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# Non ergodic particle motion in a $\mathbf{C}^{\mathbf{0}}$ potential 

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#### Abstract

Résumé. - Le comportement ordonné ou chaotique des trajectoires de particules chargées dans le champ magnétique statique du réacteur thermonucléaire Astron est étudié numériquement. Malgré le fait que la fonction hamiltonienne correspondante est de classe $\mathrm{C}^{0}$, ce qui entraîne que ce système dynamique ne possède pas de véritables tores invariants dans l'espace de phase pour aucune valeur numérique $h$ de la fonction hamiltonienne, aucun signe de comportement chaotique n'a été constaté pour des valeurs de $h$ modérées et pour des intervalles de temps d'importance physique. Nous calculons une intégrale formelle du mouvement qui peut, dans certains cas, décrire - d'une manière satisfaisante le comportement ordonné des trajectoires.


#### Abstract

The ordered or chaotic behaviour of charged particle trajectories in the static magnetic field of the Astron thermonuclear reactor is numerically investigated. Despite the fact that the corresponding Hamiltonian function is of class $\mathrm{C}^{0}$, from which it follows that this dynamical system does not possess true phase space invariant tori for any numerical value $h$ of the Hamiltonian function, no sign of chaotic behaviour is detected for moderate values of $h$ and for time intervals of physical significance. A formal integral of motion is calculated that can, in certain cases, describe in a satisfactory way the ordered trajectory behaviour.


## 1. Introduction.

Since the pioneering work of Hénon and Heiles [1] the study of ordered and of chaotic trajectory behaviour in perturbed integrable dynamical systems modelling various non-linear processes has been the topic of numerous research papers. The common general result of all this work was that for a small enough perturbation strength all the trajectories of a given system were found to be of the ordered type, suggesting that the system was integrable. When the perturbation strength, however, crossed a «critical» value, domains of chaotic behaviour began to develop in phase space demonstrating its true non-integrable nature [2]. Today the ordered picture that all these systems exhibit for mild perturbations is understood through a series of theorems, known under the collective name «K.A.M. theorem » after the initials of Kolmogorov, Arnold and Moser who formulated and
proved them. This theorem concerns the qualitative behaviour of the trajectories of perturbed conservative integrable dynamical systems that are described by Hamiltonian functions of the type

$$
\begin{equation*}
H(\underline{I}, \underline{\theta})=H_{0}(\underline{I})+\varepsilon H_{1}(\underline{I}, \underline{\theta}) \tag{1}
\end{equation*}
$$

where $I$ and $\theta$ are the action-angle variables of $H_{0}$ (which therefore is integrable), $H_{1}$ is $2 \pi$-periodic in $\theta_{1}, \theta_{2}, \ldots, \theta_{n}$ and $\varepsilon$ measures the strength of the perturbation. The K.A.M. theorem then guarantees that for $\varepsilon>0$ the majority of the phase space invariant tori of $H_{0}$ (the surfaces where lie the isoenergetic trajectories of the integrable dynamical system) are not destroyed but they do survive, being only slightly deformed, provided that the value of $\varepsilon$ is « sufficiently small». Therefore the ordered behaviour of such systems for small values of $\varepsilon$ is the manifestation of the persistence of these invariant tori ([2]-[4] and references therein).

The K.A.M. theorem is proved under the following three conditions which, as we will see, are far from trivially satisfied in dynamical systems of physical interest.
a) The Hamiltonian function $H$ has to be of the form of (1) with $H_{0}$ non-degenerate, which generally means that the zero order frequencies $\left(\frac{\partial H_{0}}{\partial I_{i}}\right)$ of the system depend explicitly on the actions.
b) $H$ has to be sufficiently differentiable in $I_{i}$ and $\theta_{i}$, of class $\mathrm{C}^{m}$. The lowest necessary and sufficient value of $m$ is not known in even one case [3], but it appears that it is a function of the number of degrees of freedom of the dynamical system [5], [6]. In the special $N=2$ case, which is the most studied, Chirikov [6] has shown that $m>2$ is necessary, while Herman [7] has shown that $m>3+\varepsilon, 0<\varepsilon<1$, is sufficient. For $m=2$ there are examples of dynamical systems showing the existence of invariant tori [6] as well as counter-examples showing the opposite [8], while Sinai [9] proved that no invariant tori exist in the case of a hard sphere Boltzmann gas, whose Hamiltonian is not even differentiable.
c) The initial conditions have to be « sufficiently far » from a resonance $\left|\underline{r} \cdot \frac{\partial H_{0}}{\partial I_{i}}\right|=0$, i.e., they have to satisfy for all $\underline{r}$ the relation

$$
\left|\underline{r} \cdot \frac{\partial H_{0}}{\partial I_{i}}\right| \geqslant \gamma|\underline{r}|^{-\tau}
$$

where $\underline{r}$ is an integer $N$-vector, $\gamma$ depends on $\varepsilon$ and the non-linearity of $H_{0}$ (see condition a) above), and $\tau$ depends on $N$ and $m$.

