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J. Phys. France **51** (1990) 2243-2261 15 OCTOBRE **1990**, PAGE 2243

Classification
Physics Abstracts
04.80 — 42.78

Analytical models of transient thermoelastic deformations of mirrors heated by high power cw laser beams

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(*Received 3 May 1990, accepted in final form 3 July 1990*)

Résumé. — Nous nous intéressons aux aberrations causées aux miroirs massifs par des faisceaux laser intenses échauffant les couches réfléchissantes ou la masse même du substrat. La distribution de température engendrée dans le substrat produit des déformations thermoélastiques dont nous donnons des modèles analytiques et des évaluations numériques dans le cas particulier de la symétrie axiale, tout d'abord dans l'état stationnaire, puis dans le régime transitoire.

Abstract. — We study the aberrations of massive mirrors heated by partial of intense laser beams in the coating or in the bulk. The non-uniform temperature field generated in the solid substrate induce thermoelastic deformations for which with the special assumption of axial symmetry, we give analytical models and numerical evaluations, first in the steady state, then in the transient regime.

1. Introduction.

In this study, we shall consider a cylindrical mass made of a transparent material which bears near its axis an optical coating and constitutes a mirror. The coating is supposed to absorb power from an impinging laser beam and to become a heat source for the substrate. The bulk material of the substrate itself may be assumed to have a linear absorption coefficient for light, and to undergo internal generation of heat. The substrate in turn undergoes thermoelastic deformations which affect the mirror quality. This problem arises from the preliminary simulations of the giant Fabry-Perot cavities involved in the Italo-French VIRGO project of interferometric gravitational wave detector ; the planned mirrors are blocks of pure silica and are suspended in a vacuum vessel : the heat losses are only due to the thermal radiation.

The resulting temperature fields have been derived in our preceding paper [1], the calculation needs the linearization of the thermal losses, the model is thus relevant also for convection losses. We have found in the axially symmetrical case an analytical solution for the time dependent temperature field as a Dini series, and we have evaluated the induced thermal lens in the two cases of absorption in the coating or in the bulk. Now with the same model for the temperature field we address the thermoelastic deformations of the mirror.

The three-dimensional thermoelastic problem has in general no analytical solution. Nevertheless, some authors have studied the deformations of mirrors heated by lasers [2-8]; but they almost always consider the static regime with sometimes fuzzy boundary conditions. Therefore we find it useful to publish our attempt to treat the boundary conditions and the time dependence precisely.

In the present paper, with the help of the Saint-Venant Principle, we first find a steady state solution for the thermal deformations of a finite circular cylinder heated on a coated face by an axially symmetrical light beam, without applied forces. The conclusion met in this first step allows us to develop a general model, valid even when the temperature is not a harmonic function, especially in the transient case of heating either in the coating or in the bulk.

In the following we will use the parameters of pure silica:

ρ is the specific mass ($2\,202\ \mathrm{kg\,m^{-3}}$),
C is the specific heat ($745\ \mathrm{J\,kg^{-1}\,K^{-1}}$),
K is the thermal conductivity ($1.38\ \mathrm{W\,m^{-1}\,K^{-1}}$),
λ is the first Lamé coefficient ($1.56 \times 10^{10}\ \mathrm{J\,m^{-3}}$),
μ is the second Lamé coefficient ($3.13 \times 10^{10}\ \mathrm{J\,m^{-3}}$),
ν is the stress temperature modulus ($5.91 \times 10^{4}\ \mathrm{J\,m^{-3}\,K^{-1}}$).
ν is related to the average thermal expansion coefficient α by:

$$\nu = \alpha (3\lambda + 2\mu).$$

In the numerical calculations, we consider a disk of radius $a = 0.30$ m and thickness $h = 0.20$ m (see Fig. 1), a laser waist of 2 cm and one Watt absorbed power.

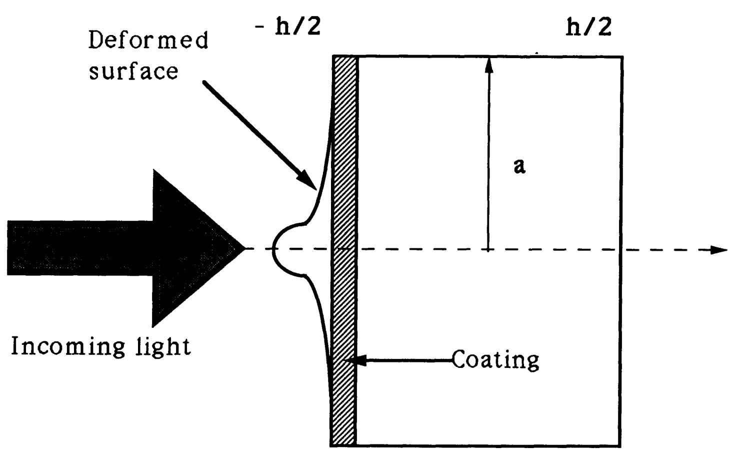

Fig. 1. — Drawing of the input mirror of the Virgo cavity.

2. The thermoelastic equations.

Let **E** be the strain tensor, **ϑ** the stress tensor and **u** the displacement vector. We confine our attention on the axially symmetrical problem: if the radial variable is denoted by r, the axial variable by z, and the azimutal angle by ϕ then, the non-zero components of **u** are

u_r and u_z, and the non-zero strain components are [9, 10] :

$$E_{rr} = \frac{\partial u_r}{\partial r} \quad E_{\phi\phi} = \frac{u_r}{r} \quad E_{zz} = \frac{\partial u_z}{\partial z} \quad E_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{2.1}$$

If a non-uniform temperature field $T(t, r, z)$ is present, the tensors $\mathbf{E}$ and $\boldsymbol{\vartheta}$ are related by [9, 10] :

$$\begin{cases} \vartheta_{rr} = -\nu T + \lambda E + 2\mu E_{rr} \\ \vartheta_{\phi\phi} = -\nu T + \lambda E + 2\mu E_{\phi\phi} \\ \vartheta_{zz} = -\nu T + \lambda E + 2\mu E_{zz} \\ \vartheta_{rz} = 2\mu E_{rz} \end{cases} \tag{2.2}$$

where E is the trace of the strain tensor : $E = E_{rr} + E_{\phi\phi} + E_{zz}$.

Finally $\boldsymbol{\vartheta}$ obeys the equilibrium equations [11] :

$$\begin{cases} \dfrac{\partial \vartheta_{rr}}{\partial r} + \dfrac{\partial \vartheta_{rz}}{\partial z} + \dfrac{\vartheta_{rr} - \vartheta_{\phi\phi}}{r} = \rho \dfrac{\partial^2 u_r}{\partial t^2} \\ \dfrac{\partial \vartheta_{rz}}{\partial r} + \dfrac{\partial \vartheta_{zz}}{\partial z} + \dfrac{\vartheta_{rz}}{r} = \rho \dfrac{\partial^2 u_z}{\partial t^2}. \end{cases} \tag{2.3}$$

We shall assume that there are no applied external forces, so that the boundary conditions reduce to :

$$\vartheta_{rr}(a, z) = 0, \quad \vartheta_{rz}(a, z) = 0, \quad \vartheta_{rz}(r, \pm h/2) = 0, \quad \vartheta_{zz}(r, \pm h/2) = 0. \tag{2.4}$$

3. Steady state solution in the case of dissipation in a coating.

We shall first consider the case where the generation of heat is located at one face of the disk, in the coating, regarded as a thin layer. Everywhere in the bulk of the substrate, the stationary temperature is a harmonic function of r and z, and we may exploit this property to derive a simple solution of the thermoelastic equations.

