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3D solutions of non-linear evolution equations with diffusion

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Résumé. — On développe une nouvelle technique en vue de résoudre pour la première fois des équations d'évolution non linéaires avec diffusion. Les résultats, qui sont obtenus au moyen d'une fonction génératrice et d'un opérateur associé, sont énoncés sous une forme générale qui se rapporte explicitement au cas sphérique avec une double déformation elliptique. Des solutions pour un plus petit nombre de dimensions ou une plus grande symétrie peuvent être directement obtenues comme cas particuliers de cette forme générale. Les solutions, qui correspondent à des conditions initiales physiquement réalistes, sont exactes dans les cas cylindrique et sphérique à symétrie radiale, ainsi que dans le cas plan, alors qu'elles sont valables asymptotiquement pour de longues durées dans les cas elliptiquement déformés. On discute la pertinence des résultats pour la description de plasmas diffusifs avec recombinaison.

Abstract. — A new technique is developed to solve for the first time non-linear evolution equations with diffusion. The results, which are obtained by means of a generating function and an associated operator, are expressed in a general form which explicitly refers to a double elliptically deformed spherical case. Solutions to cases of lower dimensionality or of higher symmetry can be directly found as special cases of this general form. The solutions, which correspond to physically realistic initial conditions are exact in the radially symmetric spherical and cylindrical cases, as well as in the plane case, whereas in elliptically deformed cases the solutions are valid asymptotically for long times. The relevance of the results for description of recombining diffusive plasmas is discussed.

1. Introduction.

It is well known that non-linear partial differential equations often lead to formidable difficulties when attempts are made to find general solutions. Even if sometimes general methods are applicable, such as the inverse scattering technique, the form of the solutions thus obtained is often so complicated that the physics of the problem is hidden and the interpretation of the formal results is not evident.

For purpose of predicting or interpreting physical effects, e.g. for a plasma in laboratory or in space it is essential to have available theoretical results which are solutions to model equations for particular initial conditions which are relevant for the physical situation.

It is the purpose of the present paper to develop a new technique which allows particular solutions of non-linear evolution equations to be obtained, taking into account also effect of diffusion. The technique consists in determining a particular form of generating function and to find an operator, which generates the solution of the equation when operating on constants of the generating function. In essence the problem amounts to finding an operator which is common to all terms of the equation and which can, accordingly, be factored out, leaving a linear operator equation to be solved. The procedure requires that commutation relations are fulfilled for space and time derivatives on the one hand, and derivatives with respect to constants of the generating function on the other hand. This can be true for any dimension in space with the choice of generating function that is here made.

The solutions which we obtain correspond to certain specific initial space distributions of e.g. an electron density. The half-width of this distribution is related to the maximum value. There exists a characteristic time (the escape time) such that for time longer than the characteristic time the solutions become asymptotically independent of the maximum initial value. This particular feature of the solutions provides extended domains of applicability of the results to physical systems.

The phenomena of physical nature which the equation is intended to describe are particle recombination and diffusion in a plasma. The recombination term is of quadratic form. For the term describing diffusion we choose for simplicity a constant diffusion coefficient.

2. Solutions in the double elliptically deformed spherical case.

We here study the case where we allow for more general deformations, namely double elliptically deformed spheroids, of which prolate and oblate ellipsoids are special cases.

The evolution equation for anisotropic diffusion can be written as

$$\frac{\partial n}{\partial t} = -\alpha n^2 + D_x \frac{\partial^2 n}{\partial x^2} + D_y \frac{\partial^2 n}{\partial y^2} + D_z \frac{\partial^2 n}{\partial z^2}. \quad (1)$$

We choose as a generating function

$$\Lambda = \frac{1}{x^2 + py^2 + sz^2 + q(t + t_0)} \quad (2)$$

where p, s and q are constants.

We introduce an operator P, such that the solution can be written

$$n = P\Lambda \quad (3)$$

where we assume that P acts on p, q and s in (2).

We have

$$\frac{\partial^2}{\partial x^2}\Lambda = 6\Lambda^2 - 8[py^2 + sz^2 + q(t + t_0)]\Lambda^3$$

$$= 2\left(3 + 2p\frac{\partial}{\partial p} + 2s\frac{\partial}{\partial s} + 2q\frac{\partial}{\partial q}\right)\Lambda^2 \quad (4)$$

$$\frac{\partial^2}{\partial y^2}\Lambda = -2p\Lambda^2 + 8p^2 y^2 \Lambda^3 = 2p\left(1 + 2p\frac{\partial}{\partial p}\right)\Lambda^2 \quad (5)$$

and similarly

$$\frac{\partial^2}{\partial z^2}\Lambda = -2s\left(1 - 2s\frac{\partial}{\partial s}\right)\Lambda^2. \quad (6)$$

We choose the following form for the operator P, namely

$$P\left(q, p, s, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}\, \frac{\partial}{\partial s}\right) = A + B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p} + D\frac{\partial}{\partial s} \quad (7)$$

where A, B, C and D are constants.

