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Covariance and stochastic mechanics

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Résumé. — La covariance de la mécanique stochastique dans les changements de temps et, simultanément, dans les changements de coordonnées dépendant du temps est rendue explicite dans le cadre des processus de diffusion, et étendue à la formulation opératorielle. Des dérivées temporelles covariantes sont définies dans les deux représentations et utilisées pour donner aux équations du mouvement une expression manifestement covariante. Elles servent aussi à fixer des contraintes générales sur la correspondance entre états quantiques et processus stochastiques.

Abstract. — Covariance of stochastic mechanics under changes of time and simultaneous time-dependent changes of coordinates is made explicit in the context of diffusion processes, and extended to the operator formulation. Covariant time derivatives are defined in both representations and used to express the equations of motion in a manifestly covariant way. They also provide general constraints on the correspondence between quantum states and stochastic processes.

1. Introduction.

Both in classical and quantum mechanics (or field theory), the equations of motion are deduced from the extremum of a Lagrangian. Such a universal procedure insures that the dynamical evolution will be invariant under a group of quite general transformations. In particular, it will contain arbitrary changes of time and simultaneous time-dependent changes of coordinates. Then any viable extension of classical or quantum mechanics must in particular preserve this group of symmetry. In the standard operator representation of quantum mechanics (or field theory) this is readily obtained by extending the classical Hamiltonian formalism, which can preserve a larger group, that of arbitrary canonical transformations. In particular De Witt [1] developed a general treatment for any Lagrangian which is quadratic in the velocities and for an arbitrary number of degrees of freedom.

As an alternative representation of quantum mechanics, stochastic mechanics extends classical mechanics by replacing its kinematics of differentiable trajectories by those of stochastic processes [2, 3]. The degrees of freedom are promoted to random variables which describe continuous but not

differentiable trajectories. This representation, in terms of probability spaces over *fluctuating* trajectories is covariant under the group mentioned above, of time dependent changes of coordinates. This appears naturally when stochastic differentials are introduced, which transform according to the rules of stochastic calculus [4]. This property did not show explicitly in the first expression which was given of stochastic dynamics, and which moreover made contact with quantum mechanics [3]: a stochastic extension of the classical acceleration is used to express a generalised Newton law which is shown to be equivalent to the Schrödinger equation. One way to insure the previous general covariance inside this representation, is to make the equations of motion result from a stochastic Lagrangian principle [5], or an equivalent stochastic Hamiltonian formalism, as that provided by stochastic control theory [6]. However, one would like to possess a description which makes explicit the role of the previous general covariance in the derivation of the dynamics, and also in the correspondence between quantum states and stochastic processes. Another approach was to define stochastic processes, with kinematics and dynamics, on a Riemannian manifold [7, 8, 11]. However this only provides direct

covariance under time-independent changes of coordinates, and for time-dependent ones, one must come back to the previous approaches.

Another central question concerning the relation between quantum and stochastic mechanics, is about the correspondence between quantum states and stochastic processes. The requirement that both representations should give the same predictions for observable variables leads to the identification of the probability densities at all times (that of the process with $|\psi|^2$, where ψ is the wave function). This still leaves the diffusion coefficient of the process undetermined [9, 10]. In the conventional approach [8, 11], the latter is identified with the metrics entering the Schrödinger equation, that is the Lagrangian. This is an intriguing prescription, as it determines a kinematical characteristic of the process from dynamical considerations.

This paper is devoted to these questions : make explicit the covariance of stochastic mechanics under time-dependent changes of coordinates, and also translate it into the operator representation. Indeed, this natural symmetry of stochastic differentials will persist when an operator formulation is given. This will be exhibited by the definition, in both representations, of covariant time derivatives which will be used to rewrite the equations of motion. Making also use of them for the definition of stochastic acceleration, will allow one to clarify the role of the previous general covariance in the choice of stochastic dynamics. This will also give a general frame for the constraints that limit the correspondence between quantum states and stochastic processes.

2. Covariance of diffusion processes.

This first part will exhibit the natural covariance which is inherent in diffusion processes. By convention, this purpose will be realised from Itô stochastic differentials, but Stratonovich differentials could be used as well [4], as only the transformation properties of the drift and diffusion coefficients will be required (an example of the alternative point of view is developed in [12]). Indeed, the underlying structure will be an extension, so that to include time-dependent changes of coordinates, of the Riemannian structure which appears naturally with diffusion processes [7, 8, 11]. Also, the number of degrees of freedom N will be arbitrary, so that our main topic will be field theory, to be recovered by an appropriate limit of infinite N , rather than the mechanics of the point particle.

On a given probability space, a general diffusion process is defined by a collection of measurable functions $x = (x^i(t))_{i=1,N}$, and $t \in]-\infty, +\infty[$ such that :

$$\begin{aligned} \langle \Delta x^i \rangle_x &= b^i(x) \Delta t + o(\Delta t) \\ \Delta x^i &= x^i(t + \Delta t) - x^i(t), \quad \Delta t \geq 0 \end{aligned} \quad (1)$$

$$\begin{aligned} \langle \Delta x^i \Delta x^j \rangle_x &= 2 \nu^{ij}(x) \Delta t + o(\Delta t) \\ \langle \Delta x^{i_1} \Delta x^{i_2} \dots \Delta x^{i_n} \rangle_x &= o(\Delta t), \quad n > 2 \end{aligned}$$

where $\langle \rangle_x$ denotes the expectation value, conditional in $x(t)$, i labels the different degrees of freedom (with N arbitrary) and t is the time parameter. These properties are summarised in a stochastic differential equation, in the sense of Itô :

$$\begin{aligned} dx^i &= b^i(x) dt + dw^i \\ \langle dw^i dw^j \rangle_x &= 2 \nu^{ij}(x) dt \\ (\langle dx^i dx^j \rangle_x &= 2 \nu^{ij}(x) dt). \end{aligned} \quad (2)$$

The diffusion process is then characterised by two fields, $b^i(x)$ the drift field and $\nu^{ij}(x)$ the diffusion field, which in general for an inhomogeneous process depend on the random variables x^i . In the following, we shall keep such general dependences for both fields b^i and ν^{ij} . In fact, this is rendered necessary by allowing arbitrary changes of variables.

