopf and Ruelle-Takens are based. The simplest model which falls in this *counterexample* category is due to Lorenz [15]. It has subsequently surfaced in several other guises.

In order to discuss this model, we shall finally have to be more specific. First let us recall again the basic equations that describe convection, the Boussinesq equations. They are the Navier-Stokes equations with a driving term due to the buoyancy force.

One equation is the statement of momentum conservation

$$\frac{\partial u}{\partial t} + u.\nabla u = - g\beta\hat{z}(\delta T) - \frac{1}{\rho}\nabla p + \nu\nabla^2 u .$$

In this equation, p is the pressure, and $- g\beta\hat{z}(\delta T)$, the buoyancy force, expressed in terms of

$$\beta = - \frac{1}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}T}\right) .$$

In an incompressible fluid, p can be eliminated in terms of u, closing the equation.

The second Boussinesq equation is the statement in terms of the temperature, of energy conservation,

$$\frac{\partial \delta T}{\partial t} + (u.\nabla)\,\delta T = \chi\,\nabla^2 \delta T .$$

The thermal diffusivity χ is given by

$$\chi = \frac{\kappa}{\rho c_p}$$

with κ, the thermal conductivity, and ρc_p the specific heat per unit volume at constant pressure.

One basic parameter in these equations, the Rayleigh number, is defined as

$$R = \frac{g\beta L^4}{\nu\chi}\left(\frac{\Delta T}{L}\right) .$$

The critical value of R for the onset of convection [16] and the critical wave number at which the instability occurs have been calculated by Lord Rayleigh. For free and rigid boundaries they are given respectively by :

Free Boundaries	Rigid Boundaries
$R_c = 27\,\pi^4/4 = 657.51$	$R_c = 1\,707.762$
$\pi^{-1}\,k_c\,L = 1/\sqrt{2} \equiv a$	$\pi^{-1}\,k_c\,L = 0.992 \equiv a.$

The Boussinesq equations also depend on a second parameter, the Prandtl number, $\sigma = \nu/\chi$. All fluids do not behave in the same fashion at all Rayleigh numbers. Some characteristic values of the Prandtl number are given in table II. The threshold for oscillatory behaviour, the onset of instabilities, and other phenomena depend on the Prandtl number.

TABLE II

Prandtl Numbers of a variety of fluids

Mercury	0.025
Air	0.7
Liquid helium	0.85
Water	7.0
Silicone oil	57.0

By far the simplest idealization that contains interesting physics is obtained by working with only the very simplest Fourier components in the truncated Boussinesq equations. The result is the Lorenz model. The modes retained and their significance are indicated in figure 5.

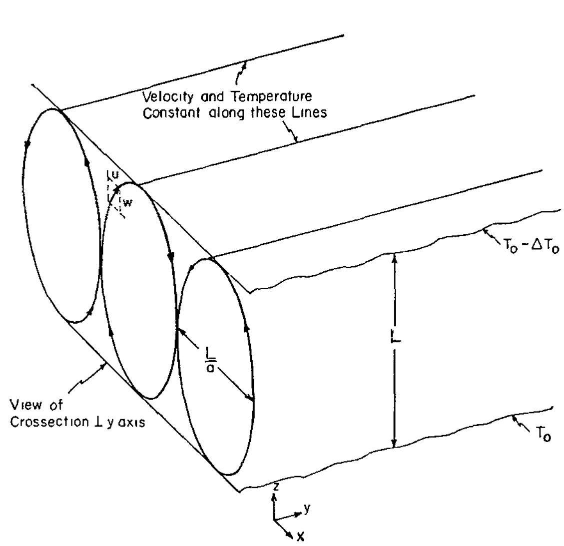


FIG. 5. — The Lorenz model and the significance of the three modes retained.