3.1 Expression as integrals of the temperature. — In the steady state equilibrium equations, the inertial force vanishes and the homogeneous equations remain

$$\begin{cases} \dfrac{\partial \theta_{rr}}{\partial r} + \dfrac{\partial \theta_{rz}}{\partial z} + \dfrac{\theta_{rr} - \theta_{\phi\phi}}{r} = 0 \\ \dfrac{\partial \theta_{rz}}{\partial r} + \dfrac{\partial \theta_{zz}}{\partial z} + \dfrac{\theta_{rz}}{r} = 0. \end{cases} \tag{3.1}$$

It will be shown that both the equilibrium equations and all but one boundary conditions are satisfied by the following expression of the displacement vector :

$$\begin{cases} u_z = \dfrac{\nu}{2(\lambda + \mu)}\left[\displaystyle\int_{-h/2}^{z} T(r, z')\,\mathrm{d}z' + \Phi(r)\right] \\ u_r = \dfrac{\nu}{2(\lambda + \mu)}\dfrac{1}{r}\displaystyle\int_0^r T(r', z)\,r'\,\mathrm{d}r' \end{cases}$$

where $\Phi(r)$ is an arbitrary function of r which will be determined later. The components of the strain tensor are (2.1) :

$$\begin{cases} E_{rr} = \dfrac{\nu}{2(\lambda + \mu)} \left[T(r, z) - \dfrac{1}{r^2} \displaystyle\int_0^r T(r', z)\, r'\, \mathrm{d}r' \right] \\ E_{\phi\phi} = \dfrac{\nu}{2(\lambda + \mu)} \dfrac{1}{r^2} \displaystyle\int_0^r T(r', z)\, r'\, \mathrm{d}r' \\ E_{zz} = \dfrac{\nu}{2(\lambda + \mu)} T(r, z) \\ E_{rz} = \dfrac{\nu}{4(\lambda + \mu)} \left[\phi'(r) + \displaystyle\int_{-h/2}^{z} \dfrac{\partial T}{\partial r}(r, z')\, \mathrm{d}z' + \dfrac{1}{r} \displaystyle\int_0^r \dfrac{\partial T}{\partial z}(r', z)\, r'\, \mathrm{d}r' \right]. \end{cases}$$

Then the trace of **E** is

$$E = \frac{\nu}{(\lambda + \mu)} T(r, z)$$

and we can derive from (2.2) the components of the stress tensor :

$$\begin{cases} \vartheta_{rr} = -\dfrac{\mu \nu}{(\lambda + \mu)} \dfrac{1}{r^2} \displaystyle\int_0^r T(r', z)\, r'\, \mathrm{d}r' \\ \vartheta_{\phi\phi} = -\dfrac{\mu \nu}{(\lambda + \mu)} \left[T(r, z) - \dfrac{1}{r^2} \displaystyle\int_0^r T(r', z)\, r'\, \mathrm{d}r' \right] \\ \vartheta_{zz} = 0 \end{cases} \tag{3.2}$$

$$\vartheta_{rz} = \frac{\mu \nu}{2(\lambda + \mu)} \left[\Phi'(r) + \int_{-h/2}^{z} \frac{\partial T}{\partial r}(r, z')\, \mathrm{d}z' + \frac{1}{r} \int_0^r \frac{\partial T}{\partial z}(r', z)\, r'\, \mathrm{d}r' \right]. \tag{3.3}$$

The equilibrium equations (3.1) allow us to determine the function Φ ; with the above expressions of ϑ_{rr}, $\vartheta_{\phi\phi}$ and ϑ_{zz}, they lead to :

$$\frac{\partial \vartheta_{rz}}{\partial z} = 0 \tag{3.4}$$

and

$$\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \vartheta_{rz} = 0 . \tag{3.5}$$

We can easily check that the first equation is identically satisfied. We have indeed from (3.3) :

$$\frac{\partial \vartheta_{rz}}{\partial z} = \frac{\mu \nu}{2(\lambda + \mu)} \left[\frac{\partial T}{\partial r}(r, z) + \frac{1}{r} \int_0^r \frac{\partial^2 T}{\partial z^2}(r', z)\, r'\, \mathrm{d}r' \right] \tag{3.6}$$

but with no internal heat generation, the temperature obeys the special Fourier equation :

$$\Delta T = 0 \tag{3.7}$$

or

$$\frac{\partial^2 T}{\delta z^2} = -\frac{1}{r} \partial_r \left(r \frac{\partial T}{\partial r} \right) \tag{3.8}$$

and equation (3.4) is obtained by substituting (3.8) in (3.6). With the expression (3.3) given for ϑ_{rz}, we obtain, using again (3.7):

$$\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \vartheta_{rz} = \frac{\mu \nu}{2(\lambda + \mu)} \left(\Phi''(r) + \frac{\Phi'(r)}{r} + \frac{\partial T}{\partial z} \left(r, -\frac{h}{2} \right) \right) .$$

Then we have to choose Φ in such a way that the preceding expression vanishes. This happens if:

$$\left(\frac{d}{dr} + \frac{1}{r} \right) \frac{d\Phi}{dr} = - \frac{\partial T}{\partial z} \left(r, -\frac{h}{2} \right)$$

the above differential equation admits the solution:

$$\Phi(r) = - \int_0^r \frac{dr'}{r'} \int_0^{r'} \frac{\partial T}{\partial z} \left(r'', -\frac{h}{2} \right) r'' \, dr'' + C$$

where C is an arbitrary constant.

Finally we obtain for the stress tensor component ϑ_{rz}:

$$\vartheta_{rz} = \frac{\mu \nu}{2(\lambda + \mu)} \left(\int_{-h/2}^{z} \frac{\partial T}{\partial r}(r, z') \, dz' + \frac{1}{r} \int_0^r \left[\frac{\partial T}{\partial z}(r', z) - \frac{\partial T}{\partial z} \left(r', -\frac{h}{2} \right) \right] r' \, dr' \right) .$$

This last form makes it clear that $\vartheta_{rz}(r, -h/2) = 0$. But it has just been shown that $\partial \vartheta_{rz}/\partial z = 0$; it follows that:

$$\vartheta_{rz} = 0 .$$

The boundary conditions

$$\vartheta_{rz}(r, \pm h/2) = \vartheta_{zz}(r, \pm h/2) = 0 = \vartheta_{rz}(a, z)$$

are thus fulfilled. The last condition is: $\vartheta_{rr}(a, z) = 0$.

From (3.2) we find:

$$\vartheta_{rr}(a, z) = - \frac{\mu \nu}{\lambda + \mu} \frac{1}{a^2} \int_0^a T(r, z) \, r \, dr .$$

The last boundary condition is thus not satisfied. A numerical plot shows however that the function $\vartheta_{rr}(a, z)$ is almost linear (see Fig. 2) if we take for T the steady state temperature computed in [1]; a minor transformation of the vector $\mathbf{u}$ will allow us to remove the resultant force on the boundary. We consider a new displacement vector $\hat{\mathbf{u}}$:

$$\hat{u}_r(r, z) = u_r(r, z) + \frac{\lambda + 2\mu}{2\mu(3\lambda + 2\mu)} (Ar + Brz)$$

$$\hat{u}_z(r, z) = u_z(r, z) - \frac{\lambda}{\mu(3\lambda + 2\mu)} \left(Az + \frac{1}{2} Bz^2 \right) - \frac{\lambda + 2\mu}{4\mu(3\lambda + 2\mu)} Br^2$$

A and B are two arbitrary constants. It is easy to check that $\hat{\mathbf{u}}$ and $\mathbf{u}$ obey the same equilibrium equations and the same boundary conditions on the planes $z = \pm h/2$. But the new radial stress is:

$$\Theta_{rr}(r, z) = \vartheta_{rr}(r, z) + A + Bz .$$

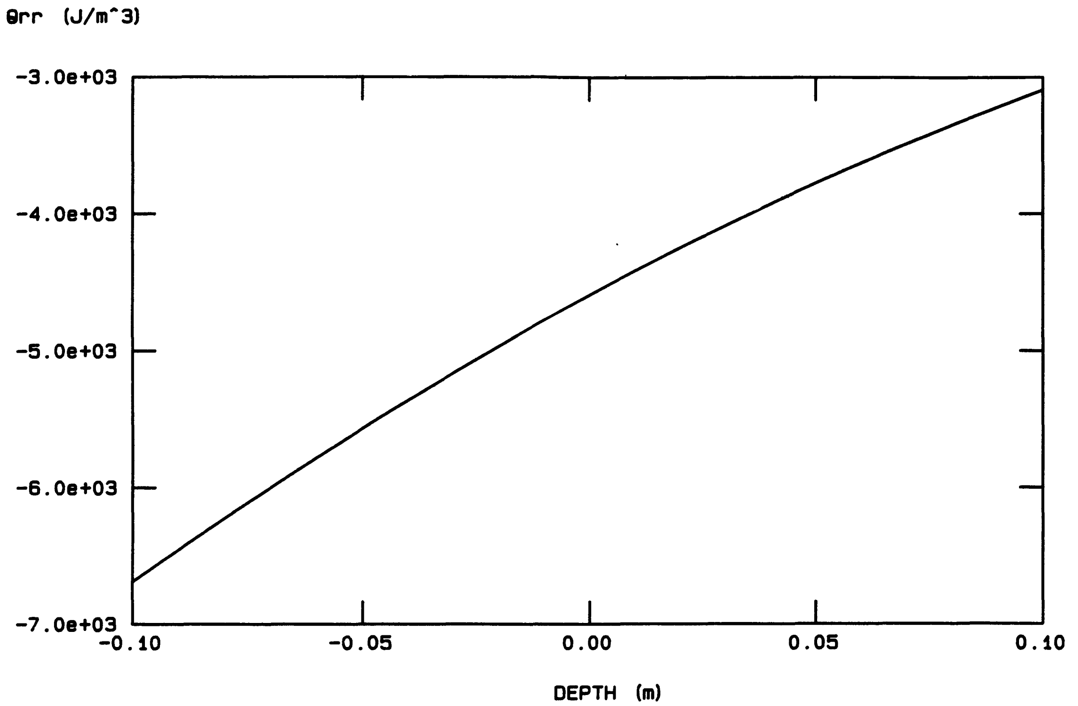

Fig. 2. — A plot of the function $\vartheta_{rr}(a, z)$ representative of the radial stress at the edge of the mirror. The shape of the curve allows us to approximate $\vartheta_{rr}(a, z)$ by a linear function. We assume, as in further figures, 1 Watt absorbed power in the coating.

Now, the arbitrary constants A and B may be chosen to make the radial resultant force and the associated mean torque at $r = a$ vanish :

$$A = -\frac{1}{h}\int_{-h/2}^{h/2} \vartheta_{rr}(a, z)\,\mathrm{d}z$$

and

$$B = -\frac{12}{h^3}\int_{-h/2}^{h/2} \vartheta_{rr}(a, z)\,z\,\mathrm{d}z\,.$$

In terms of temperature integrals, we have :

$$A = \frac{\mu\nu}{\lambda+\mu}\frac{1}{a^2 h}\int_0^a r\,\mathrm{d}r\int_{-h/2}^{h/2}\mathrm{d}z\,T(r, z) \tag{3.9}$$

$$B = \frac{12\,\mu\nu}{\lambda+\mu}\frac{1}{a^2 h^3}\int_0^a r\,\mathrm{d}r\int_{-h/2}^{h/2} z\,\mathrm{d}z\,T(r, z)\,. \tag{3.10}$$

Strictly speaking, the resulting displacement vector $\hat{\mathbf{u}}(r, z)$ is not the exact solution of the thermoelastic problem. But according to the Principle of Saint-Venant [10], $\hat{\mathbf{u}}$ will not appreciably differ from the exact solution except at the edges of the disk. However the physical radius of the substrate is much larger than the optical waist of the light beam, and we are not interested in knowing the aberrations accurately in a region where no light is present.

The preceding solution $\hat{\mathbf{u}}$ is valid for any axisymmetric and harmonic temperature field. We can go further if the temperature field is explicitly known ; in the following section we assume that the temperature can be expanded in a Dini series as we have found in [1].

3.2 EXPRESSION AS A DINI SERIES. — Let us recall the notation used in [1].

τ is the reduced radiation constant, defined by :

$$\tau = \frac{4\,\sigma'\,T_{\text{ext}}^3\,a}{K}$$

σ' is the Stefan-Boltzmann constant corrected for emissivity,
T_{ext} is the temperature of the walls of the vacuum vessel that surrounds the mirror,
ζ_m is the m-th solution of the equation

$$\zeta J_1(\zeta) - \tau J_0(\zeta) = 0$$

where the J_p are the p-th-order Bessel functions.

The axially symmetrical light intensity profile is expanded as a Dini series, on the basis of the orthogonal functions $J_0(\zeta_m\,r/a)$:

$$I(r) = \sum_m p_m\,J_0(\zeta_m\,r/a)\,.$$

For instance, for a Gaussian beam of waist w and power P, we have :

$$p_m = \frac{P}{\pi a^2}\frac{\zeta_m^2}{(\zeta_m^2 + \tau^2)\,J_0(\zeta_m)^2}\exp\left(-\frac{1}{8}\zeta_m^2\frac{w^2}{a^2}\right).$$

With this notation, the analytical expression for the stationary temperature field generated by surface absorption [1] is :

$$T_\infty(r, z) = \sum_m \left(A_m\,\mathrm{e}^{\frac{\zeta_m z}{a}} + B_m\,\mathrm{e}^{-\frac{\zeta_m z}{a}}\right) J_0\left(\frac{\zeta_m r}{a}\right).$$

The absorbing surface is the plane $z = -h/2$. The constants A_m and B_m have the following expressions :

$$A_m = \frac{\varepsilon p_m\,a}{K}\frac{(\zeta_m - \tau)\,\mathrm{e}^{-3\zeta_m h/2a}}{(\zeta_m + \tau)^2 - (\zeta_m - \tau)^2\,\mathrm{e}^{-2\zeta_m h/a}} \tag{3.11}$$

$$B_m = \frac{\varepsilon p_m\,a}{K}\frac{(\zeta_m + \tau)\,\mathrm{e}^{-\zeta_m h/2a}}{(\zeta_m + \tau)^2 - (\zeta_m - \tau)^2\,\mathrm{e}^{-2\zeta_m h/a}} \tag{3.12}$$

where ε is the rate of dissipation of light power into heat in the coating.

Using the above theory, we find the radial stress component :

$$\vartheta_{rr}(a, z) = -\frac{\mu\,\nu}{\lambda + \mu}\sum_m \left(A_m\,\mathrm{e}^{\zeta_m z/a} + B_m\,\mathrm{e}^{-\zeta_m z/a}\right)\frac{J_1(\zeta_m)}{\zeta_m}\,.$$

Integrating (3.9) gives the mean stress at $r = a$:

$$A = \frac{\mu\,\nu}{\lambda + \mu}\,\frac{\varepsilon a^2}{Kh}\sum_m \frac{p_m}{\zeta_m^2}\,\frac{1 - \mathrm{e}^{-2\gamma_m}}{(\zeta_m + \tau) - (\zeta_m - \tau)\,\mathrm{e}^{-2\gamma_m}}\,J_1(\zeta_m) \tag{3.13}$$

and from (3.10) we find the mean slope of the stress :

$$B = \frac{12\,\mu\,\nu}{\lambda + \mu}\,\frac{\varepsilon}{K}\,\frac{a^3}{h^3}\sum_m \frac{p_m}{\zeta_m^3}\,\frac{\gamma_m - 1 + (\gamma_m + 1)\,\mathrm{e}^{-2\gamma_m}}{\zeta_m + \tau + (\zeta_m - \tau)\,\mathrm{e}^{-2\gamma_m}}\,J_1(\zeta_m) \tag{3.14}$$

where we define γ_m as

$$\gamma_m = \frac{\zeta_m\,h}{2\,a}\,. \tag{3.15}$$

We can now give the final expressions of the components of $\hat{\mathbf{u}}$:

$$u_r(r, z) = \frac{\nu}{2(\lambda + \mu)}\,\frac{\varepsilon a^2}{K}\sum_m \frac{p_m}{\zeta_m}\,\frac{(\zeta_m - \tau)\,\mathrm{e}^{-3\gamma_m}\,\mathrm{e}^{\zeta_m z/a} + (\zeta_m + \tau)\,\mathrm{e}^{-\gamma_m}\,\mathrm{e}^{-\zeta_m z/a}}{(\zeta_m + \tau)^2 - (\tau_m - \tau)^2\,\mathrm{e}^{-4\gamma_m}} \times$$
$$\times J_1(\zeta_m r/a) + \frac{\lambda + 2\,\mu}{2\,\mu\,(3\,\lambda + 2\,\mu)}\,(A + Bz)\,r$$

and

$$u_z(r, z) = \frac{\nu}{2(\lambda + \mu)}\,\frac{\varepsilon a^2}{K}\sum_m \frac{p_m}{\zeta_m}\,\frac{(\zeta_m - \tau)\,\mathrm{e}^{-3\gamma_m}\,\mathrm{e}^{\zeta_m z/a} - (\zeta_m + \tau)\,\mathrm{e}^{-\gamma_m}\,\mathrm{e}^{-\zeta_m z/a}}{(\zeta_m + \tau)^2 - (\zeta_m - \tau)^2\,\mathrm{e}^{-4\gamma_m}} \times$$
$$\times J_0(\zeta_m r/a) - \frac{\lambda}{\mu\,(3\,\lambda + 2\,\mu)}\left(A + \frac{1}{2}Bz\right) z - \frac{\lambda + 2\,\mu}{4\,\mu\,(3\,\lambda + 2\,\mu)}\,Br^2 + C\,.$$