Indeed, the changes of the drift and diffusion fields under arbitrary time-dependent changes of variables are easily deduced from the stochastic differential equation. Making the following transformation :

$$\begin{aligned} t &= f(\bar{t}) \\ x^i(t) &= g^i(\bar{t}, \bar{x}), \quad \bar{x} = (\bar{x}^i(\bar{t}))_{i=1,N} \end{aligned} \quad (3)$$

changes the time parameter, and simultaneously replaces the random variables x by equivalent ones $\bar{x}$ which also satisfy a diffusion process [4] on the same probability space. Then, developing (3) to order $d\bar{t}$ and recalling (1), so that $dx^i dx^j$ is of order dt and $dx^i dx^j \dots dx^k$ of higher order, one transforms (2) into :

$$\begin{aligned} d\bar{x}^i &= \bar{b}^i(\bar{x}) d\bar{t} + d\bar{w}^i \\ \langle d\bar{w}^i d\bar{w}^j \rangle_{\bar{x}} &= 2 \bar{\nu}^{ij} d\bar{t} \quad (\langle d\bar{x}^i d\bar{x}^j \rangle_{\bar{x}} = 2 \bar{\nu}^{ij} d\bar{t}) \end{aligned}$$

with :

$$\begin{aligned} b^i &= \frac{1}{\dot{f}} (\dot{g}^i + g_j^i \bar{b}^j + g_{jk}^i \bar{\nu}^{jk}) \\ \nu^{ij} &= \frac{1}{\dot{f}} g_k^i g_l^j \bar{\nu}^{kl} \\ \dot{f} &= \frac{df}{d\bar{t}}, \quad \dot{g}^i = \frac{\partial g^i}{\partial \bar{t}}, \\ g_j^i &= \frac{\partial g^i}{\partial \bar{x}^j}, \quad g_{jk}^i = \frac{\partial^2 g^i}{\partial \bar{x}^j \partial \bar{x}^k}. \end{aligned} \quad (4)$$

The diffusion field transforms like a metrics, while the drift field almost transforms like a velocity. Thus we shall use ν^{ij} like a metrics, for rising and lowering indices, and introduce the corresponding Christoffel symbols :

$$\begin{aligned} \nu_{ij} \nu^{jk} &= \delta_{ik} \\ \Gamma_{jk}^i &= \frac{\nu^{i\ell}}{2} \left(\frac{\partial \nu_{k\ell}}{\partial x^j} + \frac{\partial \nu_{j\ell}}{\partial x^k} - \frac{\partial \nu_{jk}}{\partial x^\ell} \right) \\ \mu^i &= -\nu^{jk} \Gamma_{jk}^i. \end{aligned} \quad (5)$$

Let us note the covariant drift, which actually transforms like a velocity :

$$b^i - \mu^i = \frac{1}{f} (\dot{g}^i + g_j^i (\bar{b}^j - \bar{\mu}^j)). \quad (6)$$

Thus, inhomogeneous diffusion fields become necessary as soon as one requires that the description be covariant under changes of the coordinates representing the degrees of freedom. As will become quite clear in the following, we are dealing in particular with diffusion processes on a Riemannian manifold, but moreover with local coordinates which have arbitrary time dependence.

Conversely, the covariant changes of the drift and diffusion fields b^i and ν^{ij} being fixed by (4), intrinsic objects related to the process should have covariant expressions, whose dependence in b and ν will be fixed by (4). For instance, P the conditional (or the joint as well) probability density transforms like a density, like the probability density ρ :

$$\begin{aligned} P(x', t' | x, t) &= |g'(\bar{x}')|^{-1} \bar{P}(\bar{x}', \bar{t}' | \bar{x}, \bar{t}), \\ |g'| &= |\det(g_j'^i)| \\ \rho(x, t) &= |g'(\bar{x})|^{-1} \bar{\rho}(\bar{x}, \bar{t}). \end{aligned}$$

They satisfy the Fokker-Planck equation :

$$\begin{aligned} \partial_{t'} P + \nabla_i' (b^i(x') P) - \nabla_i' \nabla_j' (\nu^{ij}(x') P) &= 0 \\ \partial_t \rho + \nabla_i (b^i(x) \rho) - \nabla_i \nabla_j (\nu^{ij}(x) \rho) &= 0 \quad (7) \\ \partial_t &= \frac{\partial}{\partial t}, \quad \nabla_i = \frac{\partial}{\partial x^i} \end{aligned}$$

which can be seen to be invariant under general time-dependent changes of coordinates. The latter can also be written :

$$\partial_t \rho + \nabla_i (v^i \rho) = 0, \quad v^i = b^i - \frac{\nabla_j (\nu^{ij} \rho)}{\rho}$$

where v^i is a velocity field, transforming according to (6).

As an example showing how the transformation properties strongly constrain different characteristics of the process, let us consider a construction that will apply to the particular case of Markovian diffusion processes. The general infinitesimal length, with singularity of at most order dt , is of the following form, up to order dt :

$$\begin{aligned} ds &= \alpha_{ij} \frac{dx^i dx^j}{dt} + \gamma_{ijk} \frac{dx^i dx^j dx^k}{dt} + \beta_i dx^i + \\ &+ \eta_{ijkl} \frac{dx^i dx^j dx^k dx^\ell}{dt} + \varepsilon_{ij} dx^i dx^j + \delta dt. \end{aligned} \quad (8)$$

By requiring that it be invariant under (3), one obtains the transformations of all the coefficients, which lead to the following solution (up to an arbitrary factor) :

$$\begin{aligned} \alpha_{ij} &= \frac{1}{4} \nu_{ij} \\ \gamma_{ijk} &= \frac{1}{24} (\nabla_i \nu_{jk} + \nabla_j \nu_{ik} + \nabla_k \nu_{ij}) \\ \beta_i &= -\frac{1}{2} \nu_{ij} (b^j - \mu^j) \\ \eta_{ijkl} &= \frac{1}{144} [\nabla_i \nabla_j \nu_{kl} + \nabla_i \nabla_k \nu_{jl} + \nabla_i \nabla_\ell \nu_{jk} + \\ &+ \nabla_j \nabla_k \nu_{i\ell} + \nabla_j \nabla_\ell \nu_{ik} + \nabla_k \nabla_\ell \nu_{ij} - \\ &- \nu_{mn} (\Gamma_{ij}^m \Gamma_{kl}^n + \Gamma_{ik}^m \Gamma_{jl}^n + \Gamma_{i\ell}^m \Gamma_{jk}^n)] \\ \varepsilon_{ij} &= \frac{1}{8} \partial_t \nu_{ij} + \frac{1}{4} (\nabla_i \beta_j + \nabla_j \beta_i) \\ \delta &= \frac{1}{4} \nu_{ij} (b^i - \mu^i) (b^j - \mu^j) \end{aligned} \quad (9)$$

(up to a term proportional to $R dt$, where R is the curvature relative to ν). Similarly, the general infinitesimal volume has the form, up to order dt :

$$\begin{aligned} dv &= e^{\zeta_i dx^i + \lambda_{ij} dx^i dx^j + \xi dt} \frac{1}{|2\nu|^{1/2}} \prod_i \frac{dx^i}{dt} \\ x'^i &= x^i + dx^i \\ |2\nu| &= |\det(2\nu^{ij})| \end{aligned} \quad (10)$$

and is invariant when :

$$\begin{aligned} \zeta_i &= -\frac{1}{2} \nabla_i \ln |\nu| \\ \lambda_{ij} &= -\frac{1}{4} \nabla_i \nabla_j \ln |\nu| \\ \xi &= -\frac{1}{4} \partial_t \ln |\nu| - \\ &- \frac{1}{2} |\nu|^{1/2} \nabla_i (|\nu|^{-1/2} (b^i - \mu^i)). \end{aligned} \quad (11)$$

From them one constructs the obviously invariant measure :

$$dv e^{-ds} = P(x', t' | x, t) \prod_i dx^i. \quad (12)$$

The conditional probability density P of a Markovian process will have a singular behavior of the following form :

$$\ln P = -\frac{N}{2} \ln dt + ds$$

and it can easily be shown that the solution of the Fokker-Planck equation (7), whose singular behavior is of that form, is precisely given by (8)-(12). The latter then identifies with the conditional probability density of the Markovian diffusion process having

b^i and ν^{ij} as drift and diffusion fields. Let us notice, among various other ways of deriving the expression of P , the one which takes P as the kernel of the path integrals, equal to :

$$D(x', t', x, t)^{1/2} e^{\frac{i}{\hbar} S(x', t', x, t)}, \quad D = \det \left(\frac{\partial^2 S}{\partial x^i \partial x^j} \right)$$

where S is the action, $\hbar$ Planck constant, and D the Van Vleck determinant [1, 13]. These can be developed up to order dt , while taking into account

the fact that $dx^i dx^j$ is of order dt , and provide for P an expression identical to (8)-(12).