In the Lorenz model, three modes are retained : one velocity potential and one temperature mode with the fundamental cellular wave numbers, and a second temperature mode, a second harmonic in z, that has no y or x periodicity and that contributes to the mean heat flow. Specifically we have

$$\psi : \quad w = - \frac{\partial \psi}{\partial z}, \qquad u = \frac{\partial \psi}{\partial x}$$

$$\psi(xzt) = \sqrt{2}\,v(t)\left(\frac{1 + a^2}{a^2}\right)\sin\left(\frac{\pi x a}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

$$\delta T(xzt) = \frac{R_c}{\pi}\sqrt{2}\,\theta_1(t)\cos\left(\frac{\pi x a}{L}\right)\sin\left(\frac{\pi z}{L}\right) - \frac{R_c}{\pi}\,\theta_2(t)\sin\left(\frac{2\,\pi z}{L}\right) .$$

With the dimensionless variables appropriate to free boundaries

$$r = \frac{R}{R_c} \qquad \tau = \pi^2(1 + a^2)\, t$$

$$a = \frac{1}{\sqrt{2}} \qquad b = \frac{4}{1 + a^2} = \frac{8}{3}$$

the truncated equations take the form

$$\frac{\mathrm{d}v(\tau)}{\mathrm{d}\tau} = -\sigma v(\tau) + \sigma\theta_1(\tau)$$

$$\frac{\mathrm{d}\theta_1(\tau)}{\mathrm{d}\tau} = -v(\tau)\,\theta_2(\tau) + rv(\tau) - \theta_1(\tau)$$

$$\frac{\mathrm{d}\theta_2(\tau)}{\mathrm{d}\tau} = v(\tau)\,\theta_1(\tau) - b\theta_2(\tau)\,.$$

The ratio of the heat transferred to the transfer of heat by conduction, the Nusselt number, N, is given by

$$N = 1 + 2 < \theta_2(\tau) > / r\,.$$

This model has the great virtue that all time-independent solutions can be found and their stability checked. Specifically, it is easy to verify that there are three regions of r : in the first and only in the first, the solution where all variables vanish is stable ; in the second, and only in the second, the two solutions with non-vanishing time independent values are stable ; in the third no time-independent or oscillatory solution is

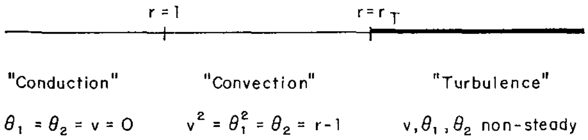


FIG. 6. — Schematic description of the transition to turbulence associated with the Lorenz model.

stable. This situation is described in figure 6. The instability of *convection* occurs for

$$r_T = \sigma \frac{(3\,\sigma + 17)}{(3\,\sigma - 11)}, \qquad \omega_T \approx \left(\frac{16\,\sigma(\sigma + 1)}{3\,\sigma - 11}\right)^{1/2}.$$

Beyond r_T there is no stable convective or conductive solution, only one which has complicated properties at all times.

Lorenz has studied how the non-steady solutions behave after a long time and has shown that the trajectory for the phase point does not settle into a periodic or time-independent orbit. Two of his graphs are reproduced in figure 7. The first shows the fluctuations in N (which are proportional to the additional heat flow). The second shows the fluctuations in time of the velocity and the corresponding left-right temperature variation in a roll.

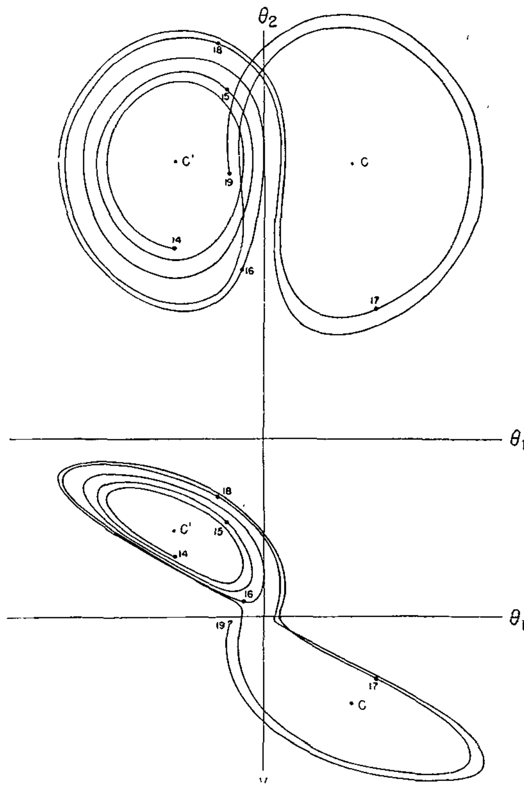


FIG. 7. — The numerically calculated variation in the heat flux in the Lorenz model and a plot of a projection of the phase space trajectory on the plane that describes the components of velocity and temperature that vary with x. The plot shows the irregular rotation of the rolls, first in one direction and then in the other.