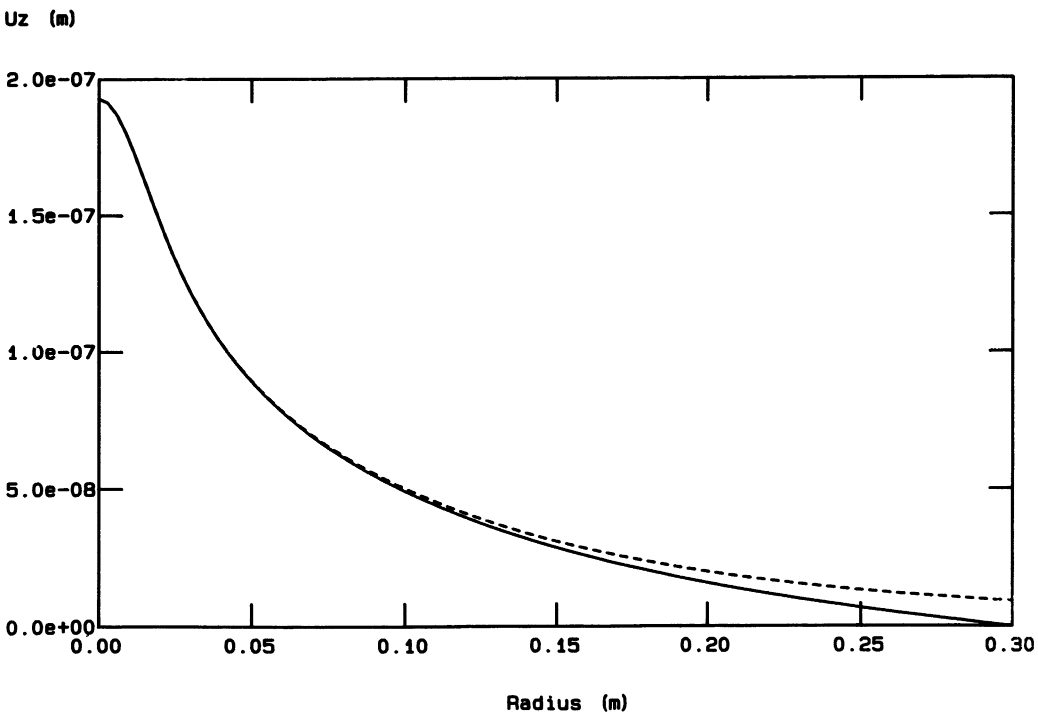

Fig. 3. — The deformed surface of the mirror with and without the Saint Venant correction (respectively in dashed and plain line) when heating occurs at the coating. Note that the mirror becomes convex. The arbitrary offset which appears through C′ (formula (3.16)) makes possible to choose the zero at the edge of the mirror.

Neglecting second order terms with respect to the displacement, the equation of the deformed surface is :

$$S(r) = u_z(r, -h/2)$$

then

$$S(r) = -\frac{\nu}{2(\lambda+\mu)}\frac{\varepsilon a^2}{K}\sum_m \frac{p_m}{\zeta_m}\frac{(\zeta_m+\tau)-(\zeta_m-\tau)\,\mathrm{e}^{-4\gamma_m}}{(\zeta_m+\tau)^2-(\zeta_m-\tau)^2\,\mathrm{e}^{-4\gamma_m}}J_0(\zeta_m r/a) - \frac{\lambda+2\mu}{4\mu(3\lambda+2\mu)}Br^2 + C'. \tag{3.16}$$

The second term will be referred to as the Saint Venant correction. The displacement is defined up to an arbitrary constant C'.

Figure 3 is a plot of the function $S(r)$ and figure 4 is a zoom on the optical zone of radius 5 cm. The constant C' allows an arbitrary offset in the plots of $S(r)$ and we use to make the displacement vanish at the edge of the representing zone. We note that in the optical zone the two solutions with and without the Saint Venant correction fit perfectly (numerically the coefficient of r^2 in (3.16) is about $10^{-7}\ \mathrm{m}^{-1}$).

This calculation is valid as long as the temperature obeys equation (3.7), i.e. the temperature is a harmonic function. This is not the case when the temperature is time dependent or if there is a generation of heat in the bulk of the mirror. Thus we have to build a more general solution. However, we point out the important result of the above section : the Saint Venant correction to the stationary solution can be neglected in the relevant part of the solid. Accordingly, we shall approximate the mirror by a mirror of infinite radius, and we do not have to take the boundary conditions on the edges into account.

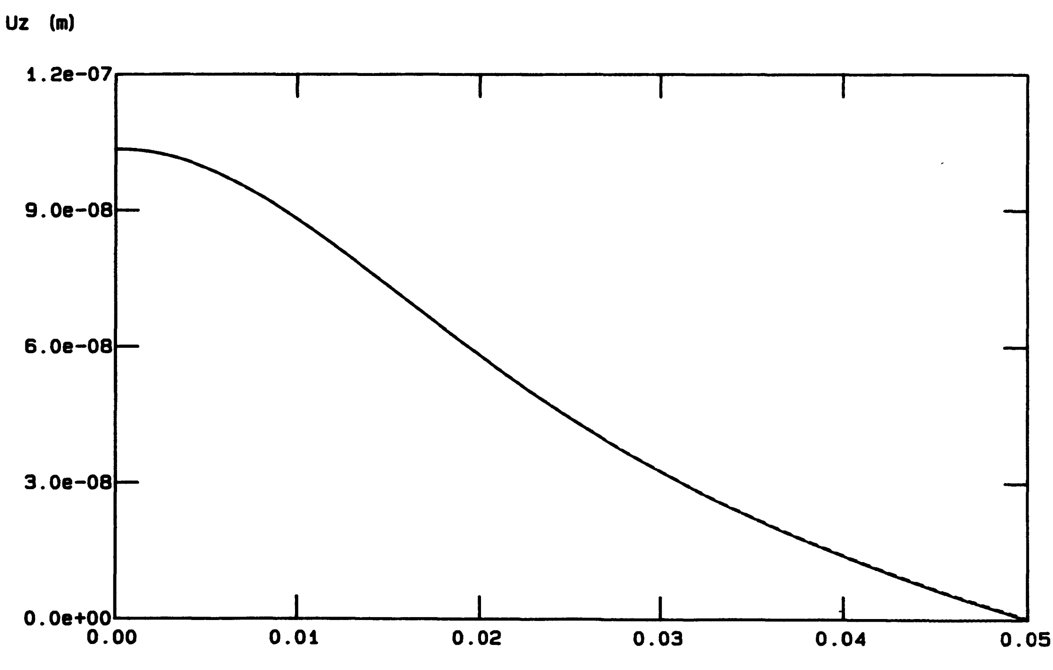

Fig. 4. — Zoom of the figure 3 on the optical zone limited by a radius of 5 cm, with a change of the offset making $S(r)$ vanish at the edge of the optical zone. The same offset will be choosen in all the following plots. In the optical zone, which is of most interest, the two curves are roughly identical : the Saint Venant correction appears negligible.