For general diffusion processes (not necessarily Markovian), the drift and diffusion fields b^i and ν^{ij} , and the probability density ρ , allow one to construct covariant time derivatives which will exhibit remarkable properties. Let us first consider an arbitrary scalar function φ of the random variables, that is a function which transforms into $\bar{\varphi}$ such that :

$$\varphi(x, t) = \bar{\varphi}(\bar{x}, \bar{t}).$$

Define the action of D and D^* by :

$$\begin{aligned} D\varphi &= [\partial_t + b^i \nabla_i + \nu^{ij} \nabla_i \nabla_j] \varphi = [\partial_t + (b^i - \mu^i) D_i + \nu^{ij} D_i D_j] \varphi \\ D^* \varphi &= [\partial_t + b^{*i} \nabla_i - \nu^{ij} \nabla_i \nabla_j] \varphi = [\partial_t + (b^{*i} + \mu^i) D_i - \nu^{ij} D_i D_j] \varphi \end{aligned}$$

and :

$$b^{*i} = b^i - 2 \frac{\nabla_j (\nu^{ij} \rho)}{\rho}$$

D_i is the covariant gradient relative to ν .

$D\varphi$ and $D^* \varphi$ are easily seen to transform also as scalar functions of the random variables. Considering now vector functions A^i or A_i , that is :

$$\begin{aligned} A^i(x, t) &= g_j^i \bar{A}^j(\bar{x}, \bar{t}) \\ \bar{A}_i(\bar{x}, \bar{t}) &= g_i^j A_j(x, t) \end{aligned}$$

one can then define the action of D and D^* by :

$$\begin{aligned} DA^i &= [\partial_t + b^j \nabla_j + \nu^{jk} \nabla_j \nabla_k] A^i + [-\nabla_j b^i - \nabla_j \nu^{kl} \Gamma_{kl}^i + 2 \nu^{kl} \Gamma_{jk}^i \nabla_l] A^j \\ &= [\partial_t + (b^j - \mu^j) D_j + \nu^{jk} D_j D_k] A^i + [-D_j (b^i - \mu^i) + \mathbf{R}_j^i] A^j \\ D^* A^i &= [\partial_t + b^{*j} \nabla_j - \nu^{jk} \nabla_j \nabla_k] A^i + [-\nabla_j b^{*i} - \nabla_j \nu^{kl} \Gamma_{kl}^i - 2 \nu^{kl} \Gamma_{jk}^i \nabla_l] A^j \\ &= [\partial_t + (b^{*j} + \mu^j) D_j - \nu^{jk} D_j D_k] A^i + [-D_j (b^{*i} + \mu^i) - \mathbf{R}_j^i] A^j \\ DA_i &= [\partial_t + b^j \nabla_j + \nu^{jk} \nabla_j \nabla_k] A_i + [\nabla_i b^j - 2 \nu^{kl} \Gamma_{ik}^j \nabla_l] A_j \\ &= [\partial_t + (b^j - \mu^j) D_j + \nu^{jk} D_j D_k] A_i + [D_i (b^j - \mu^j) - \mathbf{R}_i^j] A_j \\ D^* A_i &= [\partial_t + b^{*j} \nabla_j - \nu^{jk} \nabla_j \nabla_k] A_i + [\nabla_i b^{*j} + 2 \nu^{kl} \Gamma_{ik}^j \nabla_l] A_j \\ &= [\partial_t + (b^{*j} + \mu^j) D_j - \nu^{jk} D_j D_k] A_i + [D_i (b^{*j} + \mu^j) + \mathbf{R}_i^j] A_j \end{aligned}$$

where $b^{*i} + \mu^i$ transforms like a velocity field (6), and $\mathbf{R}_j^i$ is the Ricci tensor relative to ν :

$$\mathbf{R}_j^i = \nu^{kl} [\nabla_i \Gamma_{kl}^j - \nabla_l \Gamma_{ik}^j + \Gamma_{lm}^i \Gamma_{kl}^m - \Gamma_{ml}^j \Gamma_{ik}^m].$$

These are easily seen to transform again as vectors under time-dependent changes of coordinates. D and D^* so defined can also be seen as generalisations of the forward and backward time derivatives introduced by Dohrn and Guerra [8], where covariance is extended to include time-dependent changes. Moreover, it is straightforward to show that these time derivatives now commute with the covariant gradient :

$$\begin{aligned} DD_i \varphi &= D_i D\varphi \\ D^* D_i \varphi &= D_i D^* \varphi \end{aligned}$$

for any scalar function φ . In fact, D and D^* are particular cases of more general covariant time derivatives : let us consider any velocity field, i.e. a field a^i which transforms according to (6) :

$$a^i = \frac{1}{\dot{f}} (\dot{g}^i + g_j^i \dot{a}^j)$$

and define D_a and D_a^* by the following action on scalars and vectors :

$$D_a \varphi = [\partial_t + a^i D_i + \nu^{ij} D_i D_j] \varphi$$

$$\begin{aligned}
D_a A_i &= [\partial_t + a^j D_j + \nu^{jk} D_j D_k] A_i + \\
&\quad + [D_i a^j - \mathbf{R}_i^j] A_j \\
D_a A^i &= [\partial_t + a^j D_j + \nu^{jk} D_j D_k] A^i + \\
&\quad + [-D_j a^i + \mathbf{R}_j^i] A^j \\
D_a^* \varphi &= [\partial_t + a^i D_i - \nu^{ij} D_i D_j] \varphi \\
D_a^* A_i &= [\partial_t + a^j D_j - \nu^{jk} D_j D_k] A_i + \\
&\quad + [D_i a^j + \mathbf{R}_i^j] A_j \\
D_a^* A^i &= [\partial_t + a^j D_j - \nu^{jk} D_j D_k] A^i + \\
&\quad + [-D_j a^i - \mathbf{R}_j^i] A^j.
\end{aligned}$$

These covariant time derivatives also show the following properties :

$$\begin{aligned}
D \sum_i \lambda^i a_i &= \sum_i \lambda^i D a_i, \quad \sum_i \lambda^i = 1 \\
D_a - D_a^* &= D_{a'} - D_{a'}^*. \quad (13)
\end{aligned}$$

The latter allow one to unify the notation :

$$\begin{aligned}
D_{a+o} &= D_a, \quad D_{a-o} = D_a^* \\
D_{a+o} &= \frac{1+\lambda}{2} D_a + \frac{1-\lambda}{2} D_a^*.
\end{aligned}$$

Now, a diffusion process supports velocity fields generated by b and b^* so that, with properties (13), the covariant derivatives form an affine set generated by :

$$\begin{aligned}
D &= D_{b-\mu+o}, \quad D^* = D_{b^*+\mu-o}, \quad D' = D_{b^*+\mu+o} \\
v^i &= \frac{b^i + b^{*i}}{2}, \quad u^i = \frac{b^i - b^{*i}}{2} - \mu^i \\
D_{v+\lambda'u+\lambda_0} &= \frac{1}{2} [(1+\lambda') D + (1-\lambda) D^* + \\
&\quad + (\lambda - \lambda') D']
\end{aligned}$$

Every one of these time derivatives is easily seen to commute with the covariant gradient. Their symmetry properties will play a central role in the expression of a dynamical equation for the diffusion processes, as will be seen in part 4.