From the above three conditions the third can be usually met by adjusting the values of the parameters and the initial conditions of the system. The degree of non-linearity and of differentiability however are inherent to each system and, if conditions a) and b) are not met, only a numerical study can reveal the existence or not of invariant tori. Many dynamical systems of physical interest, for instance, do not meet the non-degeneracy condition a); their numerical investigation, however, revealed a behaviour qualitatively similar to that of non-degenerate systems, so that in general it is believed that degenerate systems do possess invariant tori of complete measure as $\varepsilon \rightarrow 0$ (e.g. $[1,10,11]$ ) although notable exceptions to this do exist (e.g. [12]). The effect of the degeneracy to the quantitative behaviour of perturbed integrable dynamical systems is discussed in detail by Lichtenberg and Lieberman [4].

Dynamical systems of physical importance described by Hamiltonian functions not analytic or even infinitely differentiable are not a common case in the literature, and thus the differentiability condition for $H$ has been up to now mainly investigated by ad hoc constructed abstract dynamical systems (e.g. [13]).

In this paper we investigate the behaviour of the trajectories of a dynamical system of physical interest which does not meet neither the non-degeneracy nor the smoothness conditions of the K.A.M. theorem and still shows numerical evidence for the existence of K.A.M.-type invariant tori. This system describes the motion of test ions in the magnetic field of the Astron thermonuclear reactor [14]. This field is static, cylindrically symmetric and has closed magnetic lines created by external coils and by a relativistic electron layer, called the E-layer, gyrating at a distance $\rho_{i}$ from the axis of symmetry of the field (Fig. 1). The curvature of the magnetic field lines inside the E-layer is very large and becomes infinite in the limit where the E-layer is considered as a surface, which is the approximation used by Christofilos. In this case the Hamiltonian function describing the single ion motion in the magnetic field of Astron is a $\mathrm{C}^{0}$ function and therefore not smooth enough to guarantee the existence of invariant tori of complete measure for a perturbation approaching to zero [6-8]. This is, however, a purely theoretical result referring to the behaviour of the ion trajectories for $t \rightarrow \infty$ which is not necessarily of physical interest, since, due to ion-ion and electron-ion collisions, single ion trajectories have a meaning for limited time intervals only. What therefore is of interest here is not the behaviour of ion trajectories for all times, but rather their behaviour for time intervals of the order of the free flight time of an ion. As we show in this paper a numerical solution of the equations of motion shows that ion trajectories seem to lie on smooth phase space surfaces (invariant tori) for a time interval $\simeq 3 \times 10^{5} \Omega_{\mathrm{c}}^{-1}$, during which the ion has crossed more than $10^{3}$ times the E-layer, where the derivatives of $H$ are discontinuous. Therefore the low differentiability of the Hamiltonian function of this system does not induce an observable stochasticity in the trajectories due to the non existence of true invariant K.A.M. surfaces, at least for a time interval of physical significance. It should be noted that similar results on dynamical systems not satisfying the K.A.M. theorem have been recently obtained by Saitô et al., Hénon and Wisdom and Contopoulos [15].


Fig. 1. - Magnetic field lines in the Astron thermonuclear reactor. Due to the symmetry, only one quadrant of the $\rho-z$ plane is shown.

This paper is organized as follows : in section 2 we give the Hamiltonian function describing the motion of an ion in the magnetic field of the Astron thermonuclear reactor and we investigate the behaviour of ion trajectories using the surface of section technique. In section 3 we calculate a typical integral of motion in series form (a new invariant) and we show that it can explain in certain cases the apparent ordered behaviour of these trajectories. Finally in section 4 we summarize and discuss our results.