4. The general transient solution.

4.1 EXPRESSION AS A DINI SERIES. — As the time dependent solution for the displacement vector, we can express u_r and u_z as series of Bessel functions :

$$u_r(t, r, z) = \sum_m A_m(t, z) J_1\left(\frac{\zeta_m r}{a}\right) \tag{4.1}$$

$$u_z(t, r, z) = \sum_m B_m(t, z) J_0\left(\frac{\zeta_m r}{a}\right). \tag{4.2}$$

We assume that the temperature T can also be expressed as a Dini series [1]

$$T(t, r, z) = \sum_m T_m(t, z) J_0\left(\frac{\zeta_m r}{a}\right).$$

Then the equilibrium equations (2.3) may be cast under the form :

$$\rho \frac{\partial^2 A_m}{\partial t^2} - \mu\left(\frac{\partial^2 A_m}{\partial z^2} - k_m^2 A_m\right) + (\lambda + \mu) k_m\left(k_m A_m + \frac{\partial B_m}{\partial z}\right) = \nu k_m T_m \tag{4.3}$$

$$\rho \frac{\partial^2 B_m}{\partial t^2} - \mu\left(\frac{\partial^2 B_m}{\partial z^2} - k_m^2 B_m\right) - (\lambda + \mu) k_m\left(k_m \frac{\partial A_m}{\partial z} + \frac{\partial^2 B_m}{\partial z^2}\right) = -\nu \frac{\partial T_m}{\partial z} \tag{4.4}$$

with

$$k_m = \frac{\zeta_m}{a}.$$

Note that the last equations have been found by Cutolo *et al.* in [4]. It is then convenient to form the z-derivative of the first equation plus k_m times the second, using

$$U_m = \frac{\partial A_m}{\partial z} + k_m B_m$$

we obtain :

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho}\frac{\partial^2}{\partial z^2} + \frac{\mu}{\rho} k_m^2\right) U_m = 0$$

which is a propagation equation for acoustic waves.

In [1] we have found a characteristic time for the evolution of temperature, of about 30 hours. We therefore expect the evolution of the deformation to be quasi-static. Thus we can neglect the inertial forces, in other words we do not take the generation of acoustic waves in the mirror into account. This is valid for cw laser beams, if the time scale is much larger than the characteristic propagation time of the waves in the silica block $t_s = h/v_s$, where v_s is the propagation speed.

It is well known that, in a solid, we can define two sound velocities : the longitudinal one

$$v_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

and the transverse one

$$v_t = \sqrt{\frac{\mu}{\rho}}\,.$$

With the numerical values given in section 1, we find the upper limit value for v_s : about 2×10^3 m/s. The acoustic time scale is thus about 10^{-4} s, which is negligible with respect to the evolution time of temperature.

Thus discarding the time derivatives in (4.3) and (4.4) we find a new system :

$$\left(\frac{\partial^2}{\partial z^2} - k_m^2\right)\left(\frac{\partial A_m}{\partial z} + k_m B_m\right) = 0 \tag{4.5}$$

and

$$\mu\left(\frac{\partial^2 A_m}{\partial z^2} - k_m^2 A_m\right) - (\lambda + \mu) k_m \left(k_m A_m + \frac{\partial B_m}{\partial z}\right) = -\nu k_m T_m\,. \tag{4.6}$$

In the last system, the time now appears as a parameter through the temperature coefficients.

Equation (4.5) has the general solution :

$$\left(\frac{\partial A_m}{\partial z} + k_m B_m\right) = k_m A_m^{(1)} \cosh(k_m z) + k_m A_m^{(2)} \sinh(k_m z)$$

which is equivalent to :

$$B_m = A_m^{(1)} \cosh(k_m z) + A_m^{(2)} \sinh(k_m z) - \frac{1}{k_m}\frac{\partial A_m}{\partial z}$$

where $A_m^{(1)}$ and $A_m^{(2)}$ are arbitrary constants.

We can now form the z-derivative of B_m and substitute it in (4.6) then we get :

$$(\lambda + 2\mu)\left(\frac{\partial^2 A_m}{\partial z^2} - k_m^2 A_m\right) = $$
$$= -\nu k_m T_m + (\lambda + \mu) k_m^2 (A_m^{(1)} \sinh(k_m z) + A_m^{(2)} \cosh(k_m z))\,.$$

The solution of this equation is the sum of the general solution of the homogeneous equation, involving new arbitrary constants $B_m^{(1)}$ and $B_m^{(2)}$, plus a special solution :

$$A_m(t, z) = B_m^{(1)} \cosh(k_m z) + B_m^{(2)} \sinh(k_m z) - \frac{\nu}{\lambda + 2\mu} k_m t_m(t, z) +$$
$$+ \frac{\lambda + \mu}{\lambda + 2\mu}\left(\frac{A_m^{(1)} k_m z}{2} \cosh(k_m z) + \frac{A_m^{(2)} k_m z}{2} \sinh(k_m z)\right)$$

where the functions $t_m(t, z)$ are arbitrary special solutions of the differential equation :

$$\frac{\partial^2 t_m}{\partial z^2} - k_m^2 t_m = T_m\,. \tag{4.7}$$

We can now express the functions B_m :

$$
\begin{aligned}
B_m(t, z) = &- B_m^{(1)} \sinh (k_m z) - B_m^{(2)} \cosh (k_m z) + \frac{\nu}{\lambda + 2 \mu} \frac{\partial t_m}{\partial z} + \\
&+ \frac{\lambda + 3 \mu}{2(\lambda + 2 \mu)} (A_m^{(1)} \cosh (k_m z) + A_m^{(2)} \sinh (k_m z)) - \\
&- \frac{\lambda + \mu}{2(\lambda + 2 \mu)} k_m z (A_m^{(1)} \sinh (k_m z) + A_m^{(2)} \cosh (k_m z)) . \qquad (4.8)
\end{aligned}
$$

Now we have to use the boundary conditions (2.4) in order to find the constants $A_m^{(1)}$, $A_m^{(2)}$, $B_m^{(1)}$ and $B_m^{(2)}$. As we have seen in section 3, we do not need to consider the boundary conditions at $r = a$; so the conditions reduce to :

$$
\vartheta_{rz}(r, \pm h/2) = 0 , \quad \vartheta_{zz}(r, \pm h/2) = 0 .
$$

But ϑ_{rz} and ϑ_{zz} have the following expressions :

$$
\begin{aligned}
\vartheta_{rz}(t, r, z) &= \mu \sum_m \left(\frac{\partial A_m}{\partial z} - k_m B_m \right) J_1(k_m r) \\
\vartheta_{zz}(t, r, z) &= - \nu T + \sum_m \left(\lambda k_m A_m + (\lambda + 2 \mu) \frac{\partial B_m}{\partial z} \right) J_0(k_m r) .
\end{aligned}
$$

We find the boundary conditions equivalent to the following linear system :

$$
\begin{aligned}
\Gamma_m^{(1)} A_m^{(1)} + \Gamma_m^{(2)} A_m^{(2)} - 2 B_m^{(1)} \sinh (\gamma_m) + 2 B_m^{(2)} \cosh (\gamma_m) &= \frac{2 \nu}{\lambda + 2 \mu} t'_m(- h/2) \\
\Gamma_m^{(1)} A_m^{(1)} - \Gamma_m^{(2)} A_m^{(2)} + 2 B_m^{(1)} \sinh (\gamma_m) + 2 B_m^{(2)} \cosh (\gamma_m) &= \frac{2 \nu}{\lambda + 2 \mu} t'_m(+ h/2) \\
\Gamma_m^{(3)} A_m^{(1)} + \Gamma_m^{(4)} A_m^{(2)} - 2 B_m^{(1)} \cosh (\gamma_m) + 2 B_m^{(2)} \sinh (\gamma_m) &= - \frac{2 \nu}{\lambda + 2 \mu} k_m t_m(- h/2) \\
- \Gamma_m^{(3)} A_m^{(1)} + \Gamma_m^{(4)} A_m^{(2)} - 2 B_m^{(1)} \cosh (\gamma_m) - 2 B_m^{(2)} \sinh (\gamma_m) &= - \frac{2 \nu}{\lambda + 2 \mu} k_m t_m(+ h/2)
\end{aligned}
$$

where γ_m has already been defined in (3.15), and

$$
\begin{aligned}
\Gamma_m^{(1)} &= - \frac{\mu}{\lambda + 2 \mu} \cosh (\gamma_m) + \frac{\lambda + \mu}{\lambda + 2 \mu} \gamma_m \sinh (\gamma_m) . \\
\Gamma_m^{(2)} &= \frac{\mu}{\lambda + 2 \mu} \sinh (\gamma_m) - \frac{\lambda + \mu}{\lambda + 2 \mu} \gamma_m \cosh (\gamma_m) \\
\Gamma_m^{(3)} &= \frac{\lambda + \mu}{\lambda + 2 \mu} \gamma_m \cosh (\gamma_m) - \sinh (\gamma_m) \\
\Gamma_m^{(4)} &= - \frac{\lambda + \mu}{\lambda + 2 \mu} \gamma_m \sinh (\gamma_m) + \cosh (\gamma_m) .
\end{aligned}
$$