3. Covariance of the operator representation.

Markovian diffusion processes can also be given an operator representation very similar to that of quantum mechanics [9, 14], which also allows one to characterise the correlation functions of the process by differential equations. Covariance under time-dependent changes of coordinates will be first extended to this operator representation of Markovian diffusion processes, and then shown to hold also for the standard operator representation of quantum mechanics.

3.1 MARKOVIAN DIFFUSION PROCESSES. — The transition probability $P(x', t' | x, t)$ of the process

determines all the time correlation functions through :

$$\begin{aligned}
\langle x(t_n) x(t_{n-1}) \dots x(t_1) \rangle &= \\
&= \int x_n x_{n-1} \dots x_1 P(x_n, t_n | x_{n-1}, t_{n-1}) \dots \\
&\quad \dots P(x_2, t_2 | x_1, t_1) \rho(x_1, t_1) dx_n dx_{n-1} \dots dx_1. \quad (14)
\end{aligned}$$

Let us first introduce a further symmetry (gauge transformations) which the correlation functions satisfy besides covariance under changes of variables. For a Markov process, expression (14) of the time correlations of the random variables is equal to any similar one, where the transition probability P has been replaced by any kernel K such that :

$$P(x', t' | x, t) = e^{-q(x', t')} K(x', t' | x, t) e^{q(x, t)}$$

with q an arbitrary function. K has then the same singular behavior as P , and satisfies the modified Fokker-Planck equation :

$$[\partial_{t'} - h(\nabla', x')] K = 0$$

with h , the Hamiltonian defined by :

$$\begin{aligned}
h &= \left(\nabla_i - \frac{1}{2} b_i - \nabla_i q \right) \left(\nabla_j - \frac{1}{2} b_j - \nabla_j q \right) \nu^{ij} + \\
&\quad + \partial_t q - \frac{1}{4} b_i b^i - \frac{1}{2} \nu^{ij} \nabla_i b_j. \quad (15)
\end{aligned}$$

The correlation functions then take the similar expression :

$$\begin{aligned}
\langle x(t_n) x(t_{n-1}) \dots x(t_1) \rangle &= \\
&= \int x_n x_{n-1} \dots x_1 \psi^*(x_n, t_n) K(x_n, t_n | x_{n-1}, t_{n-1}) \dots \\
&\quad \dots K(x_2, t_2 | x_1, t_1) \psi(x_1, t_1) dx_n dx_{n-1} \dots dx_1
\end{aligned}$$

where :

$$\psi = \rho e^q, \quad \psi^* = e^{-q}.$$

Recalling the time evolution of ρ , ψ and ψ^* are seen to satisfy :

$$\partial_t \psi = h \psi, \quad \partial_t \psi^* = -\psi^* h.$$

One can also rewrite the correlation functions as :

$$\begin{aligned}
\langle x(t_n) x(t_{n-1}) \dots x(t_1) \rangle &= \\
&= \int \psi^*(x_n, t_n) x_n e^{\int_{t_{n-1}}^{t_n} h} x_{n-1} \dots \\
&\quad \dots x_2 e^{\int_{t_1}^{t_2} h} x_1 \psi(x_1, t_1) dx_n dx_{n-1} \dots dx_1.
\end{aligned}$$

Operators x^i and ∇_i are then naturally introduced such that [14] :

$$x^i : \psi \rightarrow x^i \psi$$

$$\nabla_i : \psi \rightarrow \nabla_i \psi.$$

Using, for convenience a Dirac notation :

$$\langle \psi_t | x \rangle = \psi^*(x, t), \quad \langle x | \psi_t \rangle = \psi(x, t),$$

$$x = \int |x\rangle x dx \langle x|$$

a Heisenberg representation can be introduced, where the evolution is transferred to the operators :

$$O(t) = e^{-\int_t^t h} O e^{\int_t^t h} \text{ with } O \text{ any function of } x^i \text{ and } \nabla_i,$$

$$|\psi\rangle = e^{-\int_t^t h} |\psi_t\rangle \quad \langle\psi| = \langle\psi_t| e^{\int_t^t h}$$

being invariants, the correlation functions take the form :

$$\langle x(t_n) x(t_{n-1}) \dots x(t_1) \rangle = \langle \psi | T[x(t_n) x(t_{n-1}) \dots x(t_1)] | \psi \rangle \quad (16)$$

where T is the time-ordered product.

In this representation, the operators are characterised by their commutation relations :

$$[x^i, x^j] = [\nabla_i, \nabla_j] = 0, \quad [\nabla_i, x^j] = \delta_{ij}$$

and their time evolution :

$$\dot{O} = \frac{dO}{dt} = \partial_t O + [O, h]$$

which gives :

$$\begin{aligned} \dot{x}^i &= -2 \left(\nabla_j - \frac{1}{2} b_j - \nabla_j q \right) \nu^{ij} \\ \frac{d}{dt} (\dot{x}^j \nu_{ij}) &+ \frac{1}{2} \dot{x}^l \nu_{lj} \dot{x}^m \nu_{mk} \nabla_i \nu^{jk} + \\ &+ \frac{1}{2} (\nabla_i b_j - \nabla_j b_i) \dot{x}^j + \frac{1}{2} \dot{x}^j (\nabla_i b_j - \nabla_j b_i) \\ &- \partial_i b_j - \nabla_i \left(\nu^{jk} \nabla_j b_k + \frac{1}{2} b_j b^j \right) = 0. \quad (17) \end{aligned}$$

These operatorial equations of motion are invariant under time-dependent changes of variables and under gauge transformations, as can be directly verified (see the appendix). But this property can also be derived in another more illustrative way. Let us introduce *creation* and *annihilation* operators :

$$\begin{aligned} a_i^+ &= -\frac{1}{2} (\dot{x}^j - b^j) \nu_{ij} \\ a_i &= -\frac{1}{2} \nu_{ij} (\dot{x}^j - b^{*j}) \end{aligned}$$

such that :

$$[a_i^+, a_j^+] = [a_i, a_j] = 0, \quad [a_i, a_j^+] = \nabla_i \nabla_j \ln \rho$$

$$a_i |\psi\rangle = 0, \quad \langle\psi| a_i^+ = 0.$$

The Hamiltonian can be rewritten :

$$\begin{aligned} h &= a_i^+ \nu^{ij} a_j - \nu^i a_i + \frac{\partial_t \psi}{\psi} = \\ &= a_i^+ \nu^{ij} a_j - a_i^+ \nu^i - \frac{\partial_t \psi^*}{\psi^*} \end{aligned}$$

so that the equations of motion result immediately :

$$\begin{aligned} \dot{a}_i^+ + a_j^+ (\nabla_i b^j - a_k^+ \nabla_i \nu^{jk}) &= 0 \\ \dot{a}_i + (\nabla_i b^{*j} - \nabla_i \nu^{jk} a_k) a_j &= 0. \end{aligned} \quad (18)$$