In order to treat the n^2 term in (1) we introduce the operator Q, which fulfills

$$PQ\Lambda^2 = (P\Lambda)^2 \quad (8)$$

and which can be written

$$Q\left(q, p, s, \frac{\partial}{\partial q}, \frac{\partial}{\partial p}, \frac{\partial}{\partial s}\right) = A_1 + B_1\frac{\partial}{\partial q} + C_1\frac{\partial}{\partial p} + D_1\frac{\partial}{\partial s} \quad (9)$$

where A_1, B_1, C_1 and D_1 are constants.

The operators P and Q each commute with $\partial/\partial t$ and $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$, where the derivatives refer to the form (1), which enables us to substitute the form (3) into (1) and to perform the time and space derivatives directly on (2).

With the relation (8), the equation (1) can be transformed to such a form that each term contains the operator P to the left. It can thus formally be factored out from the whole equation. A *linear* equation remains for Q, namely

$$-q = -\alpha Q + 2D_x\left(3 + 2p\frac{\partial}{\partial p} + 2s\frac{\partial}{\partial s} + 2q\frac{\partial}{\partial q}\right) -$$

$$-2D_y p\left(1 + 2p\frac{\partial}{\partial p}\right) - 2D_z s\left(1 + 2s\frac{s}{\partial s}\right) \quad (10)$$

or

$$Q = \frac{q}{\alpha} + \frac{2D_x}{\alpha}\left(3 + 2p\frac{\partial}{\partial p} + 2s\frac{\partial}{\partial s} + 2q\frac{\partial}{\partial q}\right) -$$

$$-\frac{2D_y}{\alpha}p\left(1 + 2p\frac{\partial}{\partial p}\right) - \frac{2D_z}{\alpha}s\left(1 + 2s\frac{\partial}{\partial s}\right). \quad (11)$$

In (9) we have from (11)

$$A_1 = \frac{1}{\alpha}(6D_x + q - 2D_y p - 2D_z s) \quad (12)$$

$$B_1 = \frac{4D_x}{\alpha}q \quad (13)$$

$$C_1 = \frac{4}{\alpha}p(D_x - pD_y) \quad (14)$$

$$D_1 = \frac{4}{\alpha}s(D_x - sD_z). \quad (15)$$

Introducing (7) and (9) into (8) and identifying each type of term we obtain :

(i) For the Λ^2 terms :

$$A^2 = AA_1 + B\,\partial A_1/\partial q + C\,\partial A_1/\partial p + D\,\partial A_1/\partial s. \quad (16)$$

(ii) For the $(t + t_0)^2\Lambda^4$ terms :

$$B = 6B_1. \quad (17)$$

(iii) For the $(t + t_0)\Lambda^3$ terms :

$$AB = AB_1 + BA_1 + B\,\partial B_1/\partial q$$

or

$$A\left(1 - \frac{B_1}{B}\right) = A_1 + \partial B_1/\partial q. \quad (18)$$

(iv) For the $y^2\Lambda^3$ terms :

$$AC = AC_1 + CA_1 + C\,\partial C_1/\partial p \quad (19)$$

or

$$A\left(1 - \frac{C_1}{C}\right) = A_1 + \partial C_1/\partial p. \quad (20)$$

(v) For the $(t + t_0) y^2 A^4$ terms :

$$BC = 3\, CB_1 + 3\, BC_1 . \tag{21}$$

(vi) For the $y^4 A^4$ terms :

$$C = 6\, C_1 . \tag{22}$$

(vii) For the $z^2 A^3$ terms :

$$AD = AD_1 + DA_1 + D\, \partial D_1/\partial s \tag{23}$$

or

$$A\left(1 - \frac{D_1}{D}\right) = A_1 + \partial D_1/\partial s . \tag{24}$$

(viii) For the $z^4 A^4$ terms :

$$D = 6\, D_1 . \tag{25}$$

(ix) For the $(t + t_0) z^2 A^4$ terms :

$$BD = 3\, DB_1 + 3\, BD_1 . \tag{26}$$

(x) For the $z^2 y^2 A^4$ terms :

$$CD = 3\, CD_1 + 3\, DC_1 . \tag{27}$$

The relations (21), (26) and (27) are automatically satisfied because of (17), (22) and (25).