The solutions are :

$$
A_m^{(1)} = - \frac{\nu}{\lambda + \mu} \frac{[t'_m(- h/2) + t'_m(h/2)] \sinh (\gamma_m) + k_m [t_m(- h/2) - t_m(h/2)] \cosh (\gamma_m)}{\gamma_m - \sinh (\gamma_m) \cosh (\gamma_m)}
$$

$$A_m^{(2)} = -\frac{\nu}{\lambda+\mu}\frac{[t'_m(-h/2) - t'_m(h/2)]\cosh(\gamma_m) + k_m[t_m(-h/2) + t_m(h/2)]\sinh(\gamma_m)}{\gamma_m + \sinh(\gamma_m)\cosh(\gamma_m)}$$

$$B_m^{(1)} = -\frac{\nu}{2(\lambda+\mu)}\frac{1}{\gamma_m + \sinh(\gamma_m)\cosh(\gamma_m)} \times$$

$$\times\left(k_m[t_m(-h/2) + t_m(h/2)]\left[\frac{\mu}{\lambda+2\mu}\sinh(\gamma_m) - \frac{\lambda+\mu}{\lambda+2\mu}\gamma_m\cosh(\gamma_m)\right] + \right.$$

$$\left. + [t'_m(-h/2) - t'_m(h/2)]\left[\cosh(\gamma_m) - \frac{\lambda+\mu}{\lambda+2\mu}\gamma_m\sinh(\gamma_m)\right]\right)$$

$$B_m^{(2)} = -\frac{\nu}{2(\lambda+\mu)}\frac{1}{\gamma_m - \sinh(\gamma_m)\cosh(\gamma_m)} \times$$

$$\times\left(k_m[t_m(-h/2) - t_m(h/2)]\left[\frac{\mu}{\lambda+2\mu}\cosh(\gamma_m) - \frac{\lambda+\mu}{\lambda+2\mu}\gamma_m\sinh(\gamma_m)\right] + \right.$$

$$\left. + [t'_m(-h/2) + t'_m(h/2)]\left[\sinh(\gamma_m) - \frac{\lambda+\mu}{\lambda+2\mu}\gamma_m\cosh(\gamma_m)\right]\right).$$

Now we are able to compute the values of the functions A_m and B_m and then of u_r and u_z, everywhere in the mirror ; but our aim is the thermal aberration of the input face ; as in formula (3.16) the equation $S(t, r)$ of the transient distorted surface is

$$S(t, r) = u_z(t, r, -h/2). \tag{4.9}$$

We have thus to calculate $B_m(t, -h/2)$; after some algebra we find it from (4.8) and the values of $A_m^{(1)}$, $A_m^{(2)}$, $B_m^{(1)}$ and $B_m^{(2)}$.

Moreover it is convenient to split t_m into its odd and even parts. We use the notation :

$$\begin{aligned} e_m &= \frac{1}{2}\left[t_m\left(\frac{h}{2}\right) + t_m\left(-\frac{h}{2}\right)\right] \\ e'_m &= \frac{1}{2}\left[t'_m\left(\frac{h}{2}\right) + t'_m\left(-\frac{h}{2}\right)\right] \\ o_m &= \frac{1}{2}\left[t_m\left(\frac{h}{2}\right) - t_m\left(-\frac{h}{2}\right)\right] \\ o'_m &= \frac{1}{2}\left[t'_m\left(\frac{h}{2}\right) - t'_m\left(-\frac{h}{2}\right)\right]. \end{aligned}$$

The expression of $S(t, r)$ is then :

$$S(t, r) = \frac{-\nu}{\lambda+\mu}\sum_m \frac{\sinh(\gamma_m)}{\sinh(\gamma_m)\cosh(\gamma_m) + \gamma_m}[\cosh(\gamma_m)\,o'_m - \sinh(\gamma_m)\,k_m\,e_m]\,J_0(k_m r) +$$

$$+ \frac{\nu}{\lambda+\mu}\sum_m \frac{\cosh(\gamma_m)}{\sinh(\gamma_m)\cosh(\gamma_m) - \gamma_m}[\sinh(\gamma_m)\,e'_m - \cosh(\gamma_m)\,k_m\,o_m]\,J_0(k_m r) \tag{4.10}$$

where the time t appears implicitly through the coefficients e_m, e'_m, o_m and o'_m. Like in [4] and [1] the surface distorsion can be expanded on the Zernike polynomials $R_{2q}^{(0)}$:

$$S(t, r) = \sum_q C_{2q}\,R_{2q}^{(0)}(r/b)$$

after using the Nijboer relation [12] we get the 2 q-th coefficient :

$$C_{2q} = (-1)^q (4q+2) \sum_m B_m\left(t, -\frac{h}{2}\right) \frac{J_{2q+1}(k_m b)}{k_m b} \tag{4.11}$$

where b is the radius of the optical zone of the mirror.

In this section we have given the general solution of the transient thermoelastic problem as a Dini series and expanded the aberration on the Zernike polynomials. In the following we will specify the shape and evolution of the surface deformations and then of the aberrations in the cases of heating first by dissipation in the coating (we will find again the results of section 3 for large values of time), and then by absorption in the bulk.

4.2 CASE OF HEATING BY DISSIPATION IN THE COATING. — In the case of heating by surface absorption, we have found the transient temperature distribution when the initial condition is

$$T(t, r, z) = T_{\text{ext}} + 2\frac{\varepsilon a^2}{Kh} \sum_{p,m} p_m[1 - e^{-\alpha_{pm} t}] \frac{\cos(u_p h/2a)}{(\zeta_m^2 + u_p^2) c_p} \cos(u_p z/a) J_0(\zeta_m r/a) -$$
$$- 2\frac{\varepsilon a^2}{Kh} \sum_{p,m} p_m[1 - e^{-\beta_{pm} t}] \frac{\sin(v_p h/2a)}{(\zeta_m^2 + v_p^2) s_p} \sin(v_p z/a) J_0(\zeta_m r/a)$$

where u_p is the p-th solution of :

$$u = \tau \cot\left[\frac{uh}{2a}\right]$$

and v_p the p-th solution of :

$$v = -\tau \tan\left[\frac{vh}{2a}\right]$$

(τ is again the reduced radiation constant) recall the following definitions :

$$\alpha_{pm} = \frac{K}{\rho C a^2}[u_p^2 + \zeta_m^2], \quad \beta_{pm} = \frac{K}{\rho C a^2}[v_p^2 + \zeta_m^2]$$

also

$$c_p = 1 + \frac{a}{u_p h}\sin\left(\frac{u_p h}{a}\right) \quad \text{and} \quad s_p = 1 - \frac{a}{v_p h}\sin\left(\frac{v_p h}{a}\right).$$

From the temperature we can evaluate the functions t_m and t'_m with (4.7). If we rewrite in a condensed form $T(t, r, z)$ (modulo T_{ext} as usual) :

$$T(t, r, z) = \sum_{m,p}\left[S_{pm}\cos\left(\frac{u_p z}{a}\right) + A_{pm}\sin\left(\frac{v_p z}{a}\right)\right] J_0\left(\frac{\zeta_m r}{a}\right)$$

where the S_{pm} are identified with the even coefficients of the above expression for T and the A_{pm} with the odd ones.

Thus we can express the functions t_m and t'_m as follows :

$$t_m(t, z) = -a^2 \sum_p \left(\frac{S_{pm}}{u_p^2 + \zeta_m^2}\cos\left(\frac{u_p z}{a}\right) + \frac{A_{pm}}{v_p^2 + \zeta_m^2}\sin\left(\frac{v_p z}{a}\right)\right)$$
$$t'_m(t, z) = a \sum_p \left(\frac{S_{pm}}{u_p^2 + \zeta_m^2} u_p \sin\left(\frac{u_p z}{a}\right) - \frac{A_{pm}}{v_p^2 + \zeta_m^2} v_p \cos\left(\frac{v_p z}{a}\right)\right).$$

Then we are able to compute these two functions and thus to evaluate the aberration function $S(t, r)$ by calculating (4.10) and (4.9). We find :

$$S(t,r) = -a \frac{\nu}{\lambda+\mu} \sum_{m,p} \left[\frac{\sinh(\gamma_m)}{\sinh(\gamma_m)\cos(\gamma_m)+\gamma_m} \times \right.$$
$$\times \frac{S_{pm}}{u_p^2+\zeta_m^2} \{u_p \cosh(\gamma_m)\sin(\xi_p) + \zeta_m \sinh(\gamma_m)\cos(\xi_p)\} +$$
$$+ \frac{\cosh(\gamma_m)}{\sinh(\gamma_m)\cosh(\gamma_m)-\gamma_m} \times$$
$$\left. \times \frac{A_{pm}}{v_p^2+\zeta_m^2} \{v_p \sinh(\gamma_m)\cos(\eta_p) - \zeta_m \cosh(\gamma_m)\sin(\eta_p)\} \right] J_0(k_m r)$$

where $\xi_p = u_p h/2a$ and $\eta_p = v_p h/2a$.