But a_i^+ and a_i are just particular cases of right and left vectors, as defined by :

$$\begin{aligned} R^i &= \bar{R}^j g_j^i, \quad L^i = g_j^i \bar{L}^j \\ \bar{R}_i &= R_j g_i^j, \quad \bar{L}_i = g_i^j L_j. \end{aligned}$$

Then, as in part 2 for random scalars and vectors, one can introduce covariant time derivatives for operators which extend the ordinary time derivative, for any velocity field a :

$$\begin{aligned} \frac{D_a \varphi}{dt} &= \frac{d\varphi}{dt} = \frac{D_a^* \varphi}{dt} \\ \frac{D_a R_i}{dt} &= \frac{dR_i}{dt} + R_j [-\dot{x}^k \Gamma_{ik}^j + D_i(a^j + \mu^j)] \\ \frac{D_a R^i}{dt} &= \frac{dR^i}{dt} + R^j [\dot{x}^k \Gamma_{jk}^i - \nabla_j \nu^{kl} \Gamma_{kl}^i - \\ &\quad - D_j(a^i + \mu^i)] \\ \frac{D_a^* L_i}{dt} &= \frac{dL_i}{dt} + [-\Gamma_{ik}^j \dot{x}^k + D_i(a^j - \mu^j)] L_j \\ \frac{D_a^* L^i}{dt} &= \frac{dL^i}{dt} + [\Gamma_{jk}^i \dot{x}^k - \nabla_j \nu^{kl} \Gamma_{kl}^i - \\ &\quad - D_j(a^i - \mu^i)] L^j. \end{aligned}$$

It is straightforward to show, using the transformation properties of x^i and of functions of x^i (as in the appendix) together with the commutation rules, that these transform as scalars, right and left vectors, under time-dependent changes of variables, mixed with gauge transformations. Moreover, for operators which are functions of x^i only (then vectors are both right and left), these operatorial covariant time derivatives extend the stochastic ones :

$$\begin{aligned} \frac{D_a O}{dt} &= (\dot{x}^i - a^i - \mu^i) D_i O + D_a O \\ \frac{D_a^* O}{dt} &= D_i O (\dot{x}^i - a^i + \mu^i) + D_a^* O, \\ O &= \varphi(x), A_i(x), A^i(x). \end{aligned}$$

Then, the equations of motion (18) are obviously invariant :

$$\frac{D_{b-\mu}}{dt} [(\dot{x}^j - b^j) \nu_{ij}] = 0$$

$$\frac{D_{b^*+\mu}}{dt} [\nu_{ij}(\dot{x}^j - b^j)] = 0.$$

Finally, the operator version of the stochastic differentials :

$$\dot{x}^i = b^i(x) - 2 a_j^+ \nu^{ij}(x) = b^{*i}(x) - 2 \nu^{ij}(x) a_j$$

can be used to relate different correlation functions :

$$\langle \psi | \frac{D_{b-\mu}}{dt} [O] O' | \psi \rangle = \langle \psi | D O O' | \psi \rangle,$$

$$\langle \psi | O' \frac{D_{b^*+\mu}}{dt} [O] | \psi \rangle = \langle \psi | O' D^* O | \psi \rangle$$

$$O = \varphi(x), A_i(x), A^i(x)$$

which also remain valid when O and $D^{(*)}O$ are replaced by x^i and $b^{(*)i}$.

3.2 QUANTUM MECHANICS. — As the previous developments only rely on the transformation properties of the operators, they are applied in a straightforward way to quantum operators. In that case, the correlation functions are still given by expressions like (16), although they may not be interpreted as those of a Markov process.

The transformation rules will result from the invariance of the classical Lagrangian :

$$\mathcal{L} = \frac{1}{4} \alpha_{ij} \dot{x}^i \dot{x}^j - \frac{1}{2} \alpha_{ij} \beta^j \dot{x}^i - \mathcal{V} + \frac{dq}{dt}. \quad (19)$$

The latter defines a metrics α^{ij} , a velocity field β^i , and a scalar $\mathcal{V}$, which also transform under gauge (q) changes :

$$\alpha^{ij} = \frac{1}{f} g_k^{ii} g_l^{jj} \bar{\alpha}^{kl}$$

$$\beta^i - 2 \alpha^{ij} \nabla_j q = \frac{1}{f} [\bar{g}^i + g_j^{ii} (\bar{\beta}^j - 2 \bar{\alpha}^{jk} \bar{\nabla}_k q)]$$

$$\mathcal{V} + \frac{1}{4} \alpha_{ij} \beta^i \beta^j - \partial_t q + \nabla_i q (\alpha^{ij} \nabla_j q - \beta^i) =$$

$$= \frac{1}{f} \left[\bar{\mathcal{V}} + \frac{1}{4} \bar{\alpha}_{ij} \bar{\beta}^i \bar{\beta}^j - \partial_{\bar{t}} \bar{q} + \bar{\nabla}_i \bar{q} (\bar{\alpha}^{ij} \bar{\nabla}_j \bar{q} - \bar{\beta}^i) \right].$$

A quantum state will be determined by a probability density ρ and a scalar function S (which transforms as : $S + q = \bar{S} + \bar{q}$) :

$$\psi = \rho^{\frac{1}{2}} |\alpha|^{\frac{1}{4}} e^{\frac{i}{\hbar} S}, \quad |\alpha| = |\det(\alpha^{ij})| \quad (20)$$

obeying the Schrödinger equation :

$$\left(\partial_t - \frac{1}{4} \partial_t \ln |\alpha| \right) \psi = -\frac{i}{\hbar} H \psi$$

$$\left(\partial_t - \frac{1}{4} \partial_t \ln |\alpha| \right) \psi^* = +\frac{i}{\hbar} \psi^* H$$

$$H = \left(i \hbar \tilde{D}_i \alpha^{ij} - \frac{1}{2} \beta^j \right) \alpha_{jk} \left(i \hbar \alpha^{kl} \tilde{D}_l - \frac{1}{2} \beta^k \right) + \mathcal{V} \quad (21)$$

where $\tilde{D}_i$ is the covariant gradient relative to the metrics α (5). This equation is invariant under all the previous transformations [1]. A Heisenberg representation for operators will provide a time evolution and equations of motion which are also invariant :

$$\dot{O} = \frac{dO}{dt} = \partial_t O - \frac{i}{\hbar} [O, H] - \frac{1}{4} [\partial_t \ln |\alpha|, O].$$