While the number of rigorously proven theorems is small, it appears that the path traced out by the phase space point is ergodic on a two-dimensional subspace of three-dimensional space consisting of a surface which is locally the direct product of a simple surface and a Cantor set.

Since both Landau-Hopf and Ruelle-Takens predict oscillatory regimes before the turbulent regime, it is interesting to reconcile their predictions with the behaviour of the Lorenz model. The answer appears to be the following : if the Lorenz model is cast in the Landau framework,

$$-\frac{dB}{dt} = \frac{\partial \Phi}{\partial B}; \qquad \Phi = -\frac{1}{2} a_0 B^2 - \frac{1}{4} a_1 B^4 ;$$

then Φ is negative for large B, i. e., a_1 is positive. For a_0 negative, i. e., $R < R_c$, the *potential* Φ is volcano-shaped, and there is an unstable limit cycle. As a_0 approaches zero, the unstable limit cycle collapses onto the stable fixed point, and the eruption takes place. In the Lorenz model there does not appear to be a new minimum as there would be in the analogous (first-order) phase transition problem. [In the limit of large σ, a_0 and a_1 are given by $(r - r_T)/4\,\sigma$ and $(37/144\,\sigma)$]. It is of course possible that this behaviour is atypical but this inverted bifurcation, where an unstable limit cycle collapses onto a stable fixed point as R approaches R_c (in contrast to a stable limit cycle separating from an unstable fixed point above R_{crit}) does not seem to have been seriously considered by the suggestors of the first two pictures (Fig. 8).

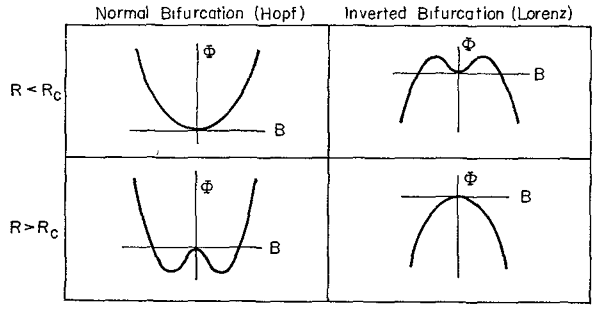


FIG. 8. — The Landau potentials appropriate to normal and inverted bifurcations below and above R_c.

Our own interest in the Lorenz model was intensified when we were told of the experiments Ahlers [17] had performed on heat flow in normal (that is, not superfluid) liquid helium as a function of the Rayleigh number. (In helium, at low temperatures, it is possible to have large Rayleigh numbers with small temperature differences, and to take advantage of relatively short relaxation times.) Ahlers observed conduction, convection, and turbulence, but he saw no indication of an oscillatory regime. The onset of turbulence, marked by the appearance of an irregularly fluctuating component of the heat flow, occurred at a reproducible value of R (Fig. 9) beyond which the intensity and characteristic time dependence of the fluctuations were studied. The behaviour of the system was reminiscent of the

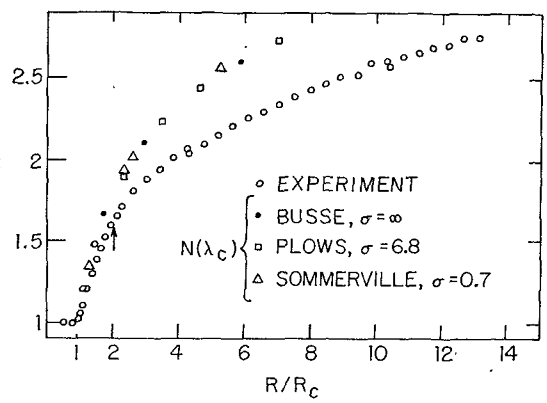


FIG. 9. — Ahlers' measurements of the heat flow as a function of Rayleigh number. The arrow indicates the position of the turbulent threshold he found. Some earlier measurements are also indicated on his figure.