When the time is larger than the characteristic time t_c ($t \gg t_c$) the transient deformation given by the above formula is the same (with an accuracy better than (10^{-15}) as the steady state deformation of section 3. This shows the very good consistency of the method. The numerical results are shown in figure 5 ; we can also see the evolution of the first Zernike coefficients computed from (4.11) in figure 6.

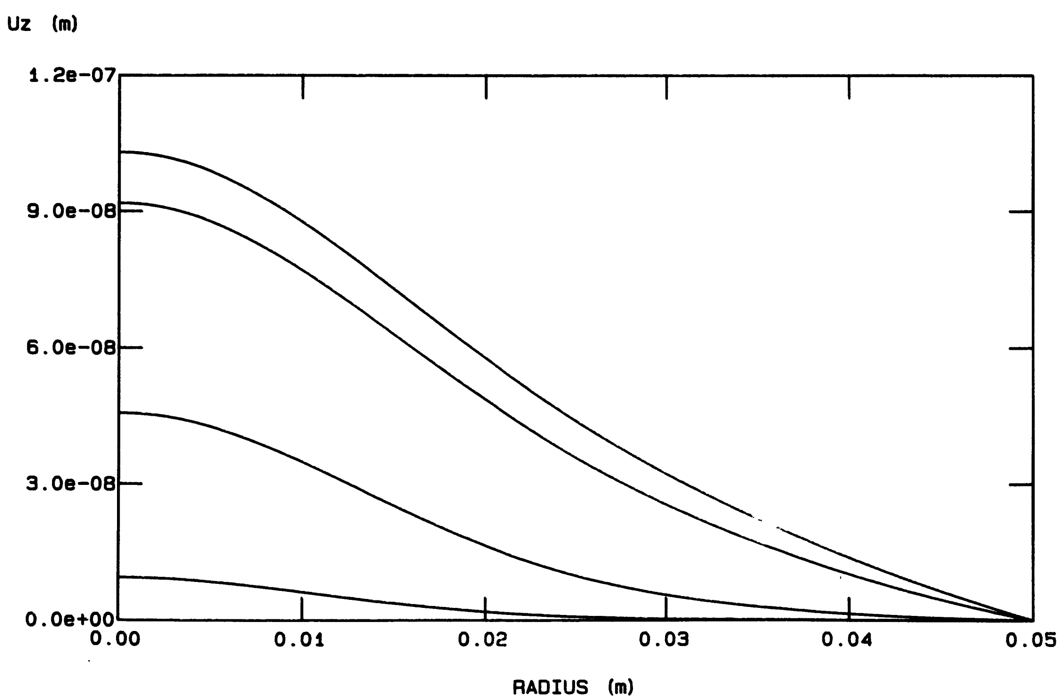

Fig. 5. — Evolution of the transient distorted surface for 1 W absorbed in the coating for different times : $10^{-4}\,t_c$, $10^{-3}\,t_c$, $10^{-2}\,t_c$ and $10^{-1}\,t_c$ ($t_c \approx 30$ hrs). We restrain the view to the optical zone.

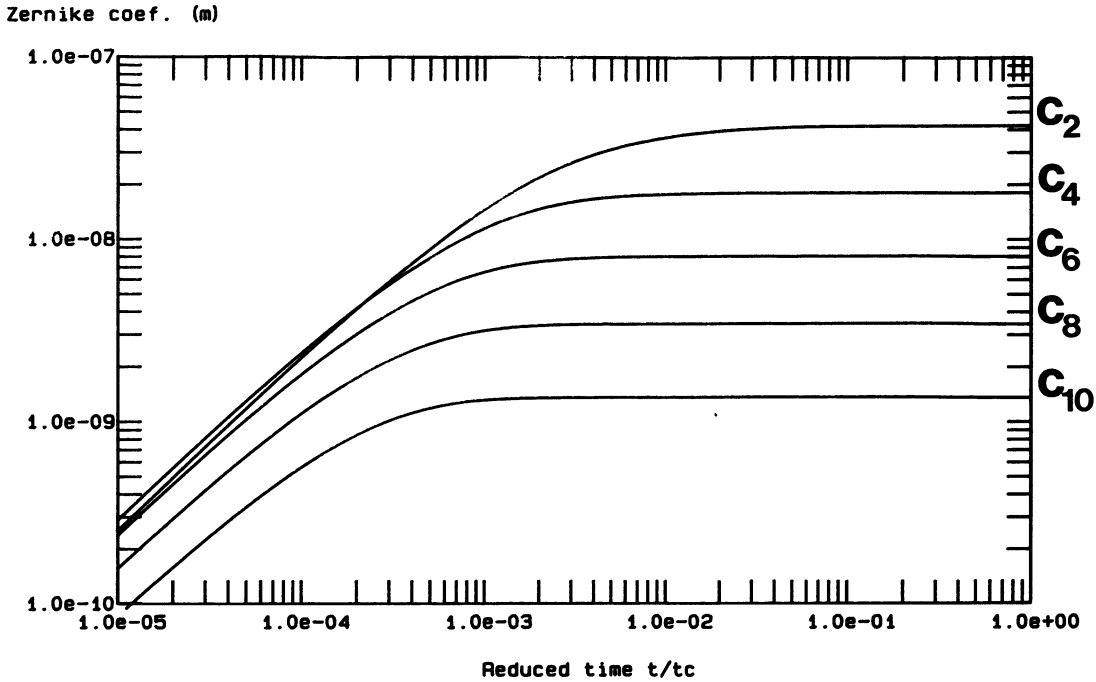

Fig. 6. — Time evolution of the first Zernike coefficients (absolute values) with respect to the time.

4.3 CASE OF HEATING BY BULK ABSORPTION. — In the case of an internal heating with a lineic absorption coefficient α we have found the transient (even) temperature distribution :

$$T(t, r, z) = T_{\text{ext}} + \sum_{p, m} T_{pm} \cos (u_p z/a) J_0(\zeta_m r/a)$$

with

$$T_{pm} = \frac{4 \alpha a^3 p_m}{Kh} \frac{}{\zeta_m^2} \left[\frac{\sin (u_p h/2 a)}{u_p} - \frac{\tau \cos (u_p h/2 a)}{\zeta_m^2 + u_p^2} \right] \frac{1}{c_p} [1 - e^{-\alpha_{pm} t}]$$

and thus, as in the preceding case, we find from (4.7) the functions t_m and t'_m :

$$t_m(t, z) = - a^2 \sum_p \frac{T_{pm}}{u_p^2 + \zeta_m^2} \cos \left(\frac{u_p z}{a} \right)$$

$$t'_m(t, z) = a \sum_p \frac{T_{pm}}{u_p^2 + \zeta_m^2} u_p \sin \left(\frac{u_p z}{a} \right) .$$

Then we obtain the equation of the distorted surface :

$$S(t, r) = - a \frac{\nu}{\lambda + \mu} \sum_{m, p} \frac{\sinh (\gamma_m)}{\sinh (\gamma_m) \cosh (\gamma_m) + \gamma_m} \times$$

$$\times \frac{T_{pm}}{u_p^2 + \zeta_m^2} \{u_p \cosh (\gamma_m) \sin (\xi_p) + \zeta_m \sinh (\gamma_m) \cos (\xi_p)\} J_0(k_m r)$$

with, as above, $\xi_p = u_p h/2 a$.