This invariance can also be derived from the following considerations. Using *creation* and *annihilation* operators defined by :

$$a_i^+ = \frac{i}{2 \hbar} (\dot{x}^j - b^{+j}) \alpha_{ij}$$

$$a_i = \frac{i}{2 \hbar} \alpha_{ij} (\dot{x}^j - b^{-j})$$

$$v^i = \beta^i + 2 \alpha^{ij} \nabla_j S, \quad b^{\pm i} = v^i \pm i \hbar \frac{\nabla_j (\alpha^{ij} \rho)}{\rho}$$

so that the Hamiltonian reads :

$$H = -\hbar^2 a_i^+ \alpha^{ij} a_j - i \hbar v^i a_i$$

$$+ i \hbar \left(\frac{\partial_t \psi}{\psi} - \frac{1}{4} \partial_t \ln |\alpha| \right)$$

$$= -\hbar^2 a_i^+ \alpha^{ij} a_j - i \hbar a_i^+ v^i -$$

$$- i \hbar \left(\frac{\partial_t \psi^*}{\psi^*} - \frac{1}{4} \partial_t \ln |\alpha| \right)$$

provides the following equations of motion :

$$\dot{a}_i^+ + a_j^+ (\nabla_i b^{+j} - i \hbar a_k^+ \nabla_i \alpha^{jk}) = 0$$

$$\dot{a}_i + (\nabla_i b^{-j} - i \hbar \nabla_i \alpha^{jk} a_k) a_j = 0. \quad (22)$$

These are obviously invariant, as they express the action on the right and left vectors a_i^+ and a_i , of the following covariant time derivatives :

$$\frac{\tilde{D}_a \varphi}{dt} = \frac{d\varphi}{dt} = \frac{\tilde{D}_a^* \varphi}{dt}$$

$$\frac{\tilde{D}_a R_i}{dt} = \frac{dR_i}{dt} + R_j [-\dot{x}^k \tilde{\Gamma}_{ik}^j + \tilde{D}_i (a^j + i \hbar \tilde{\mu}^j)]$$

$$\frac{\tilde{D}_a R^i}{dt} = \frac{dR^i}{dt} + R^j [\dot{x}^k \tilde{\Gamma}_{jk}^i - i \hbar \nabla_j \alpha^{kl} \tilde{\Gamma}_{kl}^i -$$

$$- \tilde{D}_j (a^i + i \hbar \tilde{\mu}^i)]$$

$$\frac{\tilde{D}_a^* L_i}{dt} = \frac{dL_i}{dt} + [-\tilde{\Gamma}_{ik}^j \dot{x}^k + \tilde{D}_i (a^j - i \hbar \tilde{\mu}^j)] L_j$$

$$\frac{\tilde{D}_a^* L^i}{dt} = \frac{dL^i}{dt} + [\tilde{\Gamma}_{jk}^i \dot{x}^k - i \hbar \nabla_j \alpha^{kl} \tilde{\Gamma}_{kl}^i -$$

$$- \tilde{D}_j (a^i - i \hbar \tilde{\mu}^i)] L^j.$$

The latter also lead to other covariant derivatives, restricted to functions of x only :

$$\frac{\tilde{D}_a O}{dt} = (\dot{x}^i - a^i - i\hbar\tilde{\mu}^i) \tilde{D}_i O + \tilde{D}_a O, \\ O = \varphi(x), A_i(x), A^i(x)$$

$$\frac{\tilde{D}_a^* O}{dt} = \tilde{D}_i O (\dot{x}^i - a^i + i\hbar\tilde{\mu}^i) + \tilde{D}_a^* O$$

with :

$$\tilde{D}_a = D_{a+i\hbar\alpha}, \tilde{D}_a^* = D_{a-i\hbar\alpha} \text{ (and } \alpha \text{ replacing } \nu).$$

Equations (22) are rewritten :

$$\frac{\tilde{D}_{b^+ - i\hbar\tilde{\mu}}}{dt} [(\dot{x}^j - b^{+j}) \alpha_{ij}] = 0 \\ \frac{\tilde{D}_{b^* + i\hbar\tilde{\mu}}}{dt} [\alpha_{ij} (\dot{x}^j - b^{-j})] = 0. \quad (23)$$

Let us remark that the case of quantum mechanics is formally obtained from that of Markov processes by letting $\nu^{ij} = i\hbar\alpha^{ij}$ in all equivalent expressions, so that in this limit the time correlations of the random variables $x(t)$ just become the usual quantum mechanical correlations (mean values of time-ordered products or Green functions) [15].

4. Covariance of stochastic dynamics.

The stochastic representation of quantum mechanics relies on stochastic processes to give an equivalent description of quantum objects. Quantum states are replaced by stochastic processes, and the basic condition which constrains the correspondence is that both should lead to the same predictions, that is, to the same probabilities for measurable quantities. In particular, this results in a necessary identification of the probability density of the process, and of its time evolution, with those of the quantum state :

$$\rho(t) = |\psi(t)|^2$$

or else, the state fixes the probability density ρ of the process, and a velocity field v^i such that :

$$\partial_i \rho + \nabla_i (v^i \rho) = 0 \quad (24)$$

(v^i transforming like a velocity, this equation is invariant). Thus, if one restricts the processes to diffusion ones, the quantum state fixes the probability density, and, because of the Fokker-Planck equation (7), the relation between the drift and

diffusion fields :

$$b^i = v^i + \frac{\nabla_j (\nu^{ij} \rho)}{\rho} = v^i + u^i + \mu^i, \\ u^i = \nu^{ij} \nabla_j \ln (\rho |\nu|^{1/2}). \quad (25)$$

These are just kinematical constraints, and a further (dynamical) equation is still needed to complete the time evolution of the state. Moreover the correspondence is not achieved, as the diffusion field remains arbitrary.

Let us first not fix the diffusion field and consider an arbitrary diffusion process thus associated with a given quantum state. Considering that the system has dynamics which are linked to the classical Lagrangian (19), the velocity field can equally be replaced by a scalar function S such that :

$$v^i = \beta^i + 2 \alpha^{ij} \nabla_j S \quad (25')$$

let us remark that, under gauge transformations, S transforms as : $S + q = \bar{S} + \bar{q}$. Then, the covariant time derivatives defined in part 2, allow one to construct general scalar functions which will complete the description of the time evolution of ρ and S :

$$D_{v + \lambda' u + \lambda'' (\beta - v) + \lambda_o} = \frac{1}{2} [(1 + \lambda' - \lambda'') D + \\ + (1 - \lambda - \lambda'') D^* + (\lambda - \lambda') D' + 2 \lambda'' D''] \\ D'' = \frac{1}{2} (D_\beta + D_\beta^*)$$

when acting on $\ln (\rho |\nu|^{1/2})$ and S will give scalars, under time-dependent changes of coordinates. Moreover, because of their commutation property with the covariant gradient, and of definitions (25) and (25)', the gradients of these scalars will provide various time derivatives of velocities, that is, various covariant accelerations. But, if one further requires invariance under time reversal :

$$\begin{array}{lll} D \rightarrow -D^* & D' \rightarrow D' - D - D^* & \rho \rightarrow \rho \\ D^* \rightarrow -D & D'' \rightarrow -D'' & S \rightarrow -S \end{array}$$