## 2. The Hamiltonian function.

The (non-relativistic) equations of motion of a charged particle with charge $q$ and mass $m$ in the magnetic field of Astron are, in cylindrical coordinates $\rho, \varphi, z$,

$$
\begin{gather*}
\frac{\mathrm{d} \rho}{\mathrm{~d} t}=\frac{\partial H}{\partial p_{\rho}} \quad \frac{\mathrm{d} \phi}{\mathrm{~d} t}=\frac{\partial H}{\partial p_{\phi}} \\
\frac{\mathrm{d} z}{\mathrm{~d} t}=\frac{\partial H}{\partial p_{z}}  \tag{2}\\
\frac{\mathrm{~d} p_{\rho}}{\mathrm{d} t}=-\frac{\partial H}{\partial \rho} \\
\frac{\mathrm{~d} p_{\phi}}{\mathrm{d} t}=-\frac{\partial H}{\partial \phi}
\end{gather*} \frac{\frac{\mathrm{~d} p_{z}}{\mathrm{~d} t}=-\frac{\partial H}{\partial z}}{} .
$$

In (2) $H$ is the Hamiltonian function given by [16]

$$
H=\frac{1}{2 m}\left[p_{\rho}^{2}+\left(\frac{p_{\phi}}{\rho}-\frac{q}{c} A_{\phi}\right)^{2}+p_{z}^{2}\right] .
$$

$p_{\rho}, p_{\phi}$ and $p_{z}$ are the canonical momenta and $A_{\rho}, A_{\phi}$, $A_{z}$ are the components of the vector potential of the magnetic field given by [14]
$A_{\rho}=A_{z} \equiv 0$

$$
\begin{align*}
A_{\phi}=-B_{0}\left[c_{1} J_{1}\left(k_{1} \rho\right)\right. & \left.+c_{2} J_{1}\left(k_{2} \rho\right)\right] \times \\
& \times \frac{\cosh (k z)}{k \cosh (k L)}-B_{0} \frac{E(\lambda, \rho)}{2 \lambda \rho} \tag{3}
\end{align*}
$$

where

$$
E(\lambda, \rho)=\begin{array}{lll}
\exp \left[-\lambda \rho^{2}\right] & \text { for } & 0<\rho<\rho_{i} \\
\exp \left[\lambda\left(\rho^{2}-\rho_{0}^{2}\right)\right] & \text { for } & \rho_{i} \leqslant \rho<\rho_{0}
\end{array} .
$$

In (3) $J_{1}$ is the Bessel function of the first kind and first order, $k=\frac{m_{1}}{\rho_{i}}\left(m_{1}=1.84 \ldots\right.$ is the first root of $J_{1}(x)=0$ ), $\rho_{i}$ is the radius of the E-layer, $\rho_{0}=2 \rho_{i}$ is the radius of the last closed magnetic surface, $2 L$ is the length of the E-layer, $B_{0}$ is the magnitude of the magnetic field at the centre $(\rho=0, z=0)$ of the reactor and $c_{1}, c_{2}, k_{1}, k_{2}$ and $\lambda$ are parameters defined from the geometry of the field [14], [17]. From (2) and (3) we see that $\phi$ is an ignorable coordinate of $H$, so that $p_{\phi}$ is a first integral of the motion. Therefore we can reduce by one the degrees of freedom of the system by writing (2) in the form

$$
\begin{align*}
H=\frac{1}{2}\left(p_{\rho}^{2}\right. & \left.+p_{z}^{2}\right)+ \\
& +\frac{1}{2}\left\{\frac{p_{\phi}}{\rho}+\frac{\cosh (k z)}{k \cosh (k L)}\left[c_{1} J_{1}\left(k_{1} \rho\right)\right.\right. \\
& \left.\left.+c_{2} J_{1}\left(k_{2} \rho\right)\right]+\frac{E(\lambda, \rho)}{2 \lambda \rho}\right\}^{2} \\
& \quad=\frac{1}{2}\left(p_{\rho}^{2}+p_{z}^{2}\right)+V(\rho, z)=h \tag{4}
\end{align*}
$$

$V(\rho, z)$ in (4) is an « effective » potential, $E(\lambda, \rho)$ is given by

$$
E(\lambda, \rho)=\begin{array}{ll}
\exp [-\lambda \rho]^{2} & \text { for } 0<\rho<1  \tag{5}\\
\exp \left[\lambda\left(\rho^{2}-2\right)\right] & \text { for } 1 \leqslant \rho \leqslant \sqrt{2}
\end{array}
$$

and $p_{\phi}$ is treated now as a parameter whose value depends on the initial conditions. Notice that in (4) and (5) we have used dimensionless variables by normalizing length to $\rho_{i}$, time to $\Omega_{\mathrm{c}}^{-1}=\left(q B_{0} / m c\right)^{-1}$ and mass to $m$. From (5) it is evident that the partial derivatives of the effective potential $V$ are in general discontinuous at $\rho=\rho_{i}$, so that $V$ is of class $\mathrm{C}^{0}$. Therefore, the existence of invariant tori cannot be guaranteed in this case by the K.A.M. theorem, and one has to resort to numerical calculations in order to investigate whether the trajectories of the Hamiltonian (4) show regular or stochastic behaviour. Before starting numerical integration of the trajectories of the Hamiltonian (4) one has to select a value for $p_{\phi}$, since $c_{1}, c_{2}, k_{1}, k_{2}$ and $\lambda$ are defined from the geometry of the magnetic field, which in this work we assume already given. The criterion we use for this selection concerns the topology of the zero velocity curves (ZVC) of (4). Recall that the ZVC of a Hamiltonian system of the form of (4) are defined through the equation