From (12)-(15) we have

$$\frac{\partial A_1}{\partial q} = \frac{1}{\alpha} \tag{28}$$

$$\frac{\partial B_1}{\partial q} = \frac{4\, D_x}{\alpha} \tag{29}$$

$$\frac{\partial A_1}{\partial p} = -\frac{2\, D_y}{\alpha} \tag{30}$$

$$\frac{\partial C_1}{\partial p} = \frac{4}{\alpha}(D_x - 2\, pD_y) \tag{31}$$

$$\frac{\partial D_1}{\partial s} = \frac{4}{\alpha}(D_x - 2\, sD_z) . \tag{32}$$

From (13) and (17)

$$B = \frac{24\, D_x}{\alpha}\, q . \tag{33}$$

From (14) and (22)

$$C = \frac{24}{\alpha}\, p(D_x - pD_y) . \tag{34}$$

From (15) and (25)

$$D = \frac{24}{\alpha}\, s(D_x - sD_z) . \tag{35}$$

From (18) we find by using (17) and (29)

$$\frac{5}{6} A = \frac{1}{\alpha}(10\, D_x + q - 2\, D_y\, p - 2\, D_z\, s) \tag{36}$$

whereas from (20), (22) and (32),

$$\frac{5}{6} A = \frac{1}{\alpha}(10\, D_x + q - 10\, D_y\, p - 2\, D_z\, s) \tag{37}$$

and from (24), (25) and (32)

$$\frac{5}{6} A = \frac{1}{\alpha}(10\, D_x + q - 2\, D_y\, p - 10\, D_z\, s) . \tag{38}$$

Relations (36), (37) and (38) are not fulfilled simultaneously except for $D_y\, p \equiv D_z\, s \equiv 0$.

The relations (37) and (38) can, however, as it follows from the detailed balance in the identification of terms, be neglected if

$$\frac{qD_x(t + t_0)}{p(D_x - pD_y)} \gg y^2 \tag{39}$$

and

$$\frac{qD_x(t + t_0)}{s(D_x - sD_z)} \gg z^2 \tag{40}$$

where we notice that in general

$$\frac{q}{p(D_x - pD_y)} \gg 1 \tag{41}$$

and

$$\frac{q}{s(D_x - sD_z)} \gg 1 . \tag{42}$$

Therefore it suffices that

$$D_x(t + t_0) \gtrsim y^2, z^2 . \tag{43}$$

In the limits of $p = 0$, $s = 0$ the equations (36)-(38) become identical and the solutions are exact. When $D_x - pD_y = 0$, $D_x - sD_z = 0$ the relations (37) and (38) are strictly negligible since the factors $(D_x - pD_y)$ and $(D_x - sD_z)$ enter the terms in the balance equations, which lead to the relations (37) and (38). In intermediate cases the solutions are valid in an asymptotic sense (43).

It is convenient to introduce

$$\varepsilon_p = \frac{D_x - pD_y}{D_x} \tag{44}$$

$$\varepsilon_s = \frac{D_x - sD_y}{D_x} \tag{45}$$

and

$$\delta_p \doteq p\frac{D_y}{D_x} \tag{46}$$

$$\delta_s = s\frac{D_z}{D_x} . \tag{47}$$

By using the form (36), in accordance with the above discussion, i.e.

$$A = \frac{6}{5\,\alpha}(10\,D_x + q - 2\,D_y p - 2\,D_z s)\,. \quad (48)$$

We obtain from relation (16) the following relation for q, namely

$$q = \{\,34 - 2\,\varepsilon_p - 2\,\varepsilon_s + \\ + 10\sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}\,\}\,D_x\,. \quad (49)$$

Inserting the expression (49) into (48) we have

$$A = \frac{12\,D_x}{\alpha}\{\,4 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}\,\} \quad (50)$$

and from (33)

$$B = \frac{48\,D_x^2}{\alpha} \times \\ \times \{\,17 - \varepsilon_p - \varepsilon_s + 5\sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}\,\}\,. \quad (51)$$

From (34) and (35) we write

$$C = \frac{24\,D_x}{\alpha}p\varepsilon_p = \frac{24\,D_x^2}{\alpha D_y}\delta_p\,\varepsilon_p \quad (52)$$

and

$$D = \frac{24\,D_x}{\alpha}s\varepsilon_s = \frac{24\,D_x^2}{\alpha D_z}\delta_s\,\varepsilon_s\,. \quad (53)$$

All coefficients in the operator P, (7), are determined by expressions (50)-(53).

For the characteristic time we have

$$t_0 = \frac{6}{\alpha n_0(0,0,0)}\,\frac{2 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}}{17 - \varepsilon_p - \varepsilon_s + 5\sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}}\,. \quad (54)$$

The solution of the double elliptically deformed spherical case is :

$$n(x,y,z,t) = \frac{n_0(0,0,0)}{2 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 - \delta_s)]}} \times \left\{\frac{4 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}}{N} - \frac{2\,M}{N^2}\right\} \quad (55)$$

where

$$N = 1 + \frac{t}{t_0} + \left(\frac{x^2}{D_x} + \frac{\delta_p y^2}{D_y} + \frac{\delta_s z^2}{D_z}\right)\frac{\alpha}{12}\,\frac{n_0(0,0,0)}{2 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}} \quad (56)$$

and

$$M = 1 + \frac{t}{t_0} + \left(\varepsilon_p\frac{\delta_p y^2}{D_y} + \varepsilon_s\frac{\delta_s z^2}{D_z}\right)\frac{\alpha}{12}\,\frac{n_0(0,0,0)}{2 + \sqrt{10 - 2[\varepsilon_p(1 + \delta_p) + \varepsilon_s(1 + \delta_s)]}}\,. \quad (57)$$

In (54)-(57) the deformation parameters ε_p, ε_s, δ_p, δ_s correspond to free choice of values of p and s. This is a consequence of omitting relations (37) and (38). In accordance with the above discussion the solution for the double elliptically deformed spherical case is valid in an asymptotic sense for large time (43), whereas the solution is exact in the limiting cases where

(i) $\delta_p = \delta_s = 0$, $\varepsilon_p = \varepsilon_s = 1$ (plane case);
(ii) $\delta_p = 1$, $\delta_s = 0$, $\varepsilon_p = 0$, $\varepsilon_s = 1$ (radially symmetric cylindrical case);
(iii) $\delta_p = \delta_s = 1$, $\varepsilon_p = \varepsilon_s = 0$ (radially symmetric spherical case).

From the three-dimensional and in the general form radially non-symmetric solution (54)-(57) we thus obtain exact solutions in all corresponding cases of lower dimensionality and higher symmetry.

3. Concluding remarks.

We have given exact particular solutions of the non-linear evolution equation with diffusion in the plane, cylindrical and spherical cases. In the generalized cases of elliptically deformed geometries the solutions are valid asymptotically for large time but join the exact solutions for the radially symmetric cases, or the plane case, in the limits of vanishing or infinitely increasing deformations. The reason why the results in deformed cases are valid only asymptotically for long times can be traced back to the fact that the process of identification of terms leads to several conditions which cannot all be fulfilled simultaneously, whereas asymptotically only one of these conditions survives. In the radially symmetric cases the conditions become identical and the solutions become valid for all times, as in the plane case. The solutions obtained for the double elliptically deformed case provide asymptotic solutions with density surfaces of prolate or oblate ellipsoidal shapes. By specific choices of the deformations, which are described by free parameters, the solutions of all the separate cases of lower dimensionality can be obtained from the double elliptically deformed case, which may be regarded as a dimensionally unifying solution.

The solutions here obtained can be used as a basis for consideration of effects of perturbations of various types.

In the non-linear evolution equation (1) we have here assumed that the sign of the non-linear term is such that the term contributes to a decrease of the rate of change of the field variable, e.g. the density. For the case with opposite sign of the non-linear term ($\alpha < 0$), the explosive case, the evolution of an initial distribution in space exhibits interesting features, even in the absence of diffusion. This case has recently been subject to a separate investigation [1], which is complementary to the present paper [cf. 2]. Studies related to the present work, and considering coefficients of ionization, recombination and diffusion which change in time have also recently been carried our [3, 4].

Equations of the type here studied have a considerable interest for physical applications in various contexts. In certain applications the recombination plays a major rôle, whereas the diffusion is small, even negligible, in others, e.g. for certain problems of ionospheric physics. For fusion plasmas diffusion is important, and should be described by means of some model for the dependence of the diffusion coefficient on the plasma parameters, density and temperature. In the outer layer of a fusion plasma, where the temperature is low and the plasma partially ionized, recombination effects are, however, in most cases not important for the particle balance. There are, on the other hand, other examples from fusion plasma physics, where terms of the recombination type enter the description in an essential way. Such an example is the equation used to describe the rate of increase of the ion density for neutral injection in a mirror machine, where a term, which is formally identical to a recombination type term, enters the balance equation for ions to describe the ion loss due to transport into the loss cone *via* Coulomb collisions [5]. Certain parts of the evolution equations are thus found to be more essential than others for each particular application. This is not only the case in plasma physics, but also in other fields, where non-linear evolution equations enter the description, e.g. fission and fusion reactor physics or areas of chemistry and population biology.

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