(a general property of classical and quantum mechanics, which should also be satisfied by stochastic mechanics), this set reduces to :

$$\left[(1 - \lambda'') \frac{D + D^*}{2} + \lambda'' D'' \right] S, \\ \left[\lambda \frac{D - D^*}{2} + \lambda' \left(D' - \frac{D + D^*}{2} \right) \right] \ln (\rho |\nu|^{1/2}).$$

These scalar functions should be associated with $\mathcal{U}$ (19), in a completely invariant dynamical equation. Invariance under gauge transformations constrains the equation to the following form :

$$\left[\frac{D + D^*}{4} + \frac{D''}{2} \right] S + \left[\bar{\lambda} \frac{D - D^*}{2} + \bar{\lambda}' \left(D' - \frac{D + D^*}{2} \right) \right] \ln (\rho |\nu|^{1/2}) + \mathcal{U} + \frac{1}{4} \alpha_{ij} \beta^i \beta^j = 0 \quad (26)$$

(and S has to be fixed equal to $-q$) where $\bar{\lambda}$ and $\bar{\lambda}'$ have the dimension of an action.

Let us remark that only the last two derivative terms of this expression can, and do, depend on the diffusion field. In fact, one can replace the latter by the metrics of the Lagrangian (19), which has exactly the same transformation properties, and thus express the dynamics independently of the choice of the diffusion field. But then, the covariant time derivative thus defined is not a pure kinematical object, characterised by the diffusion process only, but also refers to dynamics, through the Lagrangian. The fact is that this last choice is precisely the one which shows the desired remarkable properties. Indeed in that case, using (25)', the kinematical equation (24) can be rewritten :

$$\left[\frac{D+D^*}{4} + \frac{D''}{2} \right] \ln(\rho|\nu|^{1/2}) + \frac{\lambda}{\hbar} \left[\frac{3}{2} \frac{(D-D^*)}{2} + \frac{1}{2} \left(D' - \frac{D+D^*}{2} \right) \right] S - \frac{1}{2} \partial_t \ln|\nu| + D_i \beta^i = 0 \quad (27)$$

with :

$$\nu^{ij} = \frac{\hbar}{\lambda} \alpha^{ij},$$

$\lambda(t)$ an arbitrary function of time and for :

$$\bar{\lambda} = -\frac{3}{8} \lambda \hbar, \quad \bar{\lambda}' = -\frac{1}{8} \lambda \hbar \quad (28)$$

the two equations (26), (28) and (27) can be decoupled :

$$\left[\frac{D+D^*}{4} + \frac{D''}{2} \pm i\lambda \left(\frac{3}{4} \frac{D-D^*}{2} + \frac{1}{4} \left(D' - \frac{D+D^*}{2} \right) \right) \right] \times \\ \times \left(S \pm i \frac{\hbar}{2} \ln(\rho|\alpha|^{1/2}) \right) + \mathcal{V} + \frac{1}{4} \alpha_{ij} \beta^i \beta^j \pm i\hbar \left(-\frac{1}{4} \partial_t \ln|\alpha| + \frac{1}{2} D_i \beta^i \right) = 0 \quad (29)$$

with :

$$\frac{D+D^*}{4} + \frac{D''}{2} \pm i\lambda \left(\frac{3}{4} \frac{D-D^*}{2} + \frac{1}{4} \left(D' - \frac{D+D^*}{2} \right) \right) = \tilde{D}^{\frac{\beta+b^+-i\hbar\mu}{2}} (+), \tilde{D}^{\frac{\beta+b^--i\hbar\mu}{2}} (-) \\ b^{\pm i} \mp i\hbar\mu^i = \beta^i + 2\alpha^{ij} \nabla_j \left(S \pm i \frac{\hbar}{2} \ln(\rho|\alpha|^{1/2}) \right)$$

and then linearised, by introducing (20), under the form of the Schrödinger equation (21).

Thus, and for simplicity, the conventional approach [8, 11] fixes the diffusion field to be proportional to the metrics of the Lagrangian, with a proportion constant which has the dimension of an action, and which is fixed in a universal way to be equal to Planck constant $\hbar$. But this convention does not evade the intriguing fixing of a kinematical object on dynamical grounds. Moreover, as can easily be seen from (29), the Schrödinger equation is equally obtained with any proportion constant [9], and even any time-dependent one [10], the parameters $\bar{\lambda}$, $\bar{\lambda}'$ adjusting in consequence. (Indeed, the same arbitrariness lies in the choice of the stochastic Lagrangian which [16], through stochastic variational methods as described for instance in [11], provides the dynamical Eq. (26).) Furthermore, taking the gradient of (26), (28), and using the commutation properties, transforms (26) into a stochastic Newton law which has all the desired invariance properties :

$$\frac{\hbar}{4} \left\{ D^{\frac{\beta+v+u}{2}} \left[\nu_{ij} \left(\frac{-\beta^j + v^j}{\lambda} - \lambda u^j \right) \right] + D^{\frac{\beta+v-u}{2}} \left[\nu_{ij} \left(\frac{-\beta^j + v^j}{\lambda} + \lambda u^j \right) \right] \right\} = -\nabla_i \left(\mathcal{V} + \frac{1}{4} \alpha_{ij} \beta^i \beta^j \right).$$

Indeed, one can even adopt a more drastic point of view, leave the diffusion field completely arbitrary, and impose (26) and the Schrödinger equation as dynamics, by letting the parameters $\bar{\lambda}$, $\bar{\lambda}'$ depend on the variables x and adjust in consequence. Of course, $\bar{\lambda}$, $\bar{\lambda}'$ will now be functions which vary with the diffusion field, the metrics, and the quantum state, but such dependences can be excluded only *a priori*. Thus, in any case, requiring both the correct dynamics, through the Schrödinger equation, and that the latter be expressed by a stochastic acceler-

ation, defined on the process by itself, necessitates a new principle, capable to relate objects of different nature and to rise the arbitrariness (if only time-dependent) which remains between the diffusion field and the metrics of the Lagrangian.

On another hand, as has already been noted [9, 10], the destructive nature of measurement (the quantum state is modified between two successive measurements) renders impossible any identification of correlations between successively measured values of the variables with the correlations of the process,

and prohibits a direct measurement of the diffusion field. Moreover, there actually exists an exact identification of the correlations of the diffusion process, with time-ordered correlations of quantum mechanics: when the diffusion field is taken to be $i\hbar\alpha^{ij}$, with α^{ij} the metrics. But then, imaginary diffusion fields make it quite difficult to endow the stochastic process with other than formal existence.