$$
\begin{align*}
V(\rho, z)=\frac{1}{2}\left[\frac{p_{\phi}}{\rho}\right. & +\frac{\cosh (k z)}{\cosh (k L) k}\left(c_{1} J_{1}\left(k_{1} \rho\right)+\right. \\
& \left.\left.+c_{2} J_{1}\left(k_{2} \rho\right)\right)+\frac{E(\lambda, \rho)}{2 \lambda \rho}\right]^{2}=h \tag{6}
\end{align*}
$$

and that trajectories starting inside a closed ZVC are restricted in this region forever. If, therefore, we want to study trajectories crossing repeatedly the E-layer, we should select a value for $p_{\phi}$ giving closed ZVC that contain the E-layer. Incidentally notice that the trajectories inside these ZVC are of physical interest as well, since they lie where the « hot » plasma is supposed to be. The value we select for $p_{\phi}$, according to the above criterion, is the one giving $V(1,0)=0$, because in this case the effective potential $V$ has an absolute minimum at $\rho=1, z=0$ and (4) possesses closed ZVC for a wide range of values of $h$, in the range $0<h<h_{\text {max }}$ (e.g. for protons and for typical values of $\rho_{i}$ and $B_{0}, h_{\text {max }}$ corresponds to 250 keV [17]). For other values of $p_{\phi}$ either the ZVC restrict the particles far from the E-layer (Fig. 3) and/or there are no ZVC for small values of $h$. Note also that for
this value of $p_{\phi}$ the derivatives $\frac{\partial V}{\partial \rho}$ and $\frac{\partial V}{\partial z}$ are continuous at $\rho=1, z=0$, while their « jump » at the rest of the E-layer $(\rho=1, z \neq 0)$ is minimal.

Using the above selected value for $p_{\phi}$ and a set of values for $c_{1}, c_{2}, k_{1}, k_{2}$ and $\lambda$ calculated in [17] we have integrated numerically many trajectories of (4) for various initial conditions and for $h=0.125 \times 10^{-5}$ which, for protons and for typical values of $\rho_{i}$ and $B_{0}$, correspond to 0.125 keV [17]. Although all the integrated trajectories have been followed for a time $t_{\mathrm{traj}} \gtrsim 10^{5} \Omega_{\mathrm{c}}^{-1}$, during which they have crossed the E-layer more than $10^{3}$ times, no clear sign of chaotic behaviour has appeared yet. Namely :
a) All trajectories do not fill all the allowable space inside the corresponding ZVC (e.g. see Fig. 2), as if they were confined by a curve more restrictive than the ZVC.
b) The consequents of the trajectories on the surface of section $\rho-p_{\rho}$ (the consecutive, both ways ( $\dot{z}>0$ and $\dot{z}<0$ ), intersections of the phase space trajectory with the surface $z=0$ ) seem to lie on smooth curves which would be the intersection of the invariant tori with this surface, should these invariant tori really exist (Figs. 4 and 5). Since for typical values of the magnetic field, plasma density and temperature


Fig. 2. - Zero velocity curve, trajectory boundary from $I_{1}^{*(3)}$ and a trajectory with initial conditions $z_{0}=p_{\rho_{0}}=0$, $\rho_{0^{\circ}}=1.02, p_{\phi}=p_{\phi_{0}}$ and $h=0.125 \times 10^{-5}$. In this and the other figures the parameter values are $c_{1}=2.387, c_{2}=1.967$, $k_{1}=5.573, k_{2}=2.50$ and $\lambda=5.25$.


Fig. 3. - Zero velocity curves for $p_{\phi}<p_{\phi_{0}}$ and a) $h=10^{-3}$, b) $h=10^{-4}$, c) $h=10^{-5}$, d) $h=0.125 \times 10^{-5}$ and a typical trajectory for case d .


Fig. 4. - Surface of section plot and invariant curves drawn using $I_{1}^{*(0)}$ (dashed line) and $I_{1}^{*(3)}$ (solid line) for the trajectory in figure 2.


