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The contact density for a distribution of randomly packed spheres

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Résumé. — La densité de contacts entre des sphères empilées de façon aléatoire est calculée à l'aide de résultats obtenus sur une droite.

Abstract. — The density of contacts between randomly packed spheres is calculated in terms of data given on a random line.

Information about the distribution and shapes of close-packed objects can be obtained from observations on a random line or section through the material [1]. This is useful for example in determining the grain structure of metals from electron microscope studies of two-dimensional cross-sections, in studying percolation properties of rocks and elasticity and conductivity properties of amorphous materials.

Distributions of hard spheres can be used to model percolation properties of disordered systems [6-8]. In this paper, a homogeneous distribution of spheres will be considered and the density of contacts between spheres per unit occupied volume will be found in terms of the density of « close » contacts observed along a random line through the material.

For [2] a homogeneous distribution of spheres, the number of spheres of radius between R and $R + dR$, per unit occupied volume, $n(R) dR$ can be calculated in terms of $h(\lambda) d\lambda$, the number of chords per unit length on a random line with length between λ and $\lambda + d\lambda$.

$n(R)$ is given by

$$n(R) = -\frac{1}{\pi} \frac{d}{dR} \left(\frac{h(2R)}{R} \right). \quad (1)$$

This formula is simpler than the corresponding formula for random sections which relates $n(R)$ to the density of circles on the cross section through an Abel transform [2].

In the following, the distribution $P(R_1, R_2) dR_1 dR_2$ of spheres of radii between R_1 and $R_1 + dR_1$ and between R_2 and $R_2 + dR_2$ in contact with one another per unit volume will be determined in terms of information given on a random line. The calculation

is similar to one by Pomeau [3, 4] for the case of random sections. One considers the distribution per unit length $P(\omega, d_1, d_2)$ of neighbouring chords of lengths d_1 and d_2 on the line with separation ω . For sufficiently small ω , such a situation implies that there is a contact between the two spheres close to the line. It will turn out that $P(\omega, d_1, d_2)$ does not depend on ω for small ω and that $P(R, \alpha R)$ is related to $P(\omega, 2R, 2\alpha R)$ by

$$P(R, \alpha R) = -\frac{8}{\pi} R \frac{d}{dR} \left(\frac{P(\omega, 2R, 2\alpha R)}{R} \right) \left(\frac{\alpha + 1}{\alpha} \right) \quad (2)$$

and the density of contacts between spheres per unit occupied volume, n_c is given by

$$n_c = \frac{12 Q(0)}{\pi R} \quad (3)$$

where $Q(\omega)$ is the density of « close » contacts along the line. $P(\omega, d_1, d_2)$ and $Q(\omega)$ are independent of ω as $\omega \rightarrow 0$ in contrast to the case of random sections [3] where both distributions diverge as $1/\sqrt{\omega}$ as $\omega \rightarrow 0$ and $P(R, \alpha R)$ is related to the distribution of separations and radii of intersected spheres through an Abel transform.

In figure 1, O_1 and O_2 are the centres of the two spheres of radii R_1 and R_2 respectively. The lines $O_1 O_2$ and the random line G have a common normal. $H_1 H_2$ is the projection of G by a distance l along this normal onto the plane of $O_1 O_2$.

The cross sections of the spheres on this plane are circles of radii $d'_1/2$ and $d'_2/2$. The contact at C is at a height d and angle θ above $H_1 H_2$. This implies, for small enough d and l that a close separation $\omega'_1 \omega'_2$ is observed on the line G .

From figure 1,

$$H'_1 H'_2 = H_1 H_2 = \sqrt{(R_1 + R_2)^2 - (h_1 + h_2)^2} \quad (4)$$

$$d_1/2 = \omega'_1 H'_1 = \sqrt{R_1^2 - h_1^2 - l^2} \quad (5)$$

$$d_2/2 = \omega'_2 H'_2 = \sqrt{R_2^2 - h_2^2 - l^2}. \quad (6)$$

For small d and l ,

$$\omega = \omega'_1 \omega'_2 = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \left(\frac{d^2}{\cos^3 \theta} + \frac{l^2}{\cos \theta} \right). \quad (7)$$

For a homogeneous distribution, the line of separation between the centres $O_1 O_2$ is randomly oriented with respect to G .

Therefore,

$$\begin{aligned} P(\omega, d_1, d_2) &= \int_0^\infty dl \int_0^\infty dd \int_0^{\pi/2} \sin \theta d\theta \times \\ &\quad \times \int_0^\infty dR_1 \int_0^\infty dR_2 P(R_1, R_2) \\ &\quad \delta(d_1 - 2 R_1 \cos \theta) \delta(d_2 - 2 R_2 \cos \theta) \\ &\delta \left(\omega - \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \left(\frac{d^2}{\cos^3 \theta} + \frac{l^2}{\cos \theta} \right) \right) = \\ &= \frac{\pi}{16} \frac{d_1 d_2}{d_1 + d_2} \int_1^\infty \frac{P(d_1 x/2, d_2 x/2)}{x} dx \end{aligned} \quad (8)$$

and so $P(\omega, d_1, d_2)$ is independent of ω for small ω .

For the case of equal spheres, one can set

$$P\left(\frac{d_1}{2}x, \frac{d_2}{2}x\right) = n_c \delta\left(\frac{d_1}{2}x - R\right) \delta\left(\frac{d_2}{2}x - R\right)$$

where n_c is the number of contacts per unit volume.

Then,

$$P(\omega, d_1, d_2) = n_c \delta(d_1 - d_2) \frac{\pi}{32} \frac{d_1^2}{R^2}. \quad (9)$$

Integrating d_1 and d_2 from 0 to $2R$ gives a simple

Fig. 1. — The plane of the line $O_1 O_2$ containing the projection $H_1 H_2$ of the random line G along their common normal and the plane normal to this one containing G and $H_1 H_2$. $O_1 C = R_1$, $O_2 C = R_2$, $O_1 H_1 = h_1$, $O_2 H_2 = h_2$, $H_1 H'_1 = H_2 H'_2 = R$, $H_1 \omega_1 = H_1 \omega'_1 = d'_1/2$, $H_2 \omega_2 = H_2 \omega'_2 = d'_2/2$, $H'_1 \omega'_1 = d_1/2$, $H'_2 \omega'_2 = d_2/2$.

relation between the number of close separations per unit length $Q(\omega)$ and n_c ,

$$Q(\omega) = \frac{\pi n_c}{12} R. \quad (10)$$

Equation (8) can be inverted to obtain $P(R, \alpha R)$ for a distribution of spheres of different sizes. This leads to the result stated in equation (2).

It would be interesting to extend this result to calculate correlation functions between contacts along the line for situations in which there is long range order [5]. One would also like to generalize to distributions of objects of arbitrary shape. Provided the assumption that a sufficiently close separation on the line implies a real contact close to the line, this can be obtained from a knowledge of number of chords of given length through the object and the shape of the object near the end points of the chord.

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