Let us end with a connection between the three previous parts. Making use of the dynamical equation satisfied by the quantum state, as (29), one can rewrite the operatorial equations of motion (23), in a form which does not depend on the state any more, but only on the fields defining the Lagrangian. Using the following property of the covariant time derivatives :

$$\begin{aligned} \frac{\tilde{D}_{a+\nabla\varphi}}{dt} [(\dot{x}^j - a^j - i\hbar\tilde{\mu}^j) \alpha_{ij} - \nabla_i \varphi] &= \\ &= \frac{\tilde{D}_a}{dt} [(\dot{x}^j - a^j - i\hbar\tilde{\mu}^j) \alpha_{ij}] - \tilde{D}_{a+\frac{\nabla\varphi}{2}} (\nabla_i \varphi) \\ \frac{\tilde{D}_{a+\nabla\varphi}^*}{dt} [\alpha_{ij}(\dot{x}^j - a^j + i\hbar\tilde{\mu}^j) - \nabla_i \varphi] &= \\ &= \frac{\tilde{D}_a^*}{dt} [\alpha_{ij}(\dot{x}^j - a^j + i\hbar\tilde{\mu}^j)] - \tilde{D}_{a+\frac{\nabla\varphi}{2}}^* (\nabla_i \varphi) \end{aligned}$$

for any velocity field a and any scalar φ , one gets :

$$\begin{aligned} \frac{1}{2} \frac{\tilde{D}_\beta}{dt} [(\dot{x}^j - \beta^j - i\hbar\tilde{\mu}^j) \alpha_{ij}] + \\ + i\hbar D_i \left(\frac{1}{2} D_j \beta^j - \frac{1}{4} \partial_t \ln |\alpha| \right) \\ = \frac{1}{2} \frac{\tilde{D}_\beta^*}{dt} [\alpha_{ij}(\dot{x}^j - \beta^j + i\hbar\tilde{\mu}^j)] - \\ - i\hbar D_i \left(\frac{1}{2} D_j \beta^j - \frac{1}{4} \partial_t \ln |\alpha| \right) \\ = - \nabla_i \left(\mathcal{V} + \frac{1}{4} \alpha_{ij} \beta^i \beta^j \right). \end{aligned}$$

5. Conclusion.

It is already puzzling that the use of stochastic processes in the representation of quantum mechanics naturally introduces, at the kinematical level, a Riemannian metrics. Usually, in classical and quantum mechanics (or field theory), such a geometric structure only appears at the dynamical level, when characterising the correct time evolution among all the possible ones. Instead, in the conventional stochastic representation of quantum mechanics (or equivalently, in the path integral approach), the classical trajectories are not replaced by a summation over any (even continuous) ones, but over a subclass selected by the metrics entering the Lagrangian.

Moreover, it appears that the geometrical structure induced is even a larger one, as changes of local

maps along the trajectory do not affect the expression of kinematics and dynamics. Thus, if as usual, quantisation introduces a necessary asymmetry between time and the degrees of freedom, stochastic processes reestablish part of the symmetry, at least that which is implicit in the Lagrangian. This property is enlarged by the time-dependent arbitrariness which still subsists in the identification of the diffusion field with the metrics of the Lagrangian, when interpreting the Schrödinger equation as a stochastic Newton law. Invariance under arbitrary changes of time parametrisation seems to be an essential character of stochastic quantisation.

Interesting developments reside in further extensions of the previous covariance. In relativistic field theory, the degrees of freedom are in fact parametrised by a set of four parameters, which span the space-time manifold. Defining diffusion processes which exhibit a natural covariance under space-time dependent changes of coordinates, by for instance defining covariant space-time derivatives, might be a decisive step in formulating a relativistic stochastic quantisation [17]. In particular, this would provide a stochastic insight into the problems of quantisation in accelerated frames and in curved space [18]. Also in the relativistic context, it has been noted [19] that the identification of the diffusion field with the metrics can be replaced by a weaker condition which, while still leading to the Schrödinger equation, allows one to relate positive definite diffusion fields with non positive definite metrics.

Finally, to promote stochastic quantisation to the same status as that of the operatorial quantisation, one would also like to treat canonical variables on the same footing. For that purpose, the definition of momentum variables on the same probability space [20], although not yet accomplished in a quite satisfactory way [21] could be a preliminary step for realising covariance under general canonical transformations [22], and maybe for rising the ambiguities of the operatorial quantisation [23].

Appendix.

The definition of the changes of coordinates (3), together with the gauge transformations, lead to the following action on the operators :

$$\begin{aligned} x^i &= g^i(\bar{t}, \bar{x}) \\ \nabla_i - \nabla_i q &= (\bar{\nabla}_j - \bar{\nabla}_j q) g_i^{j'} \\ \partial_t - \partial_t q &= \frac{1}{\dot{f}} (\partial_{\bar{t}} - \partial_{\bar{t}} q - (\nabla_i - \nabla_i q) \dot{g}^i) \quad (\text{A.1}) \end{aligned}$$

so that, using (4) :

$$\partial_t - h = \frac{1}{\dot{f}} (\partial_{\bar{t}} - \bar{h}) \quad (\text{A.2})$$

where $\bar{h}$ is defined according to (15). Now, taking for any operator o :

$$\frac{do}{dt} = \frac{1}{f} \frac{do}{d\bar{t}}$$

which amounts to say that the evolution is that of the variables x , translated to time $\bar{t}$, one has for scalars ($o = \bar{o}$)

$$\frac{do}{dt} - [\partial_t - h, o] = \frac{1}{f} \left(\frac{d\bar{o}}{d\bar{t}} - [\partial_{\bar{t}} - \bar{h}, \bar{o}] \right).$$

In particular for x^i one gets :

$$\begin{aligned} \dot{x}^i &= \frac{dx^i}{dt} = \frac{1}{f} \frac{dg^i}{d\bar{t}} = \frac{1}{f} (\partial_{\bar{t}} g^i - [\bar{h}, g^i]) \\ &= \frac{1}{f} \left(\dot{g}^i + \frac{1}{2} \dot{x}^j g_j^i + \frac{1}{2} g_j^i \dot{x}^j \right) \end{aligned}$$

in agreement with (A.1) and (17). This also implies (A.2) and :

$$\left(\frac{d}{d\bar{t}} - [\partial_{\bar{t}} - \bar{h}, \] \right) g_j^i = 0.$$

For right and left vectors, there follows invariant evolutions :

$$\begin{aligned} \frac{dR^i}{dt} - \partial_t R^i + [h, R^i] &= \\ &= \frac{1}{f} \left(\frac{d\bar{R}^i}{d\bar{t}} - \partial_{\bar{t}} \bar{R}^i + [\bar{h}, \bar{R}^i] \right) g_j^i \\ \frac{dL^i}{dt} - \partial_t L^i + [h, L^i] &= \\ &= \frac{1}{f} g_j^i \left(\frac{d\bar{L}^j}{d\bar{t}} - \partial_{\bar{t}} \bar{L}^j + [\bar{h}, \bar{L}^j] \right). \quad (A.3) \end{aligned}$$

Let us remark that (A.1) implies :

$$[\partial_t, \nabla_i] = \frac{1}{f} [\partial_{\bar{t}}, \bar{\nabla}_j] g_j^i$$

which makes consistent the relation :

$$[\partial_t, \nabla_i] = [\partial_{\bar{t}}, \bar{\nabla}_j] = 0.$$

When applied to $\dot{x}^i - b^i$ and $\dot{x}^i - b^{*i}$, (A.3) identify with the equations of motion. Obviously, the same arguments apply in the quantum case, replacing identically $\partial_t - h$ with $\partial_t + \frac{i}{\hbar} H - \frac{1}{4} \partial_t \ln |\alpha|$.

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