Fig. 5. - Surface of section plot and invariant curves drawn using $I_{1}^{*(3)}$ for three trajectories with initial conditions $z_{0}=p_{\rho_{0}}=0, p_{\phi}=p_{\phi_{0}}, h=0.125 \times 10^{-5}$ and $\rho_{0}=1.04$, 1.06 and 1.09 .
$\left(B \approx 10^{4} \mathrm{G}, n \approx 10^{14}\right.$ and $T \approx 10 \mathrm{keV}$ ) the collision time $t_{\text {coll }}\left(\approx 10^{-3} \mathrm{~s}\right)$ is considerably shorter than $t_{\text {traj }}$ $\left(\approx 10^{-2} \mathrm{~s}\right)$ these trajectories should be considered as ordered for all practical purposes. The numerical evidence therefore indicates that for small values of $h$
the dynamical system described by (4) behaves as if possessing invariant tori of positive measure, in exact analogy with the dynamical systems satisfying the conditions of the K.A.M. theorem.

## 3. Calculation of a quasi-integral.

Since the Hamiltonian function (4) is only $\mathrm{C}^{0}$, we do not expect, according to what we have already mentioned, that it possesses true K.A.M. surfaces. On the other hand the numerical integration shows that the trajectories of this Hamiltonian seem to lie on smooth surfaces for long times compared to the basic time scale of the system. It is, therefore, of interest to investigate whether this regularity in the behaviour of the trajectories could be described in some way analytically, even if this cannot be justified by the smoothness of the Hamiltonian. This is done by constructing a formal integral of motion of the Hamiltonian (4) in series form by a standard perturbation algorithm based on Lie transformations [18]. In doing so we take into account that the form of the Hamiltonian in the inner part of the cylinder $\rho=1$ is different from that in the outer part, so that in fact we calculate two such integrals, one valid in the inner region $(\rho<1)$ and one in the outer $(1<\rho<\sqrt{2})$. Each series then is used, truncated at an order selected by a simple criterion as discussed later, to explain the properties of the trajectories in the region of its validity.

The algorithm we use to calculate the formal integral requires the Hamiltonian to be in the «regular» form

$$
\begin{aligned}
& H\left(p_{1}, p_{2}, q_{1}, q_{2}\right)=H_{0}\left(p_{1}, p_{2}\right)+ \\
& \quad+\sum_{i=1}^{\infty} \frac{\varepsilon^{i}}{i!} H_{i}\left(p_{1}, p_{2}, q_{1}, q_{2}\right)
\end{aligned}
$$

where $0<\varepsilon<1$ is a small parameter. Notice that the dynamical systems of the form (1) belong to the above class, having $H_{i} \equiv 0$ for $i>1$. In case where no intrinsic « small parameter » measuring the strength of the perturbation is present, as in our case, the splitting of $H$ in $H_{i}$ can be done by a Taylor expansion of $H$ around a stable equilibrium point. The dynamical system is then described (in the vicinity of the equilibrium point) by the Hamiltonian

$$
H_{N}=H-(H)_{0}=\sum_{i=0}^{\infty} \frac{H_{i}}{(i+2)!}=h_{n}
$$

where $H_{0}$ consists of the second, $H_{1}$ of the third and so on for the higher order terms of the Taylor series, and the role of the small parameter is essentially played by the numerical value $h_{N}$ of $H_{N}$ (e.g. see [4]). Applying the above method to the Hamiltonian (4) and recalling that $\rho=1, z=p_{\rho}=p_{z}=0$ is a stable equilibrium point of (4) for $p_{\phi}=p_{\phi_{0}}$, we find

$$
\begin{gather*}
H_{0}=\frac{1}{2} p_{x}^{2}+\frac{1}{2} p_{z}^{2}+\frac{1}{2} \omega_{1}^{2} x^{2}, \\
H_{1}=\frac{1}{3!}\left(\frac{\partial^{3} H}{\partial \rho^{3}}\right)_{0} x^{3}+\frac{1}{1!2!}\left(\frac{\partial^{3} H}{\partial \rho \partial z^{2}}\right)_{0} x z^{2} \tag{7}
\end{gather*}
$$

and similar expressions for the higher order terms, where $x=\rho-1$ and $\omega_{1}^{2}=\left(\frac{\partial^{2} H}{\partial \rho^{2}}\right)_{0}$. The trivial canonical transformation now

$$
\begin{gather*}
x=\left(2 I_{1}\right)^{1 / 2} \omega_{1}^{-1} \sin \omega_{1} \theta_{1} \\
p_{x}=\left(2 I_{1}\right)^{1 / 2} \cos \omega_{1} \theta_{1}, \quad z=z, \quad p_{z}=p_{z} \tag{8}
\end{gather*}
$$