Lorenz model, but the resemblance was illusory. In the Lorenz model the *turbulent* regime occurs only when the Prandtl number is larger than about 11/3 but the Prandtl number of helium is 0.85. The model, which can be described as a model in which a hot spot circulates too quickly to give up its heat in the turbulent regime, is not turbulent for fluids which are good conductors of heat.

To understand the behaviour of the convective rolls, and the nature of the modes that are strongly coupled to them to produce turbulence in low Prandtl number fluids, it is useful to consider the stability of the rolls. Busse and coworkers [18] have examined this problem as a function of Prandtl number. For low Prandtl numbers long wavelength oscillations along the convective roll axis develop instabilities for values of R only slightly above R_c. Since these motions are unstable it is natural to include them in the analysis. In the light of the cylindrical geometry of Ahlers' experiment, the rolls probably take the form of concentric toruses or doughnuts of increasing radius. The mean length of these toruses is about πR so that the fundamental mode of the torus has a wave number of about $1/R$. The wavenumbers of the higher spatial harmonics are approximately multiples of this length. For the purposes of numerical analysis, we have idealized the cylinder as a box of length $1/R$ and included the modes with wave numbers $k_y \approx n/R$ for $n = 1, ..., 4$. All of these modes are less damped near the convective threshold than harmonics with higher values of k_x and k_z, and slightly above threshold we can expect them to be the most significant. The results of this very qualitative analysis are reported in greater detail elsewhere [3]. Suffice it to say that when three or less modes are included, we find oscillatory solutions for values of r not very much larger than 1. These oscillations persist with increasing r even at values as large as $r = 20$. If a fourth mode is included, the numerical solutions are drastically altered. For values of $r \lesssim 1.6$, we find oscillating solutions in which the amplitude of oscillation is very small ($\lesssim 0.1\ \%$). Graphs of these solutions are contained in figure 10. For values of $r > 1.6$ we find solutions which are not periodic. They exhibit random behaviour. The threshold, the intensity, and the frequencies of the chaotic solution agree qualitatively with the experimental results of Ahlers. The periodic solutions for four modes when $r \lesssim 1.6$ are exhibited in figure 11. The numerical solutions and Ahlers' experimental results in the turbulent regime are compared in figure 12.

A final possibility, one that involves both the Landau-Hopf and the Ruelle-Takens pictures, may not have much bearing on fluid dynamics, but I believe it bears mention. It is embodied in the solutions of the difference equation