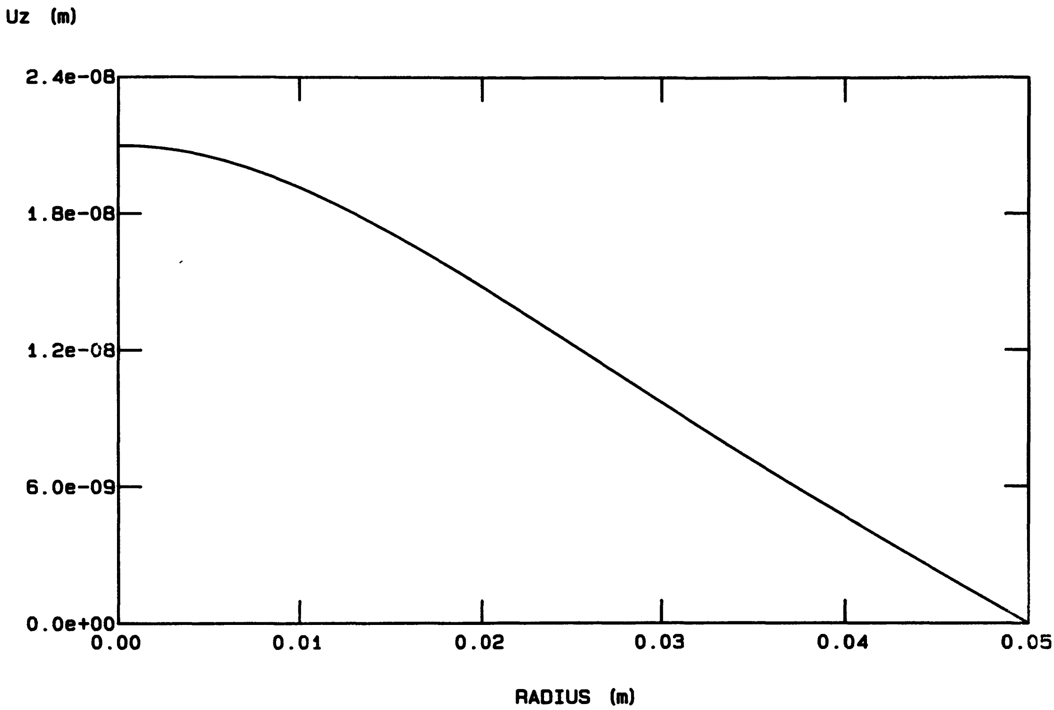

Fig. 7. — The steady state deformed surface of the mirror in the case of an absorption in the bulk (1 W). We have used here the steady state solution for the temperature (see [1]). We can see that the effect is less important (factor 5) than in the case of a coating absorption of the same power.

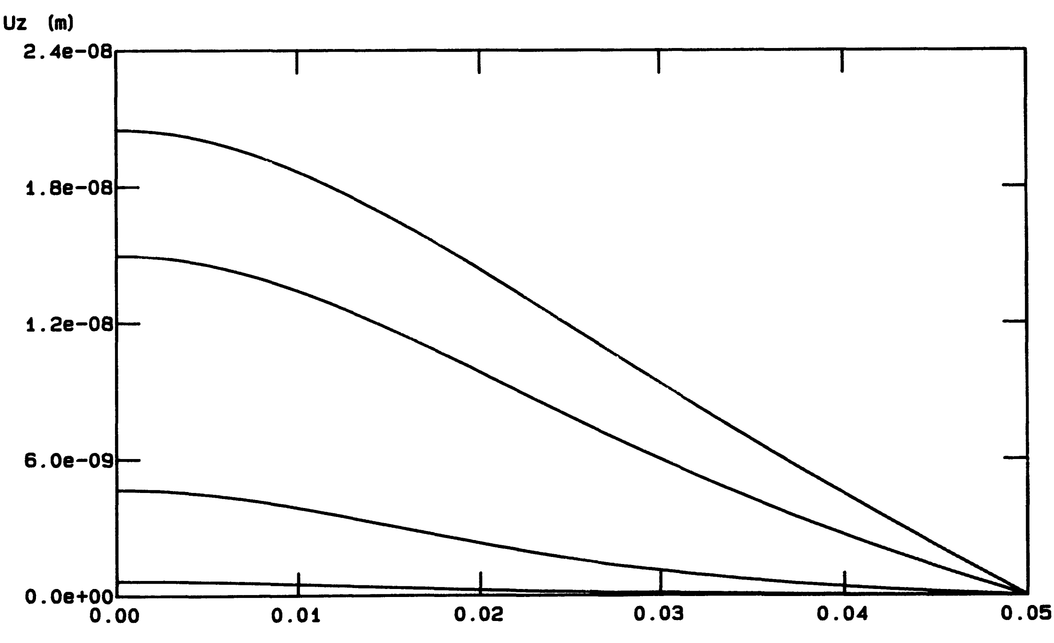

Fig. 8. — Evolution of the transient distorted surface for 1 W absorbed in the bulk for different times : $10^{-4}\,t_c$, $10^{-3}\,t_c$, $10^{-2}\,t_c$ and $10^{-1}\,t_c$.

Thus, by computing the last expression, we can evaluate the aberrations. Figure 7 shows the stationary distorted surface and figure 8 its transient evolution ; the evolution of the first Zernike coefficients are also shown in figure 9.

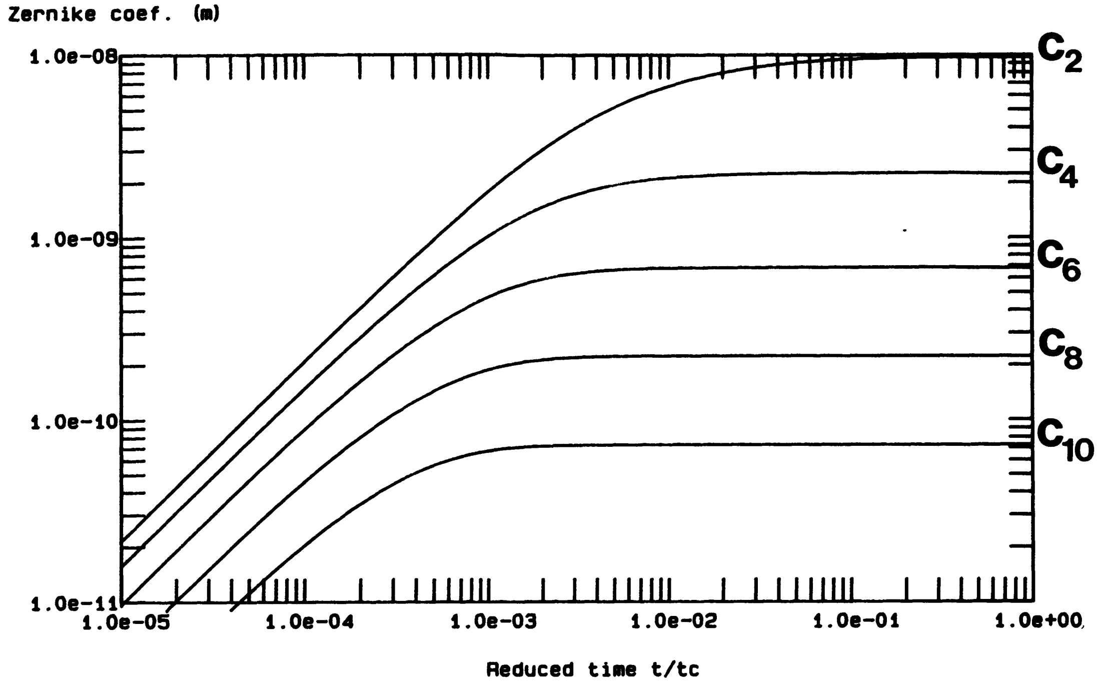

Fig. 9. — Evolution of the absolute value of the first Zernike coefficients for one Watt absorbed in the bulk.

5. Conclusion.

We have given analytical expressions for the deformed surface of a cylindrical mirror heated by an axisymmetric cw laser beam, first in the steady state, then in the transient regime. Note that the models are also valid with convection losses instead of radiative losses.

The main lesson that may be drawn from the above study is that the thermoelastic deformations caused by bulk absorption of laser light are about 5 times weaker than those caused by dissipation in the coating, for the same absorbed power, and that the resulting aberrations are at least a factor of 10 weaker than the thermal lens effect computed in [1].

When the studied mirror is the input mirror of a Fabry-Perot cavity, the thermoelastic deformations may be neglected in the transmitted beam. Their effect on the reflected beam, i.e. on the cavity mode, needs however to be dynamically investigated by modelling a whole VIRGO cavity with given initial conditions, including action of the wave front thermal distortion on the stored power, and the feedback of the stored power upon the heating rate.

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