transforms (7) to

$$
\begin{align*}
H_{0}=I_{1} & +\frac{1}{2} p_{z}^{2} \\
H_{1}= & \frac{1}{6}\left(\frac{\partial^{3} H}{\partial \rho^{3}}\right)_{0}\left(2 I_{1}\right)^{3 / 2} \omega_{1}^{-3} \sin ^{3} \omega_{1} \theta_{1}+ \\
& +\frac{1}{2}\left(\frac{\partial^{3} H}{\partial \rho \partial z^{2}}\right)_{0} z^{2}\left(2 I_{1}\right)^{1 / 2} \omega_{1}^{-1} \sin \omega_{1} \theta_{1} \tag{9}
\end{align*}
$$

where $H_{0}$ is of the form required by Deprit's algorithm. One can therefore apply directly this algorithm to construct a formal integral (e.g. see [19]). The presence of the quadratic term $p_{z}^{2}$ in $H_{0}$ however makes the calculations complex, introduces « drifting » resonances in the integral (i.e., polynomials in $p_{z}$ in the denominator) and does not contribute to the investigation of the effect of the E-layer crossings, since the latter is related to the radial part of the motion. Notice also, that $H_{0}$ is not and cannot be written in the form $H_{0}\left(I_{1}, I_{2}\right)$, as required by the K.A.M. theorem, since the zero order motion in the $z$-direction is a translation and not a libration or a rotation. Alternatively, one can try to transform $H_{0}$ in (9) to $H_{0}^{\prime}=H_{0}^{\prime}\left(I_{1}, I_{2}\right)$. A method to do this, proposed by Contopoulos and Vlahos [20], is to «create» a term $1 / 2 \omega_{2}^{2} z^{2}$ in $H_{0}$ by splitting a higher order term of $H_{1}$. By a transformation then similar to (8) $H_{0}$ in (9) becomes $H_{0}^{\prime}=I_{1}+I_{2}$ which is of the required form. In this work we follow another alternative : we consider only trajectories with a small $z$-velocity, in which case $p_{z}$ can be removed from $H_{0}$ leaving $H_{0}^{\prime}=H_{0}^{\prime}\left(I_{1}, I_{2}\right)=I_{1}$. Notice that neither $H_{0}^{\prime}=I_{1}+I_{2}$ nor $H_{0}^{\prime}=I_{1}$ do satisfy the non-degeneracy condition of the K.A.M. theorem. This alone, however, does not necessarily imply a generic chaotic behaviour, as most degenerate two degrees of freedom systems show the existence of K.A.M.-type invariant tori (e.g. the famous Hénon-Heiles system [1]; see also [21]). The removal of $p_{z}$ from $H_{0}$ is formally done through the canonical transformation [22]

$$
\begin{gathered}
x=\varepsilon^{3} x^{\prime}, \quad p_{x}=\varepsilon^{3} p_{x}^{\prime}, \quad z=\varepsilon^{2} z^{\prime} \\
p_{z}=\varepsilon^{4} p_{z}^{\prime}, \quad H=\varepsilon^{6} H^{\prime}
\end{gathered}
$$

where $\varepsilon$ is an ordering parameter used to rearrange the terms in $H_{0}$ and $H_{1}$; once the new integral $I_{i}^{*}=\sum_{i=1}^{\infty} \varepsilon^{i} I_{1 i}^{*} / i!$ has been calculated up to a desirable order, $\varepsilon$ is set equal to one. Notice that this method is similar to the method of the adiabatic ordering described in [4].

The complexity of the calculations, which increases rapidly with the order, restricted us in this work to calculate only the terms up to order 6 in $\varepsilon$, which are given in the Appendix. A partial sum

$$
I_{1}^{*(n)}=\sum_{i=0}^{n} \frac{I_{1 i}^{*}}{i!}, \quad n \leqslant 6
$$

can then be used as a true integral if it remains reasonably constant along a trajectory, something that is checked in figure 6. In this figure we see that, aside from the fact that all $I_{1}^{*(n)}$ are wildly oscillating for large $z$, which can be easily interpreted as due to the poor approximation of cosh $(k z)$ by the truncated Taylor series of $H, I_{1}^{*(3)}$ is far better conserved than either $I_{1}^{*(0)}$ or $I_{1}^{*(6)}$. Since, due to the limited smoothness of the Hamiltonian function, we do not really expect $I_{1}^{*}$ to be convergent, this behaviour could be interpreted as the manifestation of the true asymptotic nature of $I_{1}^{*}$. Alternatively one could argue that the divergence originating in the limited smoothness would appear at a higher order, in which case the divergence at $n=3$ is due to the method of calculation of $I_{1}^{*}$. To clear this point we decided to calculate another typical integral of (4), this time following the « regularization process » of Contopoulos and Vlahos [20]. The regular form of

$$
H_{0}^{\prime}=\frac{1}{2}\left(p_{x}^{2}+\omega_{1}^{2} x^{2}\right)+\frac{1}{2}\left(p_{z}^{2}+\omega_{2}^{2} z^{2}\right)
$$

in this case enabled us to use a computer program [23] to calculate the terms of this integral up to order 12 in the variables $x, z, p_{x}, p_{z}$. The partial sums $\Phi^{(n)}$