$$N_{t+1} = N_t[1 + r(1 - N_t/K)]$$

an equation describing biological populations with non-

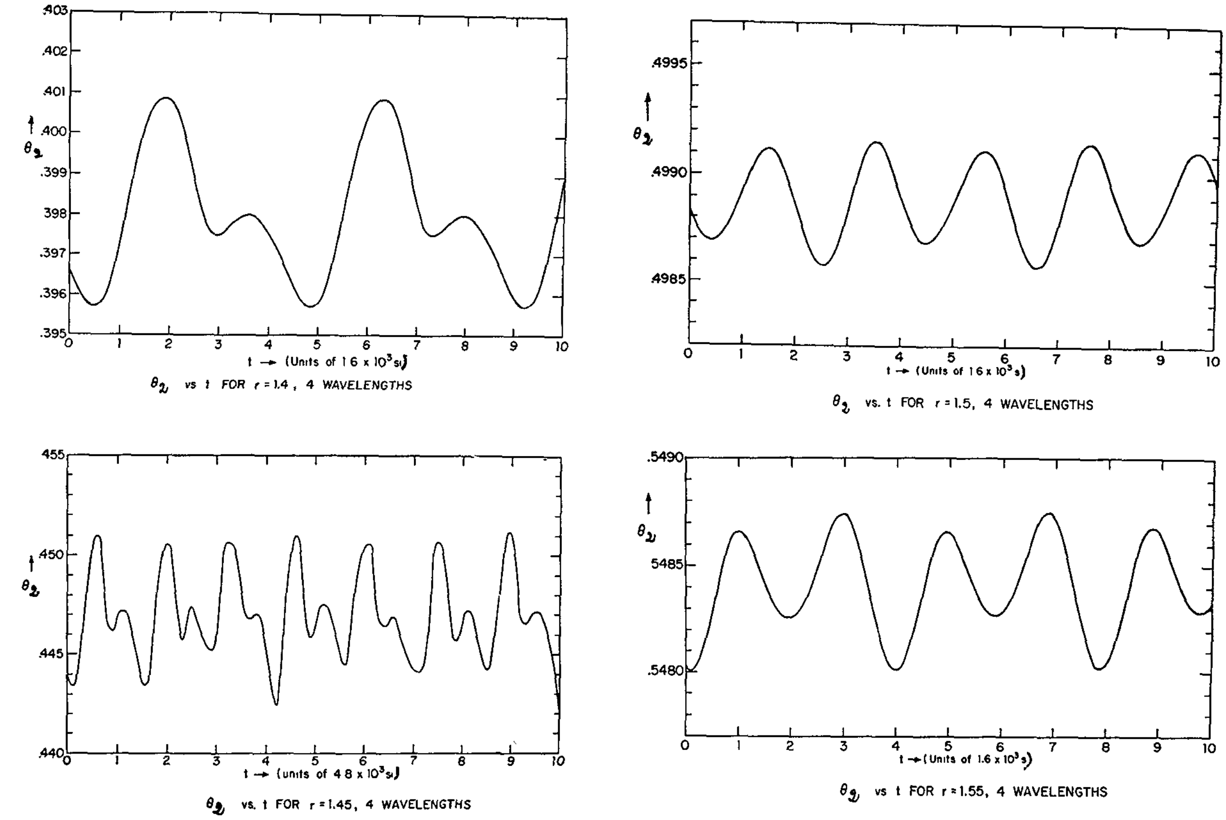


FIG. 10. — The behaviour as a function of r of the Boussinesq equations truncated to include only three wavelengths along the rolls.