Fig. 6. - Plot of the partial sums $I_{1}^{*(0)}$ (dashed line), $I_{1}^{*(3)}$ (solid line) and $I_{1}^{*(6)}$ (dotted line) versus $z$ for the trajectory in figure 2.
however showed the same divergence as the partial sums $I_{1}^{*(n)}$ at the corresponding to $\varepsilon^{3}$ order. Accordingly we decided to select $I_{1}^{*(3)}$ as the best choice and use it as a true integral of motion of the Hamiltonian (4) to calculate :
a) trajectory boundaries (by setting $p_{x}=p_{z}=0$ into $I_{1}^{*(3)}$; note that the curves calculated in this way are not envelopes but an equivalent of the ZVC, where $I_{1}^{*(3)}$ is used in place of $H$ ).
b) invariant curves (by eliminating $p_{z}$ between $H$ and $I_{1}^{*(3)}$ and then by taking the section of the resulting surface with the plane $z=0$ ).

The results are given in figures 2, 4 and 5. In figure 2 we observe that the analytically calculated boundary describes in a satisfactory way the limits of the corresponding, numerically calculated, trajectory for not very large values of $z$. This, however, should be expected since it follows directly from the decoupling at zero order of the radial and the longitudinal motions in places where $\cosh (k z) / \cosh (k L) \ll 1$. The agreement between the analytically (from $I_{1}^{*(3)}$ ) calculated invariant curves and the numerically calculated consequents of the corresponding trajectories is also very good (Figs. 4 and 5). In particular we see that $I_{1}^{*(3)}$ describes the « experimental» points better than $I_{1}^{*(0)}$ (Fig. 4) which shows that it contains « more » information. Notice, however, that, although the consequents of all trajectories on the surface of section seem to lie on smooth curves with no signs of chaotic behaviour, only the consequents of trajectories with small $x$ initial are well described by the invariant curves calculated through $I_{1}^{*(3)}$ (Fig. 5). This can be considered as an indication that perhaps a formal integral constructed by a different algorithm could describe the behaviour of the trajectories of the Hamiltonian (4) better than $I_{1}^{*(3)}$ does. Such an integral, however, has not been found in the present work.

## 4. Summary.

In this paper we studied the behaviour of single particle trajectories in the cylindrically symmetric magnetic field of the Astron thermonuclear reactor. This field is discontinuous at a cylindrical surface of radius $\rho=\rho_{i}$ where a layer of gyrating relativistic electrons, the so-called E-layer, causes the magnetic field lines to close. Due to this discontinuity the Hamiltonian function describing the single particle motion has discontinuous partial derivatives at $\rho=\rho_{i}$. The limited smoothness of the Hamiltonian and the resulting non-applicability of the K.A.M. theorem lead one to believe that the majority (in the measure sense) of the trajectories crossing the E-layer is of the chaotic type, i.e., they are not quasi-periodic on invariant tori. A numerical investigation of the trajectory behaviour however shows that this is not true since the intersections of all the integrated trajectories with the surface of section $\rho-p_{\rho}$ were found to lie on smooth invariant curves, at least for time
intervals of physical significance (an order of magnitude longer than the mean ion collision time).

The existence of true invariant curves in a two degrees of freedom dynamical system is a necessary condition for the existence of two first integrals of motion. In the present case this implies that, besides the Hamiltonian function, there exists a quasi-integral, valid for time intervals less than or equal to the integration time $t_{\text {traj. }}$. Motivated by this implication we calculated by a Lie transform algorithm a second typical integral of motion, which however showed a divergent behaviour at a low order. This behaviour could originate either in the low differentiability of the Hamiltonian function or in the method of the calculation. In any case however it is of interest to note that a partial sum $I_{1}^{*(3)}$ of this typical integral, truncated where the signs of divergence appear, describes in a satisfactory way the observed properties of the trajectories. In particular the invariant curves calculated from $I_{1}^{*(3)}$ agree with the consecutive points of intersection of the trajectories with
the surface of section $\rho-p_{\rho}$, at least for trajectories with initial condition $\rho_{0}$ near 1 , while the orbits in configuration space lie inside the level curves of $I_{1}^{*(3)}$ (calculated from the relation $\left(I_{1}^{*(3)}\right)_{p_{x}=p_{x}=0}=c$ ).