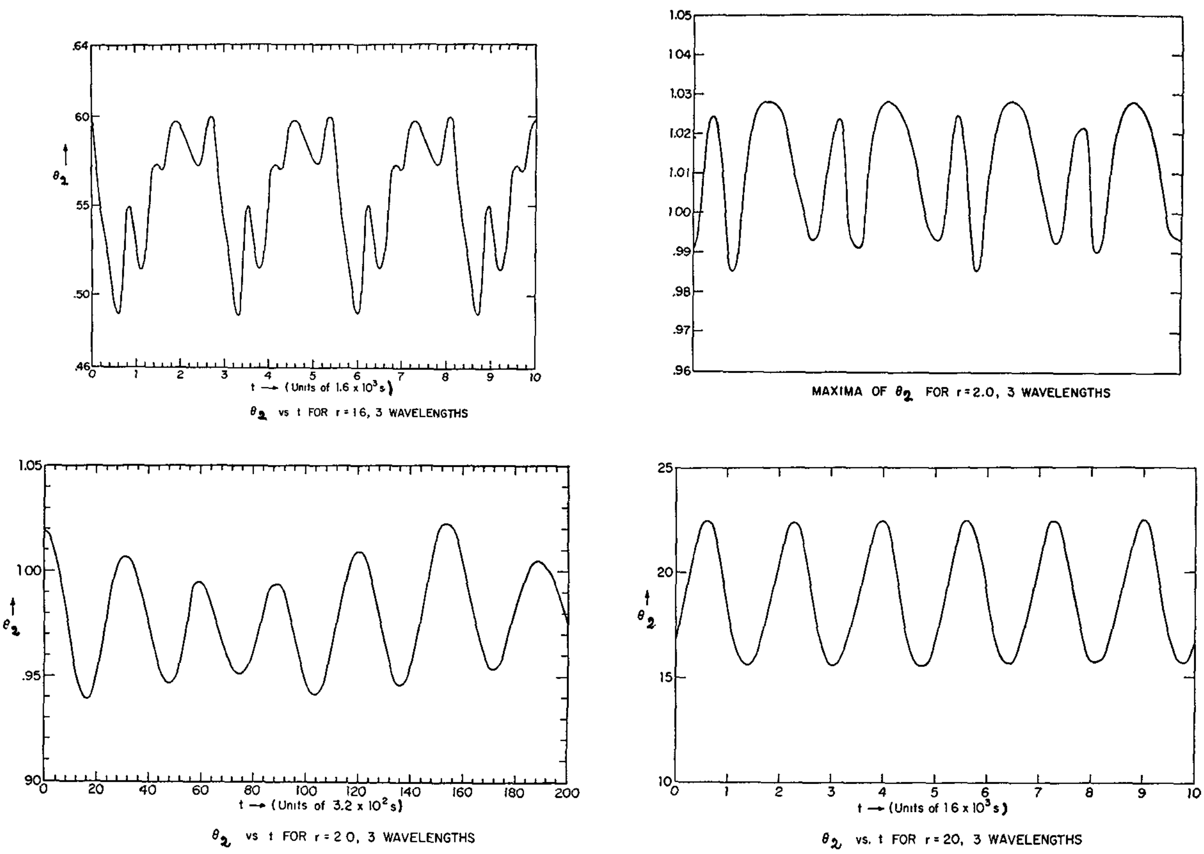


FIG. 11. — The behaviour for values of $r < 1.6$ of the Boussinesq equations truncated to include four wavelengths along the rolls.

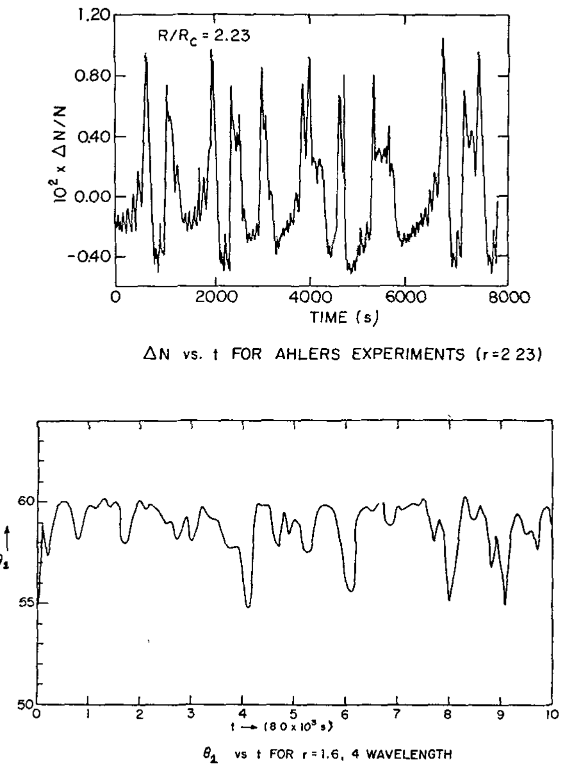


FIG. 12. — Comparison of the four wavelength calculations with Ahlers' observations on convective turbulence in helium.

overlapping generations. (The similar and more familiar, continuous equation

$$N = rN(1 - N/K)$$

for overlapping generations does not exhibit these strange properties.) The properties of this equation have been studied by May [19], and by Yorke and Li [20] and its qualitative behaviour is illustrated in figure 4.

In the discussion up to this point, the spacings of levels participating in the fluid motion has (except for the brief mention of the fluctuations in a Bénard geometry between two infinite plates) been confined to problems where the difficulty crucial to phase transition, the significant participation of infinitely many degrees of freedom, is not important. The problem in which one might hope that recent renormalization group ideas would play a significant role in improving our understanding is that of the strongly turbulent fluid, where in the limit $R \to \infty$, infinitely many modes participate. This problem is attractive to physicists and mathematicians though it has little practical interest. Indeed quantitative experimental studies are scarce [21].

The classical description is due to Kolmogorov and is based essentially on dimensional analysis. In the Kolmogorov [22] picture, the energy is fed into a fluid at long-wavelengths, L, at a specified rate, $\bar{\varepsilon}$. As a result of the non-linear interaction, these long-wavelength motions tend to feed their energy into shorter wavelength motions which tend to be more directionally homogeneous. This cascade continues until wavenumbers k are reached at which the ratio of energy dissipation to energy transfer approaches unity. These wavenumbers are ones for which $kL \sim R^x$. At these wavenumbers dissipation sets in, viscosity dissipating the energy $\bar{\varepsilon}$. If, in the inertial region $L^{-1} \ll k \ll R^x L^{-1}$, neither the length L, nor the viscosity ν plays a role, the former because the form of stirring has been forgotten, and the latter because the dissipation does not react back on the inertial regime, then dimensional analysis requires that the energy density, $E(k)$, between k and $k + \mathrm{d}k$, is

$$E(k) \simeq k^{d-1} \int \mathrm{d}^d r < u(r)\, u(0) > \mathrm{e}^{-ik.r}$$

$$\simeq k^{d-1} < u_k^2 > \simeq \bar{\varepsilon}^{2/3}/k^{5/3}$$

where the exponent d refers to the spatial dimension. The transition to the regime in which viscosity dominates must therefore occur at the Kolmogorov wavenumber, $k_K = (\bar{\varepsilon}/\nu^3)^{1/4}$ and the inertial regime must extend between $L^{-1} \ll k \ll R^{3/4} L^{-1}$.

The picture is appealing but it is not compelling. Indeed, twenty years after it was proposed by Kolmogorov and Obhukov, they each had a change of heart and proposed [23] that there would be small corrections. The basis for their arguments is not entirely clear although at least Kolmogorov refers to an objection of Landau's which refers to fluctuations in the dissipation rate.

There are many questions that may be posed in connection with a discussion of strong turbulence, and to the present, very few convincing answers. Until now, speculations based on phase transition analogies have not been very helpful although they have pointed out certain quantities that should be studied [24]. I shall list below some of these questions together with some comments. I hope they will stimulate some useful research.

1. Is there a universal law for highly developed turbulence in the region $L^{-1} \ll k \ll R^x L^{-1}$?

Unfortunately this question is difficult to answer both theoretically and experimentally. It requires the comparison of different theories or experiments. Is it clear, for example, that there is no dependence on the characteristic frequencies of the stirring force so that, for example, the velocity spectra at high wave numbers for high Reynolds numbers is the same for the static stresses we have been discussing as it is for stochastic stirring by a force distributed Gaussianly with a white noise spectrum ? Does there exist a scaling theory ?

2. Whether or not the phenomena of strong turbulence is universal, can a class of problems involving a stirring length, L, and a rate of energy dissipation, $\bar{\varepsilon}$, be analyzed for large R ?

The most promising problem to study using phase transition techniques seems to be a problem in which the stirring force is the random one described above, with the random force having spatial correlations whose Fourier transform drops off rapidly when $kL \gg 1$. What does it predict ?

3. Is the Kolmogorov prediction correct in any spatial dimension for this or any other model ? Is it correct in three dimensions ?

Although the Kolmogorov argument involves dimensional analysis it does not depend on the spatial dimension. Almost everyone agrees [25] that the argument fails in two dimensions where there are special conservation laws. But what happens near two dimensions ? We have, after all, become used to speaking of fractional spatial dimension. If the Kolmogorov law is wrong, do the corrections in the inertial range involve L, or R (or ν), or both ? Is the Kolmogorov argument particularly relevant in 8/3 dimensions ?

4. Do the phase transition techniques, or any others, explain phenomena like intermittency [21], the experimentally observed variation from homogeneity of small regions of highly developed turbulence ?

Is there a relation between intermittency and the divergence of the dissipative fluctuations of a Kolmogorov-like spectrum in more than eight-thirds spatial dimensions [24] ?

We could go one listing questions but I think I have made my point. The title of this talk applies not only to the phenomena I have been discussing but to our understanding of these phenomena. And lastly it applies to the talk itself which has now reached its final chaotic stage.

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