The main result, therefore, of the present work can be summarised as follows : a near integrable dynamical system with two degrees of freedom may not possess chaotic orbits of observable measure for small perturbation values, even if it does not satisfy the smoothness conditions of the K.A.M. theorem (as they are modified by Herman) and this for time intervals longer than any physically meaningful time scale of the system. This ordered behaviour can be described, at least in some cases, in an approximate way by a formal integral of motion truncated at an appropriate order.

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## Appendix

Here we give the seven first terms of the formal integral $I_{1}^{*}=\sum_{i} \frac{1}{i!} I_{1 i}^{*}$ calculated in section 3 of this paper. They are

$$
\begin{aligned}
& I_{10}^{*}=I_{1} \quad I_{11}^{*}=\frac{\varepsilon_{1} z^{2}\left(2 I_{1}\right)^{1 / 2}}{2 \omega_{1}} \sin \omega_{1} \theta_{1} \quad I_{12}^{*}=\frac{\varepsilon_{1}^{2} z^{4}}{4 \omega_{1}^{2}} \\
& I_{13}^{*}=\frac{3 \varepsilon_{3}\left(2 I_{1}\right)^{3 / 2}}{4 \omega_{1}^{3}}\left(3 \sin \omega_{1} \theta_{1}-\sin 3 \omega_{1} \theta_{1}\right)+\frac{6 \varepsilon_{1} z p_{z}\left(2 I_{1}\right)^{1 / 2}}{\omega_{1}^{2}} \cos \omega_{1} \theta_{1} \\
& I_{14}^{*}=-\frac{3 z^{2}\left(2 I_{1}\right)}{\omega_{1}^{2}}\left(2 \varepsilon_{4}+\frac{2 \varepsilon_{1}^{2}}{\omega_{1}^{2}}\right) \cos 2 \omega_{1} \theta_{1}+\frac{3 z^{2}\left(2 I_{1}\right)}{\omega_{1}^{4}}\left(3 \varepsilon_{1} \varepsilon_{3}-4 \varepsilon_{1}^{2}\right) \\
& I_{15}^{*}=-\frac{120 \varepsilon_{1}\left(2 I_{1}\right)^{1 / 2} p_{z}^{2}}{\omega_{1}^{3}} \sin \omega_{1} \theta_{1}+\frac{3 z^{4}\left(2 I_{1}\right)^{1 / 2}}{\omega_{1}}\left(20 \varepsilon_{5}-\frac{10 \varepsilon_{1} \varepsilon_{4}}{\omega_{1}^{2}}+\right. \\
& \\
& \left.+\frac{15 \varepsilon_{1}^{2} \varepsilon_{3}}{\omega_{1}^{4}}+\frac{80 \varepsilon_{1} \varepsilon_{2}}{\omega_{1}^{2}}-\frac{21 \varepsilon_{1}^{3}}{\omega_{1}^{4}}\right) \sin \omega_{1} \theta_{1} \\
& I_{16}^{*}=\frac{180 I_{1}^{2} \varepsilon_{6}}{\omega_{1}^{4}}\left(\cos 4 \omega_{1} \theta_{1}-\cos 2 \omega_{1} \theta_{1}\right)+\frac{30 I_{1} z p_{z}}{\omega_{1}^{3}}\left(12 \varepsilon_{4}-\frac{15 \varepsilon_{1} \varepsilon_{3}}{\omega_{1}^{2}}-\frac{100 \varepsilon_{1}^{2}}{\omega_{1}^{2}}\right) \sin 2 \omega_{1} \theta_{1}+\frac{675 \varepsilon_{3}^{2} I_{1}^{2}}{\omega_{1}^{6}} \\
& +\frac{3 \varepsilon_{1} z^{6}}{\omega_{1}^{2}}\left(20 \varepsilon_{5}-\frac{7 \varepsilon_{1} \varepsilon_{4}}{\omega_{1}^{2}}+\frac{39 \varepsilon_{1}^{2} \varepsilon_{3}}{4 \omega_{1}^{4}}+\frac{80 \varepsilon_{1} \varepsilon_{2}}{\omega_{1}^{2}}-\frac{15 \varepsilon_{1}^{3}}{\omega_{1}^{4}}\right)
\end{aligned}
$$

where by $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \varepsilon_{4}, \varepsilon_{5}$ and $\varepsilon_{6}$ we denote respectively the coefficients of the terms $x \dot{z}^{2}, x^{3}, z^{4}, x^{2} z^{2}, x z^{4}$ and $x^{4}$ of the Taylor expansion of the Hamiltonian (4) around the stable equilibrium point $x=z=p_{x}=p_{z}=